Generalized Solutions of the Capillary Problem

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Abstract: It was pointed out by FINN [2] that the capillary problem in zero gravity has not always a classical (smooth) solution in the case that the bounded domain $\Omega \subset \mathbb{R}^2$ contains cusps or corners. Here, Ω denotes the cross section of a given cylinder, in which a liquid is contained. If special energy terms could become infinite, the variational formulation is not free of limitations as well. Therefore, the concept of generalized solutions, which can be traced back to MIRANDA [11], has been developed and could be a way out.
We want to prove an existence result for such solutions under very weak preconditions. The proof is closely related to GIUSTI's paper [6], but we do not require full smoothness of the boundary. The major new difficulty is that we also want to consider domains with non-Lipschitz boundary. This excludes the application of some theorems. On the other hand, we use special geometric conditions in $\mathbb{R}^2$ and consequently, the proof cannot easily be generalized to a higher dimension.
Furthermore, we construct some generalized solutions explicitly.

Keywords: capillary problem, cusp, generalized solution

1. Preliminaries and Motivation

At first we consider the capillary equation in zero gravity

$$(1.1)\qquad \operatorname{div} Tu = 2H \qquad \text{in } \Omega$$

$$(1.2)\qquad \nu \circ Tu = \beta \qquad \text{on smooth parts of } \partial\Omega,$$

$\Omega \subset \mathbb{R}^2$ a bounded domain with piecewise smooth boundary, $Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$, ν the outer normal, H, β constant and $0 < \beta \leq 1$ the adhesion coefficient. Here, "$\circ$" denotes the euclidian scalar product. By partial integration we obtain $H = \frac{\beta|\partial\Omega|}{2|\Omega|}$.
The existence-nonexistence principle of FINN for smooth solutions of this problem is (see [2])

$$(1.3)\qquad \Phi[\Omega^*; H, \beta, \Omega] = |\Gamma_{\Omega^*}| - \beta\,|\Sigma_{\Omega^*}| + 2H|\Omega^*| > 0$$

for each Caccioppoli-set $\Omega^* \subset \Omega$, $\Omega^* \neq \emptyset, \Omega$ where $\Gamma_{\Omega^*} = \partial_*\Omega^* \cap \Omega$, $\Sigma_{\Omega^*} = \partial_*\Omega^* \cap \partial\Omega$ and $\partial_*\Omega^*$ the measure-theoretic boundary of Ω^* (see [1]). This principle is necessary (which can easily be shown by partial integration of (1.1) over Ω^*) and sufficient.
We always get a contradiction to (1.3) if Ω contains cusps.

Lemma 1.1: *No smooth solution of (1.1), (1.2) exists for domains Ω with outward cusps.*

Proof: We consider a small neighbourhood A of the cusp bounded by two functions f_1 and f_2 (see figure 1).
It follows

$$|\Gamma_A| - \beta|\Sigma_A| + \lambda|A| = -\beta x_{1,0} + o(x_{1,0})$$

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where "o" is the second Landau order symbol. For a sufficiently small $x_{1,0}$ we obtain $\Phi[A] < 0$ and by the existence-nonexistence principle no smooth solution can exist. □

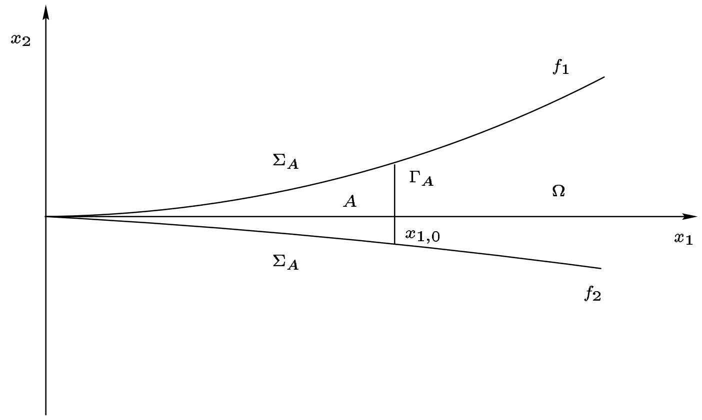

figure 1

Therefore, we have to look for a suitable generalization of the problem. From now on we will allow H and $\beta \in [0,1]$ to vary.
The variational formulation of (1.1), (1.2) is

$$(1.4)\qquad \mathcal{E}[u] = \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega \lambda(u)\,dx - \oint_{\partial\Omega} \beta(s)u\,ds \quad \to \quad \text{min.!} \qquad u \in BV(\Omega);$$

the function λ shall satisfy $\lambda'(u) = 2H(u)$ and $\lambda''(u) \geq 0$. The set $BV(\Omega)$ denotes the space of the functions of bounded variation in Ω (see [1] or [7] for an introduction to the theory of this space). The functional $\mathcal{E}$ is the sum of surface, gravitational and contact energy. See also the book of FINN [2] for further background material.
The last integral is to be understood in the sense of the trace of u on $\partial\Omega$. This trace has not always to be in L^1 if Ω contains cusps that is if $\partial\Omega$ is not Lipschitz-continuous (cf. [1], p. 177).
An observation of MIRANDA leads from the variational formulation to a generalized formulation.

Definition 1.1: Let u be a real-valued function in Ω, then the set

$$U \quad = \quad \{(x,t) \in \Omega \times \mathbb{R} \quad : \quad t < u(x)\}$$

is called the *subgraph* of u.

Theorem 1.1 ([2], p. 192 or [10])**:** *A function $u \in BV(\Omega)$ minimizes $\mathcal{E}$ in Ω (Ω Lipschitz) if and only if its subgraph U minimizes the functional*

$$(1.5)\qquad \mathcal{F}[U] \quad = \quad \int_{Q=\Omega\times\mathbb{R}} |D\varphi_U| + 2\int_Q H\varphi_U\,dxdt - \int_{\delta Q=\partial\Omega\times\mathbb{R}} \beta\varphi_U\,dsdt.$$

The function φ_U is the characteristic function of U. The minimization of (1.5) is to be understood in a special sense. Let

$$(1.6)\qquad \mathcal{F}_T[U] \quad = \quad \int_{Q_T=\Omega\times[-T,T]} |D\varphi_U| + 2\int_{Q_T} H\varphi_U\,dxdt - \int_{\delta Q_T=\partial\Omega\times[-T,T]} \beta\varphi_U\,dsdt$$

for a Caccioppoli set $U \subseteq Q$ and $T > 0$, it is defined that (cf. [2], [10], [11]):

Definition 1.2: The set U *minimizes* (or U is a *solution of*) $\mathcal{F}_T$ *in* Q_T if $\mathcal{F}_T[U] \leq \mathcal{F}_T[S]$ for every

Caccioppoli set S that coincides with U outside from Q_T.

Remark: The additional precondition on S (S coincides with U outside from Q_T) should be added in [2], p. 192 to get an equivalent to other formulations in literature (see later Lemma 1.2).

Definition 1.3: The set U *minimizes* (or U is a *local solution of*) $\mathcal{F}$ *in* Q if U minimizes the functional $\mathcal{F}_T$ in Q_T for every $T > 0$.

In this sense we can define a generalized solution, which is important for the following:

Definition 1.4: A function $u(x) : \Omega \to [-\infty, \infty]$ is called a *generalized solution* of the problem (1.1), (1.2), if its subgraph U is a Caccioppoli set and a local solution of $\mathcal{F}$ in Q.

Note that a solution can be infinite on sets of positive measure that is the solution u has even not to be an L^1-function. Furthermore, the functional $\mathcal{F}$ has not to be well-defined for a solution. It is sufficient that the functionals $\mathcal{F}_T$ are well-defined to suffice the minimization property and therefore, domains with cusps make no problem because of the boundedness of the integrals of $\mathcal{F}_T$. The solutions are in general not global minimizers for $\mathcal{F}_T$, since we can only compare them with admissible comparison sets. The solutions are only minimizers with respect to local disturbance.
We have the theorem that smooth solutions are generalized solutions ([10], see also the remarks in [2], chapter 7) and by Theorem 1.1 that every solution of the variational problem is a generalized solution.
It is clear that the definition of a generalized solution makes sense for almost arbitrary functions H and β. For the moment we only need $H \in L^1_{loc}(Q)$, $\beta \in L^1_{loc}(\delta Q)$ that means very weak preconditions.
Last we want to prove equivalence of various definitions of minimization of $\mathcal{F}_T$ in literature (which is not always trivial to see) and equivalence to a new definition. For this proof Ω can be n-dimensional.

Lemma 1.2: *With the definitions*

$$\begin{aligned} \mathcal{F}_{[a,b]}[U] &= \int_{Q_{[a,b]}=\Omega\times[a,b]} |D\varphi_U| + 2\int_{Q_{[a,b]}} H\varphi_U \, dxdt - \int_{\delta Q_{[a,b]}=\partial\Omega\times[a,b]} \beta\varphi_U \, dsdt \\ \mathcal{F}_{\mathcal{K}}[U] &= \int_{Q\cap\mathcal{K}} |D\varphi_U| + 2\int_{Q\cap\mathcal{K}} H\varphi_U \, dxdt - \int_{\delta Q\cap\mathcal{K}} \beta\varphi_U \, ds, \end{aligned}$$

$[a, b]$ an interval with $a < b$, $\mathcal{K}$ a compact subset from $\mathbb{R}^{n+1}$ the following formulations of minimization are equivalent.

Formulation A:

$$\mathcal{F}_T[U] \leq \mathcal{F}_T[S]$$

for every Caccioppoli set S that coincides with U outside of Q_T and every $T > 0$.

Formulation B:

$$\mathcal{F}_{[a,b]}[U] \leq \mathcal{F}_{[a,b]}[S]$$

for every Caccioppoli set S that coincides with U outside of $Q_{[a,b]}$ and every interval $[a, b]$ with $a < b$.

Formulation C ([6], p. 300) **:**

$$\begin{aligned} \mathcal{F}_T[U] &\leq \mathcal{F}_T[U \cup S] && (U \textit{ is a supersolution in } Q_T) \\ \mathcal{F}_T[U] &\leq \mathcal{F}_T[U \setminus S] && (U \textit{ is a subsolution in } Q_T) \end{aligned}$$

for every Caccioppoli-set $S \subseteq Q_T$ and every $T > 0$.

Formulation D ([13], p. 855) **:**

$$\mathcal{F}_{\mathcal{K}}[U] \quad \leq \quad \mathcal{F}_{\mathcal{K}}[S]$$

for every Caccioppoli set S that coincides with U outside of $\mathcal{K}$ and every compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$.

Proof: A $\Leftrightarrow$ B:
The direction "$\Leftarrow$" is trivial. For the other direction we choose for every interval $[a,b]$ the number T sufficiently large so that $[a,b] \subset [-T,T]$ and let $S = \left(U \setminus Q_{[a,b]}\right) \cup S_0$ where $S_0 \subseteq Q_{[a,b]}$ is an arbitrary Caccioppoli set. Applying the measure property of the first integral in (1.6) (e.g. [1], p. 209) we obtain formulation B.
A $\Leftrightarrow$ C:
The direction "$\Rightarrow$" is clear. Let S be a Caccioppoli set that coincides with U outside from Q_T and assume that the formulation C holds, then it follows that

$$(1.7) \qquad \mathcal{F}_T[U] \quad \leq \quad \mathcal{F}_T[U \cup S]$$

$$(1.8) \qquad \mathcal{F}_T[U] \quad \leq \quad \mathcal{F}_T[U \cap S].$$

First we want to show that

$$(1.9) \qquad \int_{Q_T} |D\varphi_{U\cup S}| + \int_{Q_T} |D\varphi_{U\cap S}| \quad \leq \quad \int_{Q_T} |D\varphi_U| + \int_{Q_T} |D\varphi_S|\,.$$

With the Coarea formula for BV-functions ([1], p. 185) it is easy to see that

$$(1.10) \qquad \int_V |D\varphi_{A\cup B}| + \int_V |D\varphi_{A\cap B}| \quad \leq \quad \int_V |D\varphi_A| + \int_V |D\varphi_B|$$

holds for an open set V and two Caccioppoli sets A, B. Note that we cannot immediately apply the Coarea formula, since Q_T is not open. The inequality (1.9) now follows from (1.10) by choosing $A = U$, $B = S$, $V = \Omega \times (-T-\varepsilon, T+\varepsilon)$ $(\varepsilon > 0)$ and by taking into consideration that $U = S$ on $\Omega \times ((-T-\varepsilon, -T) \cup (T, T+\varepsilon))$.
From inequality (1.9) the supposition follows, if we add the inequalities (1.7) and (1.8).
The other conclusions of lemma 1.2 are trivial or can be shown in the same way as direction $A \Rightarrow B$. $\square$

Obviously, the definition A is the most convenient one.

2. Proof of Existence

Although the definition of generalized solution is meaningful for almost arbitrary functions H, β, we will need some further restrictions to prove existence of such solutions. Consider the following structure conditions, which we have to assume (sometimes):

(2.1) $\quad H \in L^1_{loc}(Q),\ \beta \in L^1_{loc}(\delta Q)$

(2.2) $\quad H = H(x,t)$ non-decreasing with respect to t for almost every $x \in \Omega$

(2.3) $\quad \beta = \beta(x,t)$ non-decreasing with respect to t for almost every $x \in \partial\Omega$

(2.4) $\quad H$ locally essentially bounded on $Q \cup \delta Q$

(2.5) $\quad |\beta| \leq 1$ almost everywhere on δQ.

We will need the following convergence lemma to prove existence.

Lemma 2.1: *Let Ω be a bounded domain with piecewise smooth boundary (C^3 is sufficient), the smooth components of the boundary are of bounded curvature. Let H_j and β_j be two sequences, which terms satisfy the structure conditions (2.1), (2.4) and (2.5) and which uniformly converge to H and β. Let the subgraphs U_j minimize the functionals*

$$(2.6)\qquad \mathcal{F}_T^j(A) = \int_{Q_T} |D\varphi_A| + 2\int_{Q_T} H_j\varphi_A\,dxdt - \int_{\delta Q_T} \beta_j\varphi_A\,d\mathcal{H}_2.$$

Let us further assume that for a Caccioppoli set U it holds $U_j \to U$ in Q_T with respect to L^1-convergence of the corresponding characteristic functions and that

$$(2.7)\qquad \int_{\tilde{\partial}Q_T} |\varphi_{U_j} - \varphi_U|\,d\mathcal{H}_2 \to 0, \qquad \tilde{\partial}Q_T = \partial Q_T \setminus \delta Q_T,$$

$$(2.8)\qquad \int_{\tilde{\partial}Q_T} |D\varphi_{U_j}| = 0 \qquad \forall j \geq j_0 \in \mathbb{N},$$

$$(2.9)\qquad \int_{\tilde{\partial}Q_T} |D\varphi_U| = 0,$$

then U minimizes $\mathcal{F}_T$ in Q_T.

Equation (2.7) is to be understood in the sense of the outer trace that is the trace of $|\varphi_{U_j\setminus Q_T} - \varphi_{U\setminus Q_T}|$ on $\tilde{\partial}Q_T$.

Remark: In [6], p. 304 there is a version of this lemma for smooth domains but without the conditions (2.8), (2.9). In my opinion, these conditions should be added to complete the proof in [6]. This does not influence the further results in [6] because these conditions are fulfilled for almost every T in the situations that had been considered there (see [6]).

We will prove this lemma together with the following existence theorem.

Theorem 2.1: *Let Ω be a bounded domain with piecewise smooth boundary (C^3 is sufficient), the smooth components of the boundary are of bounded curvature. Let H, β satisfy the structure conditions (2.1) to (2.5), then there is a generalized solutions of (1.1), (1.2).*

Proof: The proof is closely related to the ideas in GIUSTI's paper [6] but here we work more directly with the functionals $\mathcal{F}_T$ instead of the energy functional $\mathcal{E}$. Our method uses special geometric properties in $\mathbb{R}^2$ and cannot simply be generalized to higher dimensions.
The proof is divided into four steps, for the moment under the precondition that lemma 2.1 holds. Finally, we will prove this lemma in the fourth step.
Step 1: (Proof of semicontinuity of the functional $\mathcal{F}_T$ with respect to L^1-convergence of characteristic functions)
Because of the preconditions the boundary of Ω can be decomposed to $\partial\Omega = (\bigcup_l \Sigma_l) \cup (\bigcup_l \{x_l\})$ ($l \in L = \{1, \ldots, n\}$). Here, Σ_l denote disjoint, smooth, connected components of $\partial\Omega$ and the points x_l ($l \neq n$) are the corners or cusps formed by the arcs Σ_l and Σ_{l+1}, x_n is the corner or cusp formed by the arcs Σ_1 and Σ_n. Without restriction of generality we can assume that $n > 1$ (see figure 2).
At first we assume that the opening angles of the corners are less than or equal to π.
In view of the precondition about the curvature and lemma 2.3 in [9], p. 234, there is for every point x_l a curve $\Gamma_l \subset \Omega$ with

$$\begin{aligned} \operatorname{dist}(x, \Sigma_l) &= \operatorname{dist}(x, \Sigma_{l+1}) && \forall x \in \Gamma_l \quad l < n \\ \operatorname{dist}(x, \Sigma_n) &= \operatorname{dist}(x, \Sigma_1) && \forall x \in \Gamma_n \end{aligned}$$

and Γ_l is locally C^2.
The argumentation in [9] holds for opening angles less than π but for opening angles π we can choose the inner normal for Γ_l.

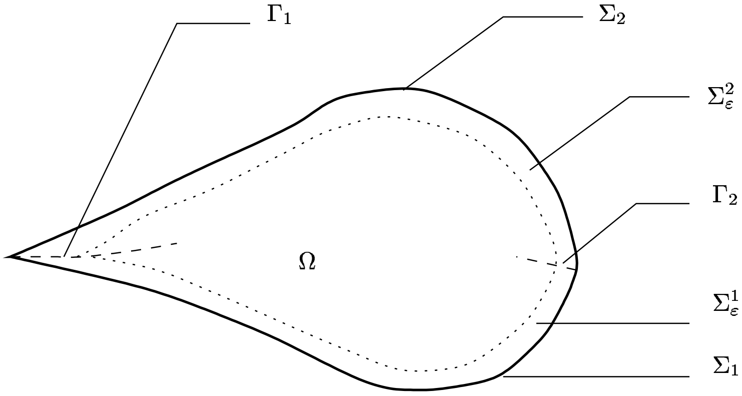

figure 2

Consider the set

$$(2.10) \qquad \Sigma_\varepsilon \;=\; \{x \in \Omega \quad : \quad \mathrm{dist}\,(x, \partial\Omega) \le \varepsilon\}$$

for a sufficiently small $\varepsilon > 0$. Choose $\varepsilon_0 > 0$ so small that for every $\varepsilon \le \varepsilon_0$ the curves Γ_l decompose Σ_ε into disjoint subdomains Σ_ε^l (see again figure 2) with the property

$$(2.11) \qquad \Sigma_\varepsilon^l \;=\; \{x \in \Sigma_\varepsilon \,:\, \mathrm{dist}(x, \partial\Omega) = \mathrm{dist}(x, \Sigma_l),\ \mathrm{dist}(x, \Sigma_k) > \mathrm{dist}(x, \Sigma_l)\ \forall k \ne l\}.$$

Because of (2.11), the precondition about the curvature and lemma 14.16 in [5], p. 355, we can choose ε_0 so small that the distance function is twice differentiable on the sets Σ_ε^l for every $0 < \varepsilon \le \varepsilon_0$, since Σ_l can be considered as a part of the boundary of a C^2-domain. With this property we can prove the following lemma (cf. lemma 1.1 in [6], p. 301).

Lemma 2.2: *Let Ω be a domain satisfying the preconditions of theorem 2.1, let $w \in BV_{loc}(Q)$ be an essentialy bounded function on Q_T, then*

$$(2.12) \qquad \int_{\delta Q_T} |w|\, d\mathcal{H}_2 \;\le\; \int_{\Sigma_\varepsilon \times (-T,T)} |Dw| + c_1 \int_{\Sigma_\varepsilon \times [-T,T]} |w|\, dxdt + \left(\sup_{x \in Q_T} |w|\right) R(\varepsilon)$$

where c_1 is a constant that only depends on Q_T and ε. Furthermore, for the quantity R independent of w it holds that $\lim_{\varepsilon \to 0} R(\varepsilon) = 0$.

Proof: Consider at first $w \ge 0$, let η be a C^∞-cutoff-function with $0 \le \eta \le 1$, $\eta = 1$ on δQ_T and $\eta = 0$ on $Q_T \setminus (\Sigma_\varepsilon \times [-T,T])$. With the above choice of ε we have for the distance function $d(x) = \mathrm{dist}(x, \delta Q_T)$ that $d \in C^2(\Sigma_{\varepsilon,T}^l)$ for every $l \in L$ where $\Sigma_{\varepsilon,T}^l = \Sigma_\varepsilon^l \times (-T,T)$.
Thus,

$$(2.13) \qquad \sum_{l=1}^n \int_{\Sigma_{\varepsilon,T}^l} w\mathrm{div}\,(\eta Dd)\, dxdt \;=\; -\sum_{l=1}^n \int_{\Sigma_{\varepsilon,T}^l} \eta Dd Dw + \sum_{l=1}^n \int_{\partial\Sigma_{\varepsilon,T}^l} w\eta\nu \circ Dd\, d\mathcal{H}_2.$$

Since d is globally Lipschitz-continuous with Lipschitz constant 1 (e.g. [5], p. 354), we can deduce

$$\sum_{l=1}^n \int_{\partial\Sigma_{\varepsilon,T}^l} w\eta\nu \circ Dd\, d\mathcal{H}_2 \;\le\; \sum_{l=1}^n \int_{\Sigma_l \times [-T,T]} w\eta\nu \circ Dd\, d\mathcal{H}_2 + \left(\sup_{x \in Q_T} |w|\right) R(\varepsilon)$$

$$(2.14) \qquad \le\; -\int_{\delta Q_T} w\, d\mathcal{H}_2 + \left(\sup_{x \in Q_T} |w|\right) R(\varepsilon).$$

For the last inequality we used that $\nu \circ Dd = -1$ on $\Sigma_l \times (-T,T)$ and $\eta = 1$ on δQ_T.
Using $|Dd| \leq 1$ we obtain:

$$-\sum_{l=1}^{n} \int_{\Sigma_{\varepsilon,T}^l} \eta Dd Dw \leq \sum_{l=1}^{n} \int_{\Sigma_{\varepsilon,T}^l} |Dw| \leq \int_{\Sigma_\varepsilon \times (-T,T)} |Dw| \tag{2.15}$$

and

$$-\sum_{l=1}^{n} \int_{\Sigma_{\varepsilon,T}^l} w \operatorname{div}(\eta Dd)\, dxdt \leq c_1(\varepsilon) \int_{\Sigma_\varepsilon \times [-T,T]} w\, dxdt \tag{2.16}$$

with $c_1(\varepsilon) = \sup_l \sup_{\Sigma_{\varepsilon,T}^l} |\operatorname{div}(\eta Dd)| < \infty$ ([5], lemma 14.17, p. 355, ε_0 sufficiently small, $0 < \varepsilon \leq \varepsilon_0$). This quantity obviously depends only on ε and Ω.
From equation (2.13) we obtain our claim in the case $w \geq 0$ by using the inequalities (2.14) to (2.16). The general case follows from the inequality

$$\int_{\Sigma_\varepsilon \times (-T,T)} |D\,|w|| \leq \int_{\Sigma_\varepsilon \times (-T,T)} |Dw| \tag{2.17}$$

(cf. [6], p. 301), which can be shown easily with the help of the Coarea formula. □

With this lemma we are able to prove lower semi-continuity for the functionals $\mathcal{F}_T$.

Lemma 2.3: *Let $U_j \subseteq Q_T$ be a sequence of Caccioppoli sets with $\varphi_{U_j} \to \varphi_U$ in $L^1(Q_T)$ for a Caccioppoli set $U \subseteq Q_T$, then*

$$\mathcal{F}_T(U) \leq \liminf_{j\to\infty} \mathcal{F}_T(U_j). \tag{2.18}$$

Proof: From the structure conditions and the definition of the functionals $\mathcal{F}_T$ we have

$$\begin{aligned}\mathcal{F}_T(U) - \mathcal{F}_T(U_j) \leq & \int_{Q_T} |D\varphi_U| - \int_{Q_T} |D\varphi_{U_j}| + C\int_{Q_T} \left|\varphi_U - \varphi_{U_j}\right| dxdt \\ & + \int_{\delta Q_T} \left|\varphi_U - \varphi_{U_j}\right| d\mathcal{H}_2.\end{aligned}$$

The constant c is independent of U, U_j. The application of lemma 2.2 for the function $w = \varphi_U - \varphi_{U_j}$ yields the following estimate

$$\begin{aligned}\mathcal{F}_T(U) - \mathcal{F}_T(U_j) \leq & \int_{Q_T} |D\varphi_U| - \int_{Q_T} |D\varphi_{U_j}| + C\int_{Q_T} \left|\varphi_U - \varphi_{U_j}\right| dxdt \\ & + \int_{\Sigma_\varepsilon \times [-T,T]} |D\varphi_U| + \int_{\Sigma_\varepsilon \times [-T,T]} |D\varphi_{U_j}| \\ & + c_1 \int_{\Sigma_\varepsilon \times [-T,T]} \left|\varphi_U - \varphi_{U_j}\right| dxdt + R(\varepsilon) \\ \leq & \int_{Q_{T,\varepsilon}} |D\varphi_U| - \int_{Q_{T,\varepsilon}} |D\varphi_{U_j}| + C\int_{Q_T} \left|\varphi_U - \varphi_{U_j}\right| dxdt \\ & + c_1 \int_{\Sigma_\varepsilon \times [-T,T]} \left|\varphi_U - \varphi_{U_j}\right| dxdt + 2\int_{\Sigma_\varepsilon \times [-T,T]} |D\varphi_U| + R(\varepsilon)\end{aligned}$$

with the notation $Q_{T,\varepsilon} = Q_T \setminus (\Sigma_\varepsilon \times [-T,T])$.
Using the lower semi-continuity of the integrals $\int |D\varphi_U|$ (e.g. [7], p. 7), we obtain by taking the limit $j \to \infty$:

$$\mathcal{F}_T(U) - \liminf_{j\to\infty} \mathcal{F}_T(U_j) \leq 2\int_{\Sigma_\varepsilon \times [-T,T]} |D\varphi_U| + R(\varepsilon). \tag{2.19}$$

Remark: At first we have lower semi-continuity for the integral $\int |D\varphi_U|$ only on open sets. With the special properties $U_j, U \subseteq Q_T$ we can extend this to our half-closed sets Q_T because of the identities $\int_{Q_T} |D\varphi_U| = \int_{\Omega\times(-T-\varepsilon_1,T+\varepsilon_1)} |D\varphi_U|$ etc. for $\varepsilon_1 > 0$. On the other side, this lemma cannot be applied for subgraphs. Compare this with a later version of lemma 2.3 (see lemma 2.4 in the fourth step)!

From (2.19) we obtain the supposition by taking the limit $\varepsilon \to 0$. □

Now we consider the case that Ω also contains corners with opening angle greater than π. We denote $\{x_{l^*}\} \subseteq \{x_l\}$ as the subset of the corners with opening angle greater than π and L^* the corresponding index subset of L. Let Γ_i^1 and Γ_i^2 $(i \in L^*)$ be the inner normals to the two the corresponding corner forming arcs Σ_i and Σ_j respectively $(i \in L^*,\ j = i+1$ if $i < n$ else $j = 1)$. These curves in combination with the curves Γ_i $(i \in L \setminus L^*)$ yield a decomposition of Σ_ε of the following form (see figure 3) for a sufficiently small ε_0 and $0 < \varepsilon \leq \varepsilon_0$:

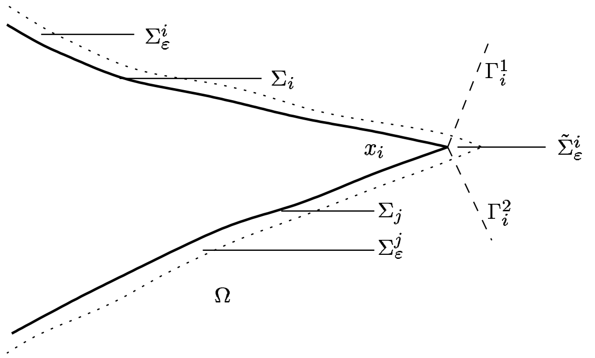

figure 3

$$(2.20)\qquad \begin{aligned} \Sigma_\varepsilon^i &= \Big\{x \in \Sigma_\varepsilon\ :\ \mathrm{dist}(x,\partial\Omega) = \mathrm{dist}(x,\Sigma_i),\ \mathrm{dist}(x,\Sigma_k) > \mathrm{dist}(x,\Sigma_i), \\ &\qquad \mathrm{dist}(x,x_l) > \mathrm{dist}(x,\Sigma_i)\ \forall k,l \in L\,,k \neq i\Big\} \\ \tilde\Sigma_\varepsilon^j &= \Big\{x \in \Sigma_\varepsilon\ :\ \mathrm{dist}(x,\partial\Omega) = \mathrm{dist}(x,x_j),\ \mathrm{dist}(x,x_j) < \mathrm{dist}(x,x_k) \\ &\qquad \forall k \in L\setminus\{j\}\Big\} \end{aligned}$$

$(i \in L,\ j \in L^*)$ with the property

$$\Sigma_\varepsilon = \left(\bigcup_{i\in L}\Sigma_\varepsilon^i\right) \cup \left(\bigcup_{i\in L\setminus L^*}\Gamma_i\cap\Sigma_\varepsilon\right) \cup \left(\bigcup_{i\in L^*}\tilde\Sigma_\varepsilon^i\right).$$

From the property (2.20) can be concluded as above that we can choose ε_0 so small that the distance function has bounded second derivatives on the sets Σ_ε^i for every $0 < \varepsilon \leq \varepsilon_0$ and $i \in L$. But on the sets $\tilde\Sigma_\varepsilon^i$ $(i \in L^*)$ the distance function has no bounded second derivatives. On the other side, an integration on these sets gives no contributions to the boundary of Ω and therefore, we can neglected them. More precisly, we substitute the set Σ_ε^*, defined as

$$(2.21)\qquad \Sigma_\varepsilon^* = \Sigma_\varepsilon \setminus \left(\bigcup_{i\in L^*}\tilde\Sigma_\varepsilon^i\right),$$

for the set Σ_ε in the previous proofs, exept for the definition of η in the proof of lemma 2.2. Analogously, we obtain an estimate of the kind of lemma 2.2 and the required lower semi-continuity of the $\mathcal{F}_T$ also in this case.

Step 2: (Construction of a minimizing sequence on Q_T)
We approximate Ω with smooth domains as below. Let Ω_n ($n \in \mathbb{N}$) be a sequence of C^2-domains so that

$$\Omega_i \subset \Omega_j \qquad \forall i,j \quad i<j$$

$$\lim_{n\to\infty} \mathcal{H}_1(\Sigma_n) = 0 \tag{2.22}$$

$$\lim_{n\to\infty} \mathcal{H}_1(\Gamma_n) = 0 \tag{2.23}$$

$$\lim_{n\to\infty} \mathcal{H}_2(\Omega_n) = \mathcal{H}_2(\Omega) \tag{2.24}$$

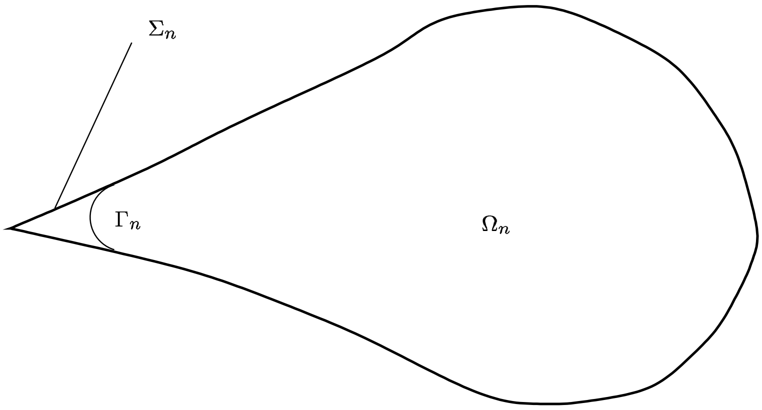

figure 4

with the notations $\Sigma_n = \partial\Omega \setminus \partial\Omega_n$ and $\Gamma_n = \partial\Omega_n \cap \Omega$ (figure 4).
We extend β to $\beta = 0$ on Γ_n. It follows from [6], p. 303 that there is a solution u_i^n on Ω_n of the functional $\mathcal{E}$ in the sets

$$V_i(\Omega_n) = \{v \in BV(\Omega_n) : |v| \le i\} \qquad i \in \mathbb{N}^+, \tag{2.25}$$

which subgraph U_i^n minimizes the functionals $\mathcal{F}_T^n$ with the definition

$$\mathcal{F}_T^n[A] = \int_{Q_T^n=\Omega_n\times[-T,T]} |D\varphi_A| + \int_{Q_T^n} 2H\varphi_A \, dxdt - \int_{\delta Q_T^n=\partial\Omega_n\times[-T,T]} \beta\varphi_A \, dsdt$$

for every $0 < T \le i$. Since if a set minimizes $\mathcal{F}_i^n$, then it obviously minimizes $\mathcal{F}_j^n$ with $0 < j \le i$.
From the set of solutions u_i^n we choose a sequence $\{u_i^n\}_{n\in\mathbb{N}}$ for a fixed i. We would like to show that we can choose a subsequence from it converging locally (that is on Q_i) to a function $u_i \in V_i(\Omega)$ and the corresponding subgraph U_i minimizes $\mathcal{F}_i$.
Obviously, we have

$$\int_{Q_i} \varphi_{U_i^n \cap Q_i^n} \le \mathcal{H}_3(Q_i) < \infty \qquad \forall n \in \mathbb{N}. \tag{2.26}$$

Since U_i^n is a solution on Ω_n for $\mathcal{F}_i^n$ we have:

$$\begin{aligned}
\mathcal{F}_i^n[U_i^n] &\le \mathcal{F}_i^n[U_i^n \setminus Q_i^n] \\
\Rightarrow \int_{Q_i^n} |D\varphi_{U_i^n}| &\le |\Omega_n| + 2\int_{Q_i^n} |H| \, dxdt + \int_{\delta Q_i^n} |\beta| \, dsdt \\
\Rightarrow \int_{Q_i^n} |D\varphi_{U_i^n \cap Q_i^n}| &\le 2|\Omega_n| + 2\int_{Q_i^n} |H| \, dxdt + \int_{\delta Q_i^n} |\beta| \, dsdt.
\end{aligned}$$

From the construction of β we obtain:

$$(2.27) \qquad \int_{Q_i^n} |D\varphi_{U_i^n \cap Q_i^n}| \;\leq\; 2|\Omega| + 2\int_{Q_i} |H|\, dxdt + \int_{\delta Q_i} |\beta|\, dsdt.$$

Thus, by using (2.26) and (2.27), we see that $\varphi_{U_i^n \cap Q_i^n}$ are uniformly bounded in BV-norm. With the help of the compactness theorem for BV-functions (e.g. [1], p. 176) we can choose a subsequence from $\{\varphi_{U_i^n \cap Q_i^n}\}_{n\in\mathbb{N}}$, which converges in $L^1(Q_i)$ to a BV-function f. This function is again a characteristic function of the intersection of a subgraph U_i of a function $u_i \in V_i$ with Q_i. We set $f = \varphi_{U_i \cap Q_i}$ and denote the upper subsequence again with $\{\varphi_{U_i^n \cap Q_i^n}\}_{n\in\mathbb{N}}$ for convenience.
We still have to show that U_i minimizes $\mathcal{F}_i$.
Let ε be a positive number, with the same reason as above and the special structure of the involved sets, it is clear that we have $\varphi_{U_i^n \cap Q_{i+\varepsilon}^n} \to \varphi_{U_i \cap Q_{i+\varepsilon}}$ in $L_1(Q_{i+\varepsilon})$. The application of lemma 2.3 for the functional $\mathcal{F}_{i+\varepsilon}$ yields:

$$(2.28) \qquad \mathcal{F}_{i+\varepsilon}[U_i \cap Q_{i+\varepsilon}] \;\leq\; \liminf_{n\to\infty} \mathcal{F}_{i+\varepsilon}[W_i^n \cap Q_{i+\varepsilon}^n]$$

where W_i^n denote the subgraphs of the functions

$$w_i^n \;=\; \begin{cases} u_i^n & : \; x \in \Omega_n \\ -\infty & : \; x \in \Omega \setminus \Omega_n \end{cases} .$$

With the equations

$$\begin{aligned} \mathcal{F}_{i+\varepsilon}[U_i \cap Q_{i+\varepsilon}] &= \mathcal{F}_i[U_i] + |\Omega| + 2\int_{\Omega\times[-i-\varepsilon,-i)} H\, dxdt - \int_{\partial\Omega\times[-i-\varepsilon,i)} \beta\, dsdt \\ \mathcal{F}_{i+\varepsilon}[W_i^n \cap Q_{i+\varepsilon}^n] &= \mathcal{F}_i[W_i^n] + |\Omega_n| + \varepsilon|\Gamma_n| + 2\int_{\Omega_n\times[-i-\varepsilon,-i)} H\, dxdt \\ &\quad - \int_{\partial\Omega_n\times[-i-\varepsilon,i)} \beta\, dsdt \end{aligned}$$

we obtain from inequality (2.28) and the conditions (2.22) to (2.24):

$$(2.29) \qquad \mathcal{F}_i[U_i] \;\leq\; \liminf_{n\to\infty} \mathcal{F}_i[W_i^n].$$

Let S be an arbitrary Caccioppoli set coinciding with U outside of Q_i, then we have because of the inner regularity of the integrals $\int |D\varphi_S|$ and the properties (2.22) to (2.24):

$$(2.30) \qquad \mathcal{F}_i[S] \;=\; \mathcal{F}_i^n[S \cap Q^n] + R_1(n) \qquad\qquad Q^n = \Omega_n \times \mathbb{R}.$$

For the quantity R_1 it holds $\lim_{n\to\infty} R_1(n) = 0$. On the other side, the set $S \cap Q^n$ is an arbitrary Caccioppoli-set that, by construction, coincides with U_i^n outside of Q_i^n. Using the minimization property of u_i^n it follows:

$$(2.31) \qquad \mathcal{F}_i^n[W_i^n] \;\leq\; \mathcal{F}_i^n[S \cap Q^n].$$

Further,

$$(2.32) \qquad \mathcal{F}_i^n[W_i^n] \;=\; \mathcal{F}_i[W_i^n] + R_2(n)$$

with $\lim_{n\to\infty} R_2(n) = 0$.
From (2.29) to (2.32) we can conclude:

$$(2.33) \qquad \mathcal{F}_i[U_i] \;\leq\; \mathcal{F}_i[S].$$

That means U_i minimizes $\mathcal{F}_i$.

Step 3: (Construction of a generalized solution)

Let $\{u_i\}_{i\in\mathbb{N}^+}$ be a sequence of the above solutions with the corresponding subgraphs U_i then we have as above:

$$\int_{Q_T} |D\varphi_{U_i}| \;\leq\; C \qquad \forall\, i > T \tag{2.34}$$

where C only depends on T. With the same argument as in the second step we can choose a subsequence from $\{u_i\}_{i\in\mathbb{N}^+}$ that converges on Q_T to a subgraph of a generalized function. We choose such a sequence for $T = 1$ and from this a subsequence for $T = 2$ and so on. The diagonal sequence, chosen from this sequence of subsequences, denoted again with $\{u_i\}_{i\in\mathbb{N}^+}$, has the property that it converges on every set Q_T to the intersection of a subgraph U with Q_T. The corresponding function u is a generalized function, which can possibly be $\pm\infty$. The subgraph U is a Borel set as the unification of the Borel sets $U \cap Q_T$ and a Caccioppoli set because of (2.34) and the semi-continuity of the measures $\int_\cdot |D\varphi_U|$.
We have still to show that u is a local solution for $\mathcal{F}$. This follows from lemma 2.1 by setting $H_j = H$ und $\beta_j = \beta$. The conditions (2.7) to (2.9) are fulfilled for almost every T, which follows from Lebesgue's theorem (e.g. [7], p. 30) and proposition 2.8 in [7], p. 36. Using that minimizers of $\mathcal{F}_T$ are also minimizers of $\mathcal{F}_{T'}$ for $0 < T' \leq T$ we obtain our supposition.

Step 4: (Proof of lemma 2.1)
We will need a further technical lemma. Compare this with lemma 2.3!

Lemma 2.4: *Let $U_j \subseteq Q$ be a sequence of Caccioppoli sets with $\varphi_{U_j} \to \varphi_U$ in $L^1_{loc}(Q)$ for a Caccioppoli set $U \subseteq Q$, assume that (2.8) and (2.9) hold, then:*

$$\mathcal{F}_T(U) \;\leq\; \liminf_{j\to\infty} \mathcal{F}_T(U_j).$$

Proof: The proof is analogous to that of lemma 2.3. Because of the conditions (2.8) and (2.9) we have relations of the form $\int_{Q_T} |D\varphi_U| = \int_{\Omega\times(-T,T)} |D\varphi_U|$ and so on. This makes it possible to apply the lower semi-continuity of the measures $\int_\cdot |D\varphi_U|$ on open sets. □

Now we can prove lemma 2.1. We first start as GIUSTI did in [6].
Let $V \subseteq Q$, $V = U$ outside of Q_T be a Caccioppoli set and let

$$V_j \;=\; \begin{cases} V & \text{in } Q_T \\ U_j & \text{outside of } Q_T \end{cases},$$

then V_j is an admissible comparison set to U_j and we have because of the minimization property of U_j:

$$\mathcal{F}^j_T(U_j) \;\leq\; \mathcal{F}^j_T(V_j). \tag{2.35}$$

Setting $Q^\circ_T = \Omega \times (-T,T)$, using that V_j coincides with V in Q_T and proposition 2.8 in [7], p. 36 respectively we obtain:

$$\begin{aligned}
\int_{Q_T} |D\varphi_{V_j}| &= \int_{Q^\circ_T} |D\varphi_{V_j}| + \int_{\tilde\partial Q_T} |D\varphi_{V_j}| \\
&= \int_{Q_T} |D\varphi_V| + \int_{\tilde\partial Q_T} |D\varphi_{V_j}| - \int_{\tilde\partial Q_T} |D\varphi_V| \\
&\leq \int_{Q_T} |D\varphi_V| + \int_{\tilde\partial Q_T} |\varphi_{U_j} - \varphi_U|\, d\mathcal{H}_2
\end{aligned}$$

$$\mathcal{F}^j_T(V_j) \;\leq\; \mathcal{F}^j_T(V) + \int_{\tilde\partial Q_T} |\varphi_{U_j} - \varphi_U|\, d\mathcal{H}_2. \tag{2.36}$$

From (2.35) and (2.36) follows that:

$$\begin{aligned}\mathcal{F}_T(U_j) + 2\int_{Q_T} (H_j - H)\,\varphi_{U_j} + \int_{\delta Q_T} (\beta - \beta_j)\,\varphi_{U_j} \\ \leq\ \mathcal{F}_T^j(V) + \int_{\tilde{\delta} Q_T} \left|\varphi_{U_j} - \varphi_U\right|\, d\mathcal{H}_2.\end{aligned}$$

Taking the lower limit $j \to \infty$ we obtain with the help of lemma 2.4, the uniform convergence of H_j, β_j and the precondition (2.7)

$$\mathcal{F}_T(U) \quad \leq \quad \mathcal{F}_T(V).$$

That is the supposition, since V is an arbitrary admissible comparison set to U. □

3. Uniqueness

Of course uniqueness of generalized solutions is of interest. But we have only the weak statement that it is not in the general case.

Lemma 3.1: *Assume that H and β are independent of t and assume that (2.1) is satisfied, then a generalized solution is determind at most up to an additive constant.*

Proof: Let u be a generalized solution then we have with lemma 1.2

$$\mathcal{F}_{[a,b]}[U] \quad \leq \quad \mathcal{F}_{[a,b]}[S]$$

for every Caccioppoli set S coinciding with U outside of $Q_{[a,b]}$ and every interval $[a,b]$ with $b > a$. Choose $\tilde{u} = u + C$ $(C \in \mathbb{R})$ and $\tilde{U}$ the corresponding subgraph, it follows by transforming the coordinates

$$\mathcal{F}_{[a+C,b+C]}[\tilde{U}] \quad \leq \quad \mathcal{F}_{[a+C,b+C]}[\tilde{S}]$$

for every Caccioppoli set $\tilde{S}$ coinciding with $\tilde{U}$ outside of $Q_{[a+C,b+C]}$ and every interval $[a+C, b+C]$ with $b > a$.
The special choice $a = -T - C$, $b = T - C$ with $T > 0$ yields the supposition. □

Nevertheless, it keeps unanswered what varieties of solutions are possible or if there is uniqueness in special cases, even if H and β are constant as in the following chapter.

4. The Special Case H, β Constant

In the following, Ω is a domain satisfying the preconditions of theorem 2.1 and H, β are constant ($\beta = \cos\gamma$, $H \geq 0$ without restriction of generality). For γ we assume $0 \leq \gamma < \frac{\pi}{2}$. From lemma 1.1 we have that there is no smooth solution of (1.1), (1.2), if Ω contains cusps but there are generalized solutions, which we want to discuss in this chapter.
We first characterize the sets where a generalized solution could become plus or minus infinity. The geometry of these sets will be very restricted.
Let $u(x)$ be a generalized solution, we define

$$\begin{aligned} P &= \{x \in \Omega \quad : \quad u(x) = +\infty\} \\ N &= \{x \in \Omega \quad : \quad u(x) = -\infty\}. \end{aligned}$$

We can change $u(x)$ on a set of measure 0 so that for every circle $B_r(x)$ it holds

$$\mathcal{H}_2\,(P \cap B_r(x)) \neq 0 \qquad \forall\, x \in P \tag{4.1}$$

$$\mathcal{H}_2\,(N \cap B_r(x)) \neq 0 \qquad \forall\, x \in N. \tag{4.2}$$

Theorem 4.1: *The set P minimizes the functional*

$$(4.3)\qquad \Phi[\Omega^*; H, \gamma, \Omega] \;=\; |\Gamma_{\Omega^*}| - |\Sigma_{\Omega^*}| \cos\gamma + 2H|\Omega^*|$$

where $\Omega^* \subseteq \Omega$ is a Caccioppoli set. The set N minimizes the functional

$$(4.4)\qquad \Psi[\Omega^*; H, \gamma, \Omega] \;=\; |\Gamma_{\Omega^*}| + |\Sigma_{\Omega^*}| \cos\gamma - 2H|\Omega^*|.$$

Remarks: For convenience we use $\Phi[\Omega^*; H, \gamma, \Omega] = \Phi[\Omega^*]$ and so on if it is clear to what quantities H, γ, Ω we refer.
An analogous version of this theorem can be found in [6], p. 306 for domains with smooth boundary. The theorem has been already proven in [2], p. 196 for global minimizers of $\mathcal{F}$. Now we want to prove it also for our local minimizers. Therefore, we adapt the idea in [2].

Proof: Let u be a generalized solution of (1.1), (1.2), then the functions $u - j$ ($j \in \mathbb{N}$) are also solutions (lemma 3.1) that is the subgraphs

$$U_j \;=\; \{(x,t) \in Q \,:\, t < u(x) - j\}$$

minimize $\mathcal{F}_T$ for every $T > 0$. We have $U_j \to U = P \times \mathbb{R}$ in $L^1_{loc}(Q)$. Furthermore, the conditions (2.7) to (2.9) are satisfied for almost every T with the same reason as in the proof of theorem 2.1 (step 3). Using lemma 2.1 we obtain that U is also a local solution.
Suppose that there is a Caccioppoli set $A \subseteq \Omega$ with $\Phi[A] < \Phi[P]$. We set $\Phi[P] - \Phi[A] = \varepsilon > 0$ and choose T sufficiently large so that $\varepsilon T > |\Omega|$ holds. Taking the admissible comparison set $S = (A \times [-T, T]) \cup (U \setminus Q_T)$ into consideration it follows, because of $\mathcal{F}_T[U] = \int_{-T}^{T} \Phi[P] = 2T\Phi[P]$, that:

$$\begin{aligned}
\mathcal{F}_T[S] &\leq \int_{-T}^{T} \Phi[A]\, dt + 2|\Omega| = 2T\Phi[A] + 2|\Omega| \\
&\leq 2T\,(-\varepsilon + \Phi[P]) + 2|\Omega| \\
&\leq \mathcal{F}_T[U] + 2\,(|\Omega| - \varepsilon T) \\
&< \mathcal{F}_T[U],
\end{aligned}$$

which means a contradiction to the minimization property of U.
The result for N follows analogously by using the property that if M is a minimizer of Φ, then $\Omega \setminus M$ is a minimizer of Ψ and vice versa. This property can easily be verified by direct calculation and motivates the name "adjoint functionals" for Φ and Ψ. □

The proof of theorem 4.1 yields a simple example for a generalized solution.

Lemma 4.1: *Let M be a minimizer of Φ, then*

$$(4.5)\qquad u(x) \;=\; \begin{cases} +\infty & : \; x \in M \\ -\infty & : \; x \in \Omega \setminus M \end{cases}$$

is a generalized solution of (1.1), (1.2).

Proof: Let U be the subgraph of u, $S = S_0 \cup (U \setminus Q_T)$ an arbitrary admissible comparison set for u, that is S_0 is an arbitrary Caccioppoli set in Q_T. We have to show that

$$(4.6)\qquad \mathcal{F}_T[U] \;\leq\; \mathcal{F}_T[S].$$

We set

$$(4.7)\qquad S^t \;=\; \{x \in \Omega \,:\, (x,t) \in S_0\}\,.$$

From the proof of lemma 9.8 in [7], p. 112 it follows that S^t is a Caccioppoli set in $\mathbb{R}^2$ for almost every $t \in [-T,T]$.
On the basis of the precondition that M is a minimizer of Φ, we have

$$(4.8) \qquad \Phi[M] \;\leq\; \Phi\left[S^t\right] \qquad \text{for almost every } t \in [-T,T].$$

We set $Q_T^\circ = \Omega \times (-T,T)$ and $\delta Q_T^\circ = \partial\Omega \times (-T,T)$. By integration of (4.8) and applying lemma 9.8 ([7], p. 112) and $\cos\gamma \geq 0$ respectively, we can deduce:

$$\int_{Q_T^\circ} |D\varphi_U| + 2H \int_{Q_T^\circ} \varphi_U \, dxdt - \cos\gamma \int_{\delta Q_T^\circ} \varphi_U \, dsdt \;\leq\; \int_{-T}^{T} \Phi\left[S^t\right] dt$$

$$(4.9) \qquad \leq\; \int_{Q_T^\circ} |D\varphi_{S_0}| + 2H \int_{Q_T^\circ} \varphi_{S_0} \, dxdt - \cos\gamma \int_{\delta Q_T^\circ} \varphi_{S_0} \, dsdt.$$

If we now go over again to the half-closed sets Q_T, we have because of the special structure of U

$$(4.10) \qquad \int_{Q_T^\circ} |D\varphi_U| \;=\; \int_{Q_T} |D\varphi_U| \, ,$$

furthermore,

$$(4.11) \qquad \int_{Q_T^\circ} |D\varphi_S| \;\leq\; \int_{Q_T} |D\varphi_S| \, .$$

From (4.9), (4.10) and (4.11) we obtain (4.6), that is the supposition. □

Lemma 4.1 is an interesting analogon to an example of generalized solutions for the minimal-surface problem discussed in [7], p. 183.
Assuming that there exists a smooth solution for (1.1), (1.2) (these solutions are determined up to an additive constant as a consequence of the comparison theorem for the capillary equation (e.g. [2], p. 135)), then the "limit solutions" $u \equiv +\infty$ and $u \equiv -\infty$ are also generalized solutions because $\emptyset$ and Ω both are minimizers of Φ (see chapter 1).
For domains with cusps it follows from chapter 1 that always $P \neq \emptyset$.
Now we want to discuss the structure of the sets M (without restriction of generality it is assumed that M is an open set satisfying a condition of the form (4.1)), which denotes a minimizer of Φ in the following. These sets have a very restricted geometry discovered by FINN. We will write these properties down for self consistence. Suppose that $\Gamma_M = \partial M \cap \Omega \neq \emptyset$ we have:

Theorem 4.2 (cf. [2], chapter 6)**:** *The set Γ_M consists of arcs of radius $R = \frac{1}{2H}$, the set M is opposite to that side, into which the curvature vector points. If an arc meets a smooth part of $\partial\Omega$, then it forms an angle γ with $\partial\Omega$. If $\gamma > 0$, then the arcs are pairwise isolated.*

There are further restrictions, e.g. that Γ_M cannot end in corners with opening angle $0 < \alpha < \pi$ as proved in [4]. This result can be easily extended also to outward cusps that is $\alpha = 0$.
With these results the set M can nearly be guessed for a given set Ω. This is important for the calculation of a special class of generalized solutions, which could be of physical interest. We will discuss them now.
Let $M \neq \Omega$ be a minimizer of Φ. FINN observed in his paper [4] that under special preconditions there is a smooth solution of the problem

$$(4.12) \qquad \operatorname{div} Tu \;=\; 2H \qquad \text{in } \Omega_0 = \Omega \setminus \bar{M}$$

$$(4.13) \qquad \nu \circ Tu \;=\; \cos\gamma \qquad \text{on smooth components of } \partial\Omega_0 \cap \partial\Omega$$

$$(4.14) \qquad \nu \circ Tu \;=\; 1 \qquad \text{on } \partial\Omega_0 \cap \Omega.$$

These solutions are called *C-singular solutions.* Note that from a theorem of FINN it follows immediately that $\lim_{x\in\Omega_0\to x_0\in\Gamma_M} u(x) = +\infty$ ([3], p. 561). The next theorem connects the C-singular solutions with our generalized solutions.

Theorem 4.3: *Let $M \neq \Omega$ be a minimizer of Φ, assume that there is a C-singular solution u_0 on $\Omega_0 = \Omega \setminus \bar{M}$, then the function*

$$(4.15) \qquad u \;=\; \begin{cases} u_0(x) & : \; x \in \Omega_0 \\ +\infty & : \; x \in \bar{M} \end{cases}$$

is a generalized solution of (1.1), (1.2) with $P = \bar{M}$.

Proof: Let $T > 0$ be an arbitrary but fixed number, we set

$$\begin{array}{rclrcl}
Q^1 & = & \Omega_0 \times \mathbb{R} & Q^2 & = & M \times \mathbb{R} \\
Q^1_T & = & \Omega_0 \times [-T,T] & Q^2_T & = & M \times [-T,T] \\
\delta Q^1 & = & \partial\Omega_0 \times \mathbb{R} & \delta Q^2 & = & \partial M \times \mathbb{R} \\
\delta Q^1_T & = & \partial\Omega_0 \times [-T,T] & \delta Q^2_T & = & \partial M \times [-T,T] \\
\beta^1(x) & = & \begin{cases} \cos\gamma & : \; x \in \partial\Omega_0 \cap \partial\Omega \\ 1 & : \; x \in \Gamma_M \end{cases} & \beta^2(x) & = & \begin{cases} \cos\gamma & : \; x \in \partial M \cap \partial\Omega \\ -1 & : \; x \in \Gamma_M \end{cases}
\end{array}$$

$$\begin{array}{rcl}
\mathcal{F}^1[A] & = & \displaystyle\int_{Q^1} |D\varphi_A| + 2\int_{Q^1} H\varphi_A \, dxdt - \int_{\delta Q^1} \beta^1 \varphi_A \, dsdt \\
\mathcal{F}^1_T[A] & = & \displaystyle\int_{Q^1_T} |D\varphi_A| + 2\int_{Q^1_T} H\varphi_A \, dxdt - \int_{\delta Q^1_T} \beta^1 \varphi_A \, dsdt \\
\mathcal{F}^2[A] & = & \displaystyle\int_{Q^2} |D\varphi_A| + 2\int_{Q^2} H\varphi_A \, dxdt - \int_{\delta Q^2} \beta^2 \varphi_A \, dsdt \\
\mathcal{F}^2_T[A] & = & \displaystyle\int_{Q^2_T} |D\varphi_A| + 2\int_{Q^2_T} H\varphi_A \, dxdt - \int_{\delta Q^2_T} \beta^2 \varphi_A \, dsdt
\end{array}$$

for a Caccioppoli set A from Q^1 and Q^2 respectively.
We denote U as the subgraph of u and S an arbitrary admissible comparison set to U with respect to T. We set once again

$$\begin{array}{rclrcl}
U^1 & = & U \cap Q^1 & S^1 & = & S \cap Q^1 \\
U^2 & = & U \cap Q^2 & S^2 & = & S \cap Q^2.
\end{array}$$

Then U^1 is the subgraph of u_0 and S^1 is an admissible comparison set to U^1. Since u_0 is a smooth solution and consequently a generalized solution, the set U^1 minimizes $\mathcal{F}^1$. That means

$$(4.16) \qquad \mathcal{F}^1_T[U^1] \;\leq\; \mathcal{F}^1_T[S^1].$$

The set U^2 is a solution for $\mathcal{F}$ (lemma 4.1), S^2 is an admissible comparison set for U^2. Thus,

$$(4.17) \qquad \mathcal{F}_T[U^2] \;\leq\; \mathcal{F}_T[S^2].$$

The special structure of U^2, S^2 as subsets of Q^2 in combination with our setting of $\mathcal{F}^2_T$ yields

$$\begin{array}{lrcl}
(4.18) & \mathcal{F}_T[U^2] & = & \mathcal{F}^2_T[U^2] \\
(4.19) & \mathcal{F}_T[S^2] & = & \mathcal{F}^2_T[S^2].
\end{array}$$

From (4.17) to (4.19) we obtain

$$(4.20) \qquad \mathcal{F}^2_T[U^2] \;\leq\; \mathcal{F}^2_T[S^2].$$

We add (4.16) and (4.20)

$$(4.21) \qquad \mathcal{F}_T^1[U^1] + \mathcal{F}_T^2[U^2] \;\; \leq \;\; \mathcal{F}_T^1[S^1] + \mathcal{F}_T^2[S^2].$$

In view of the fact that $\lim_{x\in\Omega_0\to x_0\in\Gamma_M} u_0(x) = +\infty$, the inner and outer trace of φ_U on $\Gamma_M \times [-T,T]$ are equal, thus,

$$(4.22) \qquad \mathcal{F}_T^1[U^1] + \mathcal{F}_T^2[U^2] \;\; = \;\; \mathcal{F}_T[U].$$

For the boundary integrals on $\Gamma_M \times [-T,T]$ of the functionals $\mathcal{F}_T^1$, $\mathcal{F}_T^2$ with respect to the sets S^1 and S^2 it follows with proposition 2.8 in [7], p. 36 that

$$(4.23) \qquad \int_{\Gamma_M\times[-T,T]} (\varphi_{S^2} - \varphi_{S^1})\; dsdt \;\; \leq \;\; \int_{\Gamma_M\times[-T,T]} |D\varphi_S|$$

and with it

$$(4.24) \qquad \mathcal{F}_T^1[S^1] + \mathcal{F}_T^2[S^2] \;\; \leq \;\; \mathcal{F}_T[S].$$

From (4.21), (4.22) und (4.24) we obtain

$$(4.25) \qquad \mathcal{F}_T[U] \;\; \leq \;\; \mathcal{F}_T[S],$$

that is the supposition. □

Remark: There is already a hint in [4] to the functions (4.15). Here we have shown that these functions provide really generalized solutions in the sense of MIRANDA.

We cannot expect that for every H and minimizer M of $\Phi[\Omega^*; H, \gamma, \Omega]$ there is a C-singular solution on Ω_0. Integrating (4.12) on Ω_0 we obtain:

$$(4.26) \qquad \Psi[\Omega_0; H, \gamma, \Omega] \;\; = \;\; 0.$$

We can deduce immediately

$$H \;\; > \;\; 0.$$

Since M is a minimizer of Φ and Ω_0 is a minimizer of Ψ, it follows from (4.26) by comparing Ω_0 and Ω:

$$H \;\; \leq \;\; H_0 = \frac{|\partial\Omega| \cos\gamma}{2|\Omega|},$$

thus,

$$(4.27) \qquad 0 \;\; < \;\; H \;\; \leq \;\; H_0.$$

The problem now is that for a fixed H the minimizers M of Φ are determined but then the condition (4.26) has not necessarily to be satisfied. From [4] it follows that if $\Phi[A; H_0, \gamma, \Omega] \leq 0$ for a Caccioppoli set $A \neq \emptyset, \Omega$ (which is especially fulfilled for domains with cusps) there is at least one H for which a C-singular solution exists on the complement of a minimizer of $\Phi[\Omega^*; H, \gamma, \Omega]$. That means, if there is no smooth solution of (1.1), (1.2) on Ω, there is at least one H, for which a generalized solution of the form (4.15) exists. This H has not to be uniquely defined. The argumentation in [4] works for domains without cusps but can be extended to our configurations by using our existence result theorem 2.1, which, in combination with theorem 4.1, guarantees also the existence of minimizers of Φ.
The next lemma is a pecularity of the solutions of the form (4.15).

Lemma 4.2: *Let $u(x)$ be a generalized solution of (1.1), (1.2) of the form (4.15) for a fixed H, then the function $v(x) \equiv +\infty$ on Ω is also a generalized solution for that H.*

Proof: The set M is a minimizer of $\Phi[\Omega^*; H, \gamma, \Omega]$, therefore,

$$\Phi[M] \leq \Phi[A] \tag{4.28}$$

for every Caccioppoli set $A \subseteq \Omega$.
We subtract equation (4.26) from (4.28), thus,

$$\Phi[\Omega] \leq \Phi[A]$$

for every Caccioppoli set $A \subseteq \Omega$ that means Ω is also a minimizer of $\Phi[\Omega^*; H, \gamma, \Omega]$. The supposition of that lemma now follows from lemma 4.1. □

Because of observations in practise (see e.g. [8] or [14]) the solutions of the form (4.15) could be considered as the physically interesting one. We formulate this as a hypothesis.

Hypothesis: *The solutions of the form (4.15) are the physically interesting generalized solutions of (1.1), (1.2).*

With this hypothesis the equation (4.26) provides some kind of choice condition for determination of an H of physical interest. Note that this setting provides exactly the right H in the case if a smooth solution exists on the whole domain Ω (chapter 1). This method has been already intuitively applied in [8] for some examples but without a proof that (4.15) is really a generalized solution.
We would like to point out a strategy how such examples can be calculated:

Strategy for the calculation of generalized solutions of the type (4.15):

1. For the H with (4.27) we calculate the minimizeres Ω_0 of $\Psi[\Omega^*; H, \gamma, \Omega]$ with the help of the geometric properties (theorem 4.2).
2. We check if equation (4.26) is satisfied for the calculated Ω_0 and the corresponding H. Only these pairs (H, Ω_0) we will continue to consider.
3. We check if there is really a C-singular solution on Ω_0 (this could best be done with the existence-nonexistence principle).
4. We form the function (4.15).

Further examples in [4] show that in general H has not to be uniquely defined. Finally, the author calculated also an example, which could be of interest for the understanding of the metabolism of plants (see [12]).

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