The cohomology of period domains for reductive groups over local fields

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Abstract

We compute the étale cohomology of period domains over local fields for quasisplit reductive groups. The period domains, which were introduced by Rapoport and Zink in [RZ], are open admissible rigid-analytic subsets of generalized flag varieties. They parametrize (weakly) admissible filtrations of a given isocrystal with additional structure of a reductive group.

Introduction

The goal of this paper is to give an explicit formula for the étale cohomology of period domains over local fields for quasisplit reductive groups. The concept of period domains was introduced by Griffiths [G]. They are certain open subsets of generalized flag varieties over the field of complex numbers. They parametrize polarized $\mathbb{R}$-Hodge structures of a given type. In the p-adic setting the notion of a period domain exists as well, representing an analog of the classical case. At this the notion of a weakly admissible isocrystal is an important ingredient, which was established by Fontaine [Fo]. In order to explain this idea, let L be a perfect field of characteristic p. Denote by $W(L)$ the associated Witt ring with quotient field $K_0 = K_0(L)$. A filtered isocrystal $(V, \Phi, \mathcal{F})$ is an isocrystal (V, Φ) over L together with a $\mathbb{Z}$-filtration of the vector space V defined over a field extension K of K_0. The isocrystal $(V, \Phi, \mathcal{F})$ is called weakly admissible, if for any subisocrystal $(V', \Phi', \mathcal{F}')$ we have

$$\sum_i i \dim gr^i_{\mathcal{F}'}(V' \otimes_{K_0} K) \leq \mathrm{ord}_p \det(\Phi')$$

and equality for $(V', \Phi', \mathcal{F}')$=$(V, \Phi, \mathcal{F})$. Fix an isocrystal (V, Φ) over L. Considering only filtrations of a specified type on V, the weakly admissible ones yield a rigid-analytic space $\mathcal{F}^{wa}$ over K_0. This space is an open rigid analytic submanifold of a generalized flag manifold $\mathcal{F}^{rig}$. It is called the period domain of that specified data. Applying the machinery of Tannaka formalism yields an extension of the theory above - the $GL(V)$-case - to arbitrary reductive groups $G/\mathbb{Q}_p$. For a detailled description see [RZ]. The most prominent example of a period domain is the so-called Drinfeld upper half plane $\Omega(V)$ of dimension $\dim V - 1$. This space is just the complement of all K_0-rational hyperplanes in the standard projective space $\mathbb{P}(V)$, i.e.

$$\Omega(V) = \mathbb{P}(V) \setminus \bigcup_{H \subsetneq V} \mathbb{P}(H).$$

This example is discussed in the paper [SS] by Schneider and Stuhler.

A natural problem which arises with period domains is the determination of their cohomology, in this case the étale cohomology with compact support. These cohomology groups are equipped in a natural way with actions of $J(\mathbb{Q}_p)$ and the Galois group $Gal(\overline{E_s}/E_s)$. Here J denotes the isomorphism group of the given isocrystal, which is an inner form of some Levi subgroup of G. Further E_s is the field of definition of $\mathcal{F}^{wa}$. The étale cohomology with torsion coefficients of $\Omega(V)$ has been computed in [SS]. For period domains where the considered isocrystal is basic, there exists a formula for the continious ℓ-adic Euler-Poincaré-Characteristic in the Grothendieck-group of $J(\mathbb{Q}_p) \times Gal(\overline{E_s}/E_s)$-representations due to Kottwitz and Rapoport ([R1]-[R3]). In the case of a finite base field k, in which the notion of a period domain is also available [loc.cit.], much more is known. Instead of filtered isocrystals one considers filtered vector spaces $(V, \mathcal{F})$. Subisocrystals are replaced by k-rational subspaces of V. In contrast to the p-adic case the corresponding period domain has the structure of an open subvariety of $\mathcal{F}$. The reason is that there are only finitely many k-rational subspaces of V. A precise formula for the ℓ-adic cohomology in the $GL(V)$-case has been computed in [O1]. A generalization of this computation to arbitrary reductive groups is given in [O2] using an idea of B. Totaro.

The aim of this paper is to give a precise formula for the étale cohomology of $\mathcal{F}^{wa}$ with coefficients in $\mathbb{Z}/n\mathbb{Z}$, $(n,p) = 1$, with respect to a quasisplit reductive group G (Theorem 1.1). The proof is based on the idea in [O1], [O2] which works as follows. In [loc.cit.] a complex of étale sheaves was constructed on the complement Y of a period domain, which is in that case a closed subvariety of $\mathcal{F}$. The index set of the complex corresponds to the Tits-building of the finite group $G(k)$. The étale sheaf associated to a facet is just the constant sheaf on the closed subvariety consisting of points where each vertex of the facet damages the weakly admissibility condition. The resulting spectral sequence degenerates in E_2 and computes the cohomology of Y. If one tries to adapt this idea to the p-adic case one is confrontated with two difficulties. The first one is the fact that the complement of $\mathcal{F}^{wa}$ is in general not a rigid analytic space. This problem is solved by working in the bigger category of adic spaces defined by R. Huber [H1]. In the language of adic spaces our object Y is a closed pseudo-adic subspace of $\mathcal{F}^{ad}$. The second problem consists of having infinitely many subobjects of our fixed isocrystal, where the adapted complex is not defined. The solution for this problem is to write down the favored complex as an approximation of certain complexes of étale sheaves on Y, possessing all the desired properties. By a limit argument in étale cohomology we get a spectral sequence which converges towards the cohomology of Y. Under the assumption of the the vanishing of certain Ext-groups

$$Ext^1(v_P^J, v_Q^J)$$

which is generally conjectured, we may conclude that the canonical filtration on $E_2 = E_\infty$ splits. Here P resp. Q are certain parabolic subgroups of J and v_P^J resp. v_Q^J are the corresponding generalized Steinberg representations with values in $\mathbb{Z}/n\mathbb{Z}$.. Thus we get the cohomology of Y. Applying the obvious long exact cohomology sequence we obtain finally the cohomology of $\mathcal{F}^{wa}$.

Now we come to the content of this paper. The main result is formulated in Section 1. Section 2 deals with the connection to Geometric Invariant Theory preparing the foundation for the exactness of the fundamental complex intoduced in section 3. The proof of the exactness is given in the fourth part. Finally in the last section we evaluate the spectral sequence induced by the acyclic complex.

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1 The main result

Let k be an algebraically closed field of characteristic $p > 0$. We denote by K_0=Quot($W(k)$) the corresponding quotient field of the ring of Witt vectors and by $\sigma \in \mathrm{Aut}(K_0/\mathbb{Q}_p)$ the Frobenius homomorphism. Let $\overline{\mathbb{Q}_p}$ be an algebraic closure of $\mathbb{Q}_p$ with Galois group $\Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Let G be a connected quasisplit reductive group over $\mathbb{Q}_p$. We recall briefly the theory of isocrystals with G-structure [RR], which were first introduced by Kottwitz in [K1]. An isocrystal with G-structure on k is an exact faithful tensor functor

$$Rep_{\mathbb{Q}_p}(G) \longrightarrow Isoc(K_0)$$

from the category of finite-dimensional G-representations over $\mathbb{Q}_p$ into the category of isocrystals over k. Following [RR] Remark 3.5, every such isocrystal is induced by an element $b \in G(K_0)$ in the following way. For a finite dimensional representation V of G we put

$$N_b(V) := \Big(V \otimes_{\mathbb{Q}_p} K_0, b(id_V \otimes \sigma)\Big),$$

which is an isocrystal over K_0. The map on the level of morphisms is the obvious one. This construction yields an isocrystal with G-structure N_b, where two elements $b, b' \in G(K_0)$ define the same isocrystal if and only if there exists a $g \in G(K_0)$ with $b' = gb\sigma(g)^{-1}$.

Let $b \in G(K_0)$. Consider the tensor functor

$$\begin{array}{rcl} Rep_{\mathbb{Q}_p}(G) & \longrightarrow & Grad(Vec_{K_0}, \mathbb{Q}) \\ V & \longmapsto & \oplus_{i \in \mathbb{Q}} V_i \end{array}$$

from the category $Rep_{\mathbb{Q}_p}(G)$ into the category of $\mathbb{Q}$-graded vector spaces over K_0, which is given by the slope-grading of the isocrystal N_b. Using the Tannaka formalism we obtain a rational 1-PS $\nu_b : \mathbb{D} \longrightarrow G$ defined over K_0, which induces this tensor functor. It is called the slope homomorphism of N_b. Here, $\mathbb{D}$ is the algebraic pro-torus over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. If $b' \in G(K_0)$ is σ-conjugated to b by an element $g \in G(K_0)$ then we get $\nu_{b'} = Int(g) \circ \nu_b$. For the remainder of this paper we fix a decent element b, i.e., its slope homomorphism satisfies an equation

$$(b\sigma)^s = s\nu_b(p)\sigma^s$$

for some integer s such that $s\nu$ factors through the multiplicative group $\mathbb{G}_m$. This is not really a restriction since in any σ-conjugacy class there is a decent element [K1]. Then $b \in G(\mathbb{Q}_{p^s})$ and ν_b is defined over $\mathbb{Q}_{p^s}$.

Suppose that there is given a one-parameter subgroup (1-PS) $\lambda : \mathbb{G}_m \longrightarrow G_K$ of G defined over a (finite) extension K of K_0. We obtain in a well-known way for every representation V of G a decreasing filtration $\mathcal{F}_\lambda(V)^\bullet$ on $V \otimes_{\mathbb{Q}_p} K$. Thus the pair (b, λ) yields a tensor functor from $Rep_{\mathbb{Q}_p}(G)$ to the category of filtered isocrystals over K. Following [RZ], the pair (b, λ) is called weakly admissible if for all faithful G-representations V, the filtered isocrystal $(N_b(V), \mathcal{F}_\lambda(V))$ is weakly admissible, i.e., if

$$\sum_i \dim gr^i_{\mathcal{F}_\lambda(V)}(N' \otimes_{K_0} K) \leq ord_p \det(b(\sigma \otimes id_V)|N')$$

for every subisocrystal N' of $N_b(V)$ and equality for $N' = N_b(V)$. It follows from the tensor product theorem of Faltings respectively Totaro, which says that the tensor product of two weakly admissible filtered isocrystals is again weakly admissible, that it suffices to check the weak admissibility for a single faithful representation. Although Colmez and Fontaine [CF] have proved recently that weak admissibility is the same as admissibility, we will use the former term in the sequel.

We fix a conjugacy class

$$\{\mu\} \subset X_*(G)$$

of one-parameter subgroups of G over $\overline{\mathbb{Q}_p}$. Let

$$E = \{x \in \overline{\mathbb{Q}_p};\ \tau(x) = x\ \ \forall \tau \in Stab_{\Gamma_{\mathbb{Q}_p}}(\{\mu\})\}$$

be the Shimura field of $\{\mu\}$, a finite intermediate field of $\overline{\mathbb{Q}_p}/\mathbb{Q}_p$. Put

$$E_s := E.\mathbb{Q}_{p^s}$$

and

$$\Gamma_{E_s} := Gal(\overline{\mathbb{Q}_p}/E_s).$$

Since G is quasisplit, we may apply a lemma of Kottwitz ([K3] Lemma 1.1.3) which guarantees the existence of an 1-PS $\mu \in \{\mu\}$ that is defined over E. Hence the conjugacy class $\{\mu\}$ defines a flag variety

$$\mathcal{F} := \mathcal{F}(G, \{\mu\}) := G_E/P(\mu)$$

over E. Notice that the geometric points of $\mathcal{F}$ coincide with the set

$$\{\mu\}/\sim,$$

where $\lambda_1, \lambda_2 \in \{\mu\}$ are equivalent, written $\lambda_1 \sim \lambda_2$, if they define the same filtration on $Rep_{\mathbb{Q}_p}(G)$. Let $\mathbb{C}_p := \hat{\overline{\mathbb{Q}_p}}$ be the completion of the algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, which is algebraically closed again. In the sequel we will often identify $\mathcal{F}$ with its $\mathbb{C}_p$-valued points. Following [RZ] (Proposition 1.36), the set of weakly admissible filtrations $\mathscr{F}_b^{wa}$ in $\mathscr{F}$ with respect to b has a natural structure of an open admissible rigid-analytic subspace of $(\mathscr{F} \otimes_E E_s)^{rig}$, which is called the period domain of the triple $(G, b, \{\mu\})$. In their book [RZ] Rapoport and Zink define an algebraic group J over $\mathbb{Q}_p$ such that

$$J(\mathbb{Q}_p) = \{g \in G(K_0); g(b\sigma) = (b\sigma)g\},$$

where $b\sigma$ is the product in the semi-direct product $G(K_0) \rtimes \langle\sigma\rangle$. It can be shown that J is an inner form of a Levi subgroup of G and hence a reductive group. The period domain $\mathscr{F}_b^{wa}$ is stable under the action of $J(\mathbb{Q}_p)$.

Before we can state the main result of this paper, concerning the cohomology of $\mathscr{F}_b^{wa}$, we have to introduce a few more notations. For a 1-PS $\lambda \in X_*(G)$ defined over some finite field extension $E/\mathbb{Q}_p$ we denote by $P(\lambda)$ the parabolic subgroup of G over E, whose $\overline{\mathbb{Q}_p}$−valued points are given by

$$P(\lambda)(\overline{\mathbb{Q}_p}) = \{g \in G(\overline{\mathbb{Q}_p}); \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{exists in } G(\overline{\mathbb{Q}_p})\}.$$

If besides λ is already a 1-PS of J then we will denote furthermore by $P^J(\lambda)$ the parabolic subgroup of J with geometric points

$$P^J(\lambda)(\overline{\mathbb{Q}_p}) = \{g \in J(\overline{\mathbb{Q}_p}); \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } J(\overline{\mathbb{Q}_p})\}.$$

Choose an invariant inner positive definite product on G. I.e. we have for all maximal tori T in G a non-degenerate positive definite pairing $(\,,\,)$ on $X_*(T)_\mathbb{Q} := X_*(T) \otimes_\mathbb{Z} \mathbb{Q}$, such that the natural maps

$$X_*(T)_\mathbb{Q} \longrightarrow X_*(T^g)_\mathbb{Q}$$

induced by conjugating with $g \in G(\overline{\mathbb{Q}_p})$ and

$$X_*(T)_\mathbb{Q} \longrightarrow X_*(T^\tau)_\mathbb{Q}$$

induced by conjugating with $\tau \in \Gamma_{\mathbb{Q}_p}$ are isometries for all $g \in G(\overline{\mathbb{Q}_p}), \tau \in \Gamma_{\mathbb{Q}_p}$. Here $T^g = gTg^{-1}$ is the conjugate torus resp. $T^\tau = \tau \cdot T$ is the image of T under the morphism $\tau : G \to G$ induced by τ. The inner product, together with the natural pairing

$$\langle\,,\,\rangle : X_*(T)_\mathbb{Q} \times X^*(T)_\mathbb{Q} \longrightarrow \mathbb{Q}\,,$$

induces identifications

$$\begin{array}{ccc} X_*(T)_\mathbb{Q} & \longrightarrow & X^*(T)_\mathbb{Q} \\ \lambda & \longmapsto & \lambda^* \end{array}$$

for all maximal tori T in G. We call λ^* the dual character of λ. Finally we remark that the inner product on G induces an inner product on J.

Fix a maximal $\mathbb{Q}_p$-split torus S of J and let $d := \mathbb{Q}_p\text{-rk}_{ss}(J)$ be its semisimple $\mathbb{Q}_p$-rank. Since ν factors through the centre of J, ([K2] 3.4) we may view ν as an element of $X_*(S)_{\mathbb{Q}}$. Fix a minimal parabolic subgroup P_0 of J together with its associated set of simple roots

$$\Delta = \{\alpha_1, \ldots, \alpha_d\}.$$

Let

$$\{\omega_\alpha; \alpha \in \Delta\} \subset X_*(S)_{\mathbb{Q}}$$

be the dual basis of Δ, i.e., we have

$$\langle \omega_\alpha, \beta \rangle = \delta_{\alpha,\beta} \text{ (Kronecker delta) } \forall \alpha, \beta \in \Delta.$$

The parabolic subgroups $P^J(\omega_\alpha)$ are exactly the maximal $\mathbb{Q}_p$-parabolic subgroups of J that contain P_0. We suppose that μ does not factor through the center of G. We further assume that $\mu\nu^{-1} \in X_*(G_{der})$, since otherwise $\mathcal{F}_b^{wa} = \emptyset$. Compare [FR] for a sufficiant and necessary condition such that $\mathcal{F}_b^{wa} \neq \emptyset$. Replacing μ by a conjugated element under W, we may assume that we have chosen Δ such that $\langle \mu - \nu, \omega_\alpha^* \rangle > 0, \ \forall \alpha \in \Delta$. Finally we fix a maximal torus T of G that contains S and a Borel subgroup B of G such that $B \subset P(\omega_\alpha)$ $, \forall \alpha \in \Delta$.

Let W_μ be the stabilizer of μ with respect to the action of W on $X_*(T)$. We denote by W^μ the set of Kostant-representatives with respect to W/W_μ. Consider the action of Γ_{E_s} on W. Since μ is defined over E_s, this action preserves W^μ. Denote the corresponding set of orbits by W^μ/Γ_{E_s} and its elements by $[w]$, where w is in W^μ. Clearly the length of an element in W only depends on its orbit. So the symbol $l([w])$ makes sense. Fix a number $n \in \mathbb{N}$ which is prime to p. For any orbit $[w]$ we set

$$ind_{[w]} := Ind_{Stab_{\Gamma_{E_s}}(w)}^{\Gamma_{E_s}} \mathbb{Z}/n\mathbb{Z},$$

with trivial action of $Stab_{\Gamma_{E_s}}(w)$ on $\mathbb{Z}/n\mathbb{Z}$. This induced representation is clearly independent of the specified representative.

For any subset $I \subset \Delta$ we define

$$\Omega_I := \{[w] \in W^\mu/\Gamma_{E_s};\ \langle w\mu, \omega_\alpha^* \rangle > \langle w\nu_b, \omega_\alpha^* \rangle \ \forall \alpha \notin I\}.$$

Notice that by our choice of μ and Δ the trivial element $[1] \in W^\mu/\Gamma_{E_s}$ lies in each $\Omega_I, I \subset \Delta$. We get the following inclusion relation

$$I \subset J \Rightarrow \Omega_I \subset \Omega_J.$$

In the further text we denote for $[w] \in W^\mu/\Gamma_{E_s}$ by $I_{[w]}$ the smallest subset of Δ such that $[w]$ is contained in $\Omega_{I_{[w]}}$. Obviously we have

$$I_{[w]} \subset I \Leftrightarrow [w] \in \Omega_I. \tag{1}$$

and thus

$$I_{[w]} = \{\alpha \in \Delta; \langle w\mu, \omega_\alpha^* \rangle \leq \langle w\nu_b, \omega_\alpha^* \rangle\}.$$

For a parabolic subgroup $P \subset J$ defined over $\mathbb{Q}_p$ we consider the trivial representation of $P(\mathbb{Q}_p)$ on $\mathbb{Z}/n\mathbb{Z}$. We denote by

$$i_P^J = i_{P(\mathbb{Q}_p)}^{J(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z}) = C^\infty(J(\mathbb{Q}_p)/P(\mathbb{Q}_p), \mathbb{Z}/n\mathbb{Z})$$

the resulting representation of $J(\mathbb{Q}_p)$ consisting of locally constant functions on $J/P(\mathbb{Q}_p) = J(\mathbb{Q}_p)/P(\mathbb{Q}_p)$ with values in $\mathbb{Z}/n\mathbb{Z}$. Further we set

$$v_P^J = i_P^J / \sum_{P \subsetneq Q} i_Q^J.$$

Finally if $I \subset \Delta$ we put

$$P_I := \bigcap_{\alpha \notin I} P^J(\omega_\alpha),$$

which is a parabolic subgroup of J defined over $\mathbb{Q}_p$.

Under the assumption of the vanishing of the groups

$$Ext^1(v_{P_I}^J, v_{P_{I'}}^J)$$

for $|\#I - \#I'| \neq 1$, (those extensions appear in a certain spectral sequence of the computation), we can state the following theorem. It describes the étale cohomology with compact support of the period domain $\mathcal{F}_b^{wa}$ with values in $\mathbb{Z}/n\mathbb{Z}$ as representation of the product $J(\mathbb{Q}_p) \times \Gamma_{E_s}$.

Theorem 1.1 *We have for $n \gg 0$*

$$H_c^*(\mathcal{F}_b^{wa}, \mathbb{Z}/n\mathbb{Z}) = \bigoplus_{[w] \in W^\mu/\Gamma_{E_s}} v_{P_{I_{[w]}}}^J \otimes ind_{[w]}(-l([w]))[-2l([w]) - \#(\Delta - I_{[w]})].$$

Here the symbol $(m), m \in \mathbb{N}$ means the m-th Tate twist and $[-m], m \in \mathbb{N}$ symbolizes that the corresponding module is shifted into degree m of the graded cohomology ring.

2 The relationship of period domains to GIT

In this section we want to explain the relationship between period domains and Geometric Invariant Theory. For details we refer to the papers [T] resp. [R2].

Let

$$M := P(\mu)/R_u(P(\mu))$$

be the Levi-quotient of $P(\mu)$ with center Z_M. Then μ defines an element of $X_*(Z_M)$. Let T_M be a maximal torus in M. Then we have $Z_M \subset T_M$ and T_M is the isomorphic image of a maximal torus in G. So we get an invariant inner product on M. Consider the dual character

$$\mu^* \in X^*(T_M)_{\mathbb{Q}}.$$

As μ belongs to $X_*(Z_M)$, the dual character μ^* is contained in

$$X^*(M_{ab})_{\mathbb{Q}} \cong Hom(P(\mu), \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The inverse character $-\mu^*$ induces a homogeneous line bundle $\mathcal{L}_\mu := \mathcal{L}_{-\mu^*}$ on $\mathcal{F}$. The reason for the sign is that this line bundle is ample. Applying the above machinery to the inverse of our slope homomorphism $\lambda_b := -\nu_b$, we get an ample line bundle $\mathcal{L}_b := \mathcal{L}_{-\lambda_b^*}$ on the flag variety $\mathcal{F}^b := G/P(\lambda_b)$. Consider the closed embedding

$$\mathcal{F}_{E_s} \hookrightarrow \mathcal{F}_{E_s} \times \mathcal{F}^b_{E_s},$$

given by the identity on the first factor and by the $\mathbb{Q}_{p^s}$-rational point λ_b of $\mathcal{F}^b$ on the second factor. Let $\mathcal{L}$ be the restiction of the line bundle $\mathcal{L}_\mu \times \mathcal{L}_b$ to $\mathcal{F}_{E_s}$ via the above embedding, which we consider as a J-equivariant line bundle.

For any point $x \in \mathcal{F}$ and any 1-PS $\lambda : \mathbb{G}_m \to G$ we can consider the slope $\mu^{\mathcal{L}}(x, \lambda)$ of λ in x relative to the line bundle $\mathcal{L}$ (cf. [M] Def. 2.2). The following theorem of Totaro (cf. [T] Theorem 3) describes the relation between the notion of weakly admissibility and semistability in Geometric Invariant Theory.

Theorem 2.1 *(Totaro) Let x be a point of $\mathcal{F}$. Then x is weakly admissible if and only if $\mu^{\mathcal{L}}(x, \lambda) \geq 0$ $\forall$1-PS λ of J_{der} defined over $\mathbb{Q}_p$. Here J_{der} is the derived group of J.*

In the case we have chosen a faithful representation $G \longrightarrow GL(V)$, we can compute the slope of a point explicitly. If $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ are two filtrations on a finite-dimensional vector space V we set

$$(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) = \sum_{\alpha,\beta} \alpha\beta \dim gr^\alpha_{\mathcal{F}_1}(gr^\beta_{\mathcal{F}_2}(V)).$$

The following Lemma is proved in [O2] in the case of a finite field, but the case treated here is proven similarly. It also follows from the results in [T] Lemma 6.

Lemma 2.2 *Let $G \longrightarrow GL(V)$ a faithful representation.*

(i) Let $x \in \mathcal{F}$ and $\lambda \in X_*(J)$. Denote their filtrations on $V_{\mathbb{C}_p} := V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ by $\mathcal{F}_x^\bullet$ respectively by $\mathcal{F}_\lambda^\bullet$. Furthermore let $\mathcal{F}_b^\bullet$ be the filtration on $V_{\mathbb{C}_p}$ which is induced by the slope homomorphism λ_b. Then

$$\mu^{\mathcal{L}}(x,\lambda) = -\Big((\mathcal{F}_x^\bullet, \mathcal{F}_\lambda^\bullet) + (\mathcal{F}_b^\bullet, \mathcal{F}_\lambda^\bullet)\Big).$$

(ii) Let $T \subset G$ be a maximal torus and $\lambda, \lambda' \in X_*(T)$. Then

$$(\lambda, \lambda') = (\mathcal{F}_\lambda^\bullet, \mathcal{F}_{\lambda'}^\bullet).$$

In order to investigate the GIT-semistability of points on varieties, it is useful to consider the spherical building of the given group. Let $B(J_{der})_{\mathbb{Q}_p}$ be the real $\mathbb{Q}_p$-rational spherical building of the derived group J_{der} (cf. [CLT]). It is well-known that the space $B(J_{der})_{\mathbb{Q}_p}$ is homeomorphic to the geometric realisation of the combinatorial building of J (cf. [CLT], 6.1). Thus we have a simplicial structure on $B(J_{der})_{\mathbb{Q}_p}$ which is defined as follows. For a $\mathbb{Q}_p$-rational parabolic subgroup $P \subset J$ we let

$$D(P) := \{x \in B(J_{der})_{\mathbb{Q}_p}; P(x) \supset P\}$$

be the facet corresponding to P. If P is a minimal $\mathbb{Q}_p$-parabolic subgroup, then we call $D(P)$ a chamber of $B(J_{der})_{\mathbb{Q}_p}$. If in contrast P is a proper maximal $\mathbb{Q}_p$-parabolic subgroup then $D(P)$ is called a vertex.

Consider the $\mathbb{Q}_p$-rational 1-PS $\omega_\alpha, \alpha \in \Delta$, introduced in the previous section. These 1-PS correspond to the vertices of the chamber $D_0 := D(P_0)$, since the $P^J(\omega_\alpha), \alpha \in \Delta$, are the maximal $\mathbb{Q}_p$-rational parabolic subgroups that contain P_0.

For any other chamber $D = D(P)$ in $B(J_{der})_{\mathbb{Q}_p}$, there exists a $g \in J(\mathbb{Q}_p)$, such that the conjugated elements $Int(g) \circ \omega_\alpha, \alpha \in \Delta$, correspond to the vertices of D. The element g is of course unique up to multiplication by an element of $P_0(\mathbb{Q}_p)$ from the right. We define for every chamber D in $B(J_{der})_{\mathbb{Q}_p}$ the simplex

$$\tilde{D} := \{\sum_{\alpha \in \Delta} r_\alpha \lambda_\alpha \,;\, 0 \leq r_\alpha \leq 1, \sum_{\alpha \in \Delta} r_\alpha = 1\} \subset X_*(S)_{\mathbb{R}},$$

which is the convex hull of its vertices $\lambda_\alpha \in X_*(S)_{\mathbb{R}}, \alpha \in \Delta$. The topological spaces D and $\tilde{D}$ are obviously homeomorphic.

We can extend $\mu^{\mathcal{L}}(x, \cdot)$ in a well-known way to a function on $X_*(S)_{\mathbb{R}}$ for every maximal $\mathbb{Q}_p$-split torus S in J. Notice that the slope function $\mu^{\mathcal{L}}(x, \cdot)$ is not defined on D but on $\tilde{D}$. In spite of this fact we we will say that $\mu^{\mathcal{L}}(x, \cdot)$ is affine on D if it is affine on $\tilde{D}$, i.e. if the following equality holds:

$$\mu^{\mathcal{L}}(x, \sum_{\alpha \in \Delta} r_\alpha \lambda_\alpha) = \sum_{\alpha \in \Delta} r_\alpha \mu^{\mathcal{L}}(x, \lambda_\alpha) \quad \forall \sum_{\alpha \in \Delta} r_\alpha \lambda_\alpha \in \tilde{D}.$$

The proof of the next proposition, which is the same as in the case of a finite field [O2], follows from Lemma 2.2.

Proposition 2.3 *Let $x \in \mathcal{F}$ be any point. The slope function $\mu^{\mathcal{L}}(x, \cdot)$ is affine on each chamber of $B(J_{der})_{\mathbb{Q}_p}$.*

Corollary 2.4 *Let x be a point in $\mathcal{F}$. Then x is not weakly admissible $\Leftrightarrow$ There exists an element $g \in J(\mathbb{Q}_p)$ and an $\alpha \in \Delta$ such that $\mu^{\mathcal{L}}(x, Int(g) \circ \omega_\alpha) < 0$.*

3 The fundamental complex

In the case of a finite base field we have constructed in [O1],[O2] a complex of étale sheaves on the complement of the period domain, which is there a closed subvariety of $\mathcal{F}$. Unfortunately in our situation the set $Y := \mathcal{F}^{rig} \setminus \mathcal{F}_b^{wa}$ is not an object of our category, i.e. a rigid analytic space, as one can already see in the simplest case where $\mathcal{F}_b^{wa} = \Omega^2 = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$. Therefore we can't simply adopt the proof of the finite field case. The idea to avoid this problem is to work in the category of adic spaces, which were defined by

Huber [H1]. There he defined for every complete non-archimedean field k a fully faithful functor

$$\begin{array}{rcl} {}^{ad} : \{k - \text{rigid-analytic spaces}\} & \longrightarrow & k - \text{adic spaces}, \\ X & \longmapsto & X^{ad} \end{array}$$

where the category on the RHS is a full subcategory of the category of topologically and locally ringed spaces with a distinguished valuation ring on each residue field $\kappa(x)$ of X. The above functor respects open embeddings and converts an open admissible covering $(X_i)_{i\in I}$ of a rigid analytic space X into a covering by open subsets $X^{ad} = \bigcup_{i\in I} X_i^{ad}$. Further it induces for a fixed rigid-analytic space X a bijection between the quasi-compact open subsets of X and the quasi-compact open subsets of X^{ad}. If X is an algebraic variety over k then we denote the adic space $(X^{rig})^{ad}$ simply by X^{ad}.

For both a rigid-analytic variety and an adic space X one can define étale sites $X_{ét}$ (cf. [H1], [JP]) similarly as for schemes. Let X be a rigid-analytic variety. Then the above functor induces a morphism of sites

$$X_{ét} \to X^{ad}_{ét}$$

yielding an equivalence of the associated topoi

$$\begin{array}{rcl} S(X_{ét}) & \longrightarrow & S(X^{ad}_{ét}) \\ F & \longmapsto & F^{ad}. \end{array}$$

Hence we have for every rigid-analytic variety X and every étale sheaf on X functorial isomorphisms

$$H^p_{ét}(X, F) \xrightarrow{\sim} H^p_{ét}(X^{ad}, F^{ad}).$$

The advantage of adic spaces in our situation is that in this category the subset $Y^{ad} := \mathscr{F}^{ad} \setminus (\mathscr{F}_b^{wa})^{ad}$ induces a so-called closed pseudo adic subspace of $\mathscr{F}^{ad}$ (compare Lemma 3.2). For those spaces Huber defined an étale site and hence a topos as well ([H1] 1.16).

But besides of the phenomen above, we have another difficulty in order to apply the idea of [O1], [O2]. In our case we must deal with infinitely many subobjects of a given isocrystal. Thus the fundamental complex (loc.cit.) with its summands is not well-defined. The first thought of substituating these summands by products turns out to be not a good idea. The reason is that infinite products of sheaves do not behave well with respect to localisation. The solution is to define a mixture between these two kind of sheaves, which we explain in the following.

For a 1-PS subgroup λ of J we let

$$Y_\lambda := \{x \in \mathcal{F}; \mu^{\mathcal{L}}(x,\lambda) < 0\}$$

be the closed subvariety of $\mathcal{F}$, consisting of points where λ damages the semistability condition. For any subset $I \subsetneq \Delta$ put

$$Y_I := \bigcap_{\alpha \notin I} Y_{\omega_\alpha},$$

which is a closed subvariety of $\mathcal{F}$. The following statements are proved in [loc.cit] in the case of a finite base field. The proof of the case considered here is the same.

Lemma 3.1 *Let $I \subsetneq \Delta$. The variety Y_I is defined over E_s. The natural action of $J(\mathbb{Q}_p)$ on $\mathcal{F}$ restricts to an action of $P_I(\mathbb{Q}_p)$ on Y_I.*

Fix a subset $I \subset \Delta$. Let g be a point in $J/P_I(\mathbb{Q}_p)$. Choose a representative $g' \in J(\mathbb{Q}_p)$ of g. Then the lemma above tells us that the image $g'Y_I$ of Y_I under the natural translation morphism induced by g' does not depend on the chosen representative. For this reason we set $gY_I := g'Y_I$. Consider the closed adic subspace gY_I^{ad} of $\mathcal{F}^{ad}$. Analogous to the definition of Y^{ad} we put

$$Z_I^W := \bigcup_{g \in W} gY_I^{ad},$$

for a subset $W \subset J/P_I(\mathbb{Q}_p)$. We consider it as an prepseudo adic subspace of $\mathcal{F}^{ad}$ ([H1] 1.10.1). Notice that in the case $W = J/P_I(\mathbb{Q}_p)$ we have $Z_I^W = Y^{ad}$.

Lemma 3.2 *The subset Z_I^W is a closed pseudo adic subspace of $\mathcal{F}^{ad}$ for every open and compact subset $W \subset J/P_I(\mathbb{Q}_p)$.*

Proof: We follow the construction of [RZ] 1.3.2. Let $\mathcal{H}_I$ be the closed subvariety of $J/P_I \times_E \mathcal{F}$ consisting of (geometric) points $(t,x) \in J/P_I \times_E \mathcal{F}$ such that $x \in tY_I$. Thus as a set Z_I^W is nothing but the union

$$Z_I^W = \bigcup_{t \in W} (\mathcal{H}_I^{ad})_t,$$

where $(\mathcal{H}_I^{ad})_t \subset \mathcal{F}^{ad}$ denotes the fibre through the point $t \in J/P_I(\mathbb{Q}_p)$. For a small real number $\epsilon > 0$ and a point $t \in J/P_I(\mathbb{Q}_p)$ we denote by $(\mathcal{H}_I^{ad})_t(\epsilon)$ the closed epsilon tube around $(\mathcal{H}_I^{ad})_t$, i.e.

$$(\mathcal{H}_I^{ad})_t(\epsilon) = \{x \in \mathcal{H}_I^{ad}; |f_\alpha^t(x)|_x < \epsilon\ ,\ \forall \alpha\}.$$

Here $(f^t_\alpha)_\alpha$ are the equations for the variety tY_I in some standard projective space (compare also [H3]). As in Lemma 1.33 [RZ] we conclude from the compactness of W that there exists a finite subset $S \subset J/P_I(\mathbb{Q}_p)$ with

$$\bigcup_{t\in W} (\mathcal{H}_I^{ad})_t(\epsilon) = \bigcup_{t\in S} (\mathcal{H}_I^{ad})_t(\epsilon).$$

Thus $\bigcup_{t\in W} (\mathcal{H}_I^{ad})_t(\epsilon)$ is closed in $\mathcal{H}_I^{ad}$. But

$$Z_I^W = \bigcap_{\epsilon>0} \Big(\bigcup_{t\in W} (\mathcal{H}_I^{ad})_t(\epsilon) \Big), \tag{2}$$

which follows from the fact that the function

$$\begin{array}{rcl} W & \longrightarrow & \mathbb{R}_{\geq 0} \\ t & \longmapsto & \min\{\epsilon; x \in (\mathcal{H}_I^{ad})_t(\epsilon)\} \end{array}$$

is continious for $x \in \mathcal{F}^{ad}$. So it assumes its minimum since W is compact. Thus Z_I^W is closed in $\mathcal{F}^{ad}$. Further we see from (2) that it is locally pro-constructible and closed. Hence it is a closed pseudo adic subspace of $\mathcal{F}^{ad}$. □

For an open and compact set $W \subset J/P_I(\mathbb{Q}_p)$ and a point $g \in W$ we denote by

$$\Phi_{g,I} : gY_I^{ad} \longrightarrow Y^{ad}$$

resp.

$$\tilde{\Phi}_{g,I,W} : gY_I^{ad} \longrightarrow Z_I^W$$

resp.

$$\Psi_{I,W} : Z_I^W \longrightarrow Y^{ad}$$

the natural closed embeddings of pseudo-adic spaces. Let $\mathbb{Z}/n\mathbb{Z}$ be the constant sheaf on Y^{ad}, where n is prime to p. Put

$$(\mathbb{Z}/n\mathbb{Z})_{g,I} := (\Phi_{g,I})_*(\Phi_{g,I}^*(\mathbb{Z}/n\mathbb{Z}))$$

resp.

$$(\mathbb{Z}/n\mathbb{Z})_{Z_I^W} := (\Psi_{I,W})_*(\Psi_{I,W}^*(\mathbb{Z}/n\mathbb{Z}))$$

and let

$$\tilde{\Phi}^{\#}_{g,I,W} : (\mathbb{Z}/n\mathbb{Z})_{Z_I^W} \longrightarrow (\mathbb{Z}/n\mathbb{Z})_{g,I}$$

be the adjunction homomorphism given by restriction. We denote by

$$\prod_{g\in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I}$$

the subpresheaf of $\prod_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I}$ which is defined as follows. For an element $\rho : U \to Y^{ad}$ of the étale site Y^{ad}_{et} we put

$$\Big(\prod_{g\in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I}\Big)(U) := \Big\{(s_g)_g \in \prod_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}(U); \exists \text{ (finite) disjoint covering } J/P_I(\mathbb{Q}_p) = \bigcup_j W_j \text{ of open compact subsets and } s_j \in (\mathbb{Z}/n\mathbb{Z})_{Z_I^{W_j}}(U) \text{ s.t. } \tilde{\Phi}^{\#}_{g,I,W}(s_j) = s_g\ \forall g \in W_j\Big\}.$$

Since we are dealing with an noetherian site (thus every étale covering can be refined into a finite étale covering) it is easy to see that this presheaf is in fact a sheaf. Another description of this sheaf is given by writing it as an inductive limit of sheaves. To explain this, let $\mathcal{C}_I$ be the category of open compact disjoint coverings of $J/P_I(\mathbb{Q}_p)$ ordered in the usual way, i.e., by refinement. Then we may write

$$\prod_{g\in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I} = \varinjlim_{c\in\mathcal{C}_I} G_c, \tag{3}$$

where, for a covering $c = (W_j)_j \in \mathcal{C}_I$, G_c is the sheaf defined by

$$G_c(U) := \Big\{(s_g)_g \in \prod_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}(U); \exists s_j \in (\mathbb{Z}/n\mathbb{Z})_{Z_I^{W_j}}(U) \text{ s.t. } \tilde{\Phi}^{\#}_{g,I,W}(s_j) = s_g\ \forall g \in W_j\Big\}$$

for any element $\rho : U \to Y^{ad}$ of the étale site Y^{ad}_{et}. Notice that G_c is just the image sheaf of the morphism

$$\bigoplus_{W_j\in c} (\mathbb{Z}/n\mathbb{Z})_{Z_I^{W_j}} \hookrightarrow \prod_{g\in J/P_I(\mathbb{Q}_p)} (\mathbb{Z}/n\mathbb{Z})_{g,I}.$$

We call $\prod'_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I}$ the subsheaf of locally constant sections of $\prod_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I}$.

Let $\bar{x}$ be a geometric point of Y^{ad} with underlying point $x \in Y^{ad}$. Then we have

$$\Big(\prod_{g\in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I}\Big)_{\bar{x}} = \prod_{g\in J/P_I(\mathbb{Q}_p)}{}' ((\mathbb{Z}/n\mathbb{Z})_{g,I})_{\bar{x}} \subset \prod_{g\in J/P_I(\mathbb{Q}_p)} ((\mathbb{Z}/n\mathbb{Z})_{g,I})_{\bar{x}}, \tag{4}$$

where the term in the middle is defined for abelian groups similarly as in the sheaf case. Thus we may identify $(\prod'_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I})_{\bar{x}}$ with the set of locally constant $\mathbb{Z}/n\mathbb{Z}$ valued functions on $J/P_I(\mathbb{Q}_p)$ with support $\{g \in J/P_I(\mathbb{Q}_p); x \in gY_I^{ad}\}$.

Now we are able to construct the following complex of sheaves on Y^{ad} which is defined analogously as in [loc.cit.]. Let $I \subset I'$ be two subsets of Δ. We get canonically a homomorphism

$$p_{I,I'} : \prod_{g\in J/P_{I'}(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I'} \longrightarrow \prod_{g\in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I}$$

which is induced by the closed embeddings $gY_I^{ad} \to hY_{I'}^{ad},\ g \in J/P_I(\mathbb{Q}_p), h \in J/P_{I'}(\mathbb{Q}_p)$. For two subsets $I, I' \subset \Delta$ with $\#I' - \#I = 1$ put

$$d_{I,I'} = \begin{cases} (-1)^i p_{I,I'} & : \quad I' = I \cup \{\alpha_i\} \\ 0 & : \quad I \not\subset I' \end{cases} .$$

We get a complex of sheaves on Y_{et}^{ad} :

$$(*) : 0 \to \mathbb{Z}/n\mathbb{Z} \to \bigoplus_{\substack{I\subset\Delta \\ \#(\Delta-I)=1}} \prod_{g\in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I} \to \bigoplus_{\substack{I\subset\Delta \\ \#(\Delta-I)=2}} \prod_{g\in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I} \to$$

$$\ldots \to \bigoplus_{\substack{I\subset\Delta \\ \#(\Delta-I)=d-1}} \prod_{g\in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I} \to \prod_{g\in J/P_\emptyset(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,\emptyset} \to 0.$$

Theorem 3.3 *The above complex is acyclic.*

Before we prove this theorem recall the definition of an overconvergent sheaf on an adic space X. Following [H1] Def. 8.2.1, an étale sheaf F on X is called overconvergent if for all geometric points $\bar{x}, \bar{y}$ of X such that x is a specialising point of y, the resulting specialising homomorphism $F_{\bar{x}} \to F_{\bar{y}}$ is bijective.

Lemma 3.4 *All the sheaves in the complex* $(*)$ *are overconvergent.*

Proof: Since every constant sheaf is overconvergent and the morphisms $\Phi_{g,I}$ are obviously quasi-compact, we may conclude from [H1] Prop. 8.2.3 resp. [JP] 3.5 that the étale sheaves $(\mathbb{Z}/n\mathbb{Z})_{g,I}$ are overconvergent as well. Applying of (4) yields the statement for $\prod'_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I}$. Alternatively one might use the identity (3) as well. Obviously the direct sum of overconvergent sheaves is again overconvergent, hence the claim is proved. □

Proof of Theorem 3.3: Since the complex $(*)$ consists of overconvergent sheaves, it is enough to show the exactness of $(*)$ in the maximal geometric points (with respect to the order given by specialising) of Y^{ad}. So let e be such a maximal geometric point. By definition this is just a morphism of adic spaces [H1] 2.5.1/2.

$$Spa(F_e) \longrightarrow \mathcal{F}^{ad}$$

for some separably closed affinoid field (F_e, F_e^0), which factors through Y^{ad}. Here F_e^o denotes the rank-one valuation ring consisting of power bounded elements in F_e with respect to the extended valuation of k on F_e. This morphism corresponds to a flag $\mathcal{F}_e \in \mathcal{F}(F_e)$, such that $\mathcal{F}_e \in Y^{ad}\hat{\otimes}F_e$ (Compare also [JP] for a desription of those maximal points). Localizing the above complex in e yields a chain complex with values in F_e. The chain complex corresponds to a subcomplex of the combinatorial Tits building of J. Its simplices are given by

$$\{gP_Ig^{-1}; g \in J, \mathcal{F}_e \in gY_I(F_e),\ I \subsetneq \Delta\}.$$

Let R_e be the canonical realisation of this subcomplex in $\mathcal{B}(J_{der})$. In [O2] it was shown by using Proposition 2.3 that the space R_e is contractible, and that the chain complex computes the cohomology of it. In our case we may apply exactly the same arguments. Thus the acyclicity follows, since the above complex is just the homology version of locally constant functions of R_e (Compare also the remark on p. 66 [SS]). □

4 The proof of Theorem 1.1

In this last section we compute the spectral sequence which is induced by the acyclic complex $(*)$.

Fix a subset $I \subsetneq \Delta$. The next statements are proved in [O2].

Proposition 4.1 *We have the following description of the closed varieties*

Y_I *in terms of the Bruhat cells of* G *with respect to* $P(\mu)$.

$$Y_I \;=\; \bigcup_{\substack{w\in W^\mu \\ [w]\in\Omega_I}} BwP(\mu)/P(\mu).$$

The above cell decomposition for the varieties Y_I allows us to compute their (étale) cohomology.

Proposition 4.2

$$H^*(Y_I,\mathbb{Z}/n\mathbb{Z}) = \bigoplus_{[w]\in\Omega_I} ind_{[w]}(-l[w])[-2l([w])] \qquad (5)$$

Next we want to compute for a fixed $I \subsetneq \Delta$ the cohomology of the étale sheaves $\prod'_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I}$.

Proposition 4.3 *We have*

$$H^i(Y^{ad}, \prod_{g\in J/P_I(\mathbb{Q}_p)}{}^{'} (\mathbb{Z}/n\mathbb{Z})_{g,I}) = C^\infty(J/P_I(\mathbb{Q}_p), H^i(Y_I,\mathbb{Z}/n\mathbb{Z})),\ \forall i\in\mathbb{N}.$$

Proof: We have $\varinjlim_{c\in C_I} G_c = \prod'_{g\in J/P_I(\mathbb{Q}_p)}(\mathbb{Z}/n\mathbb{Z})_{g,I}$ (compare (3)). Since the site is noetherian and $\mathcal{C}_I$ is pseudofiltered (!) we get ([H1] 2.3.13)

$$H^i(Y^{ad}, \prod_{g\in J/P_I(\mathbb{Q}_p)}{}^{'} (\mathbb{Z}/n\mathbb{Z})_{g,I}) = \varinjlim_{c\in C_I} H^i(Y^{ad}, G_c).$$

But $G_c \cong \bigoplus_{W\in c}(\mathbb{Z}/n\mathbb{Z})_{Z_I^W}$. Thus

$$H^i(Y^{ad}, G_c) = \bigoplus_{W\in c} H^i(Z_I^W, \mathbb{Z}/n\mathbb{Z})$$

Let $(W_s)_{s\in\mathbb{N}}$ be a system of open and compact neighbourhoods of the base point $1\cdot P_I$ in $J/P_I(\mathbb{Q}_p)$ s.t.

$$\bigcap_{s\in\mathbb{N}} W_s = (1\cdot P_I) \in J/P_I(\mathbb{Q}_p).$$

One can take for instance the open (and closed) tube around the base point in $J/P_I(\mathbb{Q}_p)$ of radius $|p^s|$ i.e.

$$W_s = \{t\in J/P_I(\mathbb{Q}_p);\, |g_\alpha(t)| \le |p^s|\ \forall\alpha\},$$

where $(g_\alpha)_\alpha$ is a generating system of the vanishing ideal of $1\cdot P_I$ in some standard projective space containing J/P_I.

Lemma 4.4

$$\bigcap_{s\in\mathbb{N}} Z_I^{W^s} = Y_I.$$

Proof: It is enough to show that for all $\epsilon > 0$ there exists a number $s > 0$ such that $Z_I^{W_s} \subset (\mathcal{H}_I^{ad})_1(\epsilon)$. This follows from the considerations in the proof of Lemma 3.2. □

By applying [H1] 2.4.6 we get

$$\varinjlim_{s\in\mathbb{N}} H^i(Z_I^{W^s}, \mathbb{Z}/n\mathbb{Z}) = H^i(Y_I, \mathbb{Z}/n\mathbb{Z}).$$

Combining these facts we get

$$\varinjlim_{c\in C_I} H^i(Y^{ad}, G_c) = \varinjlim_{c\in C_I} \bigoplus_{W\in c} H^i(Z_I^W, \mathbb{Z}/n\mathbb{Z}) = C^\infty(J/P_I(\mathbb{Q}_p), H^i(Y_I, \mathbb{Z}/n\mathbb{Z})).$$

Thus the proposition is proved. □

Now we will construct a complex in analogy to the complex $(*)$. Let $I \subset \Delta$ be a subset. Recall that $i^J_{P_I} = C^\infty(J/P_I(\mathbb{Q}_p), \mathbb{Z}/n\mathbb{Z})$. For two subsets $I \subset I' \subset \Delta$ with $\#(I' \setminus I) = 1$ we get a homomorphism

$$p_{I,I'} : i^J_{P_{I'}} \longrightarrow i^J_{P_I},$$

which is induced by the projection $(J/P_I)(\mathbb{Q}_p) \longrightarrow (J/P_{I'})(\mathbb{Q}_p)$. For two arbitrary subsets $I, I' \subset \Delta$ with $\#I' - \#I = 1$ we define

$$d_{I,I'} = \begin{cases} (-1)^i p_{I,I'} & I' = I \cup \{\alpha_i\} \\ 0 & I \not\subset I' \end{cases}.$$

Thus we get for every $I_0 \subset \Delta$ a $\mathbb{Z}$-indexed complex

$$K^\bullet_{I_0} : 0 \to i^J_J \to \bigoplus_{\substack{I_0\subset I\subset\Delta \\ \#(\Delta-I)=1}} i^J_{P_I} \to \bigoplus_{\substack{I_0\subset I\subset\Delta \\ \#(\Delta-I)=2}} i^J_{P_I} \to \ldots \to \bigoplus_{\substack{I_0\subset I\subset\Delta \\ \#(\Delta-I)=\#(\Delta-I_0)-1}} i^J_{P_I} \to i^J_{P_{I_0}},$$

where the differentials are induced by the above $d_{I,I'}$. The component i^J_J is in degree -1.

Proposition 4.5 *The complex $K^\bullet_{I_0}$ is exact.*

Proof: [BS] Cor. 3.3.

Now we want to evaluate the spectral sequence

$$E_1^{p,q} = H^q(Y^{ad}, \bigoplus_{\substack{I\subset\Delta \\ \#(\Delta-I)=p+1}} \prod_{g\in J/P_I(\mathbb{Q}_p)}{}' (\mathbb{Z}/n\mathbb{Z})_{g,I}) \Longrightarrow E^{p+q} = H^{p+q}(Y^{ad}, \mathbb{Z}/n\mathbb{Z})$$

given by the exact sequence $(*)$. From Proposition 4.3 we get

$$E_1^{p,q} = \bigoplus_{\substack{I\subset\Delta \\ \#(\Delta-I)=p+1}} C^\infty(J/P_I(\mathbb{Q}_p), H^q(Y_I, \mathbb{Z}/n\mathbb{Z})).$$

As in the finite field case [O1],[O2] we have a decomposition

$$E_1 = \bigoplus_{[w]\in W^\mu/\Gamma_{E_s}} E_{1,[w]}$$

into subcomplexes where $E_{1,[w]}$ is the complex

$$\Big(\bigoplus_{\substack{I_{[w]}\subset I \\ \#(\Delta-I)=1}} i^J_{P_I} \otimes ind_{[w]}(-l([w])) \to \bigoplus_{\substack{I_{[w]}\subset I \\ \#(\Delta-I)=2}} i^J_{P_I} \otimes ind_{[w]}(-l([w])) \to \ldots$$
$$\to i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w]))\Big)[-2l([w])].$$

Thus we get for E_2 the following terms:

$$\begin{array}{rcll}
I_{[w]} = \Delta & : & E_{2,[w]}^{p,q} & = 0 \;\; p\geq 0, q\geq 0 \\
\#(\Delta\setminus I_{[w]}) = 1 & : & E_{2,[w]}^{0,2l([w])} & = i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])) \\
 & & E_{2,[w]}^{p,q} & = 0 \;\; (p,q)\neq (0,2l([w])) \\
\#(\Delta\setminus I_{[w]}) > 1 & : & E_{2,[w]}^{0,2l([w])} & = i^J_J \otimes ind_{[w]}(-l([w])) \\
 & & E_{2,[w]}^{j,2l([w])} & = 0, \;\; j=1,\ldots,\#(\Delta\setminus I_{[w]})-2 \\
 & & E_{2,[w]}^{j,2l([w]))} & = v^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])), \;\; j=\#(\Delta\setminus I_{[w]})-1 \\
 & & E_{2,[w]}^{p,q} & = 0, \;\; q\neq 2l([w]) \text{ or } p > \#(\Delta\setminus I_{[w]})-1.
\end{array}$$

Since E_s is a local field we conclude as in [loc.cit.] for weight reasons that $E_2{=}E_\infty$ for $n = \ell^a, a \gg 0$, ℓ a prime number with $(\ell, p) = 1$. In fact, B. Totaro observed the validity of this equality for all $n \in \mathbb{N}$, since all the appearing objects in E_2 are free modules over their base ([SS]). Thus we have for $n \in \mathbb{N}$,

$$gr^p(H^r_{\acute{e}t}(Y^{ad}, \mathbb{Z}/n\mathbb{Z})) = E_\infty^{p,r-p} = E_2^{p,r-p} = \bigoplus_{[w]\in W^\mu} E_{2,[w]}^{p,r-p}$$

$$= \begin{cases} \bigoplus\limits_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ \#(\Delta\setminus I_{[w]})=1 \\ 2l([w])=r}} i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])) \oplus \bigoplus\limits_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ \#(\Delta\setminus I_{[w]})>1 \\ 2l([w])=r}} i^J_J \otimes ind_{[w]}(-l([w])) & : \quad p=0 \\ \bigoplus\limits_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ 2l([w])+\#(\Delta\setminus I_{[w]})-1=r \\ p=\#(\Delta\setminus I_{[w]})-1}} v^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])) & : \quad p>0 \end{cases}$$

A well-known conjecture claims that in the category of admissible representations (with coefficients in $\mathbb{Q}_\ell$) we have

$$Ext^i(v^J_{P_I}, v^J_{P_{I'}}) = \begin{cases} \mathbb{Q}_\ell & i = |\#I - \#I'| \\ 0 & \text{else} \end{cases}.$$

For naive reasons one might hope that the same formula holds for the corresponding representations with values in $\mathbb{Z}/n\mathbb{Z}$. In this case consider for instance an equality

$$2l([w]) + \#(\Delta \setminus I_{[w]}) - 1 = r = 2l([w']) + \#(\Delta \setminus I_{[w']}) - 1.$$

If $l([w]) \neq l([w'])$ then $\#(\Delta\setminus I_{[w]})$ and $\#(\Delta\setminus I_{[w']})$ differ at least by two. Hence $\#I_{[w]}$ and $\#I_{[w']}$ differ at least by two. The other case is treated similarly. Thus we get

$$H^r_{\acute{e}t}(Y^{ad}) \cong \bigoplus_{p\in\mathbb{N}} gr^p(H^r_{\acute{e}t}(Y^{ad}))$$

$$= \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ \#(\Delta\setminus I_{[w]})=1 \\ 2l([w])=r}} i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])) \oplus \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ \#(\Delta\setminus I_{[w]})>1 \\ 2l([w])=r}} i^J_J \otimes ind_{[w]}(-l([w])) \oplus \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ 2l([w])+\#(\Delta\setminus I_{[w]})-1=r \\ p=\#(\Delta\setminus I_{[w]})-1}} v^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w])).$$

Summarizing the discussion above, we get:

Theorem 4.6 *The spectral sequence $E_1^{p,q}$ degenerates in the E_2-term and we get under assumption of the validity of the conjecture above*

$$\begin{aligned} H^*(Y^{ad}, \mathbb{Z}/n\mathbb{Z}) \; = & \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ \#(\Delta - I_{[w]})=1}} \Big(i^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w]))[-2l([w])] \Big) \oplus \\ & \bigoplus_{\substack{[w]\in W^\mu/\Gamma_{E_s} \\ \#(\Delta - I_{[w]})>1}} \Big(\Big(i^J_J \otimes ind_{[w]}(-l([w]))[-2l([w])] \Big) \oplus \\ & \oplus \Big(v^J_{P_{I_{[w]}}} \otimes ind_{[w]}(-l([w]))[-2l([w]) - \#(\Delta - I_{[w]}) + 1] \Big) \Big). \end{aligned}$$

The computation of $H^*_c(\mathcal{F}^{wa}_b, \mathbb{Z}/n\mathbb{Z})$ and hence the proof of Theorem 1.1 is shown by applying the long exact sequence

$$\cdots \to H^p_c(\mathcal{F}^{wa}_b, \mathbb{Z}/n\mathbb{Z}) \to H^p_{\acute{e}t}(\mathcal{F}^{ad}, \mathbb{Z}/n\mathbb{Z}) \to H^p_{\acute{e}t}(Y^{ad}, \mathbb{Z}/n\mathbb{Z}) \to H^{p+1}_c(\mathcal{F}^{wa}, \mathbb{Z}/n\mathbb{Z}) \to \cdots.$$

This is done in [O1], [O2]. Thus Theorem 1.1 is proved.

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