Bloch-Kato Conjecture and Main Conjecture of Iwasawa Theory for Dirichlet Characters

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by Annette Huber and Guido Kings

Univ. Leipzig
Math. Institut
PF 920
04009 Leipzig
Germany
huber@mathematik.uni-leipzig.de

Univ. Regensburg
NWF-I Mathematik
93040 Regensburg
Germany
guido.kings@mathematik.uni-regensburg.de

Introduction

The Tamagawa number conjecture proposed by Bloch and Kato in [BlKa] describes the "special values" of L-functions of motives in terms of cohomological data. Special value means here the leading coefficient of the Taylor series of an L-function at an integral point.

The main conjecture of Iwasawa theory describes a p-adic L-function in terms of the structure of Galois-modules, more precisely modules for the Iwasawa algebra. We prove:

Theorem 4.4.1, 5.1.3, 1.3.1: *Let χ be a Dirichlet character. Then the Main Conjecture of Iwasawa theory holds for χ and $p \neq 2$ and the Bloch-Kato conjecture holds up ot powers of 2 for $L(\chi, r)$ with $r \in \mathbb{Z}$.*

Already in the simplest case of the Tamagawa number conjecture – the Riemann ζ-function (see [BlKa] Theorem 6.1) – the Main Conjecture due to Mazur and Wiles is used by Bloch and Kato in an essential way. It was clear to the experts that the Main conjecture for a Dirichlet character and a prime p should be used in order to prove the p-part of the Bloch-Kato conjecture (see e.g. the list in Fontaine's Bourbaki talk [Fo] § 10 and [PR3]). There are two versions of the Main Conjecture depending on parity. Both are needed in order to deduce the Bloch-Kato conjecture in general. The first version involving the p-adic L-function was proved by Mazur and Wiles. Under the condition that p does not divide $\Phi(N)$, the second version involving cyclotomic units follows from the Mazur-Wiles case. Under the same restriction, it was also proved directly by Rubin using Kolyvagin's Euler systems. The missing cases need to be addressed.

What we want to advocate strongly is the insight, due to Kato in [Ka2] and [Ka1] and Perrin-Riou [PR3], that the Bloch-Kato conjecture and the Main Conjecture are two incarnations of the same mathematical content. Knowing the p-part of the Bloch-Kato conjecture for all fields in the cyclotomic tower is equivalent to the Main Conjecture formulated appropriately. Our approach is to take Kato's viewpoint seriously and to prove the Main Conjecture without any restriction on the prime p from the Bloch-Kato conjecture. As there is one basic case in which the Bloch-Kato conjecture is known – the analytic class number formula for $\zeta_F(0)$ where F is a number field – we can start a bootstrapping process which in the end proves both conjectures for all Dirichlet characters.

The elements of the proof are not really new, but basically copy Rubin's proof using Euler systems. What is new is the right formulation in which all problems with bad primes disappear. What is the problem? If p divides $\Phi(N)$, then the projectors to χ-eigenparts are not $\mathbb{Z}_p$-integral. It is not possible to decompose for example the class group of $\mathbb{Q}(\mu_N)$ into χ-eigenparts. Following Kato's idea, we formulate the Main conjecture as it is dictated by the Bloch-Kato conjecture. It turns out that the "zeta elements" of Kato lead to Euler systems, an observation already due to Kato. We then follow Rubin's proof using the machinery of Euler systems developed by Rubin and independently by Perrin-Riou and Kato. This general machinery leads to a divisibility statement for the characteristic power series of the modules involved. To get equality, the classical class number trick is replaced by the same trick with the Bloch-Kato conjecture for the Dedekind-ζ-function.

The use of the Bloch-Kato formulation has two important advantages over the more classical approach to Iwasawa theory. On the one hand it explains clearly why, depending on parity, cyclotomic units or the p-adic L-function appear in the formulation of the Main Conjecture. The reason is precisely that they interpret the L-value of the complex L-function in the "fundamental line". On the other hand the Bloch-Kato conjecture is "rational", so that a decomposition into characters is possible, i.e., one can formulate the conjecture for the motives attached to Dirichlet characters. The isogeny invariance of the formulation allows to choose the lattices in the Galois representations according to our needs. Thus we do not have to decompose a fixed lattice into eigenspaces of the characters. This is an advantage over the classical situation and allows to resolve the technical issues connected with the decomposition into eigenspaces.

The proof uses very few ingredients and a minimum of computations. As indicated earlier, we do not use any results on the Main Conjecture from the literature. We see this also as an advantage, as the literature is not free of mistakes and the computations with cyclotomic units are hard to follow. Here is the list of key ingredients:

1. standard results for Galois cohomology like Poitou-Tate duality;
2. the analytic class number formula for Dedekind-ζ-functions;
3. the explicit reciprocity law for $\mathbb{Z}(r)$ with $r \geq 2$ over unramified cyclotomic fields as proved by Kato [Ka2] or Perrin-Riou [PR4];
4. the theory of Euler systems due to Kolyvagin, Rubin, Kato and Perrin-Riou;
5. a comparison result on the image of cyclotomic elements in Deligne-cohomology and p-adic cohomology known as Bloch-Kato Conjecture 6.2 from [BlKa] (proved in [Hu-Wi] following Beilinson and Deligne; a second proof is given in [Hu-Ki]).

Shortly before finishing the first version of this paper, we learned at the Obernai conference that Burns and Greither had been working independently on nearly the same problem. They prove the equivariant Tamagawa number conjecture for the Tate-motive $\mathbb{Q}(r)$ and $r \leq 0$ over abelian number fields. This also allows to deduce

Tamagawa number formulas for the L-functions of Dirichlet characters at negative integers. Their proof is quite different, for example they use what was previously known on the Main conjecture and difficult computations of μ-invariants.

The Bloch-Kato conjecture for all Dirichlet characters immediately implies the Bloch-Kato conjecture for Dedekind-ζ-functions of abelian number fields. For negative values this amounts to the cohomological Lichtenbaum conjecture, see theorem 1.4.1. The reduction of the conjecture at positive values to the Lichtenbaum conjecture was shown independently by Benois and Nguyen Quang Do ([BenNg]). Kolster, Nguyen Quang Do and Fleckinger consider the latter in [KNF]. Together with the correction of Euler factors worked out by Kolster and Nguyen Quang Do in [KoNg], they prove the cohomological Lichtenbaum conjecture for abelian number fields up to an explicit set of bad primes. The problem are the bad primes in the above mentioned second version of the Main Conjecture. None of the references given in [KNF] 5.2, nor [BelNg] 3.2 (quoted in the [BenNg] A.2.3) seems to address this point correctly.

Overview: The text is organized as follows: In section 1 we state the Bloch-Kato conjecture for all Artin motives. This of course includes the case of motives attached to Dirichlet characters.

Section 2 starts with the proof of the Bloch-Kato conjecture by establishing compatibility with the functional equation and the special cases $r = 0$ and $r = 1$ for Dedekind-ζ-functions. In section 3 facts about cyclotomic elements and their relation with the Bloch-Kato conjecture are assembled.

Most of section 4 is independent from the previous results. The Main conjecture is stated and proved. Finally, the Bloch-Kato conjecture is proved from the Main conjecture in section 5. The last section 6 establishes a few facts about certain Iwasawa modules. These results are needed in the discussion of the Main conjecture.

Appendix A proves a well-known lemma on p-adic cohomology of local fields but where we did not find a reference. Appendix B establishes compatibility of the conjecture with the functional equation. This would also be a consequence of an unpublished result of Kato [Ka3].

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1 The Bloch-Kato conjecture for Artin motives

Our first aim is to present a formulation of the Bloch-Kato conjecture for Artin motives and their L-values at integral points. We follow Fontaine's approach in [Fo].

1.1 Artin motives, realizations and regulators

We work over the base field $\mathbb{Q}$ and with coefficients in some number field E. Let $G_{\mathbb{Q}}$ be the absolute Galois group of $\mathbb{Q}$ and $\mathcal{O}$ be the ring of integers of E. An *Artin motive over* $\mathbb{Q}$ *with coefficients in* E is a direct summand of the motive of a number

field in the category of all Grothendieck motives with coefficients in E. They form a well-defined abelian category. The *dual motive* of V is denoted $V^\vee$.
Example: Let F be a number field. Then $h^0(F)$ itself is an Artin motive. It is self-dual.
Example: A Dirichlet character is a homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times .$$

The conductor f is the smallest number such that χ factors through $(\mathbb{Z}/f\mathbb{Z})^\times$. The character is *primitive* if $f = N$. It is convenient to extend χ to $\mathbb{Z}/N\mathbb{Z}$ by $\chi(a) = 0$ if $a \in \mathbb{Z}/N\mathbb{Z} \smallsetminus (\mathbb{Z}/N\mathbb{Z})^\times$. Consider the isomorphism

$$rec : \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \xrightarrow{\cong} (\mathbb{Z}/N\mathbb{Z})^\times$$

which maps the geometric Frobenius Fr_p at $p \nmid N$ to p. Via this isomorphism we view χ as character

$$\chi : \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \to \mathbb{C}^\times .$$

Let E be the number field generated by all values of Dirichlet characters of $(\mathbb{Z}/N\mathbb{Z})^\times$. Explicitly, it is the field of $\Phi(N)$-th roots of unity, where $\Phi(N) = \#(\mathbb{Z}/N\mathbb{Z})^\times$. The character χ defines a rank one Artin motive $V(\chi)$ with coefficients in E as follows: $V(\chi)$ is the image of the projector $p_{\chi^{-1}}$ (*sic!*) on $h^0(\mathbb{Q}(\mu_N))$

$$\begin{aligned} p_{\chi^{-1}} : h^0(\mathbb{Q}(\mu_N)) &\to h^0(\mathbb{Q}(\mu_N)) \\ \alpha &\mapsto \frac{1}{\Phi(N)} \sum_{\sigma \in G} \chi(\sigma)\sigma^*\alpha. \end{aligned}$$

where we use the identification $G := \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times$. We call these *Dirichlet motives*. In fact,

$$h^0(\mathbb{Q}(\mu_N)) = \bigoplus_\chi V(\chi)$$

where the sum is over all characters of conductor dividing N. If F is an abelian number field, its motive is a direct summand of the motive of $h^0(\mathbb{Q}(\mu_N))$ for some N and hence (after extension of coefficients to big enough E) isomorphic to a direct sum of Dirichlet motives.

Let V be an Artin motive. V has *p-adic realizations* V_p for all primes p. They are $E_p = E \otimes \mathbb{Q}_p$-modules with a continuous operation of $G_\mathbb{Q}$.
Example: The p-adic realization of $V(\chi)$ is defined as

$$V_p(\chi) = p_{\chi^{-1}} H^0_{et}(\mathrm{Spec}\, F \otimes \overline{\mathbb{Q}}, \mathbb{Q}_p)$$

where the projector is taken with respect to the action of the Galois group of F over $\mathbb{Q}$. It is a $G_\mathbb{Q}$-module via the action on $\overline{\mathbb{Q}}$ over $\mathbb{Q}$. This also factors through $G(F/\mathbb{Q})$ but gives the contragredient representation. Hence $V_p(\chi)$ is the rank one E_p-module with operation of $G_\mathbb{Q}$ via χ.

Let V_B be the *Betti-realization* and V_{DR} the *de Rham realization* of V. They are finite dimensional E-vector spaces linked by an $E \otimes_\mathbb{Q} \mathbb{C}$-linear isomorphism $V_B \otimes_\mathbb{Q} \mathbb{C} \cong V_{\mathrm{DR}} \otimes_\mathbb{Q} \mathbb{C}$.

Example: For $V = h^0(F)$ with $E = \mathbb{Q}$, the Betti-realization V_B is given by the maps from the set of embeddings $F \to \mathbb{C}$ to $\mathbb{Q}$. The operation of $G(F/\mathbb{Q})$ is via $(gf)(\sigma) = f(\sigma g)$. It has a natural self-dual lattice T_B given by the functions with values in $\mathbb{Z}$. A natural basis is given by the functions δ_τ with $\delta_\tau(\tau) = 1$ and $\delta_\tau(\sigma) = 0$ for $\sigma \neq \tau$. The de Rham realization is $V_{\mathrm{DR}} = F \otimes_{\mathbb{Q}} E$.

Motivic cohomology of a number field is given by Adams eigenspaces of K-theory of its ring of integers. Taking direct summands this also defines motivic cohomology with coefficients in V. The only non-vanishing motivic cohomology groups are $H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Z}, V(r))$ for $r \geq 1$ and $H^0_{\mathcal{M}}(\operatorname{Spec}\mathbb{Z}, V)$. For $r \geq 1$ there is the *Beilinson regulator map*

$$r_\infty : H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Z}, V(r)) \to H^1_{\mathcal{D}}(\mathbb{R}, V_{\mathbb{R}}(r))$$

with values in real Deligne cohomology. We use the identification

$$H^1_{\mathcal{D}}(\mathbb{R}, V_{\mathbb{R}}(r)) \cong V_{\mathrm{DR}}(r)_{\mathbb{R}}/V_B(r)^+_{\mathbb{R}} \cong (V_B(r-1)_{\mathbb{R}})^+$$

where $+$ denotes the invariants under complex conjugation and where the second isomorphism is induced by $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}(-1)$. $r_\infty \otimes \mathbb{R}$ is an isomorphism for $r > 1$ by [Bo] and [Ra]. For $r = 0$ the cycle class map to singular cohomology induces

$$z : H^0_{\mathcal{M}}(\operatorname{Spec}\mathbb{Z}, V) \to (V_B \otimes \mathbb{R})^+ = H^1_{\mathcal{D}}(\mathbb{R}, V_{\mathbb{R}}(1)) .$$

Let S be a finite set of rational primes and $j : \operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}[1/Sp]$ the natural inclusion. We abbreviate $H^i(\mathbb{Z}[1/Sp], V_p(r)) := H^i_{et}(\operatorname{Spec}\mathbb{Z}[1/Sp], j_*V_p(r))$ where j_* is the direct image in the étale topology (sic, *not* the derived direct image). Note that for $V = h^0(F)$, we have the equality $H^i(\mathbb{Z}[1/Sp], V_p(r)) = H^i(\mathcal{O}_F[1/Sp], \mathbb{Q}_p(r))$ as the direct image of $\mathbb{Q}_p$ under the map $\operatorname{Spec}\mathcal{O}_F[1/Sp] \to \operatorname{Spec}\mathbb{Z}[1/Sp]$ is j_*V_p.

For all primes p, there are also *p-adic regulator maps*

$$r_p : H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Z}, V(r)) \to H^1(\mathbb{Z}[1/Sp], V_p(r))$$

induced by

$$r_p : H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(r)) \to H^1(\mathcal{O}_F[1/Sp], \mathbb{Q}_p(r)).$$

The map $r_p \otimes \mathbb{Q}_p$ is again an isomorphism for $r > 1$ see [So] Theorem 1. We will need a refined version of this regulator. Let $v \neq p$ be a rational prime. We denote by I_v its inertia group and by $\mathbb{F}_v$ the residue class field. One defines

$$H^1_f(\mathbb{Q}_v, V_p(r)) = H^1(\mathbb{F}_v, V_p(r)^{I_v}) .$$

For $v = p$ we put

$$D_{\mathrm{cris}}(V_p(r)) = (B_{\mathrm{cris}} \otimes V_p(r))^{G_{\mathbb{Q}_p}} \ , \ \tan(V_p(r)) = \left((B_{\mathrm{DR}}/Fil^0 B_{\mathrm{DR}}) \otimes V_p(r)\right)^{G_{\mathbb{Q}_p}} .$$

The latter vanishes for $r \leq 0$ and is naturally isomorphic to $V_{\mathrm{DR}}(r) \otimes \mathbb{Q}_p$ for $r \geq 1$.

We consider the fundamental short exact sequence of Bloch-Kato and Fontaine

$$0 \to \mathbb{Q}_p \to B_{\mathrm{cris}} \xrightarrow{(\phi-1,\pi)} B_{\mathrm{cris}} \oplus B_{\mathrm{DR}}/Fil^0 B_{\mathrm{DR}} \to 0,$$

where ϕ is the Frobenius on B_{cris} as in [Fo] 3.2 p. 210 footnote and π is the composition $B_{\mathrm{cris}} \to B_{\mathrm{DR}} \to B_{\mathrm{DR}}/Fil^0 B_{\mathrm{DR}}$. This sequence induces after tensoring with $V_p(r)$ and taking cohomology

$$0 \to H^0(\mathbb{Q}_p, V_p(r)) \to D_{\mathrm{cris}}(V_p(r)) \to D_{\mathrm{cris}}(V_p(r)) \oplus \tan(V_p(r)) \to H^1(\mathbb{Q}_p, V_p(r))$$

The space $H^1_f(\mathbb{Q}_p, V_p(r))$ is defined as the image of the last map. Under local duality, it is dual to $H^1_{/f}(\mathbb{Q}_p, V_p^\vee(1-r))$ where $H^1_{/f} = H^1/H^1_f$. Moreover, the p-adic exponential

$$\exp_p : \tan(V_p(r)) \to H^1(\mathbb{Q}_p, V_p(r))$$

is also induced from the same sequence. By definition it induces an isomorphism of $\tan(V_p(r))$ with $H^1_f(\mathbb{Q}_p, V_p(r))$ if $r \neq 0$. The local cohomology groups $H^1_f(\mathbb{Q}_p, V_p(r))$ are used to define

$$H^1_f(\mathbb{Z}[1/Sp], V_p(r)) = \{a \in H^1(\mathbb{Z}[1/Sp], V_p(r)) \mid a \in H^1_f(\mathbb{Q}_v, V_p(r)) \text{ for all } v \in Sp\}$$

Now, the p-adic regulator factors through

$$r_p : H^1_{\mathcal{M}}(\mathbb{Z}, V(r)) \to H^1_f(\mathbb{Z}[1/Sp], V_p(r))$$

and $r_p \otimes \mathbb{Q}_p$ is in fact always an isomorphism. (This follows from the previous isomorphism for $r > 1$, compatibility of H^1_f with local duality [BlKa] 3.8 and the explicit shape of $\exp_p$ for $r = 1$, [BlKa] p.358).

1.2 The Bloch-Kato conjecture for Artin motives

Attached to an Artin motive V, there is an $E \otimes \mathbb{C}$-valued L-function

$$L(V, s) = \prod_v \frac{1}{P_v(V, v^{-s})}$$

with v a rational prime and

$$P_v(V, t) = \det_{E_l}(1 - \mathrm{Fr}_v\, t \mid V_l^{I_v})$$

the characteristic polynomial of the geometric Frobenius Fr_v operating on the l-adic realization (I_v is the inertia group at v) for any auxiliary prime $l \neq v$. It is in fact an element of the polynomial ring $E[t]$. For $V = h^0(F)$ and $E = \mathbb{Q}$ this gives the Dedekind-ζ-function of F. Let $V(\chi)$ be a Dirichlet motive with coefficients in E big enough. Let N be the conductor of χ and consider χ as a map $\chi : \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \to E^*$. We identify

$$E \otimes \mathbb{C} \cong \bigoplus_{\tau: E \to \mathbb{C}} \mathbb{C}.$$

Then

$$L(V(\chi), s) = \left(L(\tau \circ \chi, s)\right)_\tau$$

where

$$L(\tau \circ \chi, s) = \sum_{n \geq 1} \frac{\tau \circ \chi(n)}{n^s}\ .$$

Definition 1.2.1. *We write $L(V,r)^* \in (E \otimes \mathbb{C})^*$ for the leading coefficient of the Taylor expansion of the $E \otimes \mathbb{C}$-valued function $L(V,s)$ at r.*

Definition 1.2.2 ([Fo] § 1). *Let V be an Artin motive. The* fundamental line *$\Delta_f(V(r))$ is the one-dimensional E-vector space*

$$\begin{array}{ll}\det_E H^0_{\mathcal{M}}(\mathbb{Z}, V(r)) \otimes \det_E^{-1} H^1_{\mathcal{M}}(\mathbb{Z}, V^\vee(1-r)) \otimes \det_E^{-1} V_B(r)^+ & \text{if } r \leq 0 \\ \det_E H^0_{\mathcal{M}}(\mathbb{Z}, V^\vee(1-r)) \otimes \det_E^{-1} H^1_{\mathcal{M}}(\mathbb{Z}, V(r)) \otimes \det_E^{-1} V_B(r)^+ \otimes \det_E V_{\mathrm{DR}}(r) & \text{if } r \geq 1\end{array}$$

Proposition 1.2.3 ([Fo] 6.10). *There is a natural isomorphism*

$$\Delta_f(V(r)) \otimes_{\mathbb{Q}} \mathbb{R} \cong E \otimes_{\mathbb{Q}} \mathbb{R} =: E_\infty$$

induced by the short exact sequence

$$0 \to H^0_{\mathcal{M}}(\mathbb{Z}, V(r))_{\mathbb{R}} \to V_B(r)^+_{\mathbb{R}} \to H^1_{\mathcal{M}}(\mathbb{Z}, V^\vee(1-r))^\vee_{\mathbb{R}} \to 0$$

in the case $r \leq 0$ and by the short exact sequence

$$0 \to H^1_{\mathcal{M}}(\mathbb{Z}, V(r))_{\mathbb{R}} \to V_B(r-1)^+_{\mathbb{R}} \to H^0_{\mathcal{M}}(\mathbb{Z}, V^\vee(1-r))^\vee_{\mathbb{R}} \to 0$$

together with the isomorphism $V_B(r-1)^+_{\mathbb{R}} \cong V_{\mathrm{DR}} \otimes \mathbb{R}/V_B(r)^+_{\mathbb{R}}$ in the case $r \geq 1$.

Proof. The maps are the cycle maps z and the Beilinson regulator r_∞ respectively their duals. □

Let S be a finite set of primes. We are going to define an isomorphism of $E_p := E \otimes_{\mathbb{Q}} \mathbb{Q}_p$-modules

$$\Delta_f(V(r)) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \det_{E_p} R\Gamma_c(\mathbb{Z}[1/Sp], V_p(r)) ,$$

where $R\Gamma_c(\mathbb{Z}[1/Sp], V_p(r))$ is defined in [Fo], i.e., it sits in an exact triangle

$$R\Gamma_c(\mathbb{Z}[1/Sp], V_p(r)) \to R\Gamma(\mathbb{Z}[1/Sp], V_p(r)) \to V_p(r)^+ \oplus \bigoplus_{v|pS} R\Gamma(\mathbb{Q}_v, V_p(r)).$$

As in [Fo] § 4, we define $R\Gamma_f(\mathbb{Q}_v, V_p(r)) \subset \tau_{\leq 1} R\Gamma(\mathbb{Q}_v, V_p(r))$ to be the subcomplex with $H^0_f(\mathbb{Q}_v, V_p(r)) = H^0(\mathbb{Q}_v, V_p(r))$ and $H^1_f(\mathbb{Q}_v, V_p(r))$ as in section 1.1. If $v \neq p$, we have a quasi isomorphism

$$R\Gamma_f(\mathbb{Q}_v, V_p(r)) \cong [V_p(r)^{I_v} \xrightarrow{\mathrm{Fr}_v - 1} V_p(r)^{I_v}].$$

If $v = p$, recall that $D_{\mathrm{cris}}(V_p(r)) = (B_{\mathrm{cris}} \otimes V_p(r))^{G_{\mathbb{Q}_p}}$. On this space the Frobenius ϕ of B_{cris} still acts. One has a quasi isomorphism

$$R\Gamma_f(\mathbb{Q}_p, V_p(r)) \cong [D_{\mathrm{cris}}(V_p(r)) \xrightarrow{(\phi-1,\pi)} D_{\mathrm{cris}}(V_p(r)) \oplus \tan(V_p(r))],$$

where π is the canonical projection. The above quasi isomorphisms induce by the theory of determinants isomorphisms

$$\det_{E_p} R\Gamma_f(\mathbb{Q}_v, V_p(r)) \cong \det_{E_p}[V_p(r)^{I_v} \xrightarrow{\mathrm{Fr}_v - 1} V_p(r)^{I_v}],$$

for $v \neq p$ and

$$\det_{E_p} R\Gamma_f(\mathbb{Q}_p, V_p(r)) \cong \det_{E_p}[D_{\mathrm{cris}}(V_p(r)) \xrightarrow{(\phi-1,\pi)} D_{\mathrm{cris}}(V_p(r)) \oplus \tan(V_p(r))],$$

for $v = p$. The determinants of the latter complexes are equal to $\det_{E_p} V_p(r)^{I_v} \otimes \det_{E_p}^{-1} V_p(r)^{I_v}$ and $\det_{E_p} D_{\mathrm{cris}}(V_p(r)) \otimes \det_{E_p}^{-1} D_{\mathrm{cris}}(V_p(r)) \otimes \det_{E_p}^{-1} \tan(V_p(r))$ respectively. As these are tensor products of a rank one E_p-module and its dual, these are identified with E_p and $\det_{E_p}^{-1} \tan_{V_p(r)}$ respectively. We record this in a definition:

Definition 1.2.4. *We identify*

$$\alpha : \det_{E_p} R\Gamma_f(\mathbb{Q}_v, V_p(r)) \xrightarrow{\cong} \begin{cases} E_p & v \neq p \text{ or } r \leq 0 \\ \det_{E_p}^{-1} \tan(V_p(r)) & v = p, r \geq 1 \end{cases}$$

where the isomorphisms are the ones discussed above.

One has to be very careful in working with these isomorphisms. Assume that $\phi - 1$ is an isomorphism. This implies that the complex $[D_{\mathrm{cris}}(V_p(r)) \xrightarrow{(\phi-1,\pi)} D_{\mathrm{cris}}(V_p(r)) \oplus \tan(V_p(r))]$ is quasi isomorphic to $\tan(V_p(r))$. Hence the theory of determinants gives an isomorphism $\beta : [D_{\mathrm{cris}}(V_p(r)) \xrightarrow{\phi-1} D_{\mathrm{cris}}(V_p(r)) \oplus \tan(V_p(r))] \cong \tan(V_p(r))$.

Proposition 1.2.5. *Under the above conditions*

$$\alpha = P_p(V, p^{-r})\beta$$

where P_p is as before the characteristic polynomial of Fr_p.

Proof. As α and β are the identity on $\det \tan(V_p(r))$ we may as well assume it to be E_p. Let $d_1, \ldots, d_n$ be a basis of $D = D_{\mathrm{cris}}(V_p(r))$. Then $d = d_1 \wedge \cdots \wedge d_n$ is a basis of $\det D$. By definition $\alpha(d \otimes d^{-1}) = 1$ and $\beta(d \otimes (\det(\phi - 1)d)^{-1}) = 1$. This implies $\beta = \det(\phi - 1)^{-1}\alpha$. By the normalization of ϕ ([Fo] 3.2, footnote p. 210), $\det(\phi - 1)$ is the characteristic polynomial of Fr_p (up to sign). ∎

Remark: The reason for using this normalization lies in the following fact: the formulation of Fontaine and Perrin-Riou is concerned with the full L-function. As we will see later, the determinant of the (cyclotomic) elements in Galois cohomology $H^1(\mathbb{Z}[1/pS], V_p(r))$ is related to the L-function with the Euler factors at pS removed. The difference between these two L-functions are the Euler factors at pS, which are exactly introduced by the above normalization of the determinant of $R\Gamma_f(\mathbb{Q}_v, V_p(r))$.

Again as in [Fo] § 4 we put:

$$R\Gamma_{/f}(\mathbb{Q}_v, V_p(r)) = \mathrm{Cone}\,[R\Gamma_f(\mathbb{Q}_v, V_p(r)) \to R\Gamma(\mathbb{Q}_v, V_p(r))]$$
$$R\Gamma_f(\mathbb{Z}[1/Sp], V_p(r)) = \mathrm{Cone}\,\left[R\Gamma(\mathbb{Z}[1/Sp], V_p(r)) \to R\Gamma_{/f}(\mathbb{Q}_p, V_p(r))\right][-1]$$

Lemma 1.2.6 ([Fo] 4.4). *Let V be an Artin motive. Let S be a finite set of primes.*

1. *For $r > 1$, the complex $R\Gamma_f(\mathbb{Z}[1/Sp], V_p(r))$ is concentrated in degree 1 and its first cohomology is isomorphic to $H^1(\mathbb{Z}[1/Sp], V_p(r))$.*

2. *For $r = 1$, the complex is concentrated in degrees 1 and 3. There is a distinguished triangle*

$$H^1_f(\mathbb{Z}[1/Sp, V_p(1))[-1] \to R\Gamma_f(\mathbb{Z}[1/Sp], V_p(1)) \to H^0(\mathbb{Z}[1/Sp], V_p^\vee)^\vee[-3] \ .$$

3. *For $r < 0$, the complex $R\Gamma_f(\mathbb{Z}[1/Sp], V_p(r))$ is concentrated in degree 2 and its second cohomology is isomorphic to $H^1(\mathbb{Z}[1/Sp], V_p^\vee(1-r))^\vee$.*

4. *For $r = 0$, the complex $R\Gamma_f(\mathbb{Z}[1/Sp], V_p(r))$ is concentrated in degrees 0 and 2. There is a distinguished triangle*

$$H^0(\mathbb{Z}[1/Sp], V_p) \to R\Gamma_f(\mathbb{Z}[1/Sp], V_p) \to H^1_f(\mathbb{Z}[1/Sp], V_p^\vee(1))^\vee[-2]$$

Proof. See [Fo] p. 215-216. The maps are either inclusions or their duals via the localization sequence. □

Note finally, that there is a natural distinguished triangle (see [Fo] p. 215)

$$R\Gamma_c(\mathbb{Z}[1/Sp], V_p(r)) \to R\Gamma_f(\mathbb{Z}[1/Sp], V_p(r)) \to V_p(r)^+ \oplus \bigoplus_{v \in Sp} R\Gamma_f(\mathbb{Q}_v, V_p(r))$$

Definition 1.2.7. *We identify*

$$\begin{aligned}
&\det_{E_p} R\Gamma_c(\mathbb{Z}[1/Sp], V_p(r)) \\
&\qquad \cong \det_{E_p} R\Gamma_f(\mathbb{Z}[1/Sp], V_p(r)) \otimes \det^{-1}_{E_p} V_p(r)^+ \otimes \det^{-1}_{E_p} \bigoplus_{v \in Sp} R\Gamma_f(\mathbb{Q}_v, V_p(r)) \\
&\qquad \cong \det_{E_p} R\Gamma_f(\mathbb{Z}[1/Sp], V_p(r)) \otimes \det^{-1}_{E_p} V_p(r)^+ \otimes \det_{E_p} \tan(V_p(r)) \\
&\qquad \cong \Delta_f(V(r)) \otimes_{\mathbb{Q}_p} E_p
\end{aligned}$$

where the second isomorphism uses 1.2.4 and the last is by term by term comparison via the p-adic regulators (and their duals) together with the isomorphism of $\tan(V_p(r))$ *with* $V_{\mathrm{DR}}(r) \otimes \mathbb{Q}_p$ *in the case $r \geq 1$.*

We can now formulate the Tamagawa number conjecture of Bloch and Kato. The precise formulation is taken from Fontaine's Bourbaki talk [Fo] who also shows equivalence to the original statement. In his formulation a set of primes S appears, but see 1.2.9 below.

Conjecture 1.2.8 (Bloch-Kato). *Let V be an Artin motive over $\mathbb{Q}$ with coefficients in a number field E and $r \in \mathbb{Z}$. Denote by $L(V,r)^*$ the leading coefficient of the Taylor expansion of the L-function of V at r. Let $\delta \in \Delta_f(V(r)) \otimes_{\mathbb{Q}} \mathbb{R}$ be such that $L(V,r)^*\delta$ is mapped to 1 under the isomorphism of 1.2.3*

$$\Delta_f(V(r)) \otimes_{\mathbb{Q}} \mathbb{R} \cong E \otimes_{\mathbb{Q}} \mathbb{R} \ .$$

Then, $\delta \in \Delta_f(V(r))$ and for all primes p, the image of δ under the identification of 1.2.7

$$\Delta_f(V(r)) \otimes \mathbb{Q}_p \cong \det_{E_p} R\Gamma_c(\mathbb{Z}[1/p], V_p(r))$$

is a generator of $\det_{\mathcal{O}_p} R\Gamma_c(\mathbb{Z}[1/p], T_p(r))$ *where $\mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p$ and T_p is any $G_{\mathbb{Q}}$-stable $\mathcal{O}_p$-lattice in V_p.*

Remark: The element $\delta \in \Delta_f(V(r))$ is uniquely determined up to units in $\mathcal{O} = \mathcal{O}_E$ by the integrality condition. Hence the above can be read as a formula for $L(V,r)^*$. We are going to make the explicit translation at least in a special case in section 1.4.

Proposition 1.2.9. a) *The conjecture is well-posed, i.e., independent of the choice of lattice T_p.*

b) *It is equivalent to the conjecture formulated by Fontaine in [Fo] and hence to the original conjecture in [BlKa].*

Proof. a) Let $T_p' \subset T_p$ be a sublattice. The quotient is a finite group, say Q. The assertion now follows from the computation of Galois-cohomology with torsion coefficients and the fact that the Euler characteristic of $R\Gamma_c(\mathbb{Z}[1/p], Q)$ is 1, see e.g. [Mi] I§ 5 Theorem 5.1.

b) Fontaine's formulation of the conjecture coincides with the above but he replaces $R\Gamma_c(\mathbb{Z}[1/p], T_p(r))$ by $R\Gamma_c(\mathbb{Z}[1/Sp], T_p(r))$ where S is any set of primes including the primes of bad reduction of V. What we have to show is independence of S (including all primes of bad reduction or not.) Hence let v be a prime different from p which is not in S. We only have to remark that there is a localization sequence

$$R\Gamma_c(\mathbb{Z}[1/Svp], T_p(r)) \to R\Gamma_c(\mathbb{Z}[1/Sp], T_p(r)) \to R\Gamma(\mathbb{F}_v, T_p(r)^{I_v})$$

whether v is a good reduction prime or not (see e.g. [Mi] II prop. 2.3 d)). Then

$$R\Gamma(\mathbb{F}_v, V_p(r)^{I_v}) = R\Gamma_f(\mathbb{Q}_v, V_p(r))$$

and the isomorphisms in 1.2.7 are compatible with enlarging S.

The equivalence with original conjecture of Bloch and Kato was shown by Fontaine in [Fo]. $\square$

It is often useful to reformulate the p-adic condition in terms of global duality, which gives for $p \neq 2$

$$\det R\Gamma_c(\mathbb{Z}[1/p], V_p(r)) \otimes \det V_p(r)^+ \cong \det R\Gamma(\mathbb{Z}[1/p], V_p^\vee(1-r)).$$

Proposition 1.2.10. *Let T_p be a Galois stable lattice in V_p and $p \neq 2$. Then the integral structures on the left resp. right hand side of the last formula given by*

$$\det R\Gamma_c(\mathbb{Z}[1/p], T_p(r)) \otimes \det T_p(r)^+ \text{ resp. } \det R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r))$$

agree.

Remark: For $p = 2$ the complex $R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r))$ is not perfect and one has to consider $\tau_{\leq 2} R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r))$ instead. Moreover $T_p(r)^+ = H^0(\mathbb{R}, T_p(r))$ has to be substituted by the zeroeth Tate-cohomology of $T_p(r)$.

Proof. Let * denote the Pontryagin dual $\mathrm{Hom}(\cdot, E_p/\mathcal{O}_p)$. Global duality implies

$$\det R\Gamma_c(\mathbb{Z}[1/p], T_p(r)) \otimes \det T_p(r)^+ \cong \det{}^{-1} R\Gamma(\mathbb{Z}[1/p], T_p^*(1-r))^*$$

under the above rational isomorphism. Hence we have to show that the integral structures $\det R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r))$ and $\det^{-1} R\Gamma(\mathbb{Z}[1/p], T_p^*(1-r))^*$ in the E_p-module $\det R\Gamma(\mathbb{Z}[1/p], V_p^\vee(1-r))$ coincide. For a bounded complex A, we have

$$R\operatorname{Hom}(A, \mathcal{O}_p) = R\operatorname{Hom}(A, R\operatorname{Hom}(E_p/\mathcal{O}_p, E_p/\mathcal{O}_p)) \\ = R\operatorname{Hom}(A \otimes^{\mathbb{L}} E_p/\mathcal{O}_p, E_p/\mathcal{O}_p) = (A \otimes^{\mathbb{L}} E_p/\mathcal{O}_p)^*$$

Moreover,

$$R\Gamma(\mathbb{Z}[1/p], T_p^*(1-r)) = \\ R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r) \otimes^{\mathbb{L}}_{\mathcal{O}_p} E_p/\mathcal{O}_p) = R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r)) \otimes^{\mathbb{L}}_{\mathcal{O}_p} E_p/\mathcal{O}_p$$

because $R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r) \otimes^{\mathbb{L}}_{\mathcal{O}_p} E_p) = R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r)) \otimes^{\mathbb{L}}_{\mathcal{O}_p} E_p$. Together this implies

$$\det^{-1} R\Gamma(\mathbb{Z}[1/p], T_p^*(1-r))^* = \det^{-1}\left(R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r)) \otimes^{\mathbb{L}}_{\mathcal{O}_p} E_p/\mathcal{O}_p\right)^* \\ = \det^{-1} R\operatorname{Hom}(R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r)), \mathcal{O}_p) = \det R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r))$$

□

Remark: Kato formulates versions of the Tamagawa number conjecture for values at positive integers in [Ka1] and for values at negative integers in [Ka2]. The above proposition implies that his conjectures are equivalent to the ones formulated by Fontaine. Note that in the formulation of the Tamagawa number conjecture for the full L-function the integral structure of the fundamental line defined using $\det R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r))$ still involves the identification 1.2.4 of the local factor.

1.3 Main Theorem

We now state our main theorem on the Bloch-Kato conjecture and sum up the known cases.

Theorem 1.3.1. *The Bloch-Kato conjecture 1.2.8 holds up to powers of 2 for all $r \in \mathbb{Z}$ and all abelian Artin motives, i.e., for Artin motives which are direct sums of direct summands of $h^0(F)$'s where F is an abelian number field. In particular, it holds for all Dirichlet motives.*

1. Some parts of the proof certainly work for the prime 2 but we did not check whether there are serious problems or not.

2. The case $V = h^0(\mathbb{Q})$ already appears in the original paper by Bloch and Kato [BlKa] Theorem 6.1 (conjecture 6.2 has meanwhile been proved in [Hu-Wi].) In [Ka1] § 6 Kato proves the p-part of the equivariant conjecture for $\mathbb{Q}(\mu_{p^n})^+$ for $r \geq 2$ even.

3. The Beilinson conjecture for Dirichlet motives, i.e. the first part of conjecture 1.2.8, was already proved by Beilinson in [Be] (with some corrections in [Ne] and [Es]).

4. Fontaine [Fo] § 10 mentions the case of Dirichlet motives $V(\chi)$ but assumes that p^2 resp. p does divide N or $\phi(N)$ (where N is the conductor of χ) depending on parity conditions.

5. The Bloch-Kato conjecture for $V = h^0(F)$ with coefficients in $\mathbb{Q}$ and $r \leq 0$ is easily seen to be equivalent to the cohomological Lichtenbaum conjecture (see 1.4.1 below). The case $V = h^0(F)$ where F is totally real and $r \leq 0$ is odd is a direct consequence of the main conjecture due to Wiles. In [KNF] the cohomological Lichtenbaum conjecture is treated for abelian number fields. Together with the correction of Euler factors worked out by Kolster and Nguyen Quang Do in [KoNg], they prove the cohomological Lichtenbaum conjecture for abelian number fields up to an explicit set of bad primes. It follows (for all primes $p \neq 2$) as a corollary of our main theorem 1.3.1.

6. In recent work [BenNg], Benois and Nguyen show how to reduce the Bloch-Kato conjecture for $h^0(F)$ and $r \geq 1$ to the case of $r \leq 0$ by showing compatibility under the functional equation. See also section 2.2 below.

7. Recall that the Beilinson conjecture (which determines the L-value up to rational factor) is known for $h^0(F)$ where F is any number field. However, it is not known for all Artin motives!

8. There is also an equivariant version of the Bloch-Kato conjecture, formulated by Kato in the abelian case and by Burns-Flach in general (see also conjecture 1.5.2 below). A proof of the equivariant conjecture for Tate motives $\mathbb{Q}(r)$ and $r \leq 0$ over abelian number fields and $p \neq 2$ is given by Burns and Greither in [BuGr]. Our theorem 1.3.1 is equivalent to the equivariant conjecture with respect to the maximal order $\widetilde{\mathbb{Z}[G]}$ in $\mathbb{Q}[G]$ and hence for $r \leq 0$ a consequence of the result of Burns and Greither. Using a key observation by Burns and Greither, it is also possible to deduce the equivariant case from theorem 1.3.1.

1.4 Relation to Lichtenbaum conjecture

The above theorem 1.3.1 can be restated to give a formula for the L-value. For simplicity of notation, we do this in the case $E = \mathbb{Q}$, $V = h^0(F)$ where F is a number field and for $r \leq -1$.

Theorem 1.4.1 (Cohomological Lichtenbaum conjecture). *Let F be a number field, $k \geq 2$. Let $R_k(F)$ be the Beilinson regulator of F, i.e., the covolume of the image of $K_{2k-1}(F)$ in $h^0(F)^\vee(k-1)^+_{\mathbb{R}}$ under the Beilinson regulator map. Then the Bloch-Kato conjecture for $h^0(F)$ and $1-k$ implies that up to sign and up to powers of 2,*

$$\zeta_F(1-k)^* = R_k(F) \prod_p \frac{\#H^2(\mathcal{O}_F[1/p], \mathbb{Z}_p(k))}{\#H^1(\mathcal{O}_F[1/p], \mathbb{Z}_p(k))_{\mathrm{tors}}} .$$

In particular, the cohomological Lichtenbaum conjecture holds for abelian number fields F.

Proof. Let us show that conjecture 1.2.8 with $r = 1 - k \leq -1$ implies this formula for the L-value. Let $T_B^\vee(-r)^+$ be the natural integral structure in $h^0(F)^\vee(-r)^+_{\mathbb{R}}$ and $\Omega \subset H^1_{\mathcal{M}}(F, \mathbb{Q}(1-r))$ the quotient of $K_{2k-1}(F)$ by the torsion subgroup. In this case

$$\Delta_f(V(r)) = \det{}^{-1} H^1_{\mathcal{M}}(F, \mathbb{Q}(1-r)) \otimes \det{}^{-1} \left(h^0(F)_B(r)\right)^+ .$$

In the fundamental line one has the lattice $\det^{-1}\Omega \otimes \det^{-1} T_B(r)^+$. By definition

$$\det \Omega = R_{1-r} \det T_B^\vee(-r)^+ = R_{1-r} \det{}^{-1} T_B(r)^+$$

where R_{1-r} is the Beilinson regulator of F. Recall that δ is the element of the fundamental line corresponding to $1/\zeta_F(r)^*$ under its identification with $\mathbb{R}$. In terms of the above lattice

$$\delta\mathbb{Z} = \frac{R_{1-r}}{\zeta_F(r)^*} \det{}^{-1} \Omega \otimes \det{}^{-1} T_B(r)^+$$

Then conjecture 1.2.8 implies that δ is mapped to a generator of $\det R\Gamma_c(\mathbb{Z}[1/p], T_p(r))$ under the isomorphism $\Delta_f(V(r)) \otimes \mathbb{Q}_p \cong \det R\Gamma_c(\mathbb{Z}[1/p], V_p(r))$. In our case

$$R\Gamma_f(\mathbb{Z}[1/p], V_p(r)) \cong H^1(\mathbb{Z}[1/p], V_p^\vee(1-r))^\vee[-2] = R\Gamma(\mathbb{Z}[1/p], V_p^\vee(1-r))^\vee[-3]$$

The isomorphism of the fundamental line was defined by

$$\begin{aligned}\Delta_f(V(r)) \otimes \mathbb{Q}_p &\xrightarrow{\det r_p^\vee \otimes \mathrm{id}} \det H^1(\mathbb{Z}[1/p], V_p^\vee(1-r))^\vee \otimes \det{}^{-1} V_p(r)^+ \\ &\xrightarrow{\mathrm{id} \otimes \alpha^\vee} \det H^1(\mathbb{Z}[1/p], V_p^\vee(1-r))^\vee \otimes \det{}^{-1} V_p(r)^+ \otimes \det{}^{-1} R\Gamma_f(\mathbb{Q}_p, V_p(r))\end{aligned}$$

with α as in 1.2.4. By lemma 1.2.5 this isomorphism differs from the one induced by $\det r_p^\vee \otimes \mathrm{id}$ alone by the Euler factor $P_p(V, p^{-r}) = \prod_{v|p}(1 - N(v)^{k-1})$. This is a p-unit as $k \geq 2$.

Using proposition 1.2.10 this implies that $r_p(\delta)$ is mapped to a generator of

$$\det R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r)) \otimes \det{}^{-1} T_p(r)^+$$

or equivalently,

$$\begin{aligned}\frac{R_{1-r}}{\zeta_F(r)^*}\mathbb{Z}_p &= \det r_p(\Omega \otimes \mathbb{Z}_p) \otimes \det R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1-r)) \\ &= \det r_p(\Omega \otimes \mathbb{Z}_p) \otimes \det{}^{-1} H^1(\mathbb{Z}[1/p], T_p^\vee(1-r)) \otimes \det H^2(\mathbb{Z}[1/p], T_p^\vee(1-r)) \\ &= \det{}^{-1}(H^1(\mathbb{Z}[1/p], T_p^\vee(1-r))/r_p(\Omega \otimes \mathbb{Z}_p)) \otimes \det H^2(\mathbb{Z}[1/p], T_p^\vee(1-r))\end{aligned}$$

Write again $k = 1 - r$. We have $H^i(\mathbb{Z}[1/p], T_p^\vee(k)) = H^i(\mathcal{O}_F[1/p], \mathbb{Z}_p(k))$. Soulé has proved that the map $K_{2k-1}(F) \otimes \mathbb{Z}_p \to H^1(\mathcal{O}_F[1/p], \mathbb{Z}_p(r))$ is surjective and the ranks agree by the computation of Borel. Hence $r_p(\Omega \otimes \mathbb{Z}_p)$ is identified with the free part of $H^1(\mathcal{O}_F[1/p], \mathbb{Z}_p(k))$, i.e., the quotient is $H^1(\mathcal{O}_F[1/p], \mathbb{Z}_p(k))_{\mathrm{tors}}$. Note finally that a finite $\mathbb{Z}_p$-module of order h has determinant $h^{-1}\mathbb{Z}_p$. □

Many cases were known before, see the discussion after theorem 1.3.1. Lichtenbaum's original conjecture [Li] involves the Borel regulator rather than the Beilinson regulator and K-groups rather than Galois cohomology. The missing ingredient is the Quillen-Lichtenbaum conjecture

$$K_{2r-i}(F) \otimes \mathbb{Z}_p \cong H^i(\mathcal{O}_F[1/p], \mathbb{Z}_p(r))$$

for $i = 1, 2$ and $r \geq 2$.

1.5 The equivariant conjecture

There is also an equivariant version of the Tamagawa number conjecture. It was first introduced by Kato for abelian groups in [Ka1]. The case of non-abelian groups is treated by Burns and Flach in [BuFl]. We are going to treat the abelian case.

As before let M be an Artin motive over $\mathbb{Q}$ with coefficients in E. Let K be an abelian extension of $\mathbb{Q}$ with Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. Let $X(K/\mathbb{Q})$ be the set of $\mathbb{C}$-valued characters of G. We assume for simplicity that E contains all values of elements of G. Note that we have

$$E[G] = \bigoplus_{\omega \in X(K/\mathbb{Q})} E(\omega)$$

where G operates on $E(\omega)$ via ω^{-1}. For every Artin motive M let $M(\omega) := M \otimes V(\omega)$.

Definition 1.5.1. *The equivariant L-value*

$$L(K/\mathbb{Q}, M, r)^* \in E_\infty[G]^*$$

has ω-component $L(M(\omega), r)^*$. The equivariant fundamental line is defined as

$$\Delta_f(K/\mathbb{Q}, M(r)) = \bigoplus_{\omega \in X(K/\mathbb{Q})} \Delta_f(M(\omega)(r))$$

Remark:

1. $L(K/\mathbb{Q}, M, r)^*$ is an element of $E_\infty[G] = E \otimes \mathbb{R}[G]$ rather than $E \otimes \mathbb{C}[G]$ as the element is invariant under complex conjugation on coefficients.

2. Let M be an Artin motive with coefficients in E. Then $L(K/\mathbb{Q}, M, r)^*$ is indeed an element in $E_\infty[G]$, even though the ω-components have coefficients in some bigger field E'. This is seen by checking invariance of $L(K/\mathbb{Q}, M, r)^*$ under $\mathrm{Gal}(E'/E)$.

3. If K'/K is an extension such that $K'/\mathbb{Q}$ is still abelian, then $L(K'/\mathbb{Q}, M, r)^*$ is mapped to $L(K/\mathbb{Q}, M, r)$ under the norm map.

The absolute comparison morphisms for all ω together give natural isomorphisms

$$\Delta_f(K/\mathbb{Q}, M(r)) \otimes \mathbb{R} \cong E_\infty[G]$$
$$\Delta_f(K/\mathbb{Q}, M(r)) \otimes \mathbb{Q}_p \cong \det_{E_p[G]} R\Gamma_c(\mathbb{Z}[1/p], \bigoplus_\omega M_p(\omega)(r))$$

The isormophism $\bigoplus_\omega M_p(\omega) \cong M_p \otimes E_p[G]$ induces

$$R\Gamma_c(\mathbb{Z}[1/p], \bigoplus_\omega M_p(\omega)(r)) = R\Gamma_c(\mathcal{O}_K[1/p], M_p(r))$$

Now we can formulate the equivariant conjecture:

Conjecture 1.5.2 (Kato, Burns-Flach). *Let M be an Artin motive over $\mathbb{Q}$ with coefficients in E and $r \in \mathbb{Z}$. Let $K/\mathbb{Q}$ be an abelian extension with Galois group G. Let $\delta(K/\mathbb{Q}) \in \Delta_f(K/\mathbb{Q}, M(r)) \otimes \mathbb{R}$ be such that $L(K/\mathbb{Q}, M, r)^* \delta(K/\mathbb{Q})$ is mapped to 1 under the isomorphism with $E_\infty[G]$. Then $\delta(K/\mathbb{Q}) \in \Delta_f(K/\mathbb{Q}, M(r))$, and for all primes the image of $\delta(K/\mathbb{Q})$ under the isomorphism with $\det_{E_p[G]} R\Gamma_c(\mathcal{O}_K[1/p], M_p(r))$ is a generator of $\det_{\mathcal{O}_p[G]} R\Gamma_c(\mathcal{O}_K[1/p], T_p(r))$ where T_p is any $G_\mathbb{Q}$-stable lattice of M_p.*

Clearly, this is identical with conjecture 1.2.8 if $K = \mathbb{Q}$. Note that the ω-component of $\delta(K/\mathbb{Q})$ is nothing but δ for the motive $M(\omega)(r)$. Also the ω-component of the image of $\delta(K/\mathbb{Q})$ in $\det_{E_p[G]} R\Gamma(\mathcal{O}_K[1/p], M_p(r))$ is given by the image of δ (for $M(\omega)(r)$) in $\det_{E_p} R\Gamma(\mathbb{Z}[1/p], M_p(\omega)(r))$.

2 First steps in the proof

2.1 Overview

We start with a few reductions. As extension of coefficients is faithfully flat, it suffices to prove the conjecture after extension of coefficients. The conjecture is also compatible with direct sums of motives. As any abelian Artin motive decomposes into a direct sum of Dirichlet motives after a suitable extension of coefficients, it suffices to consider the case of Dirichlet motives $V(\chi)$. The proof is organized as follows:

1. We prove the Bloch-Kato conjecture directly in the case of the full motive $h^0(F)$ where F is a number field and $r = 0$, see section 2.3. By the compatibility with the functional equation (which can be checked directly in this case) the conjecture also holds for $h^0(F)$ and $r = 1$.

2. We then use Euler system methods to establish a divisibility statement for Iwasawa modules in the case $\chi(-1) = (-1)^r$, see 4.3.3. By the class number trick — with the class number formula replaced by the Bloch-Kato conjecture for $F = \mathbb{Q}(\mu_N)$ and $r = 1$ — we prove the Main Conjecture of Iwasawa theory from it, see 4.4.

3. Using Kato's explicit reciprocity law, we prove the Bloch-Kato conjecture for $r \geq 1$ and $\chi(-1) = (-1)^r$ from the Main Conjecture, see 5.1.1. The necessary computation was already used in the "class number trick" in the previous step. In fact, the argument proves the p-part of the equivariant Bloch-Kato conjecture for the same r and χ and the cyclotomic $\mathbb{Z}/p^n$-extension over $\mathbb{Q}$.

4. Using the precise understanding of the regulators of cyclotomic elements we prove the Bloch-Kato conjecture for $r \leq 0$ and $\chi(-1) = (-1)^r$ from the Main Conjecture, see 5.2.3. However, the argument does not work for $r = 0$ and $\chi(p) = 1$, the case of "trivial zeroes". The trivial zeroes will be treated in in the last step.

5. By the compatibility of the Bloch-Kato conjecture under the functional equation, we deduce the Bloch-Kato conjecture in the remaining cases for $r \neq 0, 1$, see 5.1.2 and 5.2.4. This works even equivariantly and hence shows the p-part of the equivariant conjecture for $r > 1$, $\chi(-1) = (-1)^{r-1}$ and the cyclotomic $\mathbb{Z}/p^n$-extension of $\mathbb{Q}$.

6. From the equivariant Bloch-Kato conjecture in the previous step, we can deduce the second version of the Main conjecture, see 4.2.4. Conversely, the new version allows to prove the Bloch-Kato conjecture for $r = 0$ and $r = 1$ unless there are trivial zeroes.

7. The last exceptional case $r = 0$ and $\chi(p) = 1$, i.e., the case of trivial zeroes, follows again by the functional equation see 5.4.1.

2.2 Functional equation

We first study the compatibility of the equivariant Bloch-Kato conjecture under the functional equation. The corresponding statements can already be found in [FoPR] in the absolute case ($K = \mathbb{Q}$) and in [BuFl] even for the case of non commutative coefficients.

Let $K/\mathbb{Q}$ be abelian and M an Artin motive with coefficients in E. As before let $X(K/\mathbb{Q})$ be the set of $\mathbb{C}$-valued characters of $G := \mathrm{Gal}(K/\mathbb{Q})$. For $\omega \in X(K/\mathbb{Q})$ define $M(\omega) := M \otimes V(\omega)$. Consider for each $M(\omega)$ and $r \geq 1$ the E-module of rank one

$$\Delta_{\mathrm{loc}}(M(\omega)(r)) = \det{}^{-1} M_B(\omega)(r)^+ \otimes \det M_{\mathrm{DR}}(\omega)(r) \otimes \det M_B(\omega)^\vee(1-r)^+$$

After tensoring with $\mathbb{R}$ we identify it with $E \otimes \mathbb{R}$ using the isomorphism

$$M_{\mathrm{DR}}(\omega)(r)_{\mathbb{R}} / M_B(\omega)(r)^+_{\mathbb{R}} \cong M_B(\omega)(r-1)^+_{\mathbb{R}}$$

Let $\varepsilon(\omega)(r)$ be the element of $\Delta_{\mathrm{loc}}(M(\omega)(r))$ corresponding to $\frac{L(M(\omega)^\vee, 1-r)^*}{L(M(\omega), r)^*}$ under this isomorphism. Define

$$\Delta_{\mathrm{loc}}(K/\mathbb{Q}, M(r)) := \bigoplus_{\omega \in X(K/\mathbb{Q})} \Delta_{\mathrm{loc}}(M(\omega)(r))$$

(this is a module of rank 1 over $E[G] = \bigoplus_\omega E(\omega)$) and let $\varepsilon(K/\mathbb{Q}, M(r)) := (\varepsilon(\omega)(r))_\omega \in \Delta_{\mathrm{loc}}(K/\mathbb{Q}, M(r))$. On the other hand, each $\Delta_{\mathrm{loc}}(M(\omega)(r)) \otimes E_p$ is isomorphic to

$$\det{}^{-1} R\Gamma(\mathbb{Q}_p, M_p(\omega)(r)) \otimes \det{}^{-1} M_p(\omega)(r)^+ \otimes \det M_p(\omega)^\vee(1-r)^+$$

via the p-adic exponential and using 1.2.4. Thus we get

$$\Delta_{\mathrm{loc}}(K/\mathbb{Q}, M(r)) \otimes E_p \cong \det^{-1}_{E_p[G]} R\Gamma(\mathbb{Q}_p \otimes K, M_p(r)) \otimes \det^{-1}_{E_p[G]} \bigoplus_{\omega \in X(K/\mathbb{Q})} M_p(\omega)(r)^+ \otimes \det_{E_p[G]} \bigoplus_{\omega \in X(K/\mathbb{Q})} M_p(\omega)^\vee (1-r)^+,$$

where we have used as before that $R\Gamma(\mathbb{Q}_p, \bigoplus_\omega M_p(\omega)(r)) \cong R\Gamma(\mathbb{Q}_p \otimes K, M_p(r))$. Observe that for any Galois-stable lattice T_p in M_p the $\mathcal{O}_p[G]$-module $T_p[G] := T_p \otimes \mathcal{O}_p[G]$ is a lattice in $\bigoplus_{\omega \in X(K/\mathbb{Q})} M(\omega)_p$. We also write $M_p[G] := M_p \otimes E_p[G]$ and similarly for M_B and M_{DR}.

Conjecture 2.2.1. *Let M be an Artin motive, $r \geq 1$, p a prime. Then the image of $\varepsilon(K/\mathbb{Q}, M(r))$ under the above isomorphism is a generator of*

$$\det^{-1}_{\mathcal{O}_p[G]} R\Gamma(\mathbb{Q}_p \otimes K, T_p(r)) \otimes \det^{-1}_{\mathcal{O}_p[G]} T_p[G](r)^+ \otimes \det_{\mathcal{O}_p[G]}(T_p[G])^\vee (1-r)^+$$

where T_p is any Galois-stable lattice in M_p.

Theorem 2.2.2 (Kato). *The above conjecture holds for $K/\mathbb{Q}$ abelian, M an abelian Artin motive and $r > 1$, $p \neq 2$.*

Proof. This was shown for the motive $\mathbb{Q}$ and $K = \mathbb{Q}$ by Bloch and Kato in [BlKa]. Perrin-Riou [PR1] deduced it in the case that p is unramified in M. Benois and Nguyen Quang Do in [BenNg] show it for the full motive $h^0(F)$ of an abelian number field. Finally, there is unpublished work of Kato [Ka3] which settles the conjecture for $K/\mathbb{Q}$ abelian and $M = \mathbb{Q}$, thus for all abelian Artin motives. In appendix B we give a proof in the special case for $K = \mathbb{Q}(\mu_{p^n})$, M arbitrary (abelian), $r > 1$ and $p \neq 2$ (B.1.3). □

Remark: The case $r = 1$, $K = \mathbb{Q}$ follows at the very end of our arguments from the Bloch-Kato conjecture at $r = 0$ and $r = 1$.

Proposition 2.2.3. *Let $p \neq 2$, $K/\mathbb{Q}$ abelian, M be an Artin motive and $r \geq 1$. Then two of the following assertions imply the third.*

1. *The p-part of the equivariant Bloch-Kato conjecture for $K/\mathbb{Q}$, M and r.*
2. *The p-part of the equivariant Bloch-Kato conjecture for $K/\mathbb{Q}$, $M^\vee$ and $1-r$.*
3. *Conjecture 2.2.1 for $K/\mathbb{Q}$, M, r and p.*

Proof. We give the argument for the convenience of the reader. By definition

$$\Delta_f(K/\mathbb{Q}, M(r)) \otimes \Delta_f(K/\mathbb{Q}, M^\vee(1-r))^\vee = \det^{-1}_{E[G]} M_B[G](r)^+ \otimes \det_{E[G]} M_{\mathrm{DR}}[G](r) \otimes \det_{E[G]}(M_B[G])^\vee (1-r)^+$$

The identification of the left hand side with $E[G] \otimes \mathbb{R}$ induced by 1.2.3 is compatible with the above identification of the right hand side with $E[G] \otimes \mathbb{R}$. The element $\delta(K/\mathbb{Q}, M(r)) \otimes \delta^{-1}(K/\mathbb{Q}, M^\vee, 1-r)$ is nothing but $\varepsilon(K/\mathbb{Q}, M(r))$.

On the other hand, the tensor product of the fundamental lines with E_p in 1.2.7 can be simplified using global duality (proposition 1.2.10 over K instead of $\mathbb{Q}$) and the localization sequence:

$$\Delta_f(K/\mathbb{Q}, M(r)) \otimes \Delta_f(K/\mathbb{Q}, M^\vee(1-r))^\vee \otimes E_p$$
$$\cong \det_{\mathcal{O}[G]} R\Gamma_c(\mathcal{O}_K[1/p], M_p(r)) \otimes \det^{-1}_{\mathcal{O}[G]} R\Gamma(\mathcal{O}_K[1/p], M_p(r)) \otimes \det_{\mathcal{O}[G]}(M_p[G])^\vee(1-r)^+$$
$$\cong \det^{-1}_{\mathcal{O}[G]} R\Gamma(\mathbb{Q}_p \otimes K, M_p(r)) \otimes \det^{-1}_{\mathcal{O}[G]} M_p[G](r)^+ \otimes \det_{\mathcal{O}[G]}(M_p[G])^\vee(1-r)^+.$$

This is the second term appearing in the local conjecture. The integral structures are compatible with this simplification by proposition 1.2.10 (over K). □

2.3 The class number formula case

We check the first case of the Bloch-Kato conjecture by hand. This is an old result, mentioned e.g. in [Fo] § 8, but we did not find a proof in the literature. As it is the crucial case to which all the other cases are finally reduced, we discuss it in some detail.

Proposition 2.3.1. *The Bloch-Kato conjecture holds in the case $V = h^0(F)$, $r = 0$ and $r = 1$ up to powers of 2.*

The rest of this section is devoted to the proof of this proposition. The arguments are parallel to the ones used in deducing the Lichtenbaum conjecture.

Proof. We start with the case $V = h^0(F)$ and $r = 0$. We write $H^i_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(j)) = H^i_{\mathcal{M}}(\mathbb{Z}, V(j))$. The fundamental line in this case is

$$\Delta_f(V) := \det H^0_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}) \otimes \det^{-1} H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(1)) \otimes \det^{-1} V_B^+.$$

Let T_B be the natural lattice in V_B (see section 1.1). Recall that $H^0_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}) = \mathbb{Q}$ and consider the splitting s_∞ of $(V_B^\vee)^+_{\mathbb{R}} \to H^0_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q})_{\mathbb{R}}$ which maps 1 to the dual of δ_σ for a fixed σ. Then $r_\infty \oplus s_\infty : H^1_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q}(1))_{\mathbb{R}} \oplus H^0_{\mathcal{M}}(\mathcal{O}_F, \mathbb{Q})_{\mathbb{R}} \to (V_B^\vee)^+_{\mathbb{R}}$ is an isomorphism and we consider the lattice

$$\det \mathbb{Z} \otimes \det^{-1} \mathcal{O}_F^{*,\mathrm{free}} \otimes \det^{-1} T_B^+$$

in $\Delta_f(V)$, where $\mathcal{O}_F^{*,\mathrm{free}}$ is the quotient of $\mathcal{O}_F^*$ by the group of roots of unity in F. Let $R_\infty(F) := \mathrm{vol}\left((T_B^\vee \otimes \mathbb{R})^+/(r_\infty(\mathcal{O}_F^{*,\mathrm{free}}) \oplus s_\infty(\mathbb{Z}))\right)$, then

$$\det r_\infty(\mathcal{O}_F^{*,\mathrm{free}}) \otimes \det s_\infty(\mathbb{Z}) = R_\infty(F) \det(T_B^\vee)^+ = R_\infty(F) \det^{-1} T_B^+.$$

This allows to identify the element δ which maps to $1/\zeta_F(0)^*$ under the isomorphism $\Delta_f(V) \otimes \mathbb{R} \cong \mathbb{R}$ as

$$\delta\mathbb{Z} = \frac{R_\infty(F)}{\zeta_F(0)^*} \det \mathbb{Z} \otimes \det^{-1} \mathcal{O}_F^{*,\mathrm{free}} \otimes \det^{-1} T_B^+.$$

The analytic class number formula gives

$$\zeta_F(0)^* = -\frac{R_\infty(F)h}{\#\mu(F)},$$

where $h = \#Cl(\mathcal{O}_F)$ is the class number. Hence

$$\delta\mathbb{Z} = \det Cl(\mathcal{O}_F) \otimes \det \mathbb{Z} \otimes \det^{-1} \mathcal{O}_F^* \otimes \det^{-1} T_B^+.$$

Now we define following Bloch and Kato $H^1_f(\mathbb{Q}_p, T_p) := \iota^{-1}(H^1_f(\mathbb{Q}_p, V_p))$, where $\iota : H^1(\mathbb{Q}_p, T_p) \to H^1(\mathbb{Q}_p, V_p)$. We put $H^1_{/f} := H^1/H^1_f$.

Proposition 2.3.2 (Appendix A). *a) The Poitou-Tate localization sequence induces an exact sequence*

$$\begin{aligned} 0 \to \mathcal{O}_F^* \otimes \mathbb{Z}_p \to H^1(\mathcal{O}_F[1/p], \mathbb{Z}_p(1)) \to H^1_{/f}(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(1)) \to Cl(\mathcal{O}_F) \otimes \mathbb{Z}_p \to \\ \to H^2(\mathcal{O}_F[1/p], \mathbb{Z}_p(1)) \to H^2(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(1)) \to H^0(\mathcal{O}_F[1/p], \mathbb{Q}_p/\mathbb{Z}_p)^* \to 0. \end{aligned}$$

b) Under the composition

$$\det^{-1} R\Gamma_{/f}(F \otimes \mathbb{Q}_p, \mathbb{Q}_p(1)) \cong \det R\Gamma_f(F \otimes \mathbb{Q}_p, \mathbb{Q}_p) \xrightarrow{\alpha} E_p$$

of local duality with the isomorphism of 1.2.4 the lattice $\det^{-1} R\Gamma_{/f}(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(1))$ *is identified with* $\mathbb{Z}_p$.

Proof. See A.3 and A.7. □

Let us show the Bloch-Kato conjecture for $h^0(F)$ and $r = 0$: the exact sequence in part a) of the proposition implies

$$\begin{aligned} &\det R\Gamma(\mathbb{Z}[1/p], T_p^\vee(1)) \otimes \det^{-1} R\Gamma_{/f}(\mathbb{Q}_p, T_p^\vee(1)) \cong \\ &\quad \cong \det^{-1}(\mathcal{O}_F^* \otimes \mathbb{Z}_p) \otimes \det(Cl(\mathcal{O}_F) \otimes \mathbb{Z}_p) \otimes \det^{-1} H^0(\mathbb{Z}[1/p], (T_p^\vee)^*)^*. \end{aligned}$$

If we tensor both sides with $\det^{-1} T_p^+$, use proposition 1.2.10 and the fact that $1 \in \mathbb{Q} \cong H^0_{\mathcal{M}}(\mathbb{Z}, V)$ maps to $1 \in \mathbb{Z}_p \cong H^0(\mathbb{Z}[1/p], (T_p^\vee)^*)^*$ we get

$$\det R\Gamma_c(\mathbb{Z}[1/p], T_p) \otimes \det^{-1} R\Gamma_{/f}(\mathbb{Q}_p, T_p^\vee(1)) \cong \delta\mathbb{Z}_p.$$

The isomorphism $\Delta_f(V) \otimes \mathbb{Q}_p \cong \det R\Gamma_c(\mathbb{Z}[1/p], V_p)$ used in the formulation of the Bloch-Kato conjecture is the same as the composition

$$\begin{aligned} \Delta_f(V) \otimes \mathbb{Q}_p &\cong \det R\Gamma_c(\mathbb{Z}[1/p], V_p) \otimes \det^{-1} R\Gamma_{/f}(\mathbb{Q}_p, V_p^\vee(1)) \\ &\cong \det R\Gamma_c(\mathbb{Z}[1/p], V_p) \otimes \det R\Gamma_f(\mathbb{Q}_p, V_p) \xrightarrow{\mathrm{id} \otimes \alpha} \det R\Gamma_c(\mathbb{Z}[1/p], V_p) \end{aligned}$$

Hence part b) of the proposition implies that $\delta\mathbb{Z}_p \cong \det R\Gamma_c(\mathbb{Z}[1/p], T_p)$ as required.

Now we consider the case $r = 1$. We deduce this case from the case $r = 0$ by checking the compatibility conjecture 2.2.1 for $K = \mathbb{Q}$ in this special case. We keep the same lattices as before. Let moreover

$$T_{\mathrm{DR}} = \mathcal{O}_F$$

By the functional equation

$$\frac{\zeta_F(0)^*}{\zeta_F(1)^*} = \pm \frac{d^{1/2}}{2^{r_1}(2\pi)^{r_2}}$$

where d is the absolute value of the discriminant of F. On the other hand, we consider again the natural lattice T_B in V_B and the exact sequence

$$0 \to ((2\pi i)T_B)^+ \otimes \mathbb{R} \to \mathcal{O}_F \otimes \mathbb{R} \to T_B^+ \otimes \mathbb{R} \to 0$$

The two lattices $\det \mathcal{O}_F$ and $\det T_B^+ \otimes \det T_B(1)^+$ on $\det \mathcal{O}_F \otimes \mathbb{R}$ differ by the factor $d^{1/2}/2^{r_1}(2\pi)^{r_2}$. Hence, (up to powers of 2) the element $\varepsilon(\mathbb{Q}/\mathbb{Q}, V(1))$ in conjecture 2.2.1 is just

$$\varepsilon(\mathbb{Q}/\mathbb{Q}, V(1)) = \det{}^{-1} T_B(1)^+ \otimes \det \mathcal{O}_F \otimes \det T_B^+$$

It remains to show that the image of $\det \mathcal{O}_F$ is a generator of $\det R\Gamma(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(F)(1))$ under the identification 1.2.4.

We have defined before 2.3.2 integral structures such that

$$\det R\Gamma(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(1)) = \det R\Gamma_f(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(1)) \otimes \det R\Gamma_{/f}(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(1))$$

and 2.3.2 also shows that

$$\det{}^{-1} R\Gamma_{/f}(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(1))$$

maps to $\mathbb{Z}_p$ under the identification in 1.2.4.

The map

$$\exp_p : \mathcal{O}_F \otimes \mathbb{Q}_p \to H^1_f(F \otimes \mathbb{Q}_p, \mathbb{Q}_p(1))$$

is an isomorphism. By 1.2.5 we have to check that

$$\det \mathcal{O}_{F_v} \otimes \mathbb{Z}_p = (1 - 1/q) \det{}^{-1} R\Gamma_f(F_v, \mathbb{Z}_p(1))$$

where F_v is a completion of F at a p-adic place and q is the order of the residue class field of F_v. Indeed, we have $H^1_f(F_v, \mathbb{Z}_p(1)) = \mathcal{O}^*_{F_v} \otimes \mathbb{Z}_p$. The index of $\exp_p(\mathcal{O}_{F_v}) \subset \mathcal{O}^*_{F_v} \otimes \mathbb{Z}_p$ is $(q-1)/q = (1 - 1/q)$ because $\exp_p$ is a bijection of $\pi_v^n \mathcal{O}_{F_v}$ (with π_v the uniformizer of F_v) towards the group $1 + \pi_v^n \mathcal{O}_{F_v}$ for some $n \geq 1$. This finishes the proof also in the case $r = 1$. □

3 Cyclotomic elements

Cyclotomic elements play a decisive role in our paper. First they provide explicit elements in Galois cohomology groups and second they form an Euler system, which will be used to prove the main conjecture. The importance of the cyclotomic elements lies in the close connection with L-values via the explicit reciprocity law. Our approach to the main conjecture through the Bloch-Kato conjecture explains precisely, which Euler system is needed to give the equality in the main conjecture.

3.1 The Euler system of cyclotomic elements

Recall that E is a finite extension of $\mathbb{Q}$ containing all values of the Dirichlet character χ considered below. $\mathcal{O}$ is its ring of integers. We fix a prime p and put $E_p = E \otimes \mathbb{Q}_p$, $\mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p$.

Fix a collection $(\zeta_m)_{m\geq 1}$ of primitive m-th roots of unity in $\mathbb{C}$ satisfying $\zeta_{mn}^n = \zeta_m$. To be explicit we can take $\zeta_m = \exp \frac{2\pi i}{m}$. Let $F = \mathbb{Q}(\zeta_N)$. We obtain a fixed embedding $\sigma_0 : F \to \mathbb{C}$.

Let χ be a character of conductor N. We need to fix a lattice in $V_B(\chi)$. Recall that $h^0(F)_B$ are the maps from the set of embeddings $\sigma : F \to \mathbb{C}$ to $\mathbb{Q}$ with action of $g \in G := \mathrm{Gal}(F/\mathbb{Q})$ given by $(gf)(\sigma) := f(\sigma g)$. We write δ_σ for the delta function at σ. Note that we have a distinguished element $\delta_{\sigma_0} \in h^0(F)_B$.

Definition 3.1.1. *Define the element $t_B(\chi) \in V_B(\chi)$ by*

$$t_B(\chi) := p_{\chi^{-1}}\delta_{\sigma_0}$$

and let $T_B(\chi) := \mathcal{O}t_B(\chi)$. We use this to define also $T_p(\chi) := T_B(\chi) \otimes \mathbb{Z}_p$ and $t_p(\chi) := t_B(\chi) \otimes 1 \in T_p(\chi)$.

Remark: Recall that we have by definition $V(\chi) = p_{\chi^{-1}}h^0(F)$ and with this normalization $V_p(\chi)$ is the Galois module with operation of $G_\mathbb{Q}$ via χ.

Following Soulé we define cyclotomic elements

$$c_r(\zeta_m) \in H^1(\mathbb{Z}[\zeta_m][1/p], \mathbb{Z}_p(r)),$$

for all $r \in \mathbb{Z}$ and $m \geq 1$. For every integer $m \geq 1$ and prime l the following norm compatibility is satisfied:

$$\mathrm{N}_{\mathbb{Q}(\mu_{ml})/\mathbb{Q}(\mu_m)}(1-\zeta_{ml}) = \begin{cases} 1-\zeta_m & \text{if } l|m \\ (1-\zeta_m)^{1-\mathrm{Fr}_l} & \text{if } l \nmid m \text{ and } m > 1 \\ l & \text{if } m = 1 \end{cases}$$

Here Fr_l is the (geometric) Frobenius at l in $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$. For $n \geq 1$ and $m \geq 1$ the elements $1 - \zeta_{p^n m}$ are units in $\mathbb{Z}[\zeta_{p^n m}][1/p]$ and we can consider for every $n \geq 1$

$$\begin{aligned}(1-\zeta_{p^n m}) \otimes (\zeta_{p^n m}^m)^{\otimes r-1} &\in H^1(\mathbb{Z}[\zeta_{p^n m}][1/p], \mathbb{Z}/p^n\mathbb{Z}(1)) \otimes \mathbb{Z}/p^n\mathbb{Z}(r-1) \\ &= H^1(\mathbb{Z}[\zeta_{p^n m}][1/p], \mathbb{Z}/p^n\mathbb{Z}(r)).\end{aligned}$$

We let

$$c_r(\zeta_m)_n := \mathrm{cores}_{\mathbb{Q}(\zeta_{p^n m})/\mathbb{Q}(\zeta_m)}(1-\zeta_{p^n m}) \otimes (\zeta_{p^n m}^m)^{\otimes r-1} \in H^1(\mathbb{Z}[\zeta_m][1/p], \mathbb{Z}/p^n\mathbb{Z}(r)).$$

The $c_r(\zeta_m)_n$ are compatible with respect to the maps $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$.

Definition 3.1.2. *For $r \in \mathbb{Z}$ and $m \geq 1$, the Soulé-Deligne* cyclotomic elements *$c_r(\zeta_m)$ are defined as*

$$c_r(\zeta_m) := \varprojlim_n c_r(\zeta_m)_n \in H^1(\mathbb{Z}[\zeta_m][1/p], \mathbb{Z}_p(r)).$$

We are now going to define elements $c_r(\zeta_m)(\chi) \in H^1(\mathbb{Z}[\zeta_m][1/p], T_p(\chi)(r))$. Let N be the conductor of χ and K the least common multiple of N and m. Let $F := \mathbb{Q}(\zeta_N)$, then $F(\zeta_m) = \mathbb{Q}(\zeta_K)$. Let $t_p(\chi)$ be the $\mathcal{O}_p$ generator of $T_p(\chi)$ fixed in 3.1.1.

Definition 3.1.3. *Define $c_r(\zeta_m)(\chi)$ to be the image of $c_r(\zeta_K) \otimes t_p(\chi)$ under the composition*

$$\begin{aligned} H^1(\mathbb{Z}[\zeta_K][1/p], \mathbb{Z}_p(r)) \otimes T_p(\chi) &\cong H^1(\mathbb{Z}[\zeta_K][1/p], T_p(\chi)(r)) \\ &\xrightarrow{\text{cores}} H^1(\mathbb{Z}[\zeta_m][1/p], T_p(\chi)(r)). \end{aligned}$$

Note that the elements $c_r(\zeta_m)$ are modulo torsion in the $(-1)^{r-1}$-eigenspace of complex conjugation, so that $c_r(\zeta_m)(\chi)$ can be non torsion only for $\chi(-1) = (-1)^{r-1}$. If $m = 1$ we have $c_r(1)(\chi) \in H^1(\mathbb{Z}[1/p], T_p(\chi)(r))$.
Remark: Rationally, we have the isomorphism $H^1(\mathbb{Z}[\zeta_N][1/p], \mathbb{Q}_p(r)) = \bigoplus_\chi H^1(\mathbb{Z}[1/p], V_p(\chi))$ where the sum is taken over all characters of conductor dividing N. Consider a character χ such that the primes dividing its conductor are the same as the primes dividing N with the possible exception of p. Then $c_r(1)(\chi)$ is the χ-component of $c_r(\zeta_N)$, i.e., $c_r(1)(\chi) = p_{\chi^{-1}} c_r(\zeta_N)$ up to torsion, by the norm compatibility of cyclotomic elements.

Our aim is to show that the elements $c_r(\zeta_m)(\chi)$ form an Euler system for $(T_p(\chi), pN)$. Euler systems were invented by Kolyvagin. A general theory of Euler systems was developed by Kato [Ka4], Perrin-Riou [PR4] and Rubin [Ru]. We follow Rubin, because his approach is closest to our setting.

Let us recall the definition of an Euler system:

Definition 3.1.4. *An* Euler system *for $(T_p(\chi), pN)$ is a collection of elements*

$$e_r(m) \in H^1(\mathbb{Q}(\zeta_m), T_p(\chi)(r))$$

for all $m \geq 1$, such that for all primes l

$$\mathrm{cores}_{\mathbb{Q}(\zeta_{ml})/\mathbb{Q}(\zeta_m)}(e_r(ml)) = \begin{cases} e_r(m) & \text{if } l | mpN \\ (1 - \chi^{-1}(l) l^{r-1}) e_r(m) & \text{if } l \nmid mpN. \end{cases}$$

Lemma 3.1.5. *(Soulé [So]) The elements $c_r(\zeta_m)(\chi)$ form an Euler system for $(T_p(\chi)(r), pN)$.*

Proof. This follows from proposition 2.4.2. in [Ru] and the norm compatibility of the cyclotomic units. □

We need also the following variant of $c_r(\zeta_N)$ and $c_r(1)(\chi)$, see [Ka1] § 5.

Definition 3.1.6. *Define*

$$\widetilde{c}_r(\zeta_N) = \begin{cases} c_r(\zeta_N) & \text{if } p | N \\ \sum_{i \geq 0} (p^{r-1})^i c_r(\zeta_N^{p^{-i}}) & \text{if } p \nmid N,\ r \geq 2 \\ -\sum_{i \geq 1} (p^{1-r})^i c_r(\zeta_N^{p^i}) & \text{if } p \nmid N,\ r \leq 0 \end{cases}$$

Here $\zeta_N^{p^{-i}}$ *denotes the unique* N*-th root of unity whose* p^i*-th power is* ζ_N*. Similarly we let*

$$\widetilde{c}_r(1)(\chi) = \begin{cases} c_r(1)(\chi) & \text{if } p|N \\ \sum_{i\geq 0}(p^{r-1})^i\chi^{-1}(p)^i c_r(1)(\chi) & \text{if } p\nmid N,\ r\geq 2 \\ -\sum_{i\geq 1}(p^{1-r})^i\chi(p)^i c_r(1)(\chi) & \text{if } p\nmid N,\ r\leq 0 \end{cases}$$

Note that the sums converge in $H^1(\mathbb{Z}[\zeta_N][1/p],\mathbb{Z}_p(r))$ and $H^1(\mathbb{Z}[1/p],T_p(\chi)(r))$ respectively. If $p\nmid N$,

$$(1-p^{r-1}\operatorname{Fr}_p)\widetilde{c}_r(\zeta_N) = c_r(\zeta_N)$$

and

$$(1-p^{r-1}\chi^{-1}(p))\widetilde{c}_r(1)(\chi) = c_r(1)(\chi).$$

The following lemma allows the comparison with the p-adic regulator from K-theory.

Lemma 3.1.7. *The following identity holds in* $H^1(\mathbb{Z}[\zeta_{p^nN}][1/p],\mathbb{Z}/\mathbb{Z}p^n(r))$ *if* $p\nmid N$ *and* $r>1$*:*

$$\sum_{\beta^{p^n}=\zeta_N}(1-\beta)\otimes(\beta^N)^{\otimes r-1} = \sum_{i=0}^{n}(p^{r-1})^i c_r(\zeta_N^{p^{-i}})_n.$$

For $p|N$

$$\sum_{\beta^{p^n}=\zeta_N}(1-\beta)\otimes(\beta^N)^{\otimes r-1} = c_r(\zeta_N)_n.$$

In particular the elements $\widetilde{c}_r(\zeta_N)$ *and*

$$\left(\sum_{\beta^{p^n}=\zeta_N}(1-\beta)\otimes(\beta^N)^{\otimes r-1}\right)_n$$

are the same in $H^1(\mathbb{Z}[\zeta_{p^\infty N}][1/p],\mathbb{Z}_p(r))$.

Proof. In the case $p|N$ there is nothing to show, so assume $p\nmid N$. Write $H_i(\zeta_N) := \{\beta|\beta^{p^{n-i}N}=1 \text{ and } \beta^{p^n}=\zeta_N\}$. Then

$$\sum_{H_0(\zeta_N)}(1-\beta)\otimes(\beta^N)^{\otimes r-1} = \sum_{i=0}^{n}\sum_{H_i(\zeta_N)\smallsetminus H_{i+1}(\zeta_N)}(1-\beta)\otimes(\beta^N)^{\otimes r-1}.$$

We have

$$\sum_{H_i(\zeta_N)\smallsetminus H_{i+1}(\zeta_N)}(1-\beta)\otimes(\beta^N)^{\otimes r-1} = (p^{r-1})^i\sum_{H_0(\zeta_N^{p^{-i}})\smallsetminus H_1(\zeta_N^{p^{-i}})}(1-\beta)\otimes(\beta^N)^{\otimes r-1}$$

and the last sum is nothing but $c_r(\zeta_N^{p^{-i}})_n$. □

Remark: Note that the $c_r(\zeta_N)$ are the norm compatible elements, which define Euler systems and hence are related to p-adic L-functions. They miss the Euler factor at p. On the other hand, the elements $\widetilde{c}_r(\zeta_N)$ are known to come from elements in integral K-theory (see 5.2.2). These are the elements which appear in [Hu-Wi], corollary 9.7 or [BlKa] 6.2.

3.2 Relation to L-values and non-vanishing of the Euler system

We want to relate the elements $\widetilde{c}_r(\zeta_N)$ defined in 3.1.6 to L-values via the exponential map. This will also show that they are non torsion.

Definition 3.2.1. *For α an N-th root of unity let*

$$Li_s(\alpha) := \sum_{n\geq 1} \frac{\alpha^n}{n^s}$$

be the polylogarithm function. *The sum converges for $Re\ s > 1$.*

On the other hand we have the Hurwitz zeta function, defined as follows:

Definition 3.2.2. *Let for $c \in \mathbb{Z}$*

$$\zeta_N(s,c) := \sum_{n\geq 1,\ n\equiv c \bmod N} \frac{1}{n^s}$$

be the Hurwitz zeta function *of modulus N with respect to c. The sum converges for $Re\ s > 1$.*

Theorem 3.2.3. *The functions $Li_s(\alpha)$ and $\zeta_N(s,c)$ have meromorphic continuations to $\mathbb{C}$ and for $r \in \mathbb{Z}$ and $\alpha = \exp(\frac{2\pi i c}{N})$ a primitive N-th root of unity, the equality*

$$\frac{1}{2}\left(Li_{1-r}(\alpha) + (-1)^r Li_{1-r}((\alpha^{-1}))\right) = \left(\frac{-2\pi i}{N}\right)^{-r} \lim_{s\to r} \Gamma(s)\left(\zeta_N(s,c) + (-1)^r \zeta_N(s,-c)\right)$$

holds.

Proof. This is the well known functional equation of the Hurwitz zeta function, see e.g. [Ap] thm. 12.6. □

Let $L(\chi, s)$ be the $E \otimes_{\mathbb{Q}} \mathbb{C}$-valued L-function of the primitive character χ. The above theorem immediately implies with our normalization $\chi(\mathrm{Fr}_p) = \chi(p)$:

Corollary 3.2.4. *Let χ be a character with $\chi(-1) = (-1)^r$ and $r \geq 1$, then*

$$\sum_\tau \chi^{-1}(\tau) \otimes Li_{1-r}(\tau\zeta_N) = 2\left(\frac{-2\pi i}{N}\right)^{-r} (r-1)! L(\chi, r)$$

as elements in $E \otimes_{\mathbb{Q}} \mathbb{C}$.

Consider for $r \geq 1$ the composition

$$H^1(\mathbb{Z}[\zeta_N][1/p], \mathcal{O}_p(1-r)) \to H^1(\mathbb{Q}_p \otimes \mathbb{Q}(\zeta_N), E_p(1-r)) \xrightarrow{\exp_p^*} V_{\mathrm{DR}}(\mathbb{Q}(\zeta_N))^\vee \otimes_E E_p.$$

where $\exp_p^*$ is the dual of $\exp_p$ via local duality. Let us define an element in $V_{\mathrm{DR}}(\mathbb{Q}(\zeta_N))^\vee$.

Definition 3.2.5. *Let $d_{1-r}(\zeta_N)^\vee \in V_{\mathrm{DR}}(\mathbb{Q}(\zeta_N))^\vee$ be the E-linear map*

$$x \mapsto \mathrm{Tr}_{\mathbb{Q}(\zeta_N)/\mathbb{Q}}(x Li_{1-r}(\zeta_N)).$$

The next result is crucial for our approach to the Bloch-Kato conjecture.

Theorem 3.2.6. *(Kato [Ka1] theorem 5.12) For $r \geq 1$ the image of $\widetilde{c}_{1-r}(\zeta_N)$ under the map $\exp_p^*$ is $(-1)^{r-1}N^{-r}(r-1)!^{-1}d_{1-r}(\zeta_N)^\vee$. In particular the element $\widetilde{c}_{1-r}(\zeta_N)$ is not torsion in $H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(1-r))$.*

Proof. This is a standard consequence of the explicit reciprocity law of Kato, [Ka2] II, Theorem 2.1.7. For the calculation see e.g. Kato [Ka2] chapter III, 1.1.7.

As Benois pointed out to us, Theorem 2.1.7 in loc. cit. is proved only for norm compatible systems of invertible elements of $u_n \in \mathcal{O}_{K_n}$ (notation of loc.cit). However, as we learned from Colmez, the argument also works for a norm compatible system of elements $u_n \in \mathcal{O}_{K_n} \setminus \{0\}$. Then the case $N = p^n$ (where $1 - \zeta_{p^n}$ is not a unit) is also covered by the theorem. □

Let $p_\chi := \frac{1}{\#G}\sum_\tau \chi^{-1}(\tau)\tau$ be the projector of $V_{\mathrm{DR}}(\mathbb{Q}(\zeta_N))$ onto $V_{\mathrm{DR}}(\chi^{-1})$. We consider

$$p_\chi\left(d_{1-r}(\zeta_N)^\vee(\chi^{-1})\right) \in V_{\mathrm{DR}}(\chi)^\vee.$$

From the above, we have:

Corollary 3.2.7. *For $r \geq 1$ and $\chi(-1) = (-1)^r$, the image of $\widetilde{c}_{1-r}(1)(\chi^{-1})$ under*

$$\exp_p^* : H^1(\mathbb{Z}[1/p], V_p(\chi^{-1})(1-r)) \to V_{\mathrm{DR}}(\chi)^\vee$$

maps to the linear map which maps $x \in V_{\mathrm{DR}}(\chi)$ to

$$-\operatorname{Tr}_{\mathbb{Q}(\zeta_N)/\mathbb{Q}}\left(\frac{2}{\#G}(2\pi i)^{-r}L(\chi, r)x\right).$$

In particular, as $L(\chi, r) \neq 0$ the element $\widetilde{c}_{1-r}(1)(\chi^{-1})$ is not torsion in

$$H^1(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r)).$$

Proof. $\widetilde{c}_{1-r}(1)(\chi^{-1})$ maps to $(-1)^{r-1}N^{-r}(r-1)!^{-1}p_\chi(d_{1-r}(\zeta_N)^\vee) \in V_{\mathrm{DR}}(\chi)^\vee$ by the theorem. By 3.2.4 this is the element which maps x to

$$-\operatorname{Tr}_{\mathbb{Q}(\zeta_N)/\mathbb{Q}}\left(\frac{2}{\#G}(2\pi i)^{-r}L(\chi, r)x\right).$$

□

3.3 Reformulation of the Bloch-Kato conjecture

We reformulate the Bloch-Kato conjecture for characters χ with $\chi(-1) = (-1)^r$ and $r \geq 1$ using the elements $\widetilde{c}_{1-r}(1)(\chi^{-1})$ in $H^1(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r))$ defined in 3.1.6.

Lemma 3.3.1. *Let $r \geq 1$ and $\chi(-1) = (-1)^r$, then $H^0(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r))$ and $H^2(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r))$ are finite.*

Proof. According to 3.2.7 the elements

$$(1-\chi(p)p^{-r})\widetilde{c}_{1-r}(1)(\chi^{-1}) = c_{1-r}(1)(\chi^{-1})$$

are non torsion and they are the first layer of an Euler system in $H^1(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r))$. Theorem 2.2.3 in [Ru] implies then that

$$\ker\left(H^2(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r)) \to H^2(\mathbb{Q}_p, T_p(\chi^{-1})(1-r))\right)$$

is finite. By local duality and our conditions on χ, the group $H^2(\mathbb{Q}_p, T_p(\chi^{-1})(1-r))$ is finite as well. The statement about $H^0(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r))$ is clear. □

Thus, we have an isomorphism

$$\det_{E_p} H^1(\mathbb{Z}[1/p], V_p(\chi^{-1})(1-r)) \cong \det^{-1}_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r)) \otimes \mathbb{Q}_p$$

and we can consider $c_{1-r}(1)(\chi^{-1}) \in \det^{-1}_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r)) \otimes \mathbb{Q}_p$.

Our aim is to prove the following theorem:

Theorem 3.3.2. *Let χ be a character with conductor N and with $\chi(-1) = (-1)^r$ and assume that $r \geq 1$. Then the Bloch-Kato conjecture for $V(\chi)$ and r is true up to powers of 2, if and only if for all $p \neq 2$, the cyclotomic element $c_{1-r}(1)(\chi^{-1})$ is a generator of $\det^{-1}_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], T_p(\chi)(1-r))$.*

Proof. We are in the case $H^1_{\mathcal{M}}(\mathbb{Z}, V(\chi)(r)) = 0$. Hence, the fundamental line for $V(\chi)(r)$ reduces to

$$\Delta_f(V(\chi)(r)) = \det_E V_{\mathrm{DR}}(\chi) \otimes \det_E^{-1} V_B(\chi)(r)$$

Our first aim is to describe the element δ of conjecture 1.2.8 explicitly. Let $t_B(\chi)$ be the $\mathcal{O}$-generator of $T_B(\chi)$ fixed in 3.1.1. Recall that we have a distinguished embedding $\sigma_0 : \mathbb{Q}(\zeta_N) \to \mathbb{C}$ by our choice of root of unity. Let $I_\infty : V_{\mathrm{DR}}(\chi)_{\mathbb{R}} \to V_B(\chi)(r)_{\mathbb{R}}$ be the comparison isomorphism and $f \in p_{\chi^{-1}}(\mathbb{Q}(\zeta_N) \otimes E)$. Then, for all $\sigma \in G = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, $\sigma(f) = \chi^{-1}(\sigma) f$ and f is mapped to

$$\begin{aligned}
\sum_\sigma \sigma_0\sigma(f)\delta_{\sigma_0\sigma} &= \sum_\sigma \chi^{-1}(\sigma)\sigma_0(f)\delta_{\sigma_0\sigma} \\
&= (2\pi i)^{-r}(\#G)\sigma_0(f)\left(\frac{1}{\#G}\sum_\sigma \chi^{-1}(\sigma)\sigma^{-1}\delta_{\sigma_0}(2\pi i)^r\right) \\
&= (2\pi i)^{-r}(\#G)\sigma_0(f) t_B(\chi)(r).
\end{aligned}$$

Conversely, $I_\infty^{-1}(t_B(\chi)(r)) = (2\pi i)^r(\#G)^{-1}$ where we use σ_0 to interpret elements of $V_{\mathrm{DR}}(\chi)_{\mathbb{R}}$ as complex numbers. By definition

$$\delta = \frac{1}{L(\chi,r)} I_\infty^{-1}(t_B(\chi)(r)) \otimes (t_B(\chi)(r))^{-1} = \frac{(2\pi i)^r}{(\#G)L(\chi,r)} \otimes (t_B(\chi)(r))^{-1}$$

in $\Delta_f(V(\chi)(r))_{\mathbb{R}}$. We identify $\det V_{\mathrm{DR}}(\chi) = \det^{-1} V_{\mathrm{DR}}(\chi)^\vee$. In this description, $\delta = (v^\vee)^{-1} \otimes (t_B(\chi)(r))^{-1}$ where $v^\vee$ is multiplication by $(2\pi i)^{-r}(\#G)L(\chi, r)$. By 3.2.7, the element $\exp_p^*(\widetilde{c}_{1-r}(1)(\chi^{-1})) \in V_{\mathrm{DR}}(\chi)^\vee$ maps x to

$$-\operatorname{Tr}_{\mathbb{Q}(\zeta_N)/\mathbb{Q}}\left(\frac{2}{\#G}(2\pi i)^{-r}L(\chi, r)x\right) = -2(2\pi i)^{-r}L(\chi, r)x$$

Hence we get

$$\delta = \left(-\frac{\#G}{2}\exp_p^*(\widetilde{c}_{1-r}(1)(\chi^{-1}))\right)^{-1} \otimes (t_B(\chi)(r))^{-1}$$

Duality on $h^0(\mathbb{Q}(\zeta_N))$ induces an isomorphism $V_B(\chi^{-1}) \cong V_B(\chi)^\vee$. Under this duality the standard lattice $T_B \subset h^0(\mathbb{Q}(\zeta_N))_B$ is self-dual. This implies that $t_B(\chi)^\vee = (\#G)t_B(\chi^{-1})$, i.e,

$$\delta = \left(-\frac{1}{2}\exp_p^*(\widetilde{c}_{1-r}(1)(\chi^{-1}))\right)^{-1} \otimes t_B(\chi^{-1})(-r)$$

Consider the Bloch-Kato conjecture for the lattice $T_p(\chi^{-1})^\vee(r)$ in $V_p(\chi)(r)$. It holds if and only if δ is a generator of $\det_{\mathcal{O}_p} R\Gamma_c(\mathbb{Z}[1/p], T_p(\chi^{-1})^\vee(r))$, or equivalently by proposition 1.2.10 a generator of

$$\det_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r)) \otimes \det_{\mathcal{O}_p} T_p(\chi^{-1})(-r).$$

This holds if and only if $-\frac{1}{2}\exp_p^*(\widetilde{c}_{1-r}(1)(\chi^{-1}))$ corresponds to a generator of $\det^{-1}_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r))$ via $\exp_p$ and using the map on local factors in 1.2.4. By 1.2.5 this is equivalent to

$$-\frac{1}{2}c_{1-r}(1)(\chi^{-1}) = -\frac{1}{2}(1-\chi(p)p^{-r})\widetilde{c}_{1-r}(\zeta_N)(\chi^{-1})$$

being a generator of $\det^{-1}_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r))$ using the natural map. Hence the Bloch-Kato conjecture is reduced to $-\frac{1}{2}c_{1-r}(1)(\chi^{-1})$ being a generator of $\det^{-1}_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(1-r))$. □

Let $\mathbb{Q}_n$ be the cyclotomic $\mathbb{Z}/p^n$-extension. Let $\mathbb{Z}_n$ be its ring of integers. We denote by $c_{1-r}(\mathbb{Q}_n/\mathbb{Q})(\chi^{-1})$ the image of $c_{1-r}(\zeta_{p^{n+1}})(\chi)$ under the corestriction

$$H^1(\mathbb{Z}[\zeta_{p^{n+1}}][1/p], T_p(\chi^{-1})(1-r)) \to H^1(\mathbb{Z}_n[1/p], T_p(\chi^{-1})(1-r)).$$

Note that these elements are norm-compatible for varying n.

Corollary 3.3.3. *Let χ and r be as in the theorem, $p \neq 2$ a prime. Then the p-part of the equivariant Bloch-Kato conjecture 1.5.2 is true for the motive $V(\chi)$ and r and $\mathbb{Q}_n/\mathbb{Q}$ if and only if $c_{1-r}(\mathbb{Q}_n/\mathbb{Q})(\chi^{-1})$ is a generator of $\det^{-1}_{\mathcal{O}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})]} R\Gamma(\mathbb{Z}_n[1/p], T_p(\chi^{-1})(1-r))$.*

Proof. Consider the characters $\omega\chi$ where ω runs through the characters of $G = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. As $\mathbb{Q}_n$ is totally real, the parity condition is unchanged. Let N be the conductor of χ and $N_{\omega\chi}$ the conductor of $\omega\chi$. They differ by a p-power. We can apply the arguments in the proof of the theorem to the $V(\omega\chi)$'s. Hence the image of $\delta(\mathbb{Q}_n/\mathbb{Q})$ under r_p with the local Euler factors from 1.2.4 taken into account has ω-component

$$\left(-\frac{1}{2}c_{1-r}(1)(\omega^{-1}\chi^{-1})\right)^{-1} \otimes t_p(\omega^{-1}\chi^{-1})(-r).$$

This is nothing but the ω-component of

$$\left(-\frac{1}{2}c_{1-r}(\mathbb{Q}_n/\mathbb{Q})(\chi^{-1})\right)^{-1} \otimes t_p(\mathbb{Q}_n/\mathbb{Q})(\chi^{-1})(-r)$$

with $t_p(\mathbb{Q}_n/\mathbb{Q})(\chi^{-1}) = t_p(\chi^{-1}) \otimes 1 \in T_p(\chi^{-1}) \otimes \mathcal{O}_p[G]$. The equivariant Bloch-Kato conjecture is equivalent to $\delta(\mathbb{Q}_n/\mathbb{Q})$ being a generator of

$$\det_{\mathcal{O}_p[G]} R\Gamma(\mathbb{Z}_n[1/p], T_p(\chi)^\vee(1-r) \otimes \det_{\mathcal{O}_p[G]}(T_p(\chi^{-1}) \otimes \mathcal{O}_p[G])(-r)$$

As in the absolute case this is equivalent to the assertion. □

Corollary 3.3.4. *Let X be the set of characters χ with conductor dividing a fixed N and $\chi(-1) = -1$. Then the element $\prod_{\chi\in X} c_0(1)(\chi)$ is a generator of*

$$\det_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], \bigoplus_{\chi\in X} T_p(\chi)).$$

Proof. By 2.3.1 we know that the Bloch-Kato conjecture is true for $V(\mathbb{Q}(\zeta_N))$ and $r = 1$ and for the motive of the real subfield $V(\mathbb{Q}(\zeta_N)^+)$. Thus it is true for

$$V(\mathbb{Q}(\zeta_N))^- \cong \bigoplus_{\chi\in X} V(\chi)$$

and $r = 1$ with an arbitrary choice of lattice in $V(\mathbb{Q}(\zeta_N))^-$. Repeating the proof of the last theorem for the direct sum of the $V(\chi)$ implies the claim. □

Remark: This simple argument is one of the key insights of this paper. We originally have proved the Bloch-Kato conjecture for the motive of a number field using certain choices of lattices. We now apply it with a completely different choice of lattice which is compatible with direct sum decomposition. The relation of the two lattices is non-trivial – but we do not need to know anything about the comparison factors. The extra freedom in the Bloch-Kato conjecture allows to avoid these computations.

4 The main conjecture

In this section we formulate and prove the main conjecture in Iwasawa theory for all characters. It is essential for our proof of the Bloch-Kato conjecture to have the main conjecture for all characters.

4.1 Iwasawa modules

The theory of Euler systems gives relations between certain Iwasawa modules. These modules will be defined in this section. We start by defining the Iwasawa algebra:

Recall that E is a finite extension of $\mathbb{Q}$ containing all values of the Dirichlet character χ of conductor N. Let $\mathcal{O}_p := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ be the ring of integers in E_p. This is a product of discrete valuation rings.

Definition 4.1.1. *Denote by* $\mathbb{Q}_\infty$ *the maximal* $\mathbb{Z}_p$*-extension inside* $\mathbb{Q}(\zeta_{p^\infty})$ *and by* $\mathbb{Q}_n$ *the finite extensions of* $\mathbb{Q}$ *inside* $\mathbb{Q}_\infty$ *with* $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) = \mathbb{Z}/p^n\mathbb{Z}$*. Let* $\Gamma := \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ *and* $\Lambda := \varprojlim_n \mathcal{O}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})]$*. The algebra* Λ *is the* Iwasawa algebra*.*

Then we have the standard identification

$$\Lambda \cong \mathcal{O}_p[[t]]$$

with a power-series ring in one variable over $\mathcal{O}_p$. In particular Λ is a regular ring.

The following Λ modules are our main object of study:

Let χ be a Dirichlet character and $T_p(\chi)$ an $\mathcal{O}_p$-lattice in $V_p(\chi)$. Define:

$$T_p(\chi)^* := \mathrm{Hom}_{\mathcal{O}_p}(T_p(\chi), E_p/\mathcal{O}_p)$$

the $\mathcal{O}_p$-Pontryagin dual of $T_p(\chi)$.

Definition 4.1.2. *Let* $\mathbb{Z}_n$ *be the ring of integers in the* $\mathbb{Z}/p^n\mathbb{Z}$*-extension* $\mathbb{Q}_n$ *of* $\mathbb{Q}$*. Define*

$$\mathbf{H}^q_{\mathrm{gl}}(T_p(\chi)(k)) := \varprojlim_n H^q(\mathbb{Z}_n[1/p], T_p(\chi)(k))$$
$$\mathbf{H}^q_{\mathrm{loc}}(T_p(\chi)(k)) := \varprojlim_n H^q(\mathbb{Q}_p \otimes \mathbb{Q}_n, T_p(\chi)(k))$$
$$\mathbf{H}^q_{\mathrm{gl}}(T_p(\chi)^*(1-k))^* := H^q(\mathbb{Z}_\infty[1/p], T_p(\chi)^*(1-k))^*,$$

where the limit is taken with respect to the corestriction maps.

In the sequel we collect some facts about these Λ-modules. Let

$$\varepsilon_{\mathrm{cycl}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$$

be the cyclotomic character (i.e., $\mathcal{O}_p(1) = \mathcal{O}_p(\varepsilon_{\mathrm{cycl}})$) and write $\varepsilon_{\mathrm{cycl}} = \varepsilon \times \varepsilon_\infty$ according to the decomposition $\mathbb{Z}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}_p$. We write $\mathcal{O}_p(\varepsilon_\infty)$ for the $\mathcal{O}_p$ module of rank 1 with Galois action given by ε_∞.

Lemma 4.1.3. *There are isomorphisms of* Λ*-modules*

$$\mathbf{H}^q_{\mathrm{gl}}(T_p(\chi)(k)) \otimes \mathcal{O}_p(\varepsilon_\infty) \cong \mathbf{H}^q_{\mathrm{gl}}(T_p(\chi\varepsilon^{-1})(k+1))$$

and

$$\mathbf{H}^q_{\mathrm{loc}}(T_p(\chi)(k)) \otimes \mathcal{O}_p(\varepsilon_\infty) \cong \mathbf{H}^q_{\mathrm{loc}}(T_p(\chi\varepsilon^{-1})(k+1)).$$

The parity of $\chi\varepsilon^{-1}$ *is minus the parity of* χ*.*

Proof. See [Ru] 6.2.1. ☐

Lemma 4.1.4. *The following* Λ*-modules are zero:*

$$\mathbf{H}^2_{\mathrm{gl}}(T_p(\chi)^*(1-k))^* = 0 = \mathbf{H}^0_{\mathrm{loc}}(T_p(\chi)(k)).$$

Proof. The Λ-module $\varprojlim_n H^2(\mathbb{Z}[\zeta_{p^nN}][1/p], E_p/\mathcal{O}_p(1-k))$ is independent of k and is zero by [Sch] paragraph 4, lemma 7.

As the functor $\varprojlim$ is exact in our situation and H^2 is right exact for $p \neq 2$, we get a surjection $\varprojlim_n H^2(\mathbb{Z}[\zeta_{p^nN}][1/p], E_p/\mathcal{O}_p(1-k)) \to \mathbf{H}^2_{\mathrm{gl}}(T_p(\chi)^*(1-k))$. The group $\mathbf{H}^0_{\mathrm{loc}}(T_p(\chi)(k))$ is the inverse limit of $H^0(\mathbb{Q}_p \otimes \mathbb{Q}_n, T_p(\chi)(k))$, which is zero for $k \neq 0$ for weight reasons. With lemma 4.1.3 we get the statement also for $k = 0$. ☐

We define, following Kato, a Selmer group:

Definition 4.1.5. *Let*

$$\mathbf{H}^2_{\mathrm{gl},0}(T_p(\chi)(k)) := \mathrm{Coker}\left(\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(k)) \to \mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*(1-k))^*\right)$$

and

$$\mathbf{H}^2_{\mathrm{loc},0}(T_p(\chi)(k)) := \mathrm{Coker}\left(\mathbf{H}^2_{\mathrm{gl},0}(T_p(\chi)(k)) \to \mathbf{H}^2_{\mathrm{gl}}(T_p(\chi)(k))\right).$$

The following sequence gives a connection between the Λ-modules defined above.

Lemma 4.1.6. *The Poitou-Tate localization sequence induces for all* $k \in \mathbb{Z}$ *exact sequences*

$$0 \to \mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k)) \to \mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(k)) \to \mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*(1-k))^* \to \mathbf{H}^2_{\mathrm{gl},0}(T_p(\chi)(k)) \to 0,$$
$$0 \to \mathbf{H}^2_{\mathrm{loc},0}(T_p(\chi)(k)) \to \mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(k)) \to \mathbf{H}^0_{\mathrm{gl}}(T_p(\chi)^*(1-k))^* \to 0.$$

Proof. By biduality the functor $\varprojlim$ is exact on the Poitou-Tate localization sequence. It suffices to see that

$$\mathbf{H}^2_{\mathrm{gl}}(T_p(\chi)^*(1-k))^* = 0,$$

which is the content of lemma 4.1.4. ☐

The elements

$$c_k(\mathbb{Q}_n/\mathbb{Q})(\chi) = \mathrm{cores}_{\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q}_n}\, c_k(\zeta_{p^{n+1}})(\chi) \in H^1(\mathbb{Z}_n[1/p], T_p(\chi)(k))$$

are compatible with corestriction.

Definition 4.1.7. *The* cyclotomic element *in* $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))$ *is defined to be*

$$c_k(\chi) := \varprojlim_n c_k(\mathbb{Q}_n/\mathbb{Q})(\chi) \in \mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k)).$$

Lemma 4.1.8. *Under the isomorphism* $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k)) \cong \mathbf{H}^1_{\mathrm{gl}}(T_p(\chi\varepsilon^{-1})(k+1))$ *of lemma 4.1.3 the element* $c_k(\chi)$ *maps to* $c_{k+1}(\chi\varepsilon^{-1})$.

Proof. Clear from the definition of $c_k(\chi)$. ☐

4.2 The main conjecture of Iwasawa theory

In this section we formulate the main conjecture of Iwasawa theory.

Define

$$\det_\Lambda \mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi)(k)) := \bigotimes_{i=0}^{2} \det_\Lambda^{(-1)^i} \mathbf{H}^i_{\mathrm{gl}}(T_p(\chi)(k)).$$

Denote by $Q(\Lambda)$ the total quotient ring of Λ.

Proposition 4.2.1. *Let χ be a Dirichlet character of conductor N.*
a) If $\chi(-1) = (-1)^{k-1}$, then $\mathbf{H}^i_{\mathrm{gl}}(T_p(\chi)(k)) \otimes_\Lambda Q(\Lambda) = 0$ for $i = 0, 2$, $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))$ has Λ-rank 1 and

$$\left(\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))/\Lambda c_k(\chi)\right) \otimes_\Lambda Q(\Lambda) = 0.$$

Hence,

$$\det_\Lambda \Lambda c_k(\chi) \otimes_\Lambda Q(\Lambda) \cong \det_\Lambda \mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k)) \otimes_\Lambda Q(\Lambda).$$

In particular, this gives an isomorphism

$$\Psi : \det_\Lambda \Lambda c_k(\chi) \otimes_\Lambda Q(\Lambda) \cong \det_\Lambda^{-1} \mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi)(k)) \otimes_\Lambda Q(\Lambda).$$

b) If $\chi(-1) = (-1)^k$, then for $i = 0, 1, 2$

$$\mathbf{H}^i_{\mathrm{gl}}(T_p(\chi)(k)) \otimes_\Lambda Q(\Lambda) = 0.$$

In particular, one has an isomorphism

$$\Psi : Q(\Lambda) \cong \det_\Lambda^{-1} \mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi)(k)) \otimes_\Lambda Q(\Lambda).$$

Proof. The module $\mathbf{H}^0_{\mathrm{gl}}(T_p(\chi)(k))$ is zero. Hence, $\mathbf{H}^0_{\mathrm{gl}}(T_p(\chi)(k)) \otimes_\Lambda Q(\Lambda) = 0$. The result from 4.1.4 that $\mathbf{H}^2(T_p(\chi)^*(1-k)) = 0$ implies by proposition 1.3.2 from [PR3] that $\mathbf{H}^2_{\mathrm{gl}}(T_p(\chi)(k))$ is a torsion Λ-module and that $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))$ has Λ-rank 1 if $\chi(-1) = (-1)^{k-1}$ and Λ-rank 0, if $\chi(-1) = (-1)^k$. It remains to show that $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))/\Lambda c_k(\chi)$ is Λ-torsion. It suffices to consider $k < 0$. As before $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. It suffices to to prove that $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))_\Gamma/\mathcal{O}_p c_k(1)(\chi)$ is $\mathcal{O}_p$-torsion. $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))_\Gamma \subset H^1(\mathbb{Z}[1/p], T_p(\chi))$ has $\mathcal{O}_p$-rank 1 by the formula for the Euler-Poincaré characteristic and because Γ-invariants and Γ-coinvariants of $\mathbf{H}^2_{\mathrm{gl}}(T_p(\chi)(k))$ have the same rank. Finally by corollary 3.2.7, $c_k(\chi) \in \mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))_\Gamma$ is non-torsion. □

The main conjecture can now be formulated as follows:

Main Conjecture 4.2.2. [Theorem 4.4.1] (equal parity case)
Let χ be a character with $\chi(-1) = (-1)^{k-1}$. The element $c_k(\chi)$ (4.1.7) is mapped to a generator of the free Λ-module

$$\det_\Lambda^{-1} \mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi)(k))$$

under the isomorphism Ψ in proposition 4.2.1 a). Equivalently, Ψ and the Poitou-Tate sequence induce an isomorphism

$$\det_\Lambda \left(\mathbf{R}\Gamma_{\mathrm{loc}}(T_p(\chi)(k))/\Lambda c_k(\chi)[-1]\right) \cong \det_\Lambda \left(\mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi)^*(1-k))^*\right).$$

Remark: For $p \nmid \Phi(N)$ it follows from the main conjecture as shown by Mazur and Wiles [MaWi]. Under this condition it was also proved directly by Rubin [Ru].

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. As before we decompose the cyclotomic character $\varepsilon_{\mathrm{cycl}} = \varepsilon \times \varepsilon_\infty$. For every continuous character $\tau : \Gamma \to \mathcal{O}_p^*$ define the twisting map

$$Tw(\tau) : \Lambda \to \Lambda$$

by $\gamma \mapsto \tau(\gamma)\gamma$. Denote also by $Tw(\tau)$ the map induced on $Q(\Lambda)$. The character τ also induces a map $\tau : \Lambda \to \mathcal{O}_p$. If $\varrho : \Lambda \to \mathcal{O}_p$ denotes the augmentation map, we have

$$\varrho(Tw(\tau)(f)) = \tau(f)$$

for any element $f \in \Lambda$. We extend the map τ to $Q(\Lambda)$ by $\tau(f/g) = \tau(f)/\tau(g)$, whenever $\tau(g) \neq 0$. Let χ be a Dirichlet character of conductor N and $\chi(-1) = (-1)^k$. Recall from [Wa] theorem 7.10, that there is for $k > 1$ an element $\mathcal{L}_p(\chi, 1-k) \in Q(\Lambda)$ such that $\varrho(\mathcal{L}_p(\chi, 1-k)) = (1 - \chi(p)p^{k-1})L(\chi, 1-k)$. It is characterized by the fact that for all characters $\tau : \Gamma \to \mathcal{O}_p^*$ of finite order

$$\tau^{-1}(\mathcal{L}_p(\chi, 1-k)) = (1 - \chi\tau(p)p^{k-1})L(\chi\tau, 1-k) \ .$$

Remark: The power series f in [Wa] theorem 7.10 is related to our $\mathcal{L}_p(\chi, 1-k)$ as follows. Write $\chi\varepsilon^{-k} = \theta\tau$ with θ of the first and τ of the second kind. Then $Tw(\tau^{-1}\varepsilon_\infty^{k-1})(f(t,\theta)) = \mathcal{L}_p(\chi, 1-k)$.

Definition 4.2.3. *Let χ be a Dirichlet character with $\chi(-1) = (-1)^k$. For $k > 1$ we call the above $\mathcal{L}_p(\chi, 1-k) \in Q(\Lambda)$ the* p-adic L-function *at $k-1$. For $k \leq 1$ we define $\mathcal{L}_p(\chi, 1-k) \in Q(\Lambda)$ as $Tw(\varepsilon_\infty^{k-k'})\mathcal{L}_p(\chi\varepsilon^{k'-k}, 1-k')$ for some $k' > 1$.*

Main Conjecture 4.2.4. [Theorem 5.1.3] (unequal parity case)
Let $p \neq 2$ and χ be a Dirichlet character with $\chi(-1) = (-1)^k$. Then the isomorphism ψ from 4.2.1 b)

$$Q(\Lambda) \cong \det\nolimits_\Lambda^{-1} \mathbf{R}\Gamma(T_p(\chi^{-1})(k)) \otimes_\Lambda Q(\Lambda)$$

maps the p-adic L-function $\mathcal{L}_p(\chi, 1-k)$ to a generator of $\det_\Lambda^{-1} \mathbf{R}\Gamma(T_p(\chi^{-1})(k))$.

Remark: This result is already due to Mazur and Wiles [MaWi].

4.3 Application of the Euler system to Iwasawa modules

The general theory of Euler systems as developed by Kato [Ka4], Perrin-Riou [PR4] and Rubin [Ru] allows us to prove a result about the determinants of certain Iwasawa modules. We follow the formulation by Rubin.

Theorem 4.3.1. *Let χ be a character with $\chi(-1) = (-1)^{k-1}$, then*

$$\det\nolimits_\Lambda^{-1} \left(\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k)) / \Lambda c_k(\chi)\right) \subset \det\nolimits_\Lambda^{-1} \mathbf{H}^2_{\mathrm{gl},0}(T_p(\chi)(k)).$$

Remark: This should be formulated as follows: the element c_k maps to a generator of the invertible Λ-module

$$\det_{\Lambda}^{-1}\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))\otimes\det_{\Lambda}\mathbf{H}^2_{\mathrm{gl},0}(T_p(\chi)(k))$$

in $\det_{\Lambda}^{-1}\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))\otimes_{\Lambda}Q(\Lambda)$. Observe that we have $\mathbf{H}^2_{\mathrm{gl},0}(T_p(\chi)(k))\otimes_{\Lambda}Q(\Lambda)=0$.

Proof. By proposition 4.2.1 a) $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))$ has Λ-rank 1. We check the prerequisites for theorem 2.3.3. in [Ru]. These are called $\mathrm{Hyp}(K_\infty/K)$ and $\mathrm{Hyp}(K_\infty,T_p(\chi))$ in loc. cit. As $K=\mathbb{Q}$ and $K_\infty=\mathbb{Q}_\infty$, the hypothesis $\mathrm{Hyp}(K_\infty/K)$ is trivially verified. For $\mathrm{Hyp}(K_\infty,T_p(\chi))$ we take $\tau=\mathrm{id}$. Note that although our $\mathcal{O}_p$ is a product of discrete valuation rings, the theory of Rubin still goes through, because we can apply it to every factor. The module called X_∞ in theorem 2.3.3. in [Ru] is our $\mathbf{H}^2_{\mathrm{gl},0}(T_p(\chi)(k))$. Rubin defines

$$\mathrm{ind}_{\Lambda}(c_k):=\left\{\Phi(c_k):\Phi\in\mathrm{Hom}_{\Lambda}\left(\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k)),\Lambda\right)\right\}\subset\Lambda.$$

By the classification theory of Λ-modules there is a pseudo-isomorphism

$$\varrho:\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))\to\Lambda\oplus\mathbf{H}^1_{\mathrm{tors}}$$

where $\mathbf{H}^1_{\mathrm{tors}}$ is Λ-torsion. The index $\mathrm{ind}_{\Lambda}(c_k)=\mathrm{ind}_{\Lambda}(\varrho(c_k))$ is given by the ideal $\det^{-1}\Lambda/pr_1(\varrho(c_k))\Lambda$. There is an exact sequence

$$0\to K\to\left(\Lambda\oplus\mathbf{H}^1_{\mathrm{tors}}\right)/\varrho(c_k)\to\Lambda/pr_1(\varrho(c_k))\to 0.$$

As $\det^{-1}K\subset\Lambda$ this implies

$$\det_{\Lambda}^{-1}\left(\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)(k))/\Lambda c_k(\chi)\right)\subset\mathrm{ind}_{\Lambda}(c_k).$$

Kato's observation, [Ka1] proposition 6.1., shows that $\mathrm{char}(X_\infty)=\det_{\Lambda}^{-1}(X_\infty)$. Finally theorem 2.3.3 of Rubin in [Ru] tells us that

$$\mathrm{ind}_{\Lambda}(c_k)\subset\det_{\Lambda}^{-1}(X_\infty).$$

This gives the result stated in the theorem. □

The exact sequence in lemma 4.1.6 allows to reinterpret this:

Corollary 4.3.2. *Let χ be as in the theorem, then*

$$\det_{\Lambda}^{-1}\left(\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(k))/\Lambda c_k(\chi)\right)\subset\det_{\Lambda}^{-1}\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*(1-k))^*$$

holds.

Our aim is to strengthen corollary 4.3.2 to:

Theorem 4.3.3. *Let χ be a Dirichlet character of conductor N with $\chi(-1)=(-1)^{k-1}$, then*

$$\det_{\Lambda}\left(\mathbf{R}\Gamma_{\mathrm{loc}}(T_p(\chi)(k))/\Lambda c_k(\chi)[-1]\right)\subset\det_{\Lambda}\mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi)^*(1-k))^*.$$

Equivalently,

$$\det_{\Lambda}\Lambda c_k(\chi)[-1]\supset\det_{\Lambda}\left(\mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi)(k))\right).$$

The proof of this theorem will be given at the end of this section.
Remark: a) This is to be interpreted in the same way as theorem 4.3.1.
b) Note that by lemma 4.1.4 and the second exact sequence in lemma 4.1.6, to prove the theorem it is enough to show that

$$\det_\Lambda^{-1}\left(\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(k))\,/\Lambda c_k(\chi)\right)\otimes \det_\Lambda \mathbf{H}^2_{\mathrm{loc},0}(T_p(\chi)(k)) \subset \det_\Lambda^{-1}\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*(1-k))^*.$$

Thus, it is necessary to study the module $\mathbf{H}^2_{\mathrm{loc},0}(T_p(\chi)(k))$.
c) As will be shown by the explicit computation, the existence of the error term $\mathbf{H}^2_{\mathrm{loc},0}(T_p(\chi)(k))$ is related to trivial zeroes of the p-adic L-function. The problem already appears in Rubin's case $p\nmid\Phi(N)$, see [Ru] thm. 3.2.7. If χ is purely ramified, e.g., $N=p$ (the very first case treated by Euler system methods), the error term vanishes because it cancels against a global term.

For a Λ-module M denote by $M_\mathfrak{p} := M\otimes_\Lambda \Lambda_\mathfrak{p}$the localization of M at $\mathfrak{p}$. A Dirichlet character χ induces a finite character $\chi:\Gamma\to\mathcal{O}_p^*$, which extends to a map $\chi:\Lambda\to\mathcal{O}_p$ via $\gamma\mapsto\chi(\gamma)$. Let $\mathfrak{a}_\chi$ be the kernel of $\chi^{-1}:\Lambda\to\mathcal{O}_p$. Then $\mathfrak{a}_\chi$ is a prime ideal of height 1 and a principal ideal (generated by $\chi(\gamma_0)\gamma_0-1$ where γ_0 is a topological generator of Γ).

Proposition 4.3.4 (see 6.1.1, 6.1.2). *Let χ be a Dirichlet character and $G_{\infty,p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_\infty\otimes\mathbb{Q}_p)$, then*

$$\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1)) \cong \begin{cases} \textit{finite} & \textit{if } \chi|_{G_{\infty,p}}\neq 1,\\ T_p(\chi) & \textit{if } \chi|_{G_{\infty,p}}=1.\end{cases}$$

In particular, the localization at all prime ideals $\mathfrak{p}\neq\mathfrak{a}_\chi$ of height 1 is $\det_{\Lambda_\mathfrak{p}}\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1))_\mathfrak{p}\cong\Lambda_\mathfrak{p}$.

Proposition 4.3.5 (see 6.2.1). *For all characters χ with $\chi(-1)=1$,*

$$\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*)^*_{\mathfrak{a}_\chi}=0.$$

In particular $\det^{-1}_{\Lambda_{\mathfrak{a}_\chi}}\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*)^*_{\mathfrak{a}_\chi}=\Lambda_{\mathfrak{a}_\chi}$.

Proposition 4.3.6 (see 6.3.3). *Suppose that the conductor of χ is not a p-power and that $\chi(-1)=1$, then*

$$\det^{-1}_{\Lambda_{\mathfrak{a}_\chi}}\left(\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(1))/\Lambda c_1(\chi)\right)_{\mathfrak{a}_\chi}\subset\det^{-1}_{\Lambda_{\mathfrak{a}_\chi}}\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1))_{\mathfrak{a}_\chi}.$$

Before we can prove theorem 4.3.3, we need one more observation:

Lemma 4.3.7. *Suppose that the conductor of χ is not a p-power, then*

$$\det_\Lambda\mathbf{H}^2_{\mathrm{loc},0}(T_p(\chi)(1))\cong\det_\Lambda\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1)).$$

If the conductor of χ is a p-power and $\chi(-1)=1$, then

$$\det_\Lambda\mathbf{H}^2_{\mathrm{loc},0}(T_p(\chi)(1))\cong\Lambda.$$

Proof. If the conductor of χ is not a p-power, then χ is not trivial over $\mathbb{Q}_\infty$ and the map $\mathbf{H}^2_{\text{loc},0}(T_p(\chi)(1)) \to \mathbf{H}^2_{\text{loc}}(T_p(\chi)(1))$ is a pseudo-isomorphism, because the cokernel is the finite group $\mathbf{H}^0_{\text{gl}}(T_p(\chi)^*)^*$. If the conductor of χ is a p-power, then χ is trivial over $\mathbb{Q}_\infty$ because of our assumption $\chi(-1) = 1$. Hence $\mathbf{H}^0_{\text{gl}}(T_p(\chi)^*)^* \cong T_p(\chi)$ and the map

$$\mathbf{H}^2_{\text{loc}}(T_p(\chi)(1)) \to \mathbf{H}^0_{\text{gl}}(T_p(\chi)^*)^*$$

is the identity by 4.3.4. This implies that $\mathbf{H}^2_{\text{loc},0}(T_p(\chi)(1)) = 0$. □

Proof. (of theorem 4.3.3): First of all it is enough to consider $k = 1$. We use the remark after theorem 4.3.3. If the conductor of χ is a p-power, there is nothing to show by lemma 4.3.7. From now on assume that the conductor of χ is not a p-power. Again by lemma 4.3.7 we have to show that

$$\det\nolimits_\Lambda^{-1}\left(\mathbf{H}^1_{\text{loc}}(T_p(\chi)(k))\,/\Lambda c_k(\chi)\right) \otimes \det\nolimits_\Lambda \mathbf{H}^2_{\text{loc}}(T_p(\chi)(k)) \subset \det\nolimits_\Lambda^{-1} \mathbf{H}^1_{\text{gl}}(T_p(\chi)^*(1-k))^*.$$

As Λ is a regular ring, the determinant of a Λ-module is determined by its localizations at all primes of height 1, see [Ka1] proposition 6.1. Proposition 4.3.4 and corollary 4.3.2 imply the theorem after localization at a prime ideal of height 1 different from $\mathfrak{a}_\chi$. After localization at $\mathfrak{a}_\chi$, the statement follows from 4.3.5 and 4.3.6. The equivalence with the second statement follows from the Poitou-Tate exact sequence. □

4.4 The proof of the main conjecture

In this section we reduce the main conjecture to the Bloch-Kato conjecture for $r = 1$, i.e., the class number formula proved in 2.3.1.

Theorem 4.4.1. *Let χ be a character with $\chi(-1) = (-1)^{k-1}$, then the main conjecture 4.2.2 is true, i.e., the element $c_k(\chi)$ defined in 4.1.7 maps to a generator of*

$$\det\nolimits_\Lambda^{-1}\left(\mathbf{R}\Gamma_{\text{gl}}(T_p(\chi)(k))\right).$$

The proof occupies the rest of this section. We use that the main conjecture is invariant under twist. We will prove it for $k = 0$ and not for $k = 1$. The reason is that we have to avoid zeroes of the local L-factors.

Let X be the set of characters χ with $\chi(-1) = -1$ and conductor dividing some fixed N. Given the inclusions of theorem 4.3.3 for all $\chi \in X$,

$$\det\nolimits_\Lambda\left(\Lambda c_0(\chi)[-1]\right) \supset \det\nolimits_\Lambda\left(\mathbf{R}\Gamma_{\text{gl}}(T_p(\chi))\right),$$

it is enough to prove that the inclusion

$$\det\nolimits_\Lambda\left(\bigoplus_{\chi\in X} \Lambda c_0(\chi)[-1]\right) \supset \det\nolimits_\Lambda\left(\mathbf{R}\Gamma_{\text{gl}}(\bigoplus_{\chi\in X} T_p(\chi))\right)$$

is an isomorphism. This is a standard trick in Iwasawa theory.

Lemma 4.4.2. *The above inclusion is an isomorphism if and only if it is an isomorphism after tensoring with $\Lambda/\mathfrak{a}$.*

Proof. As Λ is a local ring and $\mathfrak{a}$ is contained in its radical, this is just Nakayama's lemma. □

Note that $\mathfrak{a}$ is the augmentation ideal so that the tensor product with $\Lambda/\mathfrak{a}$ just means to take the coinvariants under $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$, thus

$$\det_\Lambda \mathbf{R}\Gamma_{\mathrm{gl}}\left(\bigoplus_{\chi\in X} T_p(\chi)\right) \otimes^{\mathbb{L}}_\Lambda \Lambda/\mathfrak{a} \cong \det_{\mathcal{O}_p} R\Gamma\left(\mathbb{Z}[1/p], \bigoplus_{\chi\in X} T_p(\chi)\right).$$

On the other hand, we know from proposition 4.2.1 that $\bigoplus_{\chi\in X}\Lambda c_0(\chi)$ is a free Λ-submodule of $\mathbf{H}^1_{\mathrm{gl}}\left(\bigoplus_{\chi\in X} T_p(\chi)\right)$, so that we have to show that

$$\det_{\mathcal{O}_p}\left(\bigoplus_{\chi\in X}\mathcal{O}_p c_0(1)(\chi)[-1]\right) \cong \det_{\mathcal{O}_p} R\Gamma\left(\mathbb{Z}[1/p], \bigoplus_{\chi\in X} T_p(\chi)\right).$$

An $\mathcal{O}_p$-generator of the left hand side is $\prod_{\chi\in X} c_0(1)(\chi)$ and the claim is just the statement of corollary 3.3.4. This proves the main conjecture.

5 Proof of the Bloch-Kato conjecture

We now want to prove the Bloch-Kato conjecture. Note that there are four cases: The character χ can have the same parity as r, this is the equal parity case or the parities are different and we are in the unequal parity case. Moreover r can be ≥ 1 or we can have $r \leq 0$. This makes four cases. If we compare with the program laid out in section 2.1, we see that points 1 and 2 have meanwhile been settled.

5.1 The Bloch-Kato conjecture in the equal parity case and $r \geq 1$

Theorem 5.1.1. *Let χ be a character with $\chi(-1) = (-1)^r$ with $r \geq 1$, $p \neq 2$, then the p-part of the equivariant Bloch-Kato conjecture is true for $V(\chi)$ and r and the cyclotomic $\mathbb{Z}/p^n$-extension $\mathbb{Q}_n/\mathbb{Q}$.*

Proof. Let $G_n = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. We have by the main conjecture 4.4.1 for χ^{-1} with $k = 1-r$

$$\det_\Lambda\left(\Lambda c_{1-r}(\chi^{-1})[-1]\right) \cong \det_\Lambda\left(\mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi^{-1})(1-r))\right)$$

and if we tensor this equality over Λ with $\mathcal{O}_p[G_n]$ and use lemma 3.3.1, we get that the element $c_{1-r}(\mathbb{Q}_n/\mathbb{Q})(\chi^{-1})$ as defined before 3.3.3 maps to a generator of

$$\det_{\mathcal{O}_p[G_n]}\left(R\Gamma(\mathbb{Z}_n[1/p], T_p(\chi^{-1})(1-r))\right)$$

(compare [Ka1] lemma 6.3.). By corollary 3.3.3 this implies the equivariant Bloch-Kato conjecture. □

Note that the theorem includes the absolute Bloch-Kato conjecture for χ up to powers of 2. Applying the functional equation (proposition B.1.3), we get from this:

Corollary 5.1.2. *Let χ be a character with $\chi(-1) = (-1)^{r-1}$ with $r < 0$, $p \neq 2$, then the p-part of the equivariant Bloch-Kato conjecture is true for $V(\chi)$ and r and the cyclotomic $\mathbb{Z}/p^n$-extension $\mathbb{Q}_n/\mathbb{Q}$.*

Corollary 5.1.3. *Let $p \neq 2$. The Main Conjecture of Iwasawa theory holds in its second form 4.2.4, i.e., for χ with $\chi(-1) = (-1)^{r-1}$ and $r \in \mathbb{Z}$, the isomorphism 4.2.1 b)*

$$Q(\Lambda) \cong \det\nolimits_\Lambda^{-1} \mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi^{-1})(1-r)) \otimes Q(\Lambda)$$

maps p-adic L-function $\mathcal{L}_p(\chi, r)$ (see 4.2.3) to a generator of $\det_\Lambda^{-1} \mathbf{R}\Gamma(T_p(\chi^{-1})(1-r))$.

Proof. It suffices to consider one r. We take $r < 0$. Let Q and P be the characteristic power series of $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi^{-1})(1-r))$ and $\mathbf{H}^2_{\mathrm{gl}}(T_p(\chi^{-1})(1-r))$. Then the lattice defined by $\det_\Lambda^{-1} \mathbf{R}\Gamma(T_p(\chi^{-1})(1-r)) \subset Q(\Lambda)$ is generated by P/Q. On the other hand $\mathcal{L}_p(\chi, r) = g/h$ with $g, h \in \Lambda$. We have to show that

$$(Ph) = (Qg)$$

as ideals of Λ.

As above let $G_n = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. Let P_n, Q_n, g_n, h_n be the images of P, Q, g, h in $\mathcal{O}_p[G_n]$. The modules $\mathbf{H}^i_{\mathrm{gl}}(T_p(\chi^{-1})(1-r))$ are not only Λ-torsion but in addition $\mathbf{H}^i_{\mathrm{gl}}(T_p(\chi^{-1})(1-r)) \otimes \mathcal{O}_p[G_n]$ is finite because all cohomology groups of $R\Gamma(\mathbb{Z}_n[1/p], T_p(\chi^{-1})(1-r))$ are finite. Hence $P_n, Q_n \in E_p[G_n]^*$ and the lattice defined by

$$\det{}^{-1} R\Gamma(\mathbb{Z}_n[1/p], T_p(\chi^{-1})(1-r)) \cong \det{}^{-1} \mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi^{-1})(1-r)) \otimes \mathcal{O}_p[G_n]$$

in $E_p[G_n]$ is given by P_n/Q_n. By definition of the p-adic L-function 4.2.3 we have for all characters τ of $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ the equality

$$\tau^{-1}(g_n/h_n) = (1 - \tau\chi(p)p^r) L(\tau\chi, r) = L_{\{p\}}(\tau\chi, r),$$

where $\tau : E_p[G_n] \to E_p$. By definition of the equivariant L-function 1.5.1 with the Euler factor at p removed, this implies $g_n/h_n = L_{\{p\}}(\mathbb{Q}_n/\mathbb{Q}, V(\chi), r)$, Taking the Euler factor from 1.2.5 into account, the equivariant Bloch-Kato conjecture in this case (5.1.2) implies

$$(P_n h_n) = (g_n Q_n)$$

as ideals in $\mathcal{O}_p[G_n]$. Recall from [Wa] proof of Theorem 7.1 that the kernel of $\mathcal{O}_p[[T]] \cong \Lambda \to \mathcal{O}_p[G_n]$ is contained in $(p, T)^{n+1}$. Hence the equality of ideals in $\mathcal{O}_p[G_n]$ implies

$$Ph \in \bigcap_n \left((gQ) + (p, T)^{n+1}\right) = (gQ)$$

because the (p, T)-adic topology on Λ is separated. Conversely, $gQ \in (Ph)$ and we have proved the claim. □

5.2 The Bloch-Kato conjecture in the equal parity case and $r < 0$

We start by recalling a theorem of Beilinson [Be] (as corrected by Neukirch [Ne] and Esnault [Es]) which gives elements in motivic cohomology mapping to the polylogarithm in Deligne cohomology. More precisely:

Theorem 5.2.1. *(Beilinson [Be], Neukirch [Ne], Esnault [Es]) There is an element for $k > 1$*

$$b_k(\zeta_N) \in H^1_{\mathcal{M}}(\mathbb{Z}, V(\mathbb{Q}(\zeta_N))(k))$$

such that

$$r_\infty(b_k(\zeta_N)) = (-Li_k(\sigma\zeta_N))_{\sigma\in G} \in V_B(\mathbb{Q}(\zeta_N))(k-1)^+.$$

The next result computes the regulator of $b_k(\zeta_N)$ in $H^1(\mathbb{Z}[\zeta_N][1/p], E_p(k))$. This was first proved by Huber and Wildeshaus [Hu-Wi] cor. 9.7 following Beilinson and Deligne. A different approach to the theorem is developed in [Hu-Ki].

Theorem 5.2.2. *([Hu-Wi] cor. 9.7.) For $k > 1$, the image of $b_k(\zeta_N) \in H^1_{\mathcal{M}}(\mathbb{Z}, V(\mathbb{Q}(\zeta_N))(k))$ under the map r_p is*

$$\frac{1}{N^{k-1}(k-1)!}\widetilde{c}_k(\zeta_N) \in H^1(\mathbb{Z}[\zeta_N][1/p], E_p(k)).$$

Proof. This is corollary 9.7. in [Hu-Wi] combined with lemma 3.1.7 and [Hu-Wi] lemma B.4.9. □

Remark: For $k = 1$ the Beilinson element is $b_1(\zeta_N) = 1 - \zeta_N$. This is not an element of $H^1_{\mathcal{M}}(\mathbb{Z}, V(\mathbb{Q}(\zeta_N))) = \mathbb{Z}[\zeta_N]^* \otimes E$ if $N = l^i$ is a prime power. However, even in this case, its χ-component for $\chi \neq 1$

$$b_1(\chi) = p_{\chi^{-1}}(1-\zeta_N) = p_{\chi^{-1}}\left(\frac{(1-\zeta_{l^i})^{l^{i-1}(l-1)}}{l} \otimes \frac{1}{l^{i-1}(l-1)}\right).$$

is in fact an element of $H^1_{\mathcal{M}}(\mathbb{Z}, V(\chi)(1))$. The formula of the last theorem holds with $\widetilde{c}_1(\zeta_N)$ the class of $1-\zeta_N$ in Galois cohomology. Again its χ-component is an element of $H^1(\mathbb{Z}[1/p], V_p(\chi)(1))$ for $\chi \neq 1$. This suffices for the computations in the sequel.

Theorem 5.2.3. *Let $r < 0$ and $\chi(-1) = (-1)^r$ (or $r = 0$, $\chi(-1) = 1$ and $\chi(p) \neq 1$), then the Bloch-Kato conjecture for $V(\chi)$ and r is true up to powers of 2.*

Proof. We first repeat the computations used in the proof of the Beilinson conjecture in this case. The fundamental line for $V(\chi)$ and $r = 1-k$ (see 1.2.2) is

$$\Delta_f(V(\chi)(1-k)) = \det_E^{-1} H^1_{\mathcal{M}}(\mathbb{Z}, V(\chi)^\vee(k)) \otimes \det_E^{-1} V_B(\chi)(1-k).$$

We identitfy $V(\chi)^\vee \cong V(\chi^{-1})$. Let $t_B(\chi^{-1})$ be as in 3.1.1. Let p_χ be the projector onto the χ-eigenspace and let $b_k(\chi^{-1}) := p_\chi b_k(\zeta_N)$, where N is the conductor of χ. We claim that the element

$$\delta = \left(-\frac{N^{k-1}(k-1)!}{2} b_k(\chi^{-1})\right)^{-1} \otimes (2\pi i)^{k-1} t_B(\chi^{-1})$$

maps to $(L(\chi, 1-k)^*)^{-1}$ under the isomorphism

$$\Delta_f(V(\chi)(1-k)) \otimes \mathbb{R} \cong E_\infty.$$

Theorem 5.2.1 implies that r_∞ maps $-\frac{N^{k-1}(k-1)!}{2} b_k(\chi^{-1})$ to

$$\frac{N^{k-1}(k-1)!}{2} \sum_{\tau \in G} \chi^{-1}(\tau) Li_k(\tau\zeta_N) t_B(\chi^{-1}).$$

Using the functional equation 3.2.3 and the fact that the Γ-function has residue $\frac{(-1)^{k-1}}{(k-1)!}$ at $s = 1-k$, this is

$$L(\chi, 1-k)^*(2\pi i)^{k-1} t_B(\chi^{-1}).$$

Thus the Bloch-Kato conjecture with the lattice $T_p(\chi^{-1})^\vee \subset V_p(\chi)$ says that δ is a generator of $\det_{\mathcal{O}_p} R\Gamma_c(\mathbb{Z}[1/p], T_p(\chi^{-1})^\vee(1-k))$. Using 1.2.5 for the local Euler factor (it applies as $(1-\chi(p)p^{k-1}) \neq 0$ if $k \neq 1$ or $\chi(p) \neq 1$) and 1.2.10, this is equivalent to

$$(1-\chi(p)p^{k-1}) r_p \left(-\frac{N^{k-1}(k-1)!}{2} b_k(\chi^{-1}) \right)$$

being a generator of $\det^{-1}_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[\zeta_N][1/p], T_p(\chi^{-1})(k))$ for all p. With theorem 5.2.2 and the relation $(1-\chi(p)p^{k-1})\tilde{c}_k(1)(\chi^{-1}) = c_k(1)(\chi^{-1})$, this means that $\frac{-1}{2} c_k(1)(\chi^{-1})$ has to be a generator of $\det_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(k))$. The main conjecture implies that

$$\det_\Lambda \left(\Lambda c_k(\chi^{-1})[-1]\right) \cong \det_\Lambda \left(\mathbf{R}\Gamma_{\mathrm{gl}}(T_p(\chi^{-1})(k))\right)$$

so that

$$\det_{\mathcal{O}_p} \left(\mathcal{O}_p c_k(1)(\chi^{-1})[-1]\right) \cong \det_{\mathcal{O}_p} \left(R\Gamma_{\mathrm{gl}}(\mathbb{Z}[1/p], T_p(\chi^{-1})(k))\right),$$

i.e., $c_k(1)(\chi^{-1})$ is a generator of $\det^{-1}_{\mathcal{O}_p} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(k))$ as claimed. □

Applying the functional equation (proposition B.1.3), we get from this:

Corollary 5.2.4. *Let χ be a character with $\chi(-1) = (-1)^{r-1}$ with $r \geq 2$, then the Bloch-Kato conjecture is true up to powers of 2 for $V(\chi)$ and r.*

5.3 The Bloch-Kato conjecture for $r = 0, 1$

Using the main conjecture in its second form we can now prove the unequal parity case $\chi(-1) = (-1)^{r-1}$ and $r = 0, 1$. For $r = 0$ we still have to assume $\chi(p) \neq 1$.

Theorem 5.3.1. *Let χ be a character with $\chi(-1) = -1$ and $\chi(p) \neq 1$. Then the Bloch-Kato conjecture for $V(\chi)$ and $r = 0$ is true up to powers of 2.*

Proof. We have
$$\Delta_f(V(\chi)) = E$$
and we have to show that the element $(1-\chi(p))L(\chi,0)$ maps to a generator of
$$\det_{\mathcal{O}_p}^{-1} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(1)).$$
Let ϱ be the augmentation map on $Q(\Lambda)$ as considered before 4.2.3. Then $\varrho(\mathcal{L}_p(\chi,k)) = (1-\chi(p)p^{-k})L(\chi,k)$ for all $k \leq 0$. By the main conjecture in the unequal parity case proved in corollary 5.1.3, we have that $\varrho(\mathcal{L}_p(\chi,0))$ maps to a generator of
$$\det_\Lambda^{-1} \mathbf{R}\Gamma(T_p(\chi^{-1})(1)) \otimes_\Lambda \mathcal{O}_p \cong \det_{\mathcal{O}_p}^{-1} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(1)).$$
This proves the result. □

Remark: If $\chi(p) = 1$, then the main conjecture gives an isomorphism between $\det_{\mathcal{O}_p} 0 = \mathcal{O}_p$ and $\det_{\mathcal{O}_p}^{-1} R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1})(1))$. This does not tell us anything about $L(\chi,0)^*$. In fact, the leading coefficients of $L(\chi,s)$ and $(1-\chi(p)p^{-s})L(\chi,s)$ at $s=0$ now differ by a transcendental factor.

Theorem 5.3.2. *Let χ be a character with $\chi(-1) = 1$. Then the Bloch-Kato conjecture for $V(\chi)$ and $r=1$ is true up to powers of 2.*

Proof. If χ is the trivial character, then $V(\chi) = V(\mathbb{Q})$ and the result follows from the class number formula 2.3.1. Suppose that $\chi \neq 1$. Let N be the conductor of χ. We have
$$\Delta_f(V(\chi)(1)) = \det_E^{-1} H^1_{\mathcal{M}}(\mathbb{Z}, V(\chi)(1)) \otimes \det V_{\mathrm{DR}}(\chi).$$
Let $b_1(\chi) = p_{\chi^{-1}}(1-\zeta_N) \in H^1_{\mathcal{M}}(\mathbb{Z}, V(\chi)(1))$ as in the remark after theorem 5.2.2. Similarly, $\widetilde{c}_1(1)(\chi) = p_{\chi^{-1}}(1-\zeta_N) \in H^1(\mathbb{Z}[1/p], V_p(\chi)(1))$. Consider
$$\delta = \left(-\frac{1}{2}b_1(\chi)\right)^{-1} \otimes p_{\chi^{-1}}(\zeta_N) \in \Delta_f(V(\chi)).$$
The regulator r_∞ maps $-\frac{1}{2}b_1(\chi)$ to $1/2\sum_\sigma \chi(\sigma)Li_1(\sigma(\zeta_N))t_B(\chi)$, which is $L(\chi,1)\tau(\chi^{-1})t_B(\chi)$, where $\tau(\chi^{-1}) := \sum_{a=1}^N \chi^{-1}(a)\zeta_N^a = \sum_\sigma \chi(\sigma)\sigma(\zeta_N)$ is the Gauss sum. The identification $V_{\mathrm{DR}}(\chi)_{\mathbb{R}} \cong V_B(\chi)_{\mathbb{R}}$ maps $p_{\chi^{-1}}(\zeta_N)$ to $\tau(\chi^{-1})t_B(\chi)$, so that δ is the element which maps to $L(\chi,1)^{-1}$ under the isomorphism $\Delta_f(V(\chi)) \otimes_{\mathbb{Q}} \mathbb{R} \cong E \otimes \mathbb{R}$. We have to compare its image with some natural integral structure of
$$\det R\Gamma_c(\mathbb{Z}[1/p], V_p(\chi)(1)) = \det R\Gamma_f(\mathbb{Z}[1/p], V_p(\chi)(1)) \otimes \det^{-1} R\Gamma_f(\mathbb{Q}_p, V_p(\chi)(1))$$
under the identification 1.2.7. Taking the Euler factor from 1.2.5 into account, the image of δ is
$$\left(-\frac{1-\chi(p)p^{-1}}{2}\widetilde{c}_1(1)(\chi)\right)^{-1} \otimes \exp_p(p_{\chi^{-1}}(\zeta_N))$$
where the first element is mapped to $H^1_f(\mathbb{Z}[1/p], V_p(\chi)(1))$ and the second to $H^1_f(\mathbb{Q}_p, V_p(\chi)(1))$. Note that we are in the case $H^i_c(\mathbb{Z}[1/p], V_p(\chi)(1)) = 0$ for all i, hence we can also compare with the lattice $\mathcal{O}_p \subset E_p = \det R\Gamma_c(\mathbb{Z}[1/p], V_p(\chi)(1))$.

The image of $-\frac{1-\chi(p)p^{-1}}{2}\widetilde{c}_1(1)(\chi)$ under the composition

$$H^1_f(\mathbb{Z}[1/p], V_p(\chi)(1)) \to H^1_f(\mathbb{Q}_p, V_p(\chi)(1)) \xrightarrow{\log_p} V_{\mathrm{DR}}(\chi)\otimes E_p$$

is

$$-\frac{1-\chi(p)p^{-1}}{2\#G}\sum_\sigma \chi(\sigma)\log_p(1-\zeta_N^\sigma) = \left(-\frac{1-\chi(p)p^{-1}}{2\tau(\chi^{-1})}\sum_\sigma \chi(\sigma)\log_p(1-\zeta_N^\sigma)\right) p_{\chi^{-1}}(\zeta_N)$$

The conjecture allows the choice of lattice in $V_p(\chi)$ and we use $T_p(\chi^{-1})^\vee$. Hence it remains to show that the element

$$-\frac{1-\chi(p)p^{-1}}{2\tau(\chi^{-1})}\sum_\sigma \chi(\sigma)\log_p(1-\zeta_N^\sigma)$$

of E_p is a generator of $\det R\Gamma_c(\mathbb{Z}[1/p], T_p(\chi^{-1})^\vee)(1)) \cong \det R\Gamma(\mathbb{Z}[1/p], T_p(\chi^{-1}))$. By the main conjecture 5.1.3 it is indeed generated by $\varrho(\mathcal{L}_p(\chi,1))$ where ϱ is the augmentation of $Q(\Lambda)$ as considered before 4.2.3. The formula from [Wa] theorem 5.18 together with our normalization of the isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ with $(\mathbb{Z}/N)^*$ gives

$$\varrho(\mathcal{L}_p(\chi,1)) = -(1-\chi(p)p^{-1})\frac{\tau(\chi)}{N}\sum_\sigma \chi(\sigma)\log_p(1-\zeta_N^\sigma).$$

The claim follows using the relation $\tau(\chi)\tau(\chi^{-1}) = N$. □

5.4 End of proof of the Bloch-Kato conjecture

Theorem 5.4.1. *The Bloch-Kato conjecture is true for all abelian Artin motives.*

Proof. Extension of coefficients is faithfully flat, hence it suffices to prove the conjecture after extension of coefficients. Every abelian Artin motive is the direct sum of motives of the form $V(\chi)$ for Dirichlet characters χ after an appropriate extension, so it suffices to prove the theorem for these. Collecting our results in theorem 5.1.1, corollary 5.1.2, theorem 5.2.3, corollary 5.2.4, theorem 5.3.1 and theorem 5.3.2, we see that only the case $r=0$ and $\chi(p)=1$ is still missing. We get the final case by proposition B.1.2 via the functional equation. □

6 Investigation of some Iwasawa modules

In this section we prove some structural results about Iwasawa modules, which are only needed for theorem 4.3.3 and thus can be skipped at a first reading. For the definitions of the Λ-modules we refer to 4.1.2.

6.1 The Iwasawa module $\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1))$

In this section we investigate $\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1))$, which is necessary for the proof of proposition 4.3.4.

Lemma 6.1.1. *Let* $G_{\infty,p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_\infty \otimes \mathbb{Q}_p)$*, then*

$$\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1)) = \begin{cases} \textit{finite} & \textit{if } \chi|_{G_{\infty,p}} \neq 1, \\ T_p(\chi) & \textit{if } \chi|_{G_{\infty,p}} = 1. \end{cases}$$

Moreover there exists n *such that* $\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1)) \cong H^2(\mathbb{Q}_n \otimes \mathbb{Q}_p, T_p(\chi)(1))$.

Proof. As $\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1)) \cong \mathbf{H}^0_{\mathrm{loc}}(T_p(\chi)^*)^*$ we get that $\mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1))$ are the coinvariants $T_p(\chi)_{G_{\infty,p}}$ and the statement of the lemma is clear. □

Corollary 6.1.2. *Let* χ *be a Dirichlet character and* $G_{\infty,p}$ *as above, then*

$$\det{}_\Lambda \mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1)) \cong \begin{cases} \Lambda & \textit{if } \chi|_{G_{\infty,p}} \neq 1 \\ \mathfrak{a}_\chi^{-1} & \textit{if } \chi|_{G_{\infty,p}} = 1, \end{cases}$$

where $\mathfrak{a}_\chi$ *is the kernel of* $\chi^{-1} : \Lambda \to \mathcal{O}_p$*. In particular, the localization* $\det_{\Lambda_\mathfrak{p}} \mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1))_\mathfrak{p} \cong \Lambda_\mathfrak{p}$ *for all prime ideals* $\mathfrak{p} \neq \mathfrak{a}_\chi$ *of height* 1.

Proof. Consider the computation in lemma 6.1.1. In the first case, the corollary follows from the fact that the determinant of a finite Λ module is trivial. In the second case, it follows from the fact that $\chi(\gamma_0)\gamma_0 - 1$ generates $\mathfrak{a}_\chi$ (where γ_0 generates Γ) and $T_p(\chi) \cong \Lambda/(\chi(\gamma_0)\gamma_0 - 1)$ as Λ-modules. □

6.2 The Iwasawa module $\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*)^*$

In this section we prove 4.3.5. Let $\mathfrak{a}_\chi$ be the kernel of $\chi : \Lambda \to \mathcal{O}_p$.

Proposition 6.2.1. *Let* χ *be a character with* $\chi(-1) = 1$*, then the module*

$$\left(\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*)^*\right)_{\mathfrak{a}_\chi} = 0$$

is zero.

Proof. Let $\chi = \theta\tau$ where θ is of the first and τ of the second kind. τ can be viewed as a finite character of $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. The statement remains invariant under twisting with τ^{-1} (which is trivial over $\mathbb{Q}_\infty$), hence it suffices to consider the case $\tau = 1$, i.e., $\mathfrak{a}_\chi = \mathfrak{a}$ is the augmentation ideal.

Let $\kappa(\mathfrak{a})$ be the residue class field of $\mathfrak{a}$. By Nakayama's lemma it is enough to show that $\kappa(\mathfrak{a}) \otimes_{\Lambda_\mathfrak{a}} (\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*)^*)_\mathfrak{a} = 0$.

As $\Lambda/\mathfrak{a} = \mathcal{O}_p$ it suffices to show that

$$\Lambda/\mathfrak{a} \otimes_\Lambda (\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*)^*)$$

is finite. By the definition of $\Lambda = \mathcal{O}_p[[\Gamma]]$ we have to show that the Γ-invariants

$$\left((\mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*))^\Gamma\right)^*$$

are finite. The Hochschild-Serre spectral sequence gives

$$0 \to H^1(\Gamma, \mathbf{H}^0_{\mathrm{gl}}(T_p(\chi)^*)) \to H^1(\mathbb{Z}[1/p], T_p(\chi)^*) \to \mathbf{H}^1_{\mathrm{gl}}(T_p(\chi)^*)^\Gamma \to 0,$$

because $\Gamma \cong \mathbb{Z}_p$ has cohomological dimension 1. The group in the middle is finitely cogenerated. It suffices to compute the corank of the cokernel. We compute the rank of $H^1(\mathbb{Z}[1/p], T_p(\chi)^\vee)$ via the Euler-Poincaré characteristic

$$\sum_{i=0}^{2}(-1)^i \operatorname{rk}_{E_p} H^i(\mathbb{Z}[1/p], V_p(\chi)^\vee)$$

which is equal to 0 by [Ja], lemma 2, as $\chi(-1) = 1$. We know by [Sch] paragraph 7, Satz 2 that $H^2(\mathbb{Z}[\zeta_N][1/p], E_p)^+$ (where $+$ denotes the invariants of complex conjugation) is zero because Leopoldt's conjecture is true for $\mathbb{Q}(\zeta_N)$. This implies that $H^2(\mathbb{Z}[1/p], V_p(\chi)^\vee)$ is zero, because it is a direct summand.

If $\chi \neq 1$ is not the trivial character $H^0(\mathbb{Z}[1/p], V_p(\chi)^\vee) = 0$ and hence $\operatorname{rk}_{E_p} H^1(\mathbb{Z}[1/p], V_p(\chi)^\vee) = 0$ in this case. Hence the corank of $H^1(\mathbb{Z}[1/p], T_p(\chi)^*)$ is also 0.

If $\chi = 1$ is the trivial character, the same argument shows that $H^1(\mathbb{Z}[1/p], T_p(\chi)^*)$ has corank 1. However, in this case $H^1(\Gamma, \mathbf{H}^0_{\mathrm{gl}}(T_p(\chi)^*))$ also has corank 1. ☐

6.3 The Iwasawa module $\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(1))$

We show proposition 4.3.6. As before let $\mathfrak{a}_\chi$ be the kernel of $\chi^{-1} : \Lambda \to \mathcal{O}_p$. Recall from section 1.1 the definition of $H^1_{/f}(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi)(1))$ and let

$$\mathbf{H}^1_{/f}(V_p(\chi)(1)) := \varprojlim_n H^1_{/f}(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi)(1)).$$

We will later consider the canonical map

$$\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(1)) \to \mathbf{H}^1_{/f}(V_p(\chi)(1)).$$

the next lemma shows that the image of this map is a finitely generated $\mathcal{O}_p$-module and computes its determinant.

Lemma 6.3.1. *The $\Lambda \otimes_{\mathcal{O}_p} E_p$-module $\mathbf{H}^1_{/f}(V_p(\chi)(1))$ is a free module of finite rank over E_p. Moreover*

$$\det{}_{\Lambda_{\mathfrak{a}_\chi}} \mathbf{H}^1_{/f}(V_p(\chi)(1))_{\mathfrak{a}_\chi} \cong \det{}_{\Lambda_{\mathfrak{a}_\chi}} \mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1))_{\mathfrak{a}_\chi}$$

Proof. We show first that the inverse limit defining $\mathbf{H}^1_{/f}(V_p(\chi)(1))$ is in fact constant. By duality

$$H^1_f(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi^{-1})) \cong H^1_{/f}(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi)(1))^\vee$$

and the exact sequence in section 1.1

$$0 \to H^0(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi^{-1})) \to D_{\mathrm{cris}}(V(\mathbb{Q}_n) \otimes V_p(\chi^{-1}))$$
$$\to D_{\mathrm{cris}}(V(\mathbb{Q}_n) \otimes V_p(\chi^{-1})) \to H^1_f(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi^{-1})) \to 0$$

implies that $\operatorname{rk}_{E_p} H^1_f(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi^{-1})) = \operatorname{rk}_{E_p} H^0(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi^{-1}))$ and the latter is constant for $n \gg 0$, because $\mathbb{Q}_n$ is totally ramified at p. This implies that

$\mathbf{H}^1_{/f}(V_p(\chi)(1))$ is a free module of finite rank over E_p. We compute the determinants using the above exact sequence for n big enough:

$$\begin{aligned}
\det_{\Lambda_{\mathfrak{a}_\chi}} \mathbf{H}^1_{/f}(V_p(\chi)(1))_{\mathfrak{a}_\chi} &\cong \det_{\Lambda_{\mathfrak{a}_\chi}} H^1_{/f}(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi)(1))_{\mathfrak{a}_\chi} \\
&\cong \det^{-1}_{\Lambda_{\mathfrak{a}_\chi}} H^1_f(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi^{-1}))_{\mathfrak{a}_\chi} \\
&\cong \det^{-1}_{\Lambda_{\mathfrak{a}_\chi}} H^0(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi^{-1}))_{\mathfrak{a}_\chi} \\
&\cong \det_{\Lambda_{\mathfrak{a}_\chi}} H^2(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi)(1))_{\mathfrak{a}_\chi}.
\end{aligned}$$

Note that $p \notin \mathfrak{a}_\chi$ so that the map $\Lambda \to \Lambda_{\mathfrak{a}_\chi}$ factors through $\Lambda[1/p] = \Lambda \otimes_{\mathcal{O}_p} E_p$. In particular

$$\det_{\Lambda_{\mathfrak{a}_\chi}} H^2(\mathbb{Q}_p \otimes \mathbb{Q}_n, V_p(\chi)(1))_{\mathfrak{a}_\chi} \cong \det_{\Lambda_{\mathfrak{a}_\chi}} H^2(\mathbb{Q}_p \otimes \mathbb{Q}_n, T_p(\chi)(1))_{\mathfrak{a}_\chi}$$

and by lemma 6.1.1 the last determinant is equal to

$$\det_{\Lambda_{\mathfrak{a}_\chi}} \mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1))_{\mathfrak{a}_\chi}.$$

This is the desired result. □

Lemma 6.3.2. *Suppose that N is not a p-power, then the element $c_1(\chi) \in \mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(1))$ maps to zero in $\mathbf{H}^1_{/f}(V_p(\chi)(1))$.*

Proof. By appendix A the map at finite level

$$H^1(\mathbb{Z}_n[\zeta_{p^nN}][1/p], \mathcal{O}_p(1)) \to H^1(\mathbb{Q}_p \otimes \mathbb{Q}(\zeta_{p^nN}), \mathcal{O}_p(1)) \to H^1_{/f}(\mathbb{Q}_p \otimes \mathbb{Q}(\zeta_{p^nN}), E_p(1))$$

maps a unit in $\mathbb{Z}[\zeta_{p^nN}][1/p]^\times \otimes \mathcal{O}_p \cong H^1(\mathbb{Z}[\zeta_{p^nN}][1/p], \mathcal{O}_p(1))$ to the system of its valuations in $\bigoplus_{v|p} E_p \cong H^1_{/f}(\mathbb{Q}_p \otimes \mathbb{Q}(\zeta_{p^nN}), E_p(1))$. As N is not a p-power the elements $c_1(\zeta_{p^nN}) = 1 - \zeta_{p^nN}$ are units in $\mathbb{Z}[\zeta_{p^nN}]$, hence have zero valuation. Using the commutative diagram

$$\begin{array}{ccc}
\varprojlim_n H^1(\mathbb{Q}_p \otimes \mathbb{Q}(\zeta_{p^nN}), \mathcal{O}_p(1)) & \longrightarrow & \varprojlim_n H^1_{/f}(\mathbb{Q}_p \otimes \mathbb{Q}(\zeta_{p^nN}), E_p(1)) \\
\downarrow & & \downarrow \\
\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(1)) & \longrightarrow & \mathbf{H}^1_{/f}(V_p(\chi)(1))
\end{array}$$

proves the claim. □

As p is invertible in $\Lambda_{\mathfrak{a}_\chi}$, we get a surjection

$$\left(\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(1))/\Lambda c_1(\chi)\right)_{\mathfrak{a}_\chi} \to \mathbf{H}^1_{/f}(V_p(\chi)(1))_{\mathfrak{a}_\chi}.$$

Corollary 6.3.3. *Suppose that N is not a p-power and $\chi(-1) = 1$, then this map induces*

$$\det^{-1}_{\Lambda_{\mathfrak{a}_\chi}} \left(\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(1))/\Lambda c_1(\chi)\right)_{\mathfrak{a}_\chi} \subset \det^{-1}_{\Lambda_{\mathfrak{a}_\chi}} \mathbf{H}^2_{\mathrm{loc}}(T_p(\chi)(1))_{\mathfrak{a}_\chi}.$$

Proof. Let

$$K := \ker\left(\left(\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(1))/\Lambda c_1(\chi)\right)_{\mathfrak{a}_\chi} \to \mathbf{H}^1_{/f}(V_p(\chi)(1))_{\mathfrak{a}_\chi}\right).$$

Then

$$\det\nolimits^{-1}_{\Lambda_{\mathfrak{a}_\chi}} K \otimes \det\nolimits^{-1}_{\Lambda_{\mathfrak{a}_\chi}} \mathbf{H}^1_{/f}(V_p(\chi)(1))_{\mathfrak{a}_\chi} \cong \det\nolimits^{-1}_{\Lambda_{\mathfrak{a}_\chi}} \left(\mathbf{H}^1_{\mathrm{loc}}(T_p(\chi)(1))/\Lambda c_1(\chi)\right)_{\mathfrak{a}_\chi}$$

and $\det^{-1}_{\Lambda_{\mathfrak{a}_\chi}} K$ is an ideal in $\Lambda_{\mathfrak{a}_\chi}$ because $\Lambda_{\mathfrak{a}_\chi}$ is a discrete valuation ring. This together with lemma 6.3.1 implies the result. □

A A local computation

The following lemmas are known to the experts but for lack of finding a precise reference we repeat the proofs here. They provide the proof of proposition 2.3.2.

We keep the notations of the main text. Let K be a finite extension of $\mathbb{Q}_p$. Following Bloch-Kato we define $H^1_f(K, \mathbb{Z}_p(1)) := \iota^{-1}(H^1_f(K, \mathbb{Q}_p(1))$, where $\iota : H^1(K, \mathbb{Z}_p(1)) \to H^1(K, \mathbb{Q}_p(1))$, and $H^1_{/f} := H^1/H^1_f$. For an abelian group A we denote by $A^\wedge := \varprojlim_n A \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ the p-adic completion of A.

Lemma A.1. *Let K be a finite extension of $\mathbb{Q}_p$. Then*

$$H^1_f(K, \mathbb{Z}_p(1)) = (\mathcal{O}_K^*)^\wedge$$

under the identification $H^1(K, \mathbb{Z}_p(1)) = (K^*)^\wedge$.

Proof. Bloch-Kato show in [BlKa] p. 358 that their exponential agrees with the usual p-adic exponential in this case. Hence $H^1_f(K, \mathbb{Q}_p(1)) \subset (K^*)^\wedge \otimes \mathbb{Q}_p$ agrees with the image of the exponential which is $(\mathcal{O}_K^*)^\wedge \otimes \mathbb{Q}_p$. The preimage of this group in $(K^*)^\wedge$ is $(\mathcal{O}_K^*)^\wedge$. □

Corollary A.2. *There is a unique isomorphism*

$$H^1_{/f}(K, \mathbb{Z}_p(1)) \to \mathbb{Z}_p$$

compatible with

$$H^1(K, \mathbb{Z}_p(1)) = (K^*)^\wedge \xrightarrow{v} \mathbb{Z}_p$$

where v is the valuation map of K normalized such that $v(\pi) = 1$ for a uniformizer.

Proof. By definition, $H^1_{/f}(K, \mathbb{Z}_p(1))$ is the image of $H^1(K, \mathbb{Z}_p(1)) = (K^*)^\wedge$ in $H^1(K, \mathbb{Q}_p(1))/H^1_f(K, \mathbb{Q}_p(1))$. Hence it can be identified with $(K^*)^\wedge/(\mathcal{O}_K^*)^\wedge$ which is isomorphic to $\mathbb{Z}_p$ via the valuation. □

Proposition A.3. *Let F be a finite extension of $\mathbb{Q}$. The Poitou-Tate localization sequence induces an exact sequence*

$$0 \to (\mathcal{O}_K^*)^\wedge \to H^1(\mathcal{O}_F[1/p], \mathbb{Z}_p(1)) \to H^1_{/f}(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(1)) \to Cl(\mathcal{O}_F)^\wedge \to$$
$$\to H^2(\mathcal{O}_F[1/p], \mathbb{Z}_p(1)) \to H^2(F \otimes \mathbb{Q}_p, \mathbb{Z}_p(1)) \to H^0(\mathcal{O}_F[1/p], \mathbb{Q}_p/\mathbb{Z}_p)^* \to 0.$$

Proof. We apply the previous corollary to the local fields in $F \otimes \mathbb{Q}_p$. In particular, we identify $H^1_{/f}(F \otimes \mathbb{Q}_p, \mathbb{Q}_p(1)) \cong \bigoplus_{v|p} \mathbb{Q}_p$ via the valuation maps. Thus the kernel of $H^1(\mathcal{O}_F[1/p], \mathbb{Z}_p(1)) \to H^1_{/f}(F \otimes \mathbb{Q}_p, \mathbb{Q}_p(1))$ is $(\mathcal{O}_F^*)^\wedge$. On the other hand it follows from [Sch] that one has an exact sequence

$$0 \to Cl(\mathcal{O}_F[1/p])^\wedge \to H^2(\mathcal{O}_F[1/p], \mathbb{Z}_p(1)) \to \\ \to H^0(F \otimes \mathbb{Q}_p, \mathbb{Q}_p/\mathbb{Z}_p)^* \to H^0(\mathcal{O}_F[1/p], \mathbb{Q}_p/\mathbb{Z}_p)^* \to 0.$$

Let $j : \operatorname{Spec} \mathcal{O}_F[1/p] \to \operatorname{Spec} \mathcal{O}_F$. The long exact sequence in Zariski-cohomology for the short exact sequence of sheaves

$$0 \to \mathcal{O}^* \to j_*\mathcal{O}^* \to \bigoplus_{v|p} \mathbb{Z} \to 0$$

yields

$$0 \to \mathcal{O}_F^* \to \mathcal{O}_F[1/p]^* \to \bigoplus_{v|p} \mathbb{Z} \to Cl(\mathcal{O}_F) \to Cl(\mathcal{O}_F[1/p]) \to 0\ .$$

If we finally observe that $H^1(\mathcal{O}_F[1/p], \mathbb{Z}_p(1)) \cong (\mathcal{O}_F[1/p]^*)^\wedge$ we get the desired result. □

Let K_0 be the maximal unramified subfield of K. Let $\mathbb{F}_q$ be its residue field, $q = p^f$. We have $D_{\text{cris}}(\operatorname{Ind}_{\mathbb{Q}_p}^K \mathbb{Q}_p) = B_{\text{cris}}^{G_K} = K_0$. As before ϕ is the Frobenius on B_{cris} normalized as in [Fo] 3.2. The fundamental short exact sequence

$$0 \to \mathbb{Q}_p \to B_{\text{cris}} \cap B_{dR}^+ \xrightarrow{\phi-1} B_{\text{cris}} \to 0,$$

induces

$$0 \to \mathbb{Q}_p \to K_0 \xrightarrow{\operatorname{Fr}-1} K_0 \to H^1_f(K, \mathbb{Q}_p) \to 0$$

where Fr is the geometric Frobenius in $G(K_0/\mathbb{Q}_p) = G(\mathbb{F}_q/\mathbb{F}_p)$ by loc. cit. footnote p. 210.

Lemma A.4. *The following sequence is exact:*

$$0 \to \mathbb{Z}_p \to \mathcal{O}_{K_0} \xrightarrow{\operatorname{Fr}-1} \mathcal{O}_{K_0} \xrightarrow{\operatorname{tr}} \mathbb{Z}_p \to 0$$

Proof. We take the point of view of $\mathbb{Z}_p[G]$-modules with $G = G(K_0/\mathbb{Q}_p)$. Then $\mathcal{O}_{K_0}$ is isomorphic to $\mathbb{Z}_p[G]$ by choice of a primitive element ω. Note that this is possible because it is possible for finite fields and K_0 is unramified. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_p & \xrightarrow{\Delta} & \mathbb{Z}_p[G] & \xrightarrow{\operatorname{Fr}-1} & \mathbb{Z}_p[G] & \xrightarrow{\Sigma} & \mathbb{Z}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \operatorname{tr}(\omega)} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \operatorname{tr}(\omega)} & & \\
0 & \longrightarrow & \mathbb{Z}_p & \longrightarrow & \mathcal{O}_{K_0} & \xrightarrow{\operatorname{Fr}-1} & \mathcal{O}_{K_0} & \xrightarrow{\operatorname{tr}} & \mathbb{Z}_p & \longrightarrow & 0
\end{array}$$

where Δ is the diagonal embedding, Σ is summation, ω means muliplication by ω on the right. The first line is exact as the image of $(\operatorname{Fr}-1)$ is nothing but the augmentation ideal. In the second line, $\mathbb{Z}_p$ is kernel of $\operatorname{Fr}-1$. Hence $\operatorname{tr}(\omega)$ is a unit of $\mathbb{Z}_p$. All vertical maps are isomorphisms, hence the second line is also exact as claimed. □

In particular:

Corollary A.5. *The boundary morphism* $\delta : K_0 \to H^1_f(K, \mathbb{Q}_p)$ *factors via the trace* $\delta = \delta' \circ \mathrm{tr}$ *and* $\delta' \,\mathrm{tr}(\mathcal{O}_{K_0})$ *is a natural lattice in* $H^1_f(K, \mathbb{Q}_p)$.

We do not need to know whether this lattice agrees with $H^1_f(K, \mathbb{Z}_p)$, but rather:

Lemma A.6. *Under the local duality isomorphism* $H^1_{/f}(K, \mathbb{Q}_p(1))^\vee \cong H^1_f(K, \mathbb{Q}_p)$ *the lattice* $H^1_{/f}(K, \mathbb{Z}_p(1))^\vee$ *is identified with* $\delta' \,\mathrm{tr}(\mathcal{O}_{K_0})$.

Proof. Recall that $H^1(K, \mathbb{Q}_p(1)) \cong (K^*)^\wedge \otimes \mathbb{Q}_p$ and $H^1(K, \mathbb{Q}_p) = \mathrm{Hom}(G_K, \mathbb{Q}_p)$. We have identified $H^1_{/f}(K, \mathbb{Z}_p(1))$ with $\mathbb{Z}_p$ via the valuation map, i.e., a generator of $H^1_{/f}(K, \mathbb{Z}_p(1))^\vee$ is the valuation map. Local duality maps the valuation map to the element $\phi^\vee_{K_0} \in \mathrm{Hom}(G_K, \mathbb{Q}_p)$ which factors through G_{K_0} and maps the arithmetic Frobenius of K_0 to 1. Hence we have to show that $\phi^\vee_{K_0}$ generates $\mathrm{tr}(\mathcal{O}_{K_0}) \subset H^1(K, \mathbb{Q}_p)$. As H^1_f and $H^1_{/f}$ are orthogonal under local duality, this is certainly true rationally.

We first consider the special case $K = K_0 = \mathbb{Q}_p$. Then Fr operates as the identity. In order to compute the $\delta(1)$ it suffices to construct a preimage under $\phi - 1$ in the Wittring $W(\overline{\mathbb{F}_p}) \subset B_{\mathrm{cris}} \cap B^+_{dR}$ and note that on this subring the operation of the Frobenius of the Galois group and of the Frobenius in B_{cris} agrees. The computation yields $\delta(1) = \phi^\vee_{\mathbb{Q}_p}$.

In general, we consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}_{K_0} & \xrightarrow{\mathrm{tr}} & \mathbb{Z}_p & \xrightarrow{\delta'} & H^1(G_{K_0}, \mathbb{Q}_p) \\
\uparrow & & \uparrow{\scriptstyle f} & & \uparrow \\
\mathbb{Z}_p & \xrightarrow{id} & \mathbb{Z}_p & \xrightarrow{\delta} & H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p)
\end{array}$$

and note that the image of $\phi^\vee_{\mathbb{Q}_p}$ in $H^1(G_{K_0}, \mathbb{Q}_p)$ is $f\phi^\vee_{K_0}$. □

Remark: This computation is prone to normalization problems between arithmetic and geometric Frobenius. However, this is irrelevant for the lattice statements we need.

Proposition A.7. *Let K be a finite extension of $\mathbb{Q}_p$. Under the composition*

$$\det{}^{-1} R\Gamma_{/f}(K, \mathbb{Q}_p(1)) \cong \det R\Gamma_f(K, \mathbb{Q}_p) \xrightarrow{\alpha} E_p$$

of local duality with the isomorphism of 1.2.4 the lattice $\det^{-1} R\Gamma_{/f}(K, \mathbb{Z}_p(1))$ *is identified with* $\mathbb{Z}_p$.

Proof. The lattice on the left is

$$\det{}^{-1} R\Gamma_{/f}(K, \mathbb{Z}_p(1)) = \det H^1_{/f}(K, \mathbb{Z}_p(1)) \otimes \det{}^{-1} H^2(K, \mathbb{Z}_p(1))$$

Local duality identifies $H^2(K, \mathbb{Z}_p(1)) \cong H^0(K, \mathbb{Q}_p/\mathbb{Z}_p)^* = H^0(K, \mathbb{Z}_p)^\vee$. The lattice in $H^1_{/f}$ was identified in lemma A.6. Hence $\det^{-1} R\Gamma_{/f}(K, \mathbb{Z}_p(1))$ is identified with

$$\det{}^{-1} \delta' \,\mathrm{tr}(\mathcal{O}_{K_0}) \otimes \det H^0(K, \mathbb{Z}_p) \overset{A.4}{=} \det \mathcal{O}_{K_0} \otimes \det{}^{-1} \mathcal{O}_{K_0}$$

where we use $\mathcal{O}_{K_0} \xrightarrow{\mathrm{Fr}-1} \mathcal{O}_{K_0}$ as integral structure of $R\Gamma_f(K, \mathbb{Z}_p)$. By definition this integral structure is mapped to $\mathbb{Z}_p$ by α. □

B The functional equation

The aim of this appendix is to prove Conjecture 2.2.1, (compatibility of the Bloch-Kato conjecture with the functional equation) in the case of Dirichlet motives and for $r > 1$. See the remarks after 2.2.1 for the cases which are in the literature.

B.1 Global considerations

Let χ be a Dirichlet character of conductor N and $r \geq 1$, $\mathcal{O}$ a finite extension of $\mathbb{Z}$ containing all values of χ, $\zeta_N = \exp(2\pi i/N)$ and G the Galois group of $\mathbb{Q}(\zeta_N)$ over $\mathbb{Q}$. In particular, we have distinguished an embedding $\sigma_0 : \mathbb{Q}(\mu_N) \to \mathbb{C}$. Recall from section 1.1 the projector

$$p_{\chi^{-1}} : V(\mathbb{Q}(\zeta_N)) \to V(\mathbb{Q}(\zeta_N))$$

whose image is $V(\chi)$. Recall also that elements of $V_B(\mathbb{Q}(\zeta_N))$ are maps from the set of embeddings $\mathbb{Q}(\zeta_N) \to \mathbb{C}$ with values in E. We choose generators

$$\begin{aligned} t_{\mathrm{DR}}(\chi) &= p_{\chi^{-1}}(\zeta_N) \in V_{\mathrm{DR}}(\chi) \subset \mathbb{Q}(\zeta_N) \otimes E \\ t_B(\chi) &= p_{\chi^{-1}}(\delta_{\sigma_0}) \in p_{\chi^{-1}}(V_B(\chi))) \\ t_p(\chi) &= t_B(\chi) \otimes 1 \in V_B(\chi) \otimes E_p = V_p(\chi) \end{aligned}$$

Note that $t_{\mathrm{DR}}(\chi)$ is a generator of $p_{\chi^{-1}}(\mathbb{Z}[\mu_N] \otimes \mathcal{O})$ as this group is generated by the set $p_{\chi^{-1}}(\zeta_N^i)$ for $i \in \mathbb{Z}/N$ and these elements either vanish or are multiples of $p_{\chi^{-1}}(\zeta_N)$.

Lemma B.1.1. *Let χ be a Dirichlet character of conductor N. The element*

$$\varepsilon \in \det{}^{-1} V_B(\chi)(r)^+ \otimes \det V_{\mathrm{DR}}(\chi)(r) \otimes \det V_B(\chi)^\vee(1-r)^+$$

which is mapped to $\frac{L(\chi^{-1},1-r)^*}{L(\chi,r)^*}$ under the isomorphism of the determinant with E_∞ defined in 2.2.1 is

$$2N^{r-1}(r-1)!\left(t_B(\chi)(r)^+\right)^{-1} \otimes t_{\mathrm{DR}}(\chi)(r) \otimes t_B(\chi)^\vee(1-r)^+$$

(up to sign depending only on r).

Proof. By the functional equation for Dirichlet-L-functions

$$\frac{L(\chi^{-1},1-r)^*}{L(\chi,r)^*} = \pm 2\frac{(r-1)!N^r}{\tau(\chi)(2\pi i)^{r-\delta}}$$

where

$$\tau(\chi) = \sum_{a=1}^{N} \chi(a)\zeta_N^a = \sum_{\sigma\in G} \chi(\sigma^{-1})\sigma(\zeta_N)$$

is the Gauss sum and $\delta = 0$ (resp. $\delta = 1$) if χ and r have the same (different) parity. On the other hand we have to consider

$$0 \to V_B(\chi)(r)^+ \otimes \mathbb{R} \to V_{\mathrm{DR}}(\chi) \otimes \mathbb{R} \to V_B(\chi)(r-1)^+ \otimes \mathbb{R} \to 0$$

If r and χ have the same (different) parity, then only the last (first) Betti-term appears. The comparison morphism maps $t_{\mathrm{DR}}(\chi)$ to

$$\begin{aligned}\sum_\sigma \sigma_0\sigma(p_{\chi^{-1}}\zeta_N)\delta_{\sigma_0\sigma} &= \sum_\sigma \chi^{-1}(\sigma)\sigma_0(\frac{\tau(\chi^{-1})}{\Phi(N)})\sigma^{-1}\delta_{\sigma_0} \\ &= \frac{\tau(\chi^{-1})}{(2\pi i)^{r-\delta}} t_B(\chi)(r-\delta) = \frac{N}{\tau(\chi)(2\pi i)^{r-\delta}} t_B(\chi)(r-\delta)\end{aligned}$$

□

Hence conjecture 2.2.1 predicts that

$$2(r-1)!N^{r-1}t_{\mathrm{DR}}(\chi)$$

is a generator of $\det^{-1} R\Gamma(\mathbb{Q}_p, T_p(\chi)(r))$ under $\exp_p$ and the local identification 1.2.4.

Proposition B.1.2. *Conjecture 2.2.1 holds in the case $K = \mathbb{Q}$ (i.e., non-equivariantly), $r = 1$, $p \neq 2$, χ restricted to the Galois group of $\mathbb{Q}_p$ is the trivial character.*

Proof. As χ restricted to the local Galois group is trival, p splits completely. Hence $\mathbb{Z}[\mu_N]\otimes\mathbb{Z}_p \cong \bigoplus_{v|p}\mathbb{Z}_p$. As part of the class number case (with $F = \mathbb{Q}$) we have checked in section 2.3 that the lemma holds for the trivial character. Taking the sum over all v, this implies even that $\det_{\mathbb{Z}_p[G]}(\mathbb{Z}[\mu_N]\otimes\mathbb{Z}_p)$ is identified with $\det^{-1}_{\mathbb{Z}_p[G]} R\Gamma(\mathbb{Q}_p, \bigoplus_{v|p}\mathbb{Z}_p(1))$ where $G = \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$. Tensoring both determinants with $\mathcal{O}(\chi)$ over $\mathbb{Z}_p[G]$ implies that a generator of $p_{\chi^{-1}}(\mathbb{Z}[\mu_N]\otimes\mathbb{Z}_p)$ is mapped to a generator of $\det^{-1}_{\mathcal{O}_p} R\Gamma(\mathbb{Q}_p, \mathcal{O}_p(\chi)(1))$. As remarked before, $t_{\mathrm{DR}}(\chi)$ is a generator of the $\mathcal{O}$-module $p_{\chi^{-1}}(\mathbb{Z}[\mu_N]\otimes\mathcal{O})$, hence also of the $\mathcal{O}_p$-module $p_{\chi^{-1}}(\mathbb{Z}[\mu_N]\otimes\mathcal{O}_p)$. □

If $K/\mathbb{Q}$ is an abelian extension, $X(K/\mathbb{Q})$ the set of characters of $\mathrm{Gal}(K/\mathbb{Q})$, then the equivariant version of conjecture 2.2.1 predicts that

$$2(r-1)!\left(N_{\omega\chi}^{r-1}t_{\mathrm{DR}}(\omega\chi)\right)_{\omega\in X(K/\mathbb{Q})}$$

is a generator of $\det^{-1}_{\mathcal{O}_p[\mathrm{Gal}(K/\mathbb{Q})]} R\Gamma(K_p, T_p(\chi)(r))$ under $\exp_p$ and the local identification 1.2.4.

Proposition B.1.3. *Conjecture 2.2.1 holds for all Dirichlet characters, $r > 1$ and $p \neq 2$ equivariantly for the cyclotomic extension $\mathbb{Q}(\mu_{p^n})/\mathbb{Q}$.*

The proof will cover the rest of this appendix.

Lemma B.1.4. *If $r > 1$, then conjecture 2.2.1 holds equivariantly if and only if*

$$\left(\frac{2(r-1)!N_{\omega\chi}^{r-1}(1-\omega^{-1}\chi^{-1}(p)p^{r-1})}{1-\omega\chi(p)p^{-r}}\exp_p(t_{\mathrm{DR}}(\omega\chi))\right)_{\omega\in X(K/\mathbb{Q})}$$

is a generator of $\det^{-1}_{\mathcal{O}_p[\mathrm{Gal}(K/\mathbb{Q})]} R\Gamma(K_p, T_p(\chi)(r))$.

Proof. It suffices to compute the image of our generator characterwise. We write χ for $\omega\chi$. Recall that $\exp_p : V_{\mathrm{DR}}(\chi)(r) \to R\Gamma(\mathbb{Q}_p, V_p(\chi)(r))[1]$ is a quasi-isomorphism. The whole point of the lemma is that the identification of determinants in 1.2.4 is *not* the one induced by the above quasi-isomorphism. It uses

$$\begin{aligned}\det{}^{-1} R\Gamma(\mathbb{Q}_p, V_p(\chi)(r)) &= \det{}^{-1} R\Gamma_f(\mathbb{Q}_p, V_p(\chi)(r)) \otimes \det{}^{-1} R\Gamma_{/f}(\mathbb{Q}_p, V_p(\chi)(r)) \\ &= \det{}^{-1} R\Gamma_f(\mathbb{Q}_p, V_p(\chi)(r)) \otimes \det R\Gamma_f(\mathbb{Q}_p, V_p(\chi^{-1})(r))\end{aligned}$$

and 1.2.4 on both factors. The first factor is quasi-isomophic to $R\Gamma(\mathbb{Q}_p, V_p(\chi)(r))$, the second to zero. By proposition 1.2.5 the lattice of 1.2.4 differs from the one given by

$$\det{}^{-1} R\Gamma(\mathbb{Q}_p, T_p(\chi)(r)) \otimes \det 0$$

by the local Euler factors $1 - \chi(p)p^{-r}$ and $(1 - \chi^{-1}(p)p^{r-1})^{-1}$. □

B.2 Homological algebra

From now on, χ is considered as a local character of conductor $N = N'p^m$ where N' is prime to p. Let $K = \mathbb{Q}_p(\mu_N)$. Let K_0 be its totally unramified subfield and $K_n = K_0(\mu_{p^n})$ (this differs from the convention in section 4). In particular, $K_0 = \mathbb{Q}_p(\mu_{N'})$.

Let $\Gamma = \mathrm{Gal}(K_\infty/K_0)$, $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$ and $H = \mathrm{Gal}(K_0/\mathbb{Q}_p)$. We put $G_n = \mathrm{Gal}(K_n/K_0) = \Gamma/\Gamma_n$. In this section we let $\Lambda := \varprojlim_n \mathcal{O}_p[[G_n]]$, which differs from the Λ in the main text. Our first step follows [BlKa] 4.2 p. 367. We put $P = \mathrm{Gal}(K_\infty^{ab}/K_\infty)$, $U = \varprojlim \mathcal{O}_{K_0}[\zeta_{p^n}]^*$ (or rather their pro-p-parts). They are natural Λ-modules. There are exact sequences

$$0 \to U \to P \to \mathbb{Z}_p \to 0$$

and

$$0 \to \mathbb{Z}_p(1) \to U \xrightarrow{\mathrm{Col}} \mathcal{O}_{K_0}[[\Gamma]] \to \mathbb{Z}_p(1) \to 0$$

where Col is the Coleman map, see [BlKa] 4.2 for more details.

Lemma B.2.1. *The following Λ-modules are pseudo-isomorphic:*

$$H^1(K_\infty, T_p(\chi)(r)) \to \mathrm{Hom}(P, T_p(\chi)(r))^H$$

There are natural isomorphism of Λ-determinants

$$\begin{aligned}\det R\Gamma(\mathbb{Q}_{p,\infty}, T_p(\chi)(r))[1] &= \det{}^{-1} H^0(\mathbb{Q}_{p,\infty}, T_p(\chi)(r)) \otimes \det \mathrm{Hom}(P, T_p(\chi)(r))^H \\ &= \det \mathrm{Hom}(U, T_p(\chi)(r))^H \\ &= \det \mathrm{Hom}_H(\mathcal{O}_{K_0}[[\Gamma]], T_p(\chi)(r))\end{aligned}$$

Proof. If H has no p-torsion, then this is precisely [BlKa] 4.2, p. 367 and follows immediately from the short exact sequences. Hence we consider only the case where H has p-torsion. This means that $\chi(r)$ is non-trivial over $\mathbb{Q}_{p,\infty}$ and that $H^0(\mathbb{Q}_{p,\infty}, T_p(\chi)(r)) = 0$.

The following arguments are very ugly, we apologize for this. We abbreviate $T = T_p(\chi)(r)$. The crucial observation is finiteness of

$$H^i(H, \mathrm{Hom}(P, T))$$

for $i > 0$. We have the short exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}_p, T) \to \mathrm{Hom}(P, T) \to \mathrm{Hom}(U, T) \to 0$$

Hence

$$H^1(H, T) \to H^1(H, \mathrm{Hom}(P, T)) \to H^1(H, \mathrm{Hom}(U, T)) \to H^2(H, T)$$

The first and last term are finite. We replace P by U. We consider the triangle

$$\mathbb{Z}_p(1) \to U \to C$$

with $C = [\mathcal{O}_{K_0}[[\Gamma]] \to \mathbb{Z}_p(1)]$ Hence

$$0 \to \mathrm{Hom}(C, T) \to \mathrm{Hom}(U, T) \to \mathrm{Hom}(\mathbb{Z}_p(1), T)$$

This implies that the quotient is at most a finitely generated $\mathbb{Z}_p$-module without H-invariants and hence

$$H^1(H, \mathrm{Hom}(C, T)) \to H^1(H, \mathrm{Hom}(U, T))$$

is again a pseudo-isomorphism. Finally, we have

$$0 = \mathrm{Hom}(\mathbb{Z}_p(1)[-1], T) \to \mathrm{Hom}(C, T) \to \mathrm{Hom}(\mathcal{O}_{K_0}[[\Gamma]], T) \to \mathrm{Hom}(\mathbb{Z}_p(1)[-1], T[1]) = 0$$

and we have reduced the question to $H^1(H, \mathrm{Hom}(\mathcal{O}_{K_0}[[\Gamma]], T))$. But this is an induced H-module as $\mathcal{O}_{K_0}$ is the free $\mathbb{Z}_p[H]$-module by existence of a normal basis. Hence its H-cohomology vanishes. The same argument works for H^2.

Now we can turn to the proof of the lemma. Recall that $H^1(K_\infty, T_p(\chi)(r)) = \mathrm{Hom}(P, T)$ as the coefficients become trivial over K_∞. Hence

$$0 \to H^1(H, \mathrm{Hom}(P, T)) \to H^1(\mathbb{Q}_{p,\infty}, T) \to H^0(H, \mathrm{Hom}(P, T)) \to H^2(H, \mathrm{Hom}(P, T))$$

This is the desired pseudo-isomorphism. Pseudo-isomorphic Λ-modules have the same determinant, so this proves the first identification of determinants. By similar arguments applying $H^i(H, \cdot)$ to the exact sequences and using $T^H = 0$, the other identifications follow. □

Corollary B.2.2. *Let χ be a local character and $r > 1$ as before. Let $G_n = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$. Then there is an isomorphism of determinants*

$$\det_{\mathcal{O}_p[G_n]} R\Gamma(\mathbb{Q}(\mu_{p^n})_p, T_p(\chi)(r))[1] \cong \det_{\mathcal{O}_p[G_n]} \mathrm{Hom}_{\Gamma_n \times H}(\mathcal{O}_{K_0}[[\Gamma]], T_p(\chi)(r))$$

which rationally is induced by the isomorphism

$$H^1(\mathbb{Q}(\mu_{p^n})_p, V_p(\chi)(r)) \xrightarrow{s_\chi} \mathrm{Hom}_{\Gamma_n \times H}(K_0[[\Gamma]], V_p(\chi)(r))$$

Proof.
$$R\Gamma(\mathbb{Q}(\mu_{p^n})_p, T_p(\chi)(r)) = R\Gamma(\Gamma_n, R\Gamma(\mathbb{Q}_{p,\infty}, T_p(\chi)(r))$$
Hence by the lemma
$$\det_{\mathcal{O}_p[G_n]} R\Gamma(\mathbb{Q}(\mu_{p^n})_p, T_p(\chi)(r))[1] = \det_{\mathcal{O}_p[G_n]} R\Gamma(\Gamma_n, \mathrm{Hom}_H(\mathcal{O}_{K_0}[[\Gamma]], T_p(\chi)(r))$$
In this case there are no Γ_n-coinvariants, hence
$$R\Gamma(\Gamma_n, \mathrm{Hom}_H(\mathcal{O}_{K_0}[[\Gamma]], T_p(\chi)(r)) = \mathrm{Hom}_H(\mathcal{O}_{K_0}[[\Gamma]], T_p(\chi)(r))^{\Gamma_n}$$

The identification of determinants of Iwasawa modules was induced by pseudo-isomorphisms hence $R\Gamma(\Gamma_n, \cdot)$ of the error terms is torsion. So rationally it is induced by a quasi-isomorphism. □

B.3 Reciprocity laws

Let $N = N'p^m$ with N' prime to p. Let $\widetilde{\zeta}_{N'}$ be a generator of the free $\mathbb{Z}_p[H]$-module $\mathcal{O}_{K_0}$. We can assume that $\widetilde{\zeta}_N$ is ζ_N plus a linear combination of roots of unity of order less than N. In particular $p_\chi(\widetilde{\zeta}_{N'}) = p_\chi(\zeta_{N'})$. Evaluation in $\widetilde{\zeta}_{N'}$ fixes an isomorphism $\mathrm{Hom}_{\Gamma\times H}(K_0[[\Gamma]], V_p(\chi)(r)) \cong V_p(\chi)(r)$.

Lemma B.3.1.
$$(s_\chi \exp_p)(t_{\mathrm{DR}}(\chi)) = \frac{N'^r(1-\chi(p)p^{-r})}{N^{r-1}(r-1)!(1-p^{r-1}\chi^{-1}(p))} t_p(\chi)(r).$$

Proof. Recall that $\Gamma/\Gamma_m = G_m$. We have $\mathbb{Z}_p[H]\widetilde{\zeta}_{N'} \cong \mathcal{O}_{K_0}$. Hence $\widetilde{\zeta}_{N'}$ also defines isomorphisms
$$\mathrm{Hom}_{\Gamma_m}(\mathcal{O}_{K_0}[[\Gamma]], E_p(r)) \cong \mathrm{Hom}(\mathcal{O}_{K_0}[G_m], E_p(r)) \cong \mathrm{Hom}(\mathbb{Z}_p[H\times G_m], E_p(r))$$
The following diagram commutes
$$\begin{array}{ccccc}
V_{\mathrm{DR}}(\chi) & \xrightarrow{s_\chi \exp_p} & \mathrm{Hom}_{\Gamma\times H}(\mathcal{O}_{K_0}[[\Gamma]], V_p(\chi)(r)) & \longrightarrow & V_p(\chi)(r) \\
\downarrow & & \downarrow \iota & & \downarrow \iota' \\
K_m & \xrightarrow{s_{K_m} \exp_p} & \mathrm{Hom}_{\Gamma_m}(\mathcal{O}_{K_0}[[\Gamma]], E_p(r)) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}_p[H\times G_m], E_p(r))
\end{array}$$
where ι' is defined via the inclusion as $p_{\chi^{-1}}$-eigenpart. Under this inclusion $\iota'(t_p(\chi)(r)) = p_{\chi^{-1}}\delta$ where $\delta(e) = t_p(r)$ (the standard generator of $E_p(r)$) and $\delta(g) = 0$ for $e \neq g \in H \times G_m$. Hence
$$\iota(t_p(\chi)(r))(\widetilde{\zeta}_{N'}) = \iota'(t_p(\chi)(r))(e) = \frac{1}{\Phi(N)}\, t_p(r).$$
This implies that for $\alpha \in V_{\mathrm{DR}}(\chi)$ we have
$$(s_\chi \exp_p)(\alpha)(\widetilde{\zeta}_{N'}) = (s_{K_m}\exp_p)(\alpha)(\widetilde{\zeta}_{N'}) \cdot \frac{\Phi(N)}{t_p(r)} \cdot t_p(\chi)(r)$$

The map $s_{K_m} : K_m \to \mathrm{Hom}_{\Gamma_m}(K_0[[\Gamma]], E_p(r))$ is computed by the explicit reciprocity law for $K = K_0(\mu_{p^m})$. In the unramified case, s_{K_0} reduces to a map $K_0 \to \mathrm{Hom}(K_0, E_p(r))$. By [BlKa] Claim 4.8 (p. 368) it is given by

$$\alpha \mapsto \left(\beta \mapsto \frac{1}{(r-1)!}\,\mathrm{Tr}_{K_0/\mathbb{Q}_p}((1-p^{r-1}\,\mathrm{Fr}_p^{-1})^{-1})(\beta)\cdot(1-\mathrm{Fr}_p^{-1}\,p^{-r})(\alpha)\; t_p(r)\right)$$

(Recall that Fr_p is geometric Frobenius whereas f in loc. cit is arithmetic Frobenius).

Evaluating the formula with $\alpha = p_{\chi^{-1}}(\zeta_N)$ and $\beta = \widetilde{\zeta}_N = \widetilde{\zeta}_{N'}$, we get

$$\frac{N(1-\chi(p)p^{-r})}{(r-1)!(1-p^{r-1}\chi^{-1}(p))\Phi(N)}\; t_p(r)$$

The factor $\Phi(N)/t_p(r)$ cancels. This yields the desired formula in the unramified case.

In the ramified case $m \geq 1$, we follow an approach also used by Benois and Nguyen in an earlier version of [BenNg]. We use Kato's higher explicit reciprocity law, [Ka2] 2.1.7. The map $s_{K_m}\exp_p$ is given by

$$\alpha \mapsto \left(\beta \mapsto \frac{1}{(r-1)!}p^{-rm}\,\mathrm{Tr}_{K_m/\mathbb{Q}_p}(\alpha\cdot(1-p^{r-1}\phi)^{-1}D^r(\beta(1+T))(\zeta_{p^m}-1)t_p(r)\right)$$

where we consider $\beta\cdot(1+T) \in \mathcal{O}_{K_0}[[T]]$ and $D = (1+T)d/dT$, ϕ Frobenius on $\mathcal{O}_{K_0}[[T]]$ (see also also [PR2] for facts about Coleman power series.) We evaluate with $\alpha = p_{\chi^{-1}}(\zeta_N)$ and $\beta = \widetilde{\zeta}_{N'}$ and get

$$\frac{1}{(r-1)!p^{rm}}\,\mathrm{Tr}_{K_m/\mathbb{Q}_p}(p_{\chi^{-1}}(\zeta_N)p_\chi y_m)\; t_p(r)$$

with $y_m = [(1-\phi p^{r-1})^{-1}\widetilde{\zeta}_{N'}(1+T)](\zeta_{p^m}-1)$. Using the geometric series for the inverse of $1-p^{r-1}\phi$, we get

$$y_m = \widetilde{\zeta}_{N'}\zeta_{p^m} + p^{r-1}\,\mathrm{Fr}_p^{-1}(\widetilde{\zeta}_{N'})\zeta_{p^m}^p + p^{2(r-1)}\,\mathrm{Fr}_p^{-2}(\widetilde{\zeta}_{N'})\zeta_{p^m}^{2p} + \dots$$

Only the first summand contributes to the χ-part as we have assumened that χ is primitive of level N. Hence

$$p_\chi(y_m) = p_\chi(\widetilde{\zeta}_{N'}\zeta_{p^m}) = p_\chi(\zeta_{N'}\zeta_{p^m}) = p_\chi(\zeta_N)$$

As $p_{\chi^{-1}}(\zeta_N)p_\chi(\zeta_N) = N/\Phi(N)^2$, this yields

$$s_{K_m}\exp_p(p_{\chi^{-1}}(\zeta_N))(\widetilde{\zeta}_{N'}) = \frac{N}{(r-1)!p^{rm}\Phi(N)}t_p(r)$$

The factor $\Phi(N)/t_p(r)$ again cancels and p^{rm} is the p-part of N^r. □

Proof of proposition 2.2.1: As before let χ be a character of conductor $N = p^m N'$ with N' prime to p, $G_n = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ and $K_0 = \mathbb{Q}_p(\zeta_{N'})$. We have to check that the element of lemma B.1.4 (call it ε') is a generator of $\det^{-1}_{\mathcal{O}_p[G_n]} R\Gamma(\mathbb{Q}(\zeta_{p^n})_p, T_p(\chi)(r)$. By lemma B.2.2 this is equivalent to showing that the image of ε' in $\mathrm{Hom}_{\Gamma_n\times H}(\mathcal{O}_{K_0}[[\Gamma]], T_p(\chi)(r)) = \mathrm{Hom}_H(\mathcal{O}_{K_0}[G_n], T_p(\chi)(r))$ is an $\mathcal{O}_p[G_n]$-generator of the latter. As

$$\mathrm{Hom}_H(\mathcal{O}_{K_0}[G_n], T_p(\chi)(r)) \subset \bigoplus_{\omega} V_p(\chi\omega)(r)$$

where ω runs through all characters of G_n, the image of ε' is uniquely determined by its ω-components. By lemma B.3.1 applied to the character $\chi\omega$ and the definition of ε'

$$(s_{K_n}\varepsilon')_\omega = 2(N')^r t_p(\chi\omega)(r).$$

(Note that N', H and $\widetilde{\zeta}_{N'}$ are the same for all these $\chi\omega$). On the other hand, $\mathrm{Hom}_H(K_0[G_n], V_p(\chi)(r))$ has a standard generator given by the function δ with $\delta(\widetilde{\zeta}_{N'}e) = t_p(\chi)(r)$, $\delta(\widetilde{\zeta}_{N'}g) = 0$ for $g \neq e \in G_n$. It has ω-component $t_p(\chi\omega)(r)$. Hence

$$s_{K_n}\varepsilon' = 2(N')^r\delta.$$

As N' is prime to p and $p \neq 2$ this finishes the proof of proposition B.1.3 for Dirichlet characters.

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