On the Asymptotic Behaviour of Capillary Surfaces in Cusps

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Abstract: FINN and CONCUS [2] discovered that capillary surfaces rise to infinity in corners with sufficiently small opening angle. They also found the leading term of an asymptotic expansion. MIERSEMANN [5] improved this result to obtain a complete asymptotic expansion. In the present paper we will apply the methods of the above authors to discuss asymptotic behaviour of capillarities in cusps, which is a corner with opening angle 0. A large variety of asymptotic formulas will be provided. The general comparison theorem from FINN and CONCUS will play an important role in the proofs.

Keywords: asymptotic expansion, capillary problem, comparison principle, cusp

1. Preliminaries and Previous Results

We consider the capillary equation in two dimensions, which describes a homogeneous liquid in a container with perpendicular walls and cross section $\Omega \subset \mathbb{R}^2$ in a constant gravity field with constant, perpendicular force component:

$$(1.1) \qquad \operatorname{div} Tu = \kappa u \qquad \text{in } \Omega$$

$$(1.2) \qquad \nu \circ Tu = \cos\gamma \qquad \text{on smooth components of } \partial\Omega.$$

Here, Ω is a piecewise smooth domain, "$\circ$" the euclidian scalar product, $Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$, ν the outer normal, $\kappa > 0$ the *capillary constant*, γ the constant *contact angle*, which is the angle between liquid and container wall. For γ we assume

$$(1.3) \qquad 0 < \gamma < \frac{\pi}{2}.$$

It can be proved that there is always a solution of (1.1), (1.2) if $\partial\Omega$ is piecewise smooth (piecewise C^4 is sufficient [8]). Furthermore, we have a general comparison theorem for this equation:

Theorem 1.1 ([1], p. 192, [4]): *Let Σ_α, Σ_β, Σ_0 be a decomposition of $\Sigma = \partial\Omega$ of the form $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$ with the properties $\Sigma_\beta \in C^1$ and Σ_0 is of one-dimensional Hausdorff measure 0 so that for two functions $u, v \in C^2(\Omega) \cap C^1(\Sigma_\beta \cup \Omega)$ it holds:*

$$\begin{array}{llcll} (i) & \operatorname{div} Tu - \kappa u & \geq & \operatorname{div} Tv - \kappa v & \text{in } \Omega \\ (ii) & u & \leq & v & \text{on } \Sigma_\alpha \\ (iii) & \nu \circ Tu & \leq & \nu \circ Tv & \text{on } \Sigma_\beta, \end{array}$$

then $v \geq u$ in Ω.

No regularity condition is needed on Σ_α. A lot of a priori properties of the solutions of (1.1), (1.2) can be generated with the help of this theorem, such as uniqueness and a priori bounds like the following, which we will need later:

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Theorem 1.2 (cf. [2], p. 208 or [3], p. 113)**:** *Let u be a solution of (1.1) for a domain Ω, let B_δ be a circle of radius δ with $B_\delta \subset \Omega$, then*

$$(1.4) \qquad u \;<\; \frac{2}{\kappa\delta} + \delta$$

in B_δ.

By applying theorem 1.1, FINN and CONCUS could also show that if Ω contains a corner bounded by lines with opening angle 2α and if $\alpha + \gamma < \frac{\pi}{2}$, then the solution is unbounded. With the notations (r, θ) as polar coordinates with origin O in the peak of the corner ($\theta \in [-\alpha, \alpha]$), $k = \frac{\sin\alpha}{\cos\gamma}$, $h(\theta) = \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{k\kappa}$, B_δ a circle with centre O and $\Omega^* = \Omega \cap B_\delta$ they got:

Theorem 1.3 (cf. [2], p. 219)**:** *It exists a constant A independent of the special solution considered (that is independent of the special global shape of Ω) so that*

$$(1.5) \qquad \left| u - \frac{h(\theta)}{r} \right| \;<\; A$$

in Ω^*.

That means the solution increases asymptotically hyperbolically if approaching the peak of the corner:

$$(1.6) \qquad u \;\sim\; \frac{h(\theta)}{r} \qquad\qquad r \to 0,$$

where "$\sim$" means asymptotic equivalence (see e.g. MURRAY's book [7] for an introduction).
MIERSEMANN ([5]) improved that result and found a complete asymptotic expansion of the solutions considered in theorem 1.3:

$$(1.7) \qquad u \;\sim\; \sum_{l=0}^{\infty} h_{4l-1}(\theta) r^{4l-1} \qquad\qquad r \to 0$$

where $h_{4l-1}(\theta)$ ($l \in \mathbb{N}$, $h_{-1}(\theta) = h(\theta)$ with the above function h) are unique solutions of boundary-value problems independent of the special solution u considered. Note that the expression (1.7) does not assume the convergence of the series on the right side, which also cannot be guaranteed in general (cf. [7], p. 13)!
Furthermore, MIERSEMANN ([6]) could generalize theorem 1.3 to corners, bounded by analytic functions instead of lines:

$$(1.8) \qquad u \;\sim\; \frac{h(\theta)}{s} + q(\theta) \qquad\qquad s \to 0$$

where (θ, s) are curvilinear coordinates, q is the unique solution of a boundary-value problem.
According to (1.6) and (1.7), the solution has an asymptotic expansion with a leading term of order $\frac{1}{s}$, but the second term is of order 1 and not of order r^3 as in (1.7).
Here, we want to discuss the case $\alpha = 0$ that is Ω contains outward cusps. The behaviour of the solution is only considered in a small neighbourhood of the cusp. Consequently, the results are independent of the special global shape of Ω, like in the above papers. They only depend on the shape of the cusp. Therefore, different solutions have the same asymptotic expansion.

2. Parametrization of a Cusp

We can assume that the peak of the cusp is in the origin of a cartesian coordinate system and is bounded by two smooth functions $f_1, f_2 : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ with the property

$$\begin{aligned} f_1(x_1) &= o(x_1) \qquad\qquad x_1 \to 0 \\ f_2(x_1) &= o(x_1) \qquad\qquad x_1 \to 0 \end{aligned}$$

where "o" is the second Landau order symbol. Furthermore, we assume that there is an $x^* > 0$, for which

$$f_1(x_1) \;>\; f_2(x_1) \qquad \text{in } (0, x^*]$$

holds. Then the set $\Omega^* = \{(x_1, x_2) \; : \; 0 < x_1 < x^*, \; f_2(x_1) < x_2 < f_1(x_1)\}$ denotes a neighbourhood of the cusp and is characterized by setting in advance two functions f_1 and f_2 with the above properties. We parametrize Ω^* by curvilinear coordinates (a, s) analogous to [6] (see figure 1).

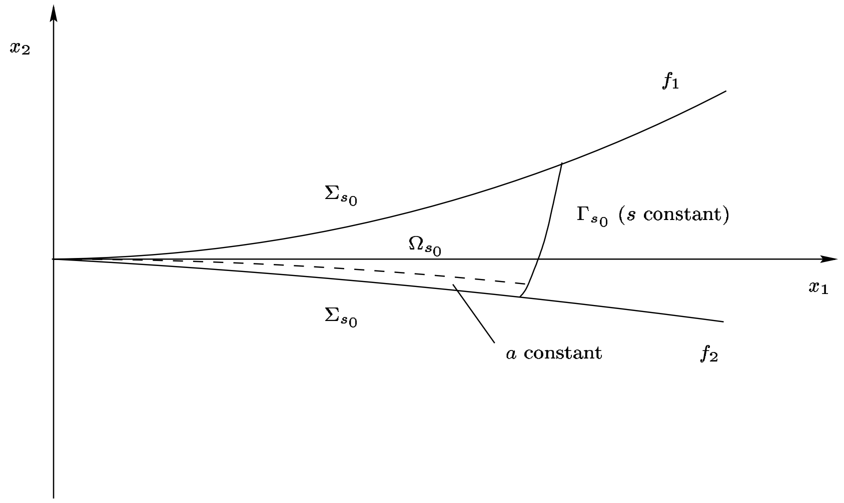

figure 1

The transformation formulas are:

(2.1) $$x_2 \;=\; x_2(a, x_1) \;=\; \frac{1+a}{2} f_1(x_1) + \frac{1-a}{2} f_2(x_1) \qquad a \in [-1, 1]$$

(2.2) $$s(x_1, a) \;=\; \int_0^{x_1} \sqrt{1 + x_{2,x_1}^2(a, \xi)}\, d\xi$$

where a has to be considered as independent of ξ in (2.2). Thus,

$$s(x_1, a) \;\sim\; x_1 \qquad x_1 \to 0.$$

Remark: Of course we can define the coordinate a in different ways, but the above seems to me the most convenient one.

We define the following quantities for the neighbourhood of the cusp (see figure 1):

(2.3) $$\Omega_{s_0} \;=\; \Big\{(a, s) \; : \quad -1 < a < 1, \quad 0 < s < s_0\Big\}$$

(2.4) $$\Sigma_{s_0} \;=\; \Big\{(a, s) \; : \quad a = \pm 1, \quad 0 < s \le s_0\Big\}$$

(2.5) $$\Gamma_{s_0} \;=\; \Big\{(a, s) \; : \quad -1 < a < 1, \quad s = s_0\Big\}$$

with $0 < s_0 \le s^* = \min_{a \in [-1,1]} s(a, x^*)$.

To calculate our asymptotic formulas we compare the solution of (1.1), (1.2) with a suitable comparison function by applying the comparison theorem (theorem 1.1) in the following form.

Corollary 2.1: *Let v be a solution of (1.1), (1.2) for a domain Ω with cusp Ω_{s^*}, $u \in C^2(\Omega_{s^*}) \cap C^1(\Sigma_{s^*} \cup \Omega_{s^*})$ (comparison function) and $0 < s_0 \le s^*$, then we have: If*

$$\begin{array}{llll} (i) & \operatorname{div} Tu - \kappa u & \ge 0 & \text{in } \Omega_{s_0} \\ (ii) & u & \le v & \text{on } \Gamma_{s_0} \\ (iii) & \nu \circ Tu - \cos\gamma & \le 0 & \text{on } \Sigma_{s_0}, \end{array}$$

then $v \geq u$ *in* Ω_{s_0}.
If

$$\begin{array}{llcll} (i) & \operatorname{div} Tu - \kappa u & \leq & 0 & \text{in } \Omega_{s_0} \\ (ii) & u & \geq & v & \text{on } \Gamma_{s_0} \\ (iii) & \nu \circ Tu - \cos\gamma & \geq & 0 & \text{on } \Sigma_{s_0}, \end{array}$$

then $v \leq u$ *in* Ω_{s_0}.

But it is not easy to find a suitable comparison function, since it must already reflect the right asymptotic behaviour. It is also not sufficient to "guess" the leading term because this does not always yield a suitable comparison function.

3. Estimation of the Leading Term

We consider two analytic functions, which form the cusp:

$$(3.1) \qquad \begin{array}{rcll} f_1(x_1) & = & a_1 x_1^n + b_1 x_1^{n+1} + O(x_1^{n+2}) & n \in \mathbb{N}, \quad n > 1 \\ f_2(x_1) & = & a_2 x_1^n + b_2 x_1^{n+1} + O(x_1^{n+2}) & a_i, b_i \in \mathbb{R} \end{array}$$

where "O" denotes the first Landau order symbol. Without restriction of generality we can assume

$$(3.2) \qquad a_1 \quad > \quad a_2.$$

Otherwise, this can be generated by a simple transformation.
At first we can estimate the solution on the sets Γ_{s_0}. With this result the condition (ii) of our comparison theorem can be proved later:

Lemma 3.1: *Let* v *be a solution of (1.1), (1.2) for a domain* Ω *with a cusp of the form (3.1), then there is a positive constant* A *independent of the special solution* v *considered so that*

$$(3.3) \qquad |v| \quad \leq \quad A \qquad\qquad \text{on } \Gamma_{s_0}$$

with $0 < s_0 < s^*$ *arbitrary but fixed.*

Consequently, the solutions v for domains with the same cusp Ω^* are uniformly bounded on Γ_{s_0}. The bound A is independent of the global shape of Ω. It depends only on the shape of the cusp Ω^*, κ and the choice of s_0. No grow condition for v is needed.

Proof: This result follows easily by using a suitable covering of Γ_{s_0} with circles and applying theorem 1.2. □

We now estimate the leading term of an asymptotic expansion of a solution of (1.1), (1.2) for a domain Ω containing a cusp of the form (3.1).

Theorem 3.1a: *Let* v *be a solution of (1.1), (1.2) for a domain* Ω *with a cusp of the form (3.1), then there are positive constants* $s_0 < s^*$, A *independent of the special solution* v *considered so that*

$$(3.4) \qquad \left| v - \frac{C}{s^n} \right| \quad \leq \quad \frac{A}{s^{n-1}} \qquad\qquad \text{in } \Omega_{s_0}$$

where

$$(3.5) \qquad C \quad = \quad \frac{2\cos\gamma}{(a_1 - a_2)\kappa}.$$

In asymptotic notation:

$$v \sim \frac{C}{s^n} \qquad s \to 0. \tag{3.6}$$

That means, the solution has a stronger singularity then it has in corners (cf. (1.8)). It depends on the order of contact of the two arcs, which form the cusp.

Proof: We transform the differential operators into curvilinear coordiantes with respect to our cusp.

$$\frac{\partial}{\partial x_1} = \left(1 + c_1(a)s^{2n-2} + O(s^{2n-1})\right)\frac{\partial}{\partial s} + \left(\frac{c_2(a)}{s} + c_3(a) + O(s)\right)\frac{\partial}{\partial a} \tag{3.7}$$

$$\frac{\partial}{\partial x_2} = \left(c_4(a)s^{n-1} + c_5(a)s^n + O(s^{n+1})\right)\frac{\partial}{\partial s} + \left(\frac{2}{a_1 - a_2}\frac{1}{s^n} + \frac{c_6(a)}{s^{n-1}} + O\left(\frac{1}{s^{n-2}}\right)\right)\frac{\partial}{\partial a}. \tag{3.8}$$

Here and in the following, the c_i denote on $[-1,1]$ analytic functions of a, which we do not need to clarify more detailed. They are independent of other quantities of the comparison function.
We consider the general comparison function

$$u(a,s) = \frac{C}{s^n} + \frac{D}{s^{n-1}} + E + \frac{h(a)}{s} + g(a) \tag{3.9}$$

with for the moment indetermined, on $[-1,1]$ analytic functions $h(a)$, $g(a)$ and constants $C > 0$, D and E.
We calculate the expressions $\operatorname{div} Tu - \kappa u$ in Ω_{s_0} and $\nu \circ Tu - \cos\gamma$ on Σ_{s_0} to apply the comparison theorem. It is necessary to distinct two cases.

Case 1: $n \geq 3$
After some calculation we obtain:

$$\operatorname{div} Tu - \kappa u = \left(\frac{4}{(a_1 - a_2)^2}\left(\frac{h'}{W}\right)' - \kappa C\right)\frac{1}{s^n} + (T_6 - \kappa D)\frac{1}{s^{n-1}} - \kappa E + O\left(\frac{1}{s^{n-2}}\right) \tag{3.10}$$

with the following abbreviations

$$\begin{aligned}
W &= \sqrt{n^2C^2 + \frac{4h'^2}{(a_1 - a_2)^2}}\\
T_1 &= 2n(n-1)CD + \frac{4h'}{a_1 - a_2}\left(\frac{2}{a_1 - a_2}g' + c_6(a)h'\right)\\
T_2 &= \frac{nCT_1}{2W^3} - \frac{(n-1)D}{W}\\
T_3 &= T_2 + c_2(a)T_2' - c_3(a)\left(\frac{nC}{W}\right)'\\
T_4 &= \frac{2g'}{a_1 - a_2} + c_6(a)h'\\
T_5 &= -\frac{h'T_1}{(a_1 - a_2)W^3} + \frac{T_4}{W}\\
T_6 &= \frac{2T_5'}{a_1 - a_2} + c_6(a)\left(\frac{2h'}{(a_1 - a_2)W}\right)'.
\end{aligned}$$

Obviously, the term $O\left(\frac{1}{s^{n-2}}\right)$ in (3.10) is independent of E.
Furthermore,

$$\nu \circ Tu - \cos\gamma = \left(\pm\frac{2h'}{a_1 - a_2}\frac{1}{W} - \cos\gamma\right) \pm T_5 s + O(s^2). \tag{3.11}$$

We have to take the upper sign for $a = 1$ and the lower sign for $a = -1$. The term $O(s^2)$ is again independent of E.

We determine the function h so that the leading term in (3.10) vanishs. The constant C and one integration constant of h can be chosen so that the leading term in (3.11) also vanishs. We choose the second integration constant, which is an additive constant, of h as 0. Thus,

$$\begin{aligned} C &= \frac{2\cos\gamma}{(a_1-a_2)\,\kappa} \\ h(a) &= -\frac{n}{\kappa}\sqrt{1-a^2\cos^2\gamma}. \end{aligned} \tag{3.12}$$

We choose g so that

$$T_6 - \frac{\kappa D}{2} = 0.$$

Let F_i $(i=1,2)$ be on $[-1,1]$ analytic functions of a independent of D, E, then we get from this equation.

$$\begin{aligned} \Rightarrow \qquad \frac{2T_5'}{a_1-a_2} &= F_1(a) + \frac{\kappa D}{2} \\ \Rightarrow \qquad T_5 &= F_2(a) + \frac{a_1-a_2}{4}\kappa D a + C_{g,1} \end{aligned}$$

We choose $C_{g,1}=0$. After rewriting the term T_5, a further integration of the last equation yields g by choosing again the additive integration constant as 0. But the function g is still dependent on D.
With these results we can simplify (3.10) and (3.11):

$$\operatorname{div} Tu - \kappa u = -\frac{\kappa D}{2}\frac{1}{s^{n-1}} - \kappa E + O\left(\frac{1}{s^{n-2}}\right) \tag{3.13}$$

$$\nu \circ Tu - \cos\gamma = \left(\frac{a_1-a_2}{4}\kappa D \pm F_2(\pm 1)\right) s + O(s^2). \tag{3.14}$$

We choose $D>0$ sufficiently large, so that

$$\frac{a_1-a_2}{4}\kappa D \pm F_2(\pm 1) > 0.$$

With this D we can determine g. Choose E positive, e.g. at first $E=1$. Because of the estimates (3.13) and (3.14) the positive quantity $s_0 < s^*$ can be chosen so small that

$$\operatorname{div} Tu - \kappa u \le 0 \qquad \text{in } \Omega_{s_0} \tag{3.15}$$

$$\nu \circ Tu - \cos\gamma \ge 0 \qquad \text{on } \Sigma_{s_0}. \tag{3.16}$$

Since the solutions v are uniformly bounded on Γ_{s_0} (lemma 3.1), we can choose $E \ge 1$ sufficiently large so that

$$u \ge v \qquad \text{on } \Gamma_{s_0}. \tag{3.17}$$

This does not influence the validity of the inequality (3.15), since the remainder term of (3.10) is independent of E; and therefore, the left side of (3.15) becomes at most smaller by this choice. The inequality (3.16) keeps also true because all terms are independent of E.
Applying theorem 3.1 we obtain form (3.15), (3.16) and (3.17):

$$\begin{aligned} v &\le u \\ v - \frac{C}{s^n} &\le \frac{A_1}{s^{n-1}} \qquad \text{in } \Omega_{s_0} \end{aligned} \tag{3.18}$$

with $A_1 = D + Es_0^{n-1} + \max_{a\in[-1,1]}\left\{|h(a)|s_0^{n-2} + |g(a)|s_0^{n-1}\right\} < \infty$.
Similarly, we get a lower bound by choosing D and E suitable negative; and with this result we obtain our supposition.

Case 2: $n = 2$
Here it is sufficient to consider a simpler comparison function:

$$(3.19) \qquad u = \frac{C}{s^2} + \frac{h(a)}{s} + g(a) + D.$$

The proof is then analogous to case 1. □

The result of theorem 3.1a can be generalized. Consider the case that the cusp is bounded by two container walls of *different* but homogeneous material, then we have, instead of (1.2), the boundary condition:

$$(3.20) \qquad \begin{aligned} \nu \circ Tu &= \cos\gamma_1 \qquad \text{on } \Sigma^o_{s^*} \\ \nu \circ Tu &= \cos\gamma_2 \qquad \text{on } \Sigma^u_{s^*} \end{aligned}$$

with

$$\begin{aligned} \Sigma^o_{s^*} &= \Big\{(a,s) \ : \ a = 1, \quad 0 < s \le s^*\Big\} \\ \Sigma^u_{s^*} &= \Big\{(a,s) \ : \ a = -1, \quad 0 < s \le s^*\Big\}. \end{aligned}$$

The quantities γ_1 and γ_2 denote the corresponding contact angles with the considered liquid. We assume

$$(3.21) \qquad \begin{aligned} 0 &< \gamma_1 < \pi \\ 0 &< \gamma_2 < \pi \end{aligned}$$

and without restriction of generality

$$(3.22) \qquad \gamma_1 \le \pi - \gamma_2.$$

We have:

Theorem 3.1b: *Let v be a solution of (3.1) with the generalized boundary condition (3.20) on Σ_{s^*} for a domain Ω with the cusp Ω^* of the form (3.1). Assume that $\gamma_1 < \pi - \gamma_2$, then there exist positive constants $s_0 < s^*$, A independent of the special solution considered so that*

$$(3.23) \qquad \left| v - \frac{C}{s^n} \right| \le \frac{A}{s^{n-1}} \qquad \text{in } \Omega_{s_0}$$

where

$$(3.24) \qquad C = \frac{\cos\gamma_1 + \cos\gamma_2}{(a_1 - a_2)\kappa}.$$

Proof: The proof is completely analogous to that of theorem 3.1a. We obtain for the function h, which we will need later:

$$(3.25) \qquad h(a) = -\frac{n}{\kappa}\sqrt{1 - \left(\frac{\cos\gamma_1 + \cos\gamma_2}{2}a + \frac{\cos\gamma_1 - \cos\gamma_2}{2}\right)^2} + C_{h,2}$$

where $C_{h,2}$ is an integration constant. □

Remarks: Our proof fails if $\gamma_1 = \pi - \gamma_2$, since in this case our comparison function is not suitable. I suppose that the asymptotic behaviour is different from (3.6).
It is not surprising that $\gamma_1 = \pi - \gamma_2$ causes trouble. We have also difficulties in this case for the corresponding problem in corners (FINN).

We do not need full analyticity of the functions f_1 and f_2 to prove asymptotic results as (3.6), but note that we have to be very careful, if we differentiate order relations (cf. [7]).
Let us show another result. Consider the functions

$$(3.26)\qquad \begin{aligned} f_1(x_1) &= a_1 x_1^\alpha \\ f_2(x_1) &= a_2 x_1^\alpha \end{aligned} \qquad \alpha > 1, \quad a_1, a_2 \in \mathbb{R}, \quad a_1 > a_2.$$

We obtain

Theorem 3.2: *We define the function $h(a)$ on $[-1,1]$ as follows:*

$$(3.27)\qquad h(a) = -\frac{\alpha}{\kappa}\sqrt{1 - a^2\cos^2\gamma}.$$

Assume that $\gamma_1 < \pi - \gamma_2$ holds; let v be a solution of (1.1), (3.20) for a domain Ω with a cusp Ω^* of the form (3.26), then there exist positive constants $s_0 < s^*$, A so that it holds on Ω_{s_0}:

$$(3.28)\qquad \left| v - \left(\frac{C}{s^\alpha} + \frac{h(a)}{s}\right)\right| \le A \qquad \text{if } \alpha > 2$$

$$(3.29)\qquad \left| v - \left(\frac{C}{s^\alpha} + \frac{h(a)}{s}\right)\right| \le A s^{\alpha-2} \qquad \text{if } 1 < \alpha \le 2$$

with the constant C from theorem 3.1b. The quantities s_0 and A are independent of the special solution considered.

In asymptotic notation ($s \to 0$):

$$(3.30)\qquad v \sim \frac{C}{s^\alpha} + \frac{h(a)}{s}.$$

Proof: The proof is similar to that of theorem 3.1a or b. Here, we use the general comparison function:

$$(3.31)\qquad u(a,s) = Cs^{-\alpha} + h(a)s^{-1} + g(a)s^{\alpha-2} + f(a)s^{2\alpha-3} + D.$$

The constant C and the function h are again determined by a condition that leading terms of $\operatorname{div} Tu - \kappa u$ and $\nu \circ Tu - \cos\gamma$ should vanish. □

We now discuss a possible practicle benefit of our results.

Example: Two circular cylinders in contact with radius 1 of the same, homogeneous material are in a sufficiently large basin with a liquid, which forms a contact angle γ with the material of the cylinders. Let the origin of a cartesian coordinate system be in the point of contact of the cylinders at the bottom of the basin and let the x_1-axis be in tangential direction, then we have:

$$\begin{aligned} f_1(x_1) &= 1 - \sqrt{1 - x_1^2} \\ f_2(x_1) &= -1 + \sqrt{1 - x_1^2}. \end{aligned}$$

By Taylor expansion we get:

$$\begin{aligned} f_1(x_1) &= \frac{1}{2}x_1^2 + O(x_1^4) \\ f_2(x_1) &= -\frac{1}{2}x_1^2 + O(x_1^4). \end{aligned}$$

Consequently, from theorem 3.1a it follows for the liquid surface v:

$$v \sim \frac{2\cos\gamma}{\kappa}\frac{1}{s^2} \qquad s \to 0.$$

4. Complete Asymptotic Expansion

The disadvantage of our estimates (3.4), (3.23), (3.29) is that the error is admittedly of lower order than the solution but still it could become infinite if $s \to 0$. To improve the estimates we construct a complete asymptotic expansion for the solution v from (1.1), (3.20) for a domain with a cusp of the shape (3.1). We assume in the following:

$$(4.1) \qquad \gamma_1 \quad < \quad \pi - \gamma_2.$$

For $k \geq 0$, $k \in \mathbb{N}$ we set

$$(4.2) \qquad u_k(a,s) \quad = \quad \sum_{i=-n}^{-n+k} C_i s^i + \sum_{i=-1}^{-1+k} h_i(a) s^i$$

where C_i are constants and $h_i(a)$ are on $[-1,1]$ analytical functions.
We can prove the following lemma.

Lemma 4.1: *There exist constants C_i $(i = -n, -n+1, \ldots)$ and on $[-1,1]$ analytical functions $h_j(a)$ $(j = -1, 0, \ldots)$ so that*

$$(4.3) \qquad \operatorname{div} Tu_k - \kappa u_k \quad = \quad O(s^{k-n+1}) \qquad \text{in } \Omega_{s_0} \quad (s_0 < s^*)$$

$$(4.4) \qquad \nu \circ Tu_k - \cos\gamma_{1,2} \quad = \quad O(s^{k+1}) \qquad \text{on } \Sigma_{s_0}$$

for every $k \geq 0$.

The estimate (4.4) is to be understood in the sense that we take γ_1 on $\Sigma_{s_0}^o$ and γ_2 on $\Sigma_{s_0}^u$.
That means the functions u_k are approximated solutions of (1.1), (3.20). We prove this result with the method of complete induction. Theorem 3.1b is the basis of our induction that is $C_{-n} = C$ and $h_{-1} = h(a)$ with the notations from theorem 3.1b (see equations (3.24), (3.25)). In every induction step we will determine a constant C_i and a function h_j, which is a solution of an ordinary second order differential equation, up to an additive constant. Because of the special notation of u_k (4.2) this additive constant can be chosen to be 0.
Otherwise, the sums in (4.2) could be simplified with the definition $\tilde{h}_i(a) = h_i(a) + C_i$. Then the functions $\tilde{h}_i$ are uniquely determined. But the notation (4.2) will turn out to be especially advantageous, since we do not need any distinction of cases.
The proof will be done together with the following theorem. □

Theorem 4.1: *Let the functions u_k satisfy (4.3), (4.4) and let $\tilde{v}$ be a solution of (1.1), (3.20) for a domain Ω with cusp Ω^* of the shape (3.1), then for every $k \geq 0$ there are positive constants $s_0 < s^*$ and A independent of the special solution considered so that*

$$(4.5) \qquad |\tilde{v} - u_k| \quad \leq \quad A s^{-n+k+1} \qquad \text{in } \Omega_{s_0}.$$

This theorem implies the complete asymptotic Laurent series expansion of the solution v if approaching the peak of the cusp:

$$(4.6) \qquad \tilde{v} \quad \sim \quad \sum_{i=-n}^{-2} C_i s^i + \sum_{i=-1}^{\infty} \tilde{h}_i(a) s^i \qquad s \to 0$$

with $\tilde{h}_i(a) = h_i(a) + C_i$ and C_i, $h_i(a)$ as in Lemma 4.1.
The first $n-1$ terms of the expansion are independent of a!

Proof: For the proof we will need a suitable decomposition of the operators div$T$ and T. Furthermore, it is necessary to estimate the remainders more precisely than in the proofs of theorem 3.1a or b. We apply again our comparison theorem but now, the condition *(ii)* follows not from the choice of an additive constant of the comparison function, but by using the result of theorem 3.1b. Only such a procedure yields better estimates then e.g. one of order 1.
We set

$$\begin{aligned}
&(4.7) \qquad & u &= u_k \\
&(4.8) \qquad & C &= C_{-n} \\
&(4.9) \qquad & h(a) &= h_{-1}(a) \\
&(4.10) \qquad & v &= Ds^{k-n+1} + q(a)s^k \\
&(4.11) \qquad & w &= u + v
\end{aligned}$$

where D is a constant and $q(a)$ an analytical function on $[-1,1]$. For the proof of lemma 4.1 we construct v, so that w is a new approximative solution; and for the proof of theorem 3.1 we construct v, so that w is a suitable comparison function. The case $k = 0$ is theorem 3.1b. Therefore, we can assume $k \geq 1$.
We first prove theorem 4.1 under the assumption that lemma 4.1 holds. We redefine the Landau order symbol as follows $F(a,s) = O(s^\beta) \Leftrightarrow |F| \leq cs^\beta$ and c is independent of $x \in \Omega_{s_0}$ for a sufficiently small s_0, independent of the constant D and independent of a quantity η, which we will later introduce. We further assume

$$\begin{aligned}
&(4.12) \qquad & q, q', q'' &= D \cdot O(1) \\
&(4.13) \qquad & |D| &\geq 1.
\end{aligned}$$

Remark: The condition (4.13) allows estimates of the form $D \cdot O(s^\beta) + O(s^\beta) = D \cdot O(s^\beta)$. It is also possible to assume $|D| \geq \varepsilon$ and ε is another fixed, positive number.

We have the differential operators in curvilinear coordinates like in the proof of theorem 3.1a.

$$(4.14) \qquad \frac{\partial}{\partial x_1} = \left(1 + c_1(a)s^{2n-2} + O(s^{2n-1})\right)\frac{\partial}{\partial s} + \left(\frac{c_2(a)}{s} + c_3(a) + O(s)\right)\frac{\partial}{\partial a}$$

$$(4.15) \qquad \frac{\partial}{\partial x_2} = \left(c_4(a)s^{n-1} + O(s^n)\right)\frac{\partial}{\partial s} + \left(\frac{2}{a_1 - a_2}\frac{1}{s^n} + \frac{c_5(a)}{s^{n-1}} + O\left(\frac{1}{s^{n-2}}\right)\right)\frac{\partial}{\partial a}.$$

The functions c_i are independent of D and q. Compare with the proof of theorem 3.1a!
Consequently,

$$\begin{aligned}
&(4.16) \qquad & u_{x_1} &= -nCs^{-n-1} + c_6(a)s^{-n} + O(s^{-n+1}) \\
&(4.17) \qquad & u_{x_2} &= \frac{2h'}{a_1 - a_2}s^{-n-1} + c_7(a)s^{-n} + O(s^{-n+1}) \\
&(4.18) \qquad & v_{x_1} &= D(k-n+1)s^{k-n} + D \cdot O(s^{k-1}) \\
&(4.19) \qquad & v_{x_2} &= \frac{2q'}{a_1 - a_2}s^{k-n} + D \cdot O(s^{k-n+1}).
\end{aligned}$$

We need a decomposition of divTw:

$$\begin{aligned}
\operatorname{div} Tw &= \left(\frac{w_{x_1}}{\sqrt{1+|\nabla u|^2}}\sqrt{\frac{1+|\nabla u|^2}{1+|\nabla w|^2}}\right)_{x_1} + \left(\frac{w_{x_2}}{\sqrt{1+|\nabla u|^2}}\sqrt{\frac{1+|\nabla u|^2}{1+|\nabla w|^2}}\right)_{x_2} \\
(4.20) \qquad &= \left[\frac{w_{x_1}}{\sqrt{1+|\nabla u|^2}}\left(1 + \frac{Q}{1+|\nabla u|^2}\right)^{-\frac{1}{2}}\right]_{x_1} + \left[\frac{w_{x_2}}{\sqrt{1+|\nabla u|^2}}\left(1 + \frac{Q}{1+|\nabla u|^2}\right)^{-\frac{1}{2}}\right]_{x_2}
\end{aligned}$$

where

$$(4.21) \qquad Q = 2u_{x_1}v_{x_1} + 2u_{x_2}v_{x_2} + v_{x_1}^2 + v_{x_2}^2.$$

Let the numbers D and s_0 satisfy

$$(4.22) \qquad |D|s_0^{k+1} \leq \eta$$

for a sufficiently small positive number $\eta \leq 1$.
Because of the special structure of u_k we have from lemma 4.1

$$\begin{aligned} \operatorname{div} Tu_k - \kappa u_k &= g(a)s^{k-n+1} + O(s^{k-n+2}) \\ \nu \circ Tu_k - \cos\gamma_{1,2} &= Z^{\pm}s^{k+1} + O(s^{k+2}) \end{aligned}$$

where $g(a)$ is an analytical function on $[-1,1]$ and $Z^{\pm}$ are two numbers. For the upper boundary $(a=1)$ we have to take Z^+ and γ_1, for the lower boundary $(a=-1)$ Z^- and γ_2. Clearly, the numbers $Z^{\pm}$ are independent of q, D, s_0 and η.
From the proof of theorem 3.1a (and b respectively) we have

$$\begin{aligned} \frac{4}{(a_1-a_2)^2}\left(\frac{h'}{W}\right)' &= \kappa C \\ \pm\frac{2h'}{(a_1-a_2)W} &= \cos\gamma_{1,2}. \end{aligned}$$

Using these results we obtain after a longer calculation:

$$\begin{aligned} (4.23) \qquad \operatorname{div} Tw - \kappa w = &\left(\frac{4n^2C^2}{(a_1-a_2)^2}\left(\frac{q'}{W^3}\right)' + \frac{4n(k-n+1)CD}{(a_1-a_2)^2}\left(\frac{h'}{W^3}\right)' + g(a) - \kappa D\right)s^{k-n+1} \\ &+ D\cdot O(s^{k-n+2}) + D^2\cdot O(s^{2k-n+2}). \end{aligned}$$

Similarly, the term Tw can be decomposed:

$$(4.24) \qquad Tw = \begin{pmatrix} \dfrac{w_{x_1}}{\sqrt{1+|\nabla u|^2}}\left(1+\dfrac{Q}{1+|\nabla u|^2}\right)^{-\frac{1}{2}} \\ \dfrac{w_{x_2}}{\sqrt{1+|\nabla u|^2}}\left(1+\dfrac{Q}{1+|\nabla u|^2}\right)^{-\frac{1}{2}} \end{pmatrix}.$$

After some calculation we get:

$$\begin{aligned} (4.25) \qquad \nu\circ Tw - \cos\gamma_{1,2} = &\left(\pm\frac{2n^2C^2}{a_1-a_2}\frac{q'}{W^3} \pm \frac{2nCD(k-n+1)h'}{(a_1-a_2)W^3} + Z^{\pm}\right)s^{k+1} \\ &+ D\cdot O(s^{k+2}) + D^2\cdot O(s^{2k+2}). \end{aligned}$$

To prove theorem 4.1 we determine D and q so that the preconditions of the comparison theorem are satisfied. We choose D:

$$(4.26) \qquad D = D_0 s_0^{-k}$$

where $D_0 \neq 0$ is still an arbitrary constant. The conditions (4.13) and (4.22) keep satisfiable for a sufficiently small s_0.
Let $\tilde{v}$ be a solution of (1.1), (3.20) for one of the considered domains, then we get by applying theorem 3.1b

$$\tilde{v} \leq u + Bs^{-n+1} \qquad \text{in } \Omega_{s_{0,1}}$$

for a sufficiently small $s_{0,1} < s^*$ ($s_{0,i} > 0$ in the following). B is a positive constant independent of D_0, $s_{0,1}$, η, q and $\tilde{v}$.
Thus,

$$\tilde{v} \leq w + Bs^{-n+1} - Ds^{k-n+1} - q(a)s^k \qquad \text{in } \Omega_{s_{0,1}}.$$

For a sufficiently small $s_{0,2} \leq s_{0,1}$ we can conclude

$$\tilde{v} \leq w + (B - D_0)\, s_0^{-n+1} + c|D_0| \qquad \text{on } \Gamma_{s_0} \quad \forall s_0 \leq s_{0,2}$$

by using precondition (4.12). The positive constant c is independent of D_0, s_0 and η. We choose $D_0 > B > 0$. In addition, $s_{0,3} \leq s_{0,2}$ can be chosen so small that

$$\tilde{v} \leq w \qquad \text{on } \Gamma_{s_0} \quad \forall s_0 \leq s_{0,3} \tag{4.27}$$

$$D \geq 1 \qquad \forall s_0 \leq s_{0,3}. \tag{4.28}$$

We determine q so that

$$\frac{4n^2C^2}{(a_1 - a_2)^2}\left(\frac{q'}{W^3}\right)' + \frac{4n(k-n+1)CD}{(a_1-a_2)^2}\left(\frac{h'}{W^3}\right)' + g(a) - \frac{\kappa D}{2} = 0. \tag{4.29}$$

After integration it follows that:

$$\frac{2n^2C^2}{a_1 - a_2}\frac{q'}{W^3} + \frac{2nCD(k-n+1)h'}{(a_1-a_2)W^3} = \frac{(a_1-a_2)\kappa D}{4}a + G(a) + C_{q,1}. \tag{4.30}$$

We set the integration constant $C_{q,1} = 0$ also the second integration constant of q ($C_{q,2}$), which arises after a further integration. By construction, the function $G(a)$ is independent of q, D_0, s_0, η and analytical on $[-1, 1]$. Consequently, q is also analytical; and it obviously satisfies the condition (4.12).
From (4.23) and (4.29) we get

$$\operatorname{div} Tw - \kappa w = -\frac{\kappa D}{2}s^{k-n+1} + \eta_1 + \eta_2.$$

For the quantities η_1, η_2 it holds

$$|\eta_1| \leq Dc_1 s^{k-n+2}$$
$$|\eta_2| \leq D^2 c_2 s^{2k-n+2}$$

on Ω_{s_0} for a sufficiently small s_0. The numbers c_1 and c_2 are positive constants and independent of D_0, s_0, η. Consequently, $s_{0,4} \leq s_{0,3}$ can be chosen so small that

$$\begin{aligned}
\operatorname{div} Tw - \kappa w &\leq -\frac{\kappa D}{2}s^{k-n+1} + Dc_1 s^{k-n+2} + D^2 c_2 s^{2k-n+2} \\
&\leq Ds^{k-n+1}\left(-\frac{\kappa}{2} + c_1 s_0 + Dc_2 s_0^{k+1}\right) \\
&\leq Ds^{k-n+1}\left(-\frac{\kappa}{2} + c_1 s_0 + c_2 \eta\right) \qquad (4.31)
\end{aligned}$$

on Ω_{s_0} $\forall s_0 \leq s_{0,4}$.
Analogously, from (4.25) and (4.30) we have

$$\nu \circ Tw - \cos\gamma_{1,2} = \left(\frac{(a_1 - a_2)\kappa D}{4} \pm G(\pm 1) + Z^{\pm}\right) s^{k+1} + \eta_3 + \eta_4$$

where the quantities η_3, η_4 satisfy

$$|\eta_3| \leq Dc_3 s^{k+2}$$
$$|\eta_4| \leq D^2 c_4 s^{2k+2}$$

on Σ_{s_0} for a sufficiently small s_0. The numbers c_3 and c_4 are again positive constants, independent of D_0, s_0, η.
Because of the choice (4.26) $s_{0,5} \leq s_{0,4}$ can be chosen so small that

$$\frac{(a_1 - a_2)\kappa D}{8} \pm G(\pm 1) + Z^{\pm} \ \geq \ 0 \qquad \forall s_0 \leq s_{0,5}.$$

With it, we can choose $s_{0,6} \leq s_{0,5}$ so small that

$$\begin{aligned} \nu \circ Tw - \cos\gamma_{1,2} &\geq \frac{(a_1 - a_2)\kappa D}{8} s^{k+1} - Dc_3 s^{k+2} - D^2 c_4 s^{2k+2} \\ &\geq Ds^{k+1}\left(\frac{(a_1 - a_2)\kappa}{8} - c_3 s_0 - c_4 \eta\right) \end{aligned} \tag{4.32}$$

on Σ_{s_0} $\forall s_0 \leq s_{0,6}$.
We choose η:

$$\eta = \min\left\{\frac{\kappa}{4c_2}, \frac{(a_1 - a_2)\kappa}{16 c_4}, 1\right\}. \tag{4.33}$$

We choose $s_0 \leq s_{0,6}$ so small that condition (4.22) is satisfied and that

$$\begin{aligned} -\frac{\kappa}{4} + c_1 s_0 &\leq 0 \\ \frac{(a_1 - a_2)\kappa}{16} - c_3 s_0 &\geq 0. \end{aligned}$$

With this s_0 all preconditions (4.13), (4.22) are satisfied and from the inequalities (4.27), (4.31), (4.32) it follows that:

$$\operatorname{div} Tw - \kappa w \leq 0 \qquad \text{in } \Omega_{s_0} \tag{4.34}$$

$$w \geq \tilde{v} \qquad \text{on } \Gamma_{s_0} \tag{4.35}$$

$$\nu \circ Tw - \cos\gamma_{1,2} \geq 0 \qquad \text{on } \Sigma_{s_0}. \tag{4.36}$$

Applying theorem 1.1 we obtain:

$$w \geq \tilde{v} \qquad \text{in } \Omega_{s_0}. \tag{4.37}$$

Consequently, there is a positive constant A_1 with

$$\tilde{v} - u_k \leq A_1 s^{k-n+1} \qquad \text{in } \Omega_{s_0}. \tag{4.38}$$

Analogously, we get a lower bound for $\tilde{v}$, if we use theorem 3.1b in the form

$$\tilde{v} \geq u - Bs^{-n+1} \qquad \text{in } \Omega_{s_{0,1}};$$

and if we choose $D_0 < -B < 0$. That is we can further choose s_0 so small that for a negative constant A_2 it holds additionaly:

$$\tilde{v} - u_k \geq A_2 s^{k-n+1} \qquad \text{in } \Omega_{s_0}. \tag{4.39}$$

The precondition now follows from (4.38) and (4.39) by setting $A = \max\{A_1, -A_2\}$.

We now prove lemma 4.1. The notations (4.7) to (4.11) can be taken over, but “O” is again the ordinary Landau order symbol without further restrictions. We also do not need the conditions (4.12), (4.13) and (4.22).
We use complete induction. From theorem 3.1b it is clear that with

$$\begin{aligned} C_{-n} &= \frac{\cos\gamma_1 + \cos\gamma_2}{(a_1 - a_2)\kappa} \\ h_{-1}(a) &= -\frac{n}{\kappa}\sqrt{1 - \left(\frac{\cos\gamma_1 + \cos\gamma_2}{2} a + \frac{\cos\gamma_1 - \cos\gamma_2}{2}\right)^2} \end{aligned}$$

it holds

$$\begin{aligned} \operatorname{div} Tu_0 - \kappa u_0 &= O(s^{-n+1}) \\ \nu \circ Tu_0 - \cos\gamma_{1,2} &= O(s) \end{aligned}$$

that is u_0 satisfies the basis of our induction.
Let the supposition be true for $k \geq 0$, we show, that then it holds also for $k+1$. Using the proof of theorem 4.1 and our inductional assumption, we get

$$\begin{aligned} \operatorname{div} Tw - \kappa w &= \Bigg(\frac{4n^2C^2}{(a_1-a_2)^2} \left(\frac{q'}{W^3}\right)' + \frac{4n(k-n+1)CD}{(a_1-a_2)^2} \left(\frac{h'}{W^3}\right)' + g(a) \\ &\quad - \kappa D \Bigg) s^{k-n+1} + O(s^{k-n+2}) \end{aligned} \tag{4.40}$$

$$\nu \circ Tw - \cos\gamma_{1,2} = \left(\pm \frac{2n^2C^2}{a_1-a_2}\frac{q'}{W^3} \pm \frac{2nCD(k-n+1)h'}{(a_1-a_2)W^3} + Z^{\pm} \right) s^{k+1} + O(s^{k+2}). \tag{4.41}$$

Again the quantities $g(a)$ and $Z^{\pm}$ are independent of q and D.
We determine q so that the coefficient of s^{k-n+1} in (4.40) vanishs:

$$\frac{4n^2C^2}{(a_1-a_2)^2} \left(\frac{q'}{W^3}\right)' + \frac{4n(k-n+1)CD}{(a_1-a_2)^2} \left(\frac{h'}{W^3}\right)' + g(a) - \kappa D = 0.$$

An integration of this equation yields

$$\frac{2n^2C^2}{a_1-a_2}\frac{q'}{W^3} + \frac{2nCD(k-n+1)h'}{(a_1-a_2)W^3} = \frac{(a_1-a_2)\kappa D}{2} a + G(a) + C_{q,1}. \tag{4.42}$$

We choose D and $C_{q,1}$ so that the coefficient of s^{k+1} in (4.41) vanishs:

$$\frac{(a_1-a_2)\kappa D}{2} \pm G(\pm 1) \pm C_{q,1} + Z^{\pm} = 0.$$

This equation system has always a unique solution. Again we integrate (4.42) to obtain q. Then q satisfies all assumptions with the same argument as in the proof of theorem 4.1. The additive integration constant can be chosen as 0, as already mentioned.
With these choices it follows that

$$\operatorname{div} Tw - \kappa w = O(s^{k-n+2}) \tag{4.43}$$

$$\nu \circ Tw - \cos\gamma_{1,2} = O(s^{k+2}). \tag{4.44}$$

Setting

$$C_{-n+k+1} = D \tag{4.45}$$

$$h_k(a) = q \tag{4.46}$$

we have

$$w = u_{k+1}. \tag{4.47}$$

From (4.43), (4.44) and (4.47) the supposition follows for $k+1$. The principle of induction completes the proof.
With the same argument as in the proof of theorem 3.1, the quantities s_0, A, h_i and C_i are independent of the special considered solution $\tilde{v}$. □

5. Summary

With the results of theorem 3.1a and b, theorem 3.2 and theorem 4.1, the asymptotic behaviour of capillarities can be characterized for a large class of objects. From a heuristic point of view it is interesting that for the considered problems we can summarize the results in one statement:

The solution rises with the same order like the order of contact of the two arcs, which form the cusp.

The complete asymptotic expansion from theorem 4.1 grants a gradually better estimate of the solution in a neighbourhood of the cusp.
The case $\gamma_1 = \pi - \gamma_2$ keeps an open question.

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