# An Attempt to Explain the Ascent of Sap in Defoliated Trees

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**Abstract:** The biologist Zimmermann has criticized in his book [12] that the effect of capillarity has been neglected in the past to explain the ascent of sap in trees. We want to propose a method how the sap ascent can be calculated, based on generalized solutions of the capillary equation. Although the theory of such solutions is not simple, the extracted method of calculating the sap storage in given cross sections of xylems can be applied also without having a large mathematical knowledge.

**Keywords:** ascent of sap, capillary problem, cusp, generalized solution

## 1. Motivation and the Model

The ascent of sap in trees has been traced back mainly to transpiration suction. This has been criticized by Zimmermann in his book [12]. He did dendrometer measurements and found that defoliated trees can also absorb water, which cannot be explained by transpiration suction. Further, he observed that water rises in intercellular spaces formed by cylindrical fibers, which are arranged in form of honeycombs. That means we have a lot of cylinders with a cross section bounded by 3 circular arcs (see later figure 2) and therefore containing 3 cusps.

The fibers are very small (about $5\mu$m [12], p. 49). Consequently, the model of capillarities in zero gravity should be a good one, since we have, roughly speaking, the following property

narrow vessels $\Leftrightarrow$ low gravity.

In both cases the gravitational energy can be neglected against the surface and contact energies. We want to discuss this fact more detailed. We assume at first that the liquid forms a surface $u$ over the bounded domain $\Omega \subseteq \mathbb{R}^2$ with piecewise smooth boundary, that is, the liquid is a graph solution over the cross section $\Omega$ of a cylindrical container with perpendicular walls. Then we can consider the following variational problem

$$\mathcal{E}[u] = \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega \lambda(u)\,dx - \oint_{\partial\Omega} \beta u\,ds \quad \to \quad \text{min.!}$$

where $\beta$ is the so called *adhesion coefficient*, which measures the adhesion force between liquid and container wall. It is determined by the involved substances and constant if these substances are homogeneous. According to [8], $\mathcal{E}$ is the sum of the three participated energies: the surface, gravitational and contact energy, which has to be counted as negative (see the model in [8], p. 239). For a gravity field with constant, perpendicular force component we have

$$\lambda(u) \quad = \quad \frac{\kappa}{2}u^2$$

where $\kappa = \frac{g\rho}{\sigma}$ is the *capillary constant* ($g$ the gravity-field intensity, $\rho$ the density difference between the liquid and the upper medium, $\sigma$ the surface tension). We transform our problem to a dimension-less one by introducing a *representative length* $a$ of our system and using $U := \frac{u}{a}$, $\mathbf{X} := \frac{\mathbf{x}}{a}$. We get

$$\tilde{\mathcal{E}}[u] = \int_{\tilde\Omega} \sqrt{1+|DU|^2} + \int_{\tilde\Omega} \frac{\kappa}{2}a^2U^2\,dX - \oint_{\partial\tilde\Omega} \beta U\,dS \quad \to \quad \text{min.!}.$$

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We see that the term $\int_{\tilde{\Omega}} \frac{\kappa}{2} a^2 U^2 \, dX$ can be neglected against the other terms if $a$ is very small. The influence of this term tends to zero even quadratically with $a$. On the other side, the dependence on $g$ is only linear. We write again untilded quantities and small letters for convenience. Consequently, we consider the problem

$$(1.1) \qquad \mathcal{E}[u] = \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega 2Hu \, dx - \oint_{\partial\Omega} \beta u \, ds \quad \to \quad \text{min.!.}$$

Here, the middle term arises from a volume restriction, which we have to assume for the gravity-free case to get a meaningful physical problem. Therefore, $2H$ is a Lagrange multiplier. According to [3], a volume restricition does not influence the shape of the solution. It only fixes the shift of the liquid surface.
We summarize our assumptions:

**Physical and Biological Assumptions for the Model:**

1. The xylems are cylindrical vessels (for the special definition of $\mathcal{E}$).
2. The cylindrical fibers that form the intercellular spaces, in which the liquid rises, have a very small radius ($a$ is small).
3. There is no water scarcity (to get a graph solution).
4. The xylems have constant wetting behaviour ($\beta$ is constant).

From (1.1) we obtain the Euler-Lagrange equations:

$$\begin{aligned} &(1.2) \qquad \operatorname{div} Tu &&= 2H && \text{in } \Omega \\ &(1.3) \qquad \nu \circ Tu &&= \beta = \cos\gamma && \text{on smooth parts of the boundary } \partial\Omega. \end{aligned}$$

Here "$\circ$" denotes the euclidian scalar product, $Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$, $\nu$ the outer normal, $H$, $\beta$ constant and $0 < \beta \leq 1$ (the case $\beta = 0$ is trivial). The quantity $\gamma$ is the *contact angle*, which is the angle between the liquid surface and the container wall. Because of equation (1.2) solutions are surfaces of constant mean curvature.
If there is a smooth solution of our problem, then $H$ is not arbitrary. By integration of (1.2) and applying the Gauss theorem we obtain

$$(1.4) \qquad H = \frac{|\partial\Omega| \cos\gamma}{2|\Omega|}$$

where $|\cdot|$ denotes the measure of the corresponding quantity.
Unfortunately our problem has no smooth solution if $\Omega$ contains cusps (cf. [11]). Also the variational problem is not free of limitations, see [3] for further details. Therefore, the concept of generalized solutions has been developed (originally Miranda used it in the theory of minimal surfaces cf. [9], [10]). Such solutions exist under very weak conditions ([11]) and have the property that they can be plus or minus infinity on sets of positive measure.

## 2. Generalized Solutions of the Problem

Observation shows (e.g. [6], [12]) that only a special class of generalized solutions should be of physical interest. Such solutions are of the type

$$(2.1) \qquad u = \begin{cases} u_0(x) & : \quad x \in \Omega_0 \\ +\infty & : \quad x \in \bar{M} \end{cases}$$

where $\bar{\cdot}$ is the closure of a set and $\Omega_0$, $M$ are disjoint subsets of $\Omega$ with $\Omega_0 \cup M = \Omega$. The set $M$ is a minimizer of the functional (see [11])

$$\Phi[\Omega^*; H, \gamma, \Omega] \quad = \quad |\Gamma_{\Omega^*}| - |\Sigma_{\Omega^*}| \cos\gamma + 2H|\Omega^*|, \tag{2.2}$$

$\Omega^* \subseteq \Omega$ denotes a Caccioppoli set (roughly speaking a sufficiently regular set). We define $\Gamma_{\Omega^*} = \partial\Omega^* \cap \Omega$, $\Sigma_{\Omega^*} = \partial\Omega^* \cap \partial\Omega$ (more exactly $\partial$ here denotes the measure-theoretic boundary, see [2]).
It is easy to see that then $\Omega_0 = \Omega \setminus \bar{M}$ minimizes the functional

$$\Psi[\Omega^*; H, \gamma, \Omega] \quad = \quad |\Gamma_{\Omega^*}| + |\Sigma_{\Omega^*}| \cos\gamma - 2H|\Omega^*|. \tag{2.3}$$

The set $M$ is most interesting: *On this set the sap rises theoretically infinitely high.*

**Remark:** For convenience we use $\Phi[\Omega^*; H, \gamma, \Omega] = \Phi[\Omega^*]$ and so on if it is clear to what quantities $H$, $\gamma$, $\Omega$ we refer.

Conversely, minimizers $M$ of $\Phi$ have a very restricted geometry.

**Theorem 2.1** (cf. [3], [4], [7], [11])**:** *Suppose that $M \neq \emptyset, \Omega$ then the set $\Gamma_M = \partial M \cap \Omega$ is analytic and consists of arcs of radius $R = \frac{1}{2H}$, the set $M$ is opposite to that side, into which the curvature vector points. If an arc meets a smooth part of $\partial\Omega$ then it forms an angle $\gamma$ with $\partial\Omega$. If $\gamma > 0$ then the arcs are pairwise isolated. An arc of $\Gamma_M$ cannot end in a corner of opening angle $0 \leq \alpha < \pi$.*

It is not difficult to find further restrictions.

**Lemma 2.1:** *The set $M$ contains the (outward) cusps of $\Omega$.*

**Proof:** We prove that a small neighbourhood $A$ of a cusp always gives a negative contribution to $\Phi$. Without restriction of generality, we can assume that the peak of the cusp is in the origin of a coordinate system and the cusp is formed by two functions $f_1$ and $f_2$ with $f_1(x_1) = o(x_1)$ and $f_2(x_1) = o(x_1)$ where "$o$" is the second Landau order symbol. Consider figure 1 we obtain

$$\begin{aligned}
|\Gamma_A| &= f_1(x_{1,0}) - f_2(x_{1,0}) \quad = \quad o(x_{1,0}) \\
|\Sigma_A| &= \int_0^{x_{1,0}} \sqrt{1 + f_1'^2(\xi)}\, d\xi + \int_0^{x_{1,0}} \sqrt{1 + f_2'^2(\xi)}\, d\xi \quad = \quad x_{1,0} + o(x_{1,0}) \\
|A| &= \int_0^{x_{1,0}} (f_1(\xi) - f_2(\xi))\, d\xi \quad = \quad o(x_{1,0}^2).
\end{aligned}$$

Thus,

$$\Phi[A] \quad = \quad -\cos\gamma x_{1,0} + o(x_{1,0}).$$

If we choose $x_{1,0}$ sufficiently small, we can deduce that $\Phi[A] < 0$, which means a negative contribution to $\Phi$ and therefore, $A$ must be contained in a minimizer of $\Phi$. $\square$

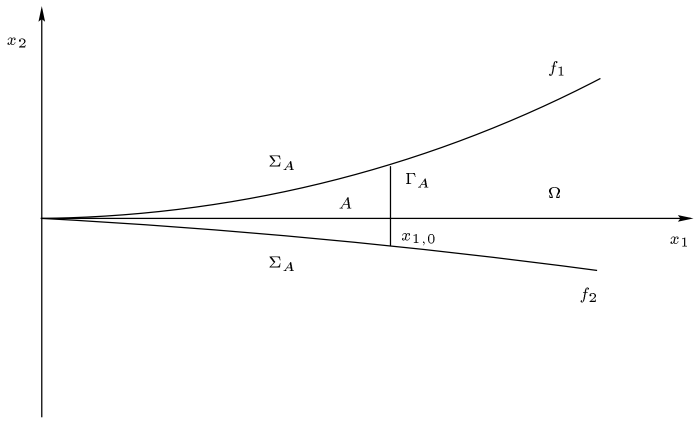


figure 1

**Lemma 2.2:** *Assume that* $0 < \gamma < \frac{\pi}{2}$ *and* $\Omega$ *contains only corners with opening angle less than* $\pi$ *then the set* $\Gamma_M$ *cannot contain complete circles* $B_R(x_0)$.

**Proof:** Suppose that $\Gamma_M$ contains such a circle. This circle cannot meet the boundary of $\Omega$ because then the angle condition of theorem 2.1 is not satisfied. It can also not meet other components of $\Gamma_M$ because of the analytizity of $\Gamma_M$. Consequently, we can expand the circle by a factor $1+\varepsilon$ ($\varepsilon$ sufficiently small) and the expanded circle still satisfies the above conditions. We compare this new set $M'$ with $M$. If follows

$$\begin{aligned} |\Gamma_{M'}| &= |\Gamma_M| + 2\pi R\varepsilon \\ |\Sigma_{M'}| &= |\Sigma_M| \\ |M'| &= |M| - \pi R^2(2\varepsilon + \varepsilon^2). \end{aligned}$$

In view of $R = \frac{1}{2H}$ we have:

$$\Phi[M'] < \Phi[M],$$

which is a contradiction to the minimization property of $M$. □

With theorem 2.1, lemma 2.1, 2.2 we can nearly guess where possible minimizers appear for a given set $\Omega$.
On the other hand, the function $u_0$ in (2.1) has to be a so called *C-singular solution* ([4]) that is a solution of

$$\begin{aligned} &(2.4) \qquad \operatorname{div} Tu = 2H && \text{in } \Omega_0 = \Omega \setminus \bar{M} \\ &(2.5) \qquad \nu \circ Tu = \cos\gamma && \text{on smooth components of } \partial\Omega_0 \cap \partial\Omega \\ &(2.6) \qquad \nu \circ Tu = 1 && \text{on } \partial\Omega_0 \cap \Omega. \end{aligned}$$

We cannot expect that for every $H$ and minimizer $M$ there is a C-singular solution in $\Omega_0$. Integrating (2.4) on $\Omega_0$ and applying again the Gauss theorem we obtain:

$$(2.7) \qquad \Psi[\Omega_0; H, \gamma, \Omega] = 0.$$

From this identity we can conclude (see [11]) that

$$(2.8) \qquad 0 < H \leq H_0 = \frac{|\partial\Omega| \cos\gamma}{2|\Omega|}.$$

Equation (2.7) can be interpreted as some kind of choice condition for the “correct” mean curvature $H$ of the liquid that is the “physically interesting” one. This yields exactly the $H$ from equation (1.4) if a smooth solution exists on the whole domain $\Omega$ (e.g. if $\partial\Omega$ is smooth) that means $\Omega_0 = \Omega$. We still want to remark that there is always a solution of the type (2.1) with a special $H$ (which does not have to be uniquely determined) whenever a smooth solution of (1.2), (1.3) fails to exist (see [4], [11]). The next lemma is another property of the corresponding minimizers to our solutions.

**Lemma 2.3:** *Let $M \neq \Omega$ be a minimizer of $\Phi[\Omega^*; H, \gamma, \Omega]$ and assume that (2.7) is satisfied for $\Omega_0 = \Omega \setminus \bar{M}$, then the set $\Omega$ is also a minimizer of $\Phi[\Omega^*; H, \gamma, \Omega]$ and the empty set is a minimizer of $\Psi[\Omega^*; H, \gamma, \Omega]$.*

**Proof:** This lemma follows immediately by the property that if $M$ minimizes $\Phi$, then $\Omega_0$ minimizes $\Psi$ and vice versa. □

We can fix a strategy to calculate the solutions (2.1) (cf. [11]).

**Strategy for the Calculation of Generalized Solutions of the Type (2.1)**:

1. For the $H$ with (2.8) we calculate the minimizers $\Omega_0$ of $\Psi[\Omega^*; H, \gamma, \Omega]$ with the help of the geometric properties (theorem 2.1, lemma 2.1, 2.2).
2. We check if equation (2.7) is satisfied for the calculated $\Omega_0$ and the corresponding $H$. Only these pairs $(H, \Omega_0)$ we will continue to consider.
3. We check if there is really a C-singular solution on $\Omega_0$.
4. We form the function (2.1).

The third step can best be done with the existence-nonexistence principle discovered by FINN.

**Theorem 2.2** ([4] or [3], p. 206)**:** *Let $\beta \in L^1(\partial\Omega)$, $|\beta| \leq 1$ and $H = \frac{1}{2|\Omega|}\int_{\partial\Omega} \beta\, ds$ then a smooth solution of (1.2), (1.3) exists if and only if the functional*

$$(2.9) \qquad \tilde{\Phi}[\Omega^*] \quad = \quad |\Gamma_{\Omega^*}| - \int_{\partial\Omega} \varphi_{\Omega^*} \cdot \beta\, ds + 2H|\Omega^*|$$

*is positive for every Caccioppoli set $\Omega^* \subset \Omega$ with $\Omega^* \neq \emptyset, \Omega$.*

In the theorem $\beta$ is allowed to vary. For our case we use the theorem in the situation of problem (2.4) to (2.6) that means $\beta$ is piecewise constant. Obviously, the condition about $H$ in theorem 2.2 here is the same as equation (2.7).

## 3. Calculation for our Example

In this chapter we want to apply the strategy to the xylem structures observed in the book [12], p. 49 (see chapter 1). That means we have a large number of xylem vessels with a cross section bounded by three circular arcs of radius $r$ (see figure 2), which in the following will be a parameter and can be interpreted as our representative length. We consider one of these vessels.

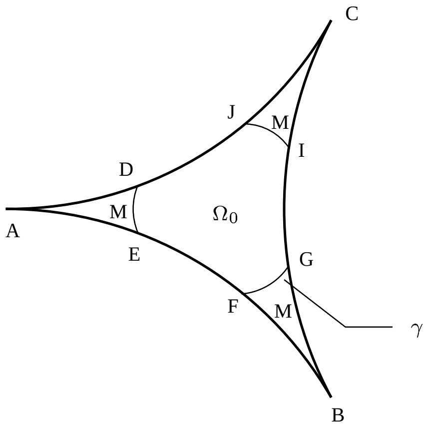


figure 2

We use the geometric conditions to determine a minimizer of $\Phi$ for a fixed $H$ for which (2.8) holds. Note that if $M$ is a minimizer for $\Phi$ then $\Omega_0 = \Omega \setminus \bar{M}$ is a minimizer of $\Psi$.
The application of congruence theorems yields that the components of $\Gamma_M$ (arc$(D, E)$, arc$(F, G)$ and so on) are circular arcs, whose centres are on the tangent through $A$, $B$ and so on to satisfy the angle condition of theorem 2.1.
We consider at first the case $\gamma > 0$. The geometric conditions of theorem 2.1 in combination with lemma 2.1, 2.2 enable only the shape of $\Omega_0$ drawn as in the figures 2 or 3 or it holds $M = \Omega$. A set of the shape of fig$(D, E, F, G)$ from figure 3 can also occur in two or three of the cusps.

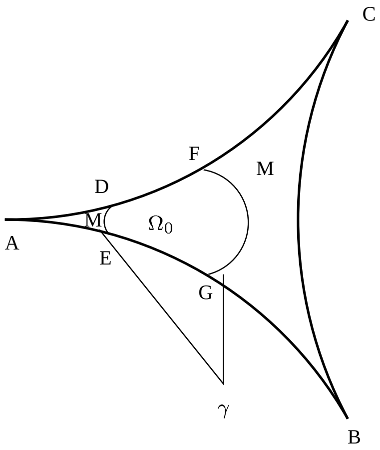


figure 3

However, the constellations of figure 3 can be excluded because we calculate that for every $H$ it holds $\Psi(\Omega_0) > 0$. That means our choice condition (2.7) is not satisfied and no solution of the type (2.1) can exist.

**Proof:** From the angle condition of theorem 2.1 we can conclude that the arcs arc$(D, E)$ and arc$(F, G)$ are from one and the same circle of radius $R = \frac{1}{2H}$. For the description of the set $\Omega_0$ we consider the next figure 4.

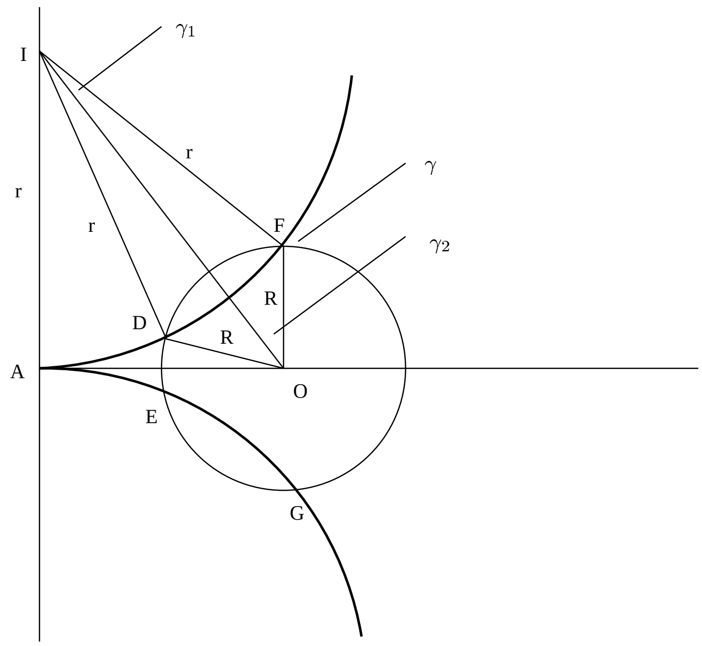


figure 4

We have

$$\gamma_1 + \gamma_2 = \gamma \tag{3.1}$$

and because of $\gamma > 0$

$$0 < \gamma_1 < \min\left(\gamma, \frac{\pi}{6}\right). \tag{3.2}$$

The calculation of the height of the triangle fig$(F, I, O)$ with respect to the edge $\overline{IO}$ yields:

$$r \sin \gamma_1 = R \sin \gamma_2. \tag{3.3}$$

We determine the quantities for the functional $\Psi$ with respect to $\Omega_0$.

$$|\Sigma_{\Omega_0}| = 4\gamma_1 r \tag{3.4}$$

$$|\Gamma_{\Omega_0}| = R\,(2\pi - 4\gamma_2) \tag{3.5}$$

$$|\Omega_0| = \pi R^2 - r^2\,(2\gamma_1 - \sin 2\gamma_1) - R^2\,(2\gamma_2 - \sin 2\gamma_2)\,. \tag{3.6}$$

Thus,

$$\begin{aligned} \Psi[\Omega_0] &= 4\gamma_1 r \cos\gamma + R\,(2\pi - 4\gamma_2) - \pi R + \frac{r^2}{R}\,(2\gamma_1 - \sin 2\gamma_1) \\ &\quad + R\,(2\gamma_2 - \sin 2\gamma_2) \\ &= r\Bigg(4\gamma_1 \cos\gamma + (\pi - 2\gamma)\,\frac{R}{r} + 2\gamma_1\left(\frac{R}{r} + \frac{r}{R}\right) \\ &\quad - \frac{r}{R}\sin 2\gamma_1 - \frac{R}{r}\sin 2\gamma_2\Bigg). \end{aligned} \tag{3.7}$$

Because of

$$\begin{aligned} \frac{r}{R}\sin 2\gamma_1 + \frac{R}{r}\sin 2\gamma_2 &= 2\sin(\gamma_1 + \gamma_2) \\ &= 2\sin\gamma \\ \frac{r}{R} &= \frac{\sin(\gamma - \gamma_1)}{\sin\gamma_1} \end{aligned}$$

$$\begin{aligned} &= \sin\gamma\cot\gamma_1 - \cos\gamma \\ \frac{R}{r} &> \frac{1}{\sin\gamma\cot\gamma_1} \end{aligned}$$

we can estimate (3.7)

$$\begin{aligned} \Psi[\Omega_0] &> r\Bigg(2\gamma_1\cos\gamma + (\pi - 2\gamma)\frac{R}{r} + 2\gamma_1\left(\sin\gamma\cot\gamma_1 + \frac{1}{\sin\gamma\cot\gamma_1}\right) \\ &\qquad - 2\sin\gamma\Bigg) \\ &> 2r\sin\gamma\,(\gamma_1\cot\gamma_1 + \gamma_1\tan\gamma_1 - 1)\,. \qquad (3.8) \end{aligned}$$

It is not difficult to prove that for $\gamma_1$ with the property (3.2) it holds that

$$\gamma_1\cot\gamma_1 + \gamma_1\tan\gamma_1 \quad > \quad 1. \qquad (3.9)$$

From (3.8) we can deduce by using the estimate (3.9)

$$\Psi[\Omega_0] \quad > \quad 0. \qquad (3.10)$$

Thus, there is no C-singular solution on $\Omega_0$. We can analogously exclude constellations, in which a set of the type fig$(D, E, F, G)$ from figure 3 occurs in more then one of the cusps. □

Next we consider the case $\gamma = 0$ separately. By the geometric conditions a connected component of $\Omega_0$ could be a circle that contacts at most one of the arcs arc$(A, B)$, arc$(B, C)$, arc$(A, C)$. From an inspection of the proof of lemma 2.2 it is easy to see that this cannot provide a minimizer. A connected component of $\Omega_0$ could be a circle, which contacts exactly two of the arcs. We can translate this circle along the corresponding tangent through $A$, $B$ or $C$ and enlarge it, so that it again contacts the boundary in exactly two points. From the proof of lemma 2.2 we see that this decreases the functional $\Psi$ and that $\Omega_0$ cannot have been a minimizer. From this it follows that also in the case $\gamma = 0$ the set $\Omega_0$ must be of the type fig$(D, E, F, G, I, J)$ of figure 2 or $\Omega_0 = \emptyset$ or $\Omega_0$ is the inscribed circle of fig$(A, B, C)$. A short calculation shows that for the last case we have again $\Psi(\Omega_0) > 0$ and no solution of the required type can exist.
We now calculate the exact shape of the set fig$(D, E, F, G, I, J)$ for a fixed $H$ and check if the condition (2.7) is satisfied. This leads to a choice of $H$. It will turn out that there is one and only one $H$ and $\Omega_0$ of the shape of fig$(D, E, F, G, I, J)$ for every $\gamma$ so that (2.7) holds. According to lemma 2.3, this set $\Omega_0$ together with the empty set are the only two minimizeres of $\Psi$.
For the calculation we consider figure 5. Obviously, the set $\Omega_0$ is already determined for a given $\gamma_1$.
We easily see from the quadrangle fig$(A, O, D, K)$ that

$$\gamma_1 + \gamma_2 + \gamma \quad = \quad \frac{\pi}{2}. \qquad (3.11)$$

By decomposition of the lines $\overline{AK}$ and $\overline{AO}$ it can be obtained that

$$\begin{aligned} r &= r\cos\gamma_1 + R\sin\gamma_2 \qquad (3.12) \\ x_0 &= r\sin\gamma_1 + R\cos\gamma_2. \qquad (3.13) \end{aligned}$$

We set

$$\begin{aligned} b &= \frac{R}{r} \\ &= \frac{1 - \cos\gamma_1}{\sin\gamma_2}. \qquad (3.14) \end{aligned}$$

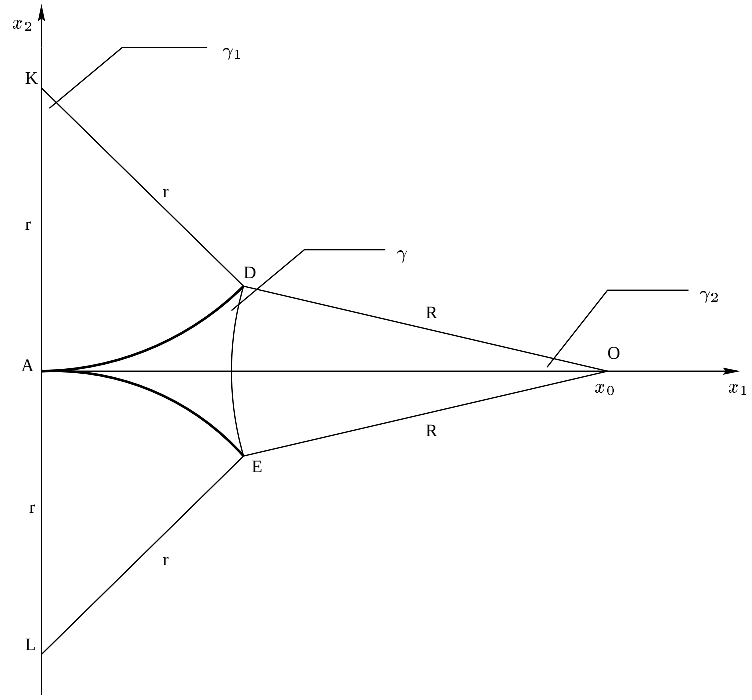


figure 5

We still want to show that the mapping $\gamma_1 \to R$ is one-to-one. One direction is clear because of (3.14). Furthermore, from (3.14) follows from the intermediate-point theorem that for every $R$ there is at least one corresponding $\gamma_1 \in \left(0, \frac{\pi}{2} - \gamma\right)$.
After a short calculation we obtain from the equations (3.12), (3.13):

$$(3.15) \qquad \cos\gamma_1 = \frac{r^2 + R\,(r\cos\gamma + x_0 \sin\gamma)}{r^2 + x_0^2}.$$

From (3.14) and (3.15) it follows that for every $R$ there is one and only one corresponding angle $\gamma_1 \in \left(0, \frac{\pi}{2} - \gamma\right)$. Therefore, we can choose $\gamma_1$ freely instead of $R$ and $H$ ($H = \frac{1}{2R}$) and $R$, $H$ dependent on $\gamma_1$. It must hold that

$$(3.16) \qquad \gamma_1 \in \left(0, \frac{\pi}{6}\right] \qquad \text{if } \gamma < \frac{\pi}{3}$$

$$(3.17) \qquad \gamma_1 \in \left(0, \frac{\pi}{2} - \gamma\right) \qquad \text{if } \gamma \geq \frac{\pi}{3}$$

to obtain really a set of the shape of fig$(D, E, F, G, I, J)$. Otherwise, there is no solution of the required type.
Now we can determine the quantities for the calculation of $\Psi[\Omega_0]$.

$$(3.18) \qquad |\Sigma_{\Omega_0}| = (\pi - 6\gamma_1)\,r$$

$$(3.19) \qquad |\Gamma_{\Omega_0}| = 6\gamma_2 R.$$

To determine the quantity $|\Omega_0|$ we calculate the area bounded by the arcs arc$(A, D)$, arc$(D, E)$ and arc$(A, E)$.
Because of

$$|\Omega| = \left(\sqrt{3} - \frac{\pi}{2}\right) r^2$$

we obtain after calculation

$$|\Omega_0| = \left(\sqrt{3}-\frac{\pi}{2}\right) r^2 - 6r^2\sin\gamma_1 + \frac{3}{2}r^2\sin 2\gamma_1 + 3\gamma_1 r^2 + 3\gamma_2 R^2 - \frac{3}{2}R^2\sin 2\gamma_2. \tag{3.20}$$

Now we get for the functional $\Psi$ of $\Omega_0$:

$$\begin{aligned}\Psi[\Omega_0] = \frac{r}{b}\Big(& 3b^2\gamma_2 + b\cos\gamma\,(\pi-6\gamma_1) - \left(\sqrt{3}-\frac{\pi}{2}\right) + 6\sin\gamma_1 - \frac{3}{2}\sin 2\gamma_1 - 3\gamma_1 \\ & + \frac{3}{2}b^2\sin 2\gamma_2\Big).\end{aligned} \tag{3.21}$$

The right-hand side of the equation depends only on $\gamma_1$ (see relations (3.11) and (3.14)). We set the term in the outer parenthesis equal to $f(\gamma_1)$ and check the condition (2.7), that is

$$f(\gamma_1) = 0. \tag{3.22}$$

We want to find out whether this non-linear equation has a solution for $\gamma_1$. At first we have

$$\begin{aligned}\lim_{\gamma_1\to 0+0} f(\gamma_1) &= -\left(\sqrt{3}-\frac{\pi}{2}\right) \\ &< 0.\end{aligned} \tag{3.23}$$

We have to consider two cases to determine the behaviour of $f$ on the right bound of the interval of definition.

**Case 1:** $\gamma < \frac{\pi}{3}$
Consequently,

$$\begin{aligned} f\left(\frac{\pi}{6}\right) &= 3b^2\gamma_2 - \left(\sqrt{3}-\frac{\pi}{2}\right) + 3 - \frac{3\sqrt{3}}{4} - \frac{\pi}{2} + \frac{3}{2}b^2\sin 2\gamma_2 \\ &> 3b^2\gamma_2 + 3 - \frac{7\sqrt{3}}{4}.\end{aligned}$$

Since $\frac{\gamma_2}{\sin^2\gamma_2} \geq 1$, from the last equation it can be concluded that

$$\begin{aligned} f\left(\frac{\pi}{6}\right) &> 3\left(1-\frac{\sqrt{3}}{2}\right)^2 + 3 - \frac{7\sqrt{3}}{4} \\ &= \left(3-\sqrt{3}\right)\frac{7-\sqrt{48}}{4} \\ &> 0.\end{aligned} \tag{3.24}$$

From (3.23) and (3.24) it follows, by using the intermediate-point theorem, that there is at least one solution of (3.22) in the corresponding domain of definition.

**Case 2:** $\gamma \geq \frac{\pi}{3}$
We obtain

$$\lim_{\gamma_1\to\frac{\pi}{2}-\gamma-0} f(\gamma_1) = +\infty$$

and, with the same reasoning as above, also in this case at least one solution for (3.22).
Finally, it shall be shown that there is exactly one solution of (3.22). Therefore, we examine the monotonicity of $f$.

$$\frac{d\gamma_2}{d\gamma_1} = -1$$

$$\begin{aligned}
\frac{db}{d\gamma_1} &= \frac{\cos\gamma_2 - \sin\gamma}{\sin^2\gamma_2} \\
\frac{df}{d\gamma_1} &= 2bb'\left(3\gamma_2 + \frac{3}{2}\sin 2\gamma_2\right) + b^2\left(-3 - 3\cos 2\gamma_2\right) + b'\cos\gamma\left(\pi - 6\gamma_1\right) \\
&\quad - 6b\cos\gamma + 6\cos\gamma_1 - 3\cos 2\gamma_1 - 3 \\
&= T + \frac{\cos\gamma_2 - \sin\gamma}{\sin^2\gamma_2}\cos\gamma\left(\pi - 6\gamma_1\right) \qquad (3.25)
\end{aligned}$$

because of

$$\begin{aligned}
-3 - 3\cos 2\gamma_2 &= -6\cos^2\gamma_2 \\
6\cos\gamma_1 - 3\cos 2\gamma_1 - 3 &= 6\cos\gamma_1\left(1 - \cos\gamma_1\right) \\
&= 6b\cos\gamma_1\sin\gamma_2
\end{aligned}$$

we have used for abbreviation

$$\begin{aligned}
T &= \frac{6b}{\sin\gamma_2}\Bigg(\left(\cos\gamma_1 - 1\right)\cos^2\gamma_2 + \frac{\gamma_2}{\sin\gamma_2}\left(\cos\gamma_2 - \sin\gamma\right) \\
&\quad + \cos\gamma_2\left(\cos\gamma_2 - \sin\gamma\right) - \cos\gamma\sin\gamma_2 + \cos\gamma_1\sin^2\gamma_2\Bigg). \qquad (3.26)
\end{aligned}$$

Equation (3.26) can be simplified:

$$\begin{aligned}
T &= \frac{6b}{\sin\gamma_2}\left(\frac{\gamma_2}{\sin\gamma_2}\left(\cos\gamma_2 - \sin\gamma\right) + \cos\gamma_1 - \sin\left(\gamma + \gamma_2\right)\right) \\
&= \frac{6b\gamma_2}{\sin^2\gamma_2}\left(\cos\gamma_2 - \sin\gamma\right). \qquad (3.27)
\end{aligned}$$

From (3.11) and (3.16), (3.17) we obtain:

$$\cos\gamma_2 \quad > \quad \sin\gamma. \qquad (3.28)$$

From (3.25), (3.27) and (3.28) it follows that:

$$\frac{df}{d\gamma_1} \quad > \quad 0. \qquad (3.29)$$

Thus, equation (3.22) has one and only one solution in our domain of definition. Consequently, there is exactly one $H$, for which there is a set $\Omega_0$ of the type of fig$(D, E, F, G, I, J)$ from figure 2, satisfying the choice condition (2.7). This set is also uniquely defined and is, together with the empty set, the only minimizer of $\Psi[\Omega^*; H, \gamma, \Omega]$.

We can now solve equation (3.22) numerically. The next table shows the results. We rounded to four decimals.

| $\gamma$ in $^o$ | $\gamma_1$ in $^o$ | $\gamma_2$ in $^o$ | $Hr$ | $H_0 r$ | $V_{st}$ in $lm^{-3}$ | Remark |
|---|---|---|---|---|---|---|
| 0 | 23.2444 | 66.7556 | 5.6599 | 9.7411 | 25.5204 | water and glass by $18^oC$ [1] |
| 30 | 20.2713 | 39.7287 | 5.1597 | 8.4360 | 21.1808 | |
| 45 | 18.2651 | 26.7349 | 4.4644 | 6.8880 | 16.6372 | |
| 54 | 16.7455 | 19.2545 | 3.8882 | 5.7257 | 13.2779 | water and cellulose acetat |
| 60 | 15.5113 | 14.4887 | 3.4346 | 4.8706 | 10.7733 | |
| 68 | 13.4133 | 08.5867 | 2.7367 | 3.6491 | 07.1372 | water und cellulose nitrat |
| 89 | 00.9971 | 00.0029 | 0.1672 | 0.1700 | 00.0030 | |
| 89.99 | 00.0100 | 00.0000 | 0.0017 | 0.0017 | 00.0000 | |

[1] all data taken from [1], p. 474-479

The first column is the parameter $\gamma$, given by the involved substances. The second and third column give the shape of the set $\Omega_0$ and $M$ respectively, as described in figure 2 with the notations from figure 5. For the quantities $\gamma_1$ and $\gamma_2$ holds (3.11).
Since

$$\begin{aligned} |\partial\Omega| &= \pi r \\ |\Omega| &= \left(\sqrt{3} - \frac{\pi}{2}\right) r^2, \end{aligned}$$

we have for $H_0$ the relation

$$H_0 = \frac{\pi \cos\gamma}{\left(2\sqrt{3} - \pi\right) r}. \tag{3.30}$$

In the fourth column we calculated our unique $H$ of physical interest for the corresponding $\gamma$. When comparing the fourth and fifth column, we can see that (2.8) is always satisfied. For $\gamma \to \frac{\pi}{2}$, which means vanishing wetting, we obtain, as expected, $H \to 0$, $\Omega_0 \to \Omega$.
The sixth column is the minimum liquid storage $V_{st}$ for the considered configuration. Since the fibers are arranged in form of honeycombs (chapter 1), we cover the plane with hexagons and calculate the ratio between the sets $M$ in the numerous vessels of cross section fig$(A, B, C)$ and the entire plane. The unit of this quantity is litre of liquid per cubic metre of xylem tissue. We can see from (3.20)

$$V_{st} = \frac{1000}{\sqrt{3}} \left(6 \sin\gamma_1 - \frac{3}{2}\sin 2\gamma_1 - 3\gamma_1 - \frac{3}{4}\gamma_2 \frac{1}{(Hr)^2} + \frac{3}{8}\frac{1}{(Hr)^2}\sin 2\gamma_2\right). \tag{3.31}$$

Note that $V_{st}$ is independent of $r$, since $Hr$ is independent of $r$. We will discuss this in the next chapter.
We still have to show, that there is really a smooth solution of (2.4) to (2.6) on the calculated $\Omega_0$ (step 3 of the strategy in chapter 2). Therefore, we apply theorem 2.2. as described there. Let $\Omega^* \subseteq \Omega_0$ with

$$\tilde{\Phi}[\Omega^*] \leq 0. \tag{3.32}$$

Let $M = \Omega \setminus \bar{\Omega}_0$. In view of the above construction $M$ is a minimizer of $\Phi$ so that for every Caccioppoli set $A \subseteq \Omega$ it holds

$$\Phi[M] \leq \Phi[A]. \tag{3.33}$$

Adding the inequalities (3.32) and (3.33) it can be concluded that

$$\Phi[M \cup \Omega^*] \leq \Phi[A]. \tag{3.34}$$

Thus, $M \cup \Omega^*$ is also a minimizer of $\Phi$. However, because of the above observations, only $M$ and $\Omega$ are minimizers of $\Phi$. Consequently, $\Omega^* = \Omega_0$ or $\Omega^* = \emptyset$. The application of theorem 2.2 guarantees the existence of a smooth solution $u_0$ of the problem (2.4) to (2.6) on $\Omega_0$.

## 4. Analysis of the Results

The results can be interpreted as follows. The effect of the liquid rising practically by itself could be of significance for explaining the water supply of trees.
If we assume that in nature we have optimal wetting, that is $\gamma = 0$, then our figure 2 is exactly the same as observed in the book of ZIMMERMANN ([12], p. 50). Note that such an optimal wetting should be an advantage in evolution.
The amount of liquid, which is stored in the xylems, can be estimated by our calculation. Note that this amount is independent of the radius $r$ of the fibers. However, the model fits better with a small $r$ because this is our represantative length (see chapter 1).

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