# On Stokes flow with variable and degenerate surface tension coefficient

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**Abstract:** Short-time existence, uniqueness, and regularity results are shown for the moving boundary problem of a free drop of liquid governed by the Stokes equations and driven by surface tension. The value of the surface tension coefficient is variable, not necessarily strictly positive, and transported with the flow on the moving surface.
By a perturbation of identity approach, the problem is transformed into a nonlinear, nonlocal first order degenerate parabolic evolution equation on a fixed reference manifold. Its solvability is proved by deriving a priori estimates and using Galerkin approximations.

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## 1 Introduction and problem formulation

It is the aim of this paper to give short-time existence, uniqueness and smoothness results for the free boundary problem of Stokes flow of a liquid drop driven by surface tension with nonconstant surface tension coefficient $\gamma \geq 0$. More precisely, our problem can be formulated (in dimensionless form) as follows: For given $\Omega(0)$, one looks for a family of bounded domains $\Omega(t) \subset \mathbb{R}^N$ parametrized by time $t \geq 0$ with sufficiently smooth boundaries $\Gamma(t)$ which move with normal velocity $V_n(t)$ in the direction of the outer normal $n(t)$ and for corresponding functions $\widetilde{u}(\cdot,t) \in C^2(\Omega(t),\mathbb{R}^N)$, $\widetilde{p} \in C^1(\Omega(t))$, such that

$$\left.\begin{array}{rcll} -\Delta\widetilde{u}(\cdot,t) + \nabla\widetilde{p}(\cdot,t) &=& 0 & \text{in } \Omega(t), \\ \operatorname{div}\widetilde{u}(\cdot,t) &=& 0 & \text{in } \Omega(t), \\ \left(T(\widetilde{u}(\cdot,t),\widetilde{p}(\cdot,t))n(t)\right)_i &=& \operatorname{div}_{\Gamma(t)}(\widetilde{\gamma}(\cdot,t)\nabla_{\Gamma(t)}x_i(t)) & \text{on } \Gamma(t), \\ \int_{\Omega(t)} \widetilde{u}(\cdot,t)\,dx &=& 0, & \\ \int_{\Omega(t)} \operatorname{rot}\widetilde{u}(\cdot,t)\,dx &=& 0, & \\ V_n(t) &=& \widetilde{u}(\cdot,t)\cdot n(t) & \text{on } \Gamma(t). \end{array}\right\} \quad (1.1)$$

Here, $\Omega(t)$ represents the domain occupied by the liquid drop at time $t \geq 0$, $\widetilde{u}$ and $\widetilde{p}$ represent the velocity and pressure field, $T$ is the deformation tensor given by

$$(T(u,p))_{ij} = \partial_i u_j + \partial_j u_i - p\delta_{ij},$$

and $\widetilde{\gamma}(\cdot,t) \geq 0$ is a scalar function on $\Gamma(t)$ representing the surface tension coefficient. The differential operators $\Delta$, $\nabla$, and div are applied with respect to the spatial coordinates, the operators $\mathrm{div}_{\Gamma(t)}$ and $\nabla_{\Gamma(t)}$ are the divergence and gradient on $\Gamma(t)$ with respect to its Riemannian metric induced from the ambient space, and $x(t) : \Gamma(t) \hookrightarrow \mathbb{R}^N$ denotes the embedding of $\Gamma(t)$ into $\mathbb{R}^N$. It follows from the Green formula for the Stokes equations that the dynamic boundary condition $(1.1)_3$ is of Neumann type (cf. [12]). Solvability of the fixed-time problem $(1.1)_1$–$(1.1)_5$ will be discussed below.

The model of Stokes or "creeping" flow is applied as an approximation of the full Navier-Stokes equations in the case of small Reynolds numbers. Mathematically, a parabolic equation in the interior of the domain is replaced by an elliptic one, i.e. we consider a quasistationary problem. As a consequence, the model reduces to a nonlocal (hyper)surface motion law for the boundary $\Gamma(t)$; in particular, no initial velocity field has to be prescribed.

For a precise formulation of our results, we refer to Theorem 4.3 below and to the remarks after Equation (2.9).

In the existing literature on this problem, most attention has been given to the case $\widetilde{\gamma}(\cdot,t) \equiv \gamma = \text{const} > 0$. For $N = 2$, families of explicit solutions have been constructed by Hopper [9], and general existence results for the case of analytic boundaries have been proved by Antanovskii [1, 2] and Prokert [13]. In [1], an exterior liquid domain (bubble) is discussed, and a case of nonconstant, but strictly positive $\widetilde{\gamma}$ arising from thermocapillarity is also treated. All these result rely crucially on complex analysis methods which are not available in higher dimensions.

For $N = 3$ and in the general case, existence results as well as additional statements on smoothing of the boundary and on equilibria and their stability have been shown in [8] and [18]. In the case $\widetilde{\gamma} \equiv 0$, sources or sinks (or other inhomogeneities) in the interior of the liquid domain have to be included as driving forces in the model. For this problem, existence and uniqueness results have been shown in [16]. In the case $N = 2$, a family of conserved quantities is identified in [4], see also [10].

It is a common property of most of the work on the problem (1.1) that it is reformulated as a nonlinear, nonlocal evolution equation. Depending on the setting, in some cases these evolution equations are seen to be of order one and parabolic for $\gamma > 0$ and hyperbolic for $\widetilde{\gamma} \equiv 0$ (cf. e.g. [16]).

If one considers nonconstant $\widetilde{\gamma}$, problem (1.1) has to be complemented by a description of the evolution of $\widetilde{\gamma}$ on the moving boundary $\Gamma(t)$. As the surface tension coefficient is a local physical property of the interface between the liquid and its environment, it is natural both from a mathematical and a physical point of view to assume that its value is transported along with the liquid particles at the boundary, i.e. to demand

$$\left.\begin{array}{rcll} \frac{d}{dt}\widetilde{\gamma}(X(t,\cdot),t) & = & 0 & \text{on } \Gamma(0), \\ \widetilde{\gamma}(\cdot,0) & = & \widetilde{\gamma}_0 & \text{on } \Gamma(0). \end{array}\right\} \tag{1.2}$$

Here $X(\cdot,t)$ denotes the parametrization of $\Gamma(t)$ by Lagrangian coordinates arising from the motion according to the velocity field $\widetilde{u}$, i.e. $X$ is defined by

$$\dot{X}(t,\xi) \quad = \quad \widetilde{u}(X(t,\xi),t),$$

$$X(0, \xi) \;=\; \xi$$

for $\xi \in \Gamma(0)$, $t \geq 0$. Note that $(1.1)_6$ guarantees that $X(t, \Gamma(0)) = \Gamma(t)$, hence $(1.2)_1$ makes sense.

For example, (1.1), (1.2) can be considered as a model for thermocapillary creeping flow where the heat transport is slow compared to the deformation. Of course, in more precise models one would have to include diffusive behavior of $\widetilde{\gamma}$ or consider $\widetilde{\gamma}$ as dependent on the density of a surfactant which is distributed on the surface. It seems reasonable, however, to consider the simpler situation described by (1.2) as a model problem which already contains the essential aspects and difficulties.

The structure of (1.2) strongly suggests a so-called "perturbation of identity" approach to the moving boundary problem. We will describe the moving boundary by an $\mathbb{R}^N$-valued function $\phi(\cdot, t)$ defined on the boundary of a fixed reference domain such that $\phi(\eta, t)$ represents the Eulerian coordinates at time $t$ of a particle with Lagrangian coordinates $\eta$. This approach is different from the one used in [8, 15, 16], where the deformation of the moving domain is described only by variations of the boundary in normal direction. Our approach avoids the unnatural geometric restrictions on the shape of the moving domain which had to be imposed there. Thus, the results presented here are generalizations of the earlier ones even for $\gamma = \text{const}$. It deserves explicit mentioning, however, that the $\mathbb{R}^N$-valued evolution equation which we will obtain here (Eq. (2.9) below) instead of a scalar one (as in the earlier papers) is degenerate parabolic even for strictly positive $\widetilde{\gamma}$. This is due to the fact that the propagator $\mathcal{F}'(\phi)$ of the evolution equation is degenerate elliptic where the degeneracy occurs in the directions tangential to the boundary while we have elliptic behavior in normal direction as long as $\widetilde{\gamma}$ is positive.

(Let us digress here for a moment to make some remarks about a modification of the model: Assume that $\widetilde{\gamma} = \sigma(\rho)$ is a function of the concentration $\rho$ of a surfactant at the surface $\Gamma(t)$ where the surfactant is transported analogously to (1.2). In this case, $\widetilde{\gamma}$ varies not only due to transport of the surfactant along the boundary but also due to local variations of the surface area. The resulting evolution equation is also degenerate parabolic for $\sigma$ nonnegative and nonincreasing. However, if $\sigma$ is positive and strictly decreasing, the degeneracy occurs only in the direction of divergence free tangential vector fields. In this case, the evolution equation is parabolic for $N = 2$. It is intended to discuss the details of this in a forthcoming paper.)

As in the earlier papers [8, 16], the existence proof will be based on the derivation of a priori estimates in a (standard) scale of $L^2$-type Sobolev spaces $H^r(\Gamma, \mathbb{R}^N)$ on the reference manifold $\Gamma$ (see below) and on Galerkin approximations. This is oriented at the work of Kato and Lai on the Euler equations [11]. No results about parabolic smoothing are obtained because of the degeneracy. Compared to [8] and to [16], some slightly more subtle arguments and calculations are needed in the situation considered here. Due to the degeneracy, no perturbation arguments like "freezing of coefficients" can be applied. On the other hand, the principal part of the propagator is nonlocal which is not the case for $\widetilde{\gamma} \equiv 0$. Technically, this has the following consequences:

1. The calculation of the linearization $\mathcal{F}'(\phi)$ and all corresponding estimates have to be carried out not only at one fixed $\phi = \phi^*$, i.e. around one fixed domain, but for all $\phi$ in a suitable neighborhood. This is the reason for the introduction of the metrics $\boldsymbol{g}$ and $\widetilde{\boldsymbol{g}}$ below which depend on $\phi$ and for the use of covariant tensor calculus.

2. In view of the weak formulation of $(1.1)_1$-$(1.1)_5$, it appears natural to give an a priori estimate for $\mathcal{F}'(\phi)$ first in the space $H^1(\Gamma)$ (cf. [8]). However, the corresponding natural scalar product involves differential operators that depend on $\phi$ which leads to additional difficulties in the existence proof. To avoid this, we use the close relationship of the propagator and the well-known Dirichlet-to-Neumann operator of the Laplacian (with respect to $\boldsymbol{g}$) which has a natural $L^2$-a priori estimate. Furthermore, the use of low norms necessitates estimates in norms with negative index. These are given by duality arguments, using symmetry properties of the occurring operators.

3. One has to use the following fact on the "Neumann-to-Dirichlet operator" for the (homogeneous) Stokes equations which links the normal component of the stress tensor at the boundary to the velocity (see Lemma 3.5 below): In highest order, the dependence is decoupled with respect to tangential and normal components, i.e. tangential components of the Neumann boundary data contribute only in lower order to the normal component of the velocity at the boundary and vice versa. This is seen easily both from a Fourier analysis approach to the halfspace problem (cf. [18]) and from the method of hydrodynamic potentials (cf. [12]), but we have to give a different proof here which also provides uniformity of our estimates with respect to $\boldsymbol{g}$.

Parallel to [8, 15], we obtain a priori estimates in higher norms using a generalized chain rule (Eq. (4.3) below) which is based on the geometric invariance of the problem under reparametrizations. It directly provides the necessary commutator estimates for the occurring nonlocal operators.

Although the propagator $\mathcal{F}'(\phi)[\cdot]$ is a pseudodifferential operator, the calculus of such operators is not used in this paper because of the considerable technical difficulties that arise when one has to treat "quasilinear" pseudodifferential operators with symbols of finite smoothness on manifolds.

# 2 Transformation and evolution equation

The main aims of this section are to construct the nonlocal operator $\mathcal{F}$, to prove mapping properties for it and to show that for appropriate data, (2.9) is equivalent to (1.1), (1.2).

We fix the following notation: Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and $\gamma \geq 0$ a smooth function on $\Gamma := \partial\Omega$. Let $\nu^0$ denote the outer unit normal on $\Gamma$ and $\mathrm{Tr}_\Gamma$ the trace operator from function spaces on $\Omega$ to the corresponding spaces on $\Gamma$.

For $t \in \mathbb{R}$, we denote by $H^t(\Gamma)$ and $H^t(\Gamma, \mathbb{R}^N)$ the usual $L^2$-based Sobolev spaces of order $t$ with values in $\mathbb{R}$ and $\mathbb{R}^N$, respectively. The norms of these spaces will be denoted by $||\cdot||_t^\Gamma$. If $z$ is a function defined on $\Omega$, we will write $||u||_t^\Gamma$ instead of $||\mathrm{Tr}_\Gamma u||_t^\Gamma$. For $t \geq 0$, $H^t(\Omega)$, $H^t(\Omega, \mathbb{R}^N)$, and $||\cdot||_t^\Gamma$ are defined analogously, but, differing from the usual conventions, for $z \in L^2(\Omega)$ we define

$$||z||_{-t}^\Omega := \sup_{||v||_t^\Omega = 1} \left| \int_\Omega zv\, dx \right|,$$

i.e. $||z||_{-t}^\Omega := ||j^* z||_{(H^t(\Omega))'}$ where $j^*$ is the dual of the embedding operator $j:\ H^t(\Omega) \hookrightarrow$

$L^2(\Omega)$. Note that for $t \le -1$, $i = 1, \dots, N$, and $z \in H^1(\Omega)$

$$||\partial_i z||_t^\Omega \le C(||z||_{t+1}^\Omega + ||z||_{t+\frac{1}{2}}^\Gamma) \tag{2.1}$$

because of

$$\begin{aligned} \left| \int_\Omega \partial_i z v \, dx \right| &\le \left| \int_\Omega z \partial_i v \, dx \right| + \left| \int_\Gamma z v \nu_i^0 \, d\Gamma \right| \\ &\le ||z||_{t+1}^\Omega ||\partial_i v||_{-t-1}^\Omega + C||z||_{t+\frac{1}{2}}^\Gamma ||v||_{-t-\frac{1}{2}}^\Gamma \le C(||z||_{t+1}^\Omega + ||z||_{t+\frac{1}{2}}^\Gamma)||v||_{-t}^\Omega. \end{aligned}$$

Concerning pointwise multiplication, the following estimates will be used throughout the paper repeatedly, sometimes without explicit mentioning:

$$||zv||_t^M \le C||z||_t^M ||v||_s^M \tag{2.2}$$

for $|t| \le s$, $s > \frac{m}{2}$, $z \in L^2(M)$, $v \in H^s(M)$,

$$||z_1 z_2 \dots z_n||_t^M \le C \prod_{i=1}^n ||z_i||_{s_i}^M \tag{2.3}$$

for $0 \le t \le s_i$, $t < \sum_{i=1}^n s_i - (n-1)\frac{m}{2}$, $z_i \in H^{s_i}(M)$, where $M$ is $\Omega$ or $\Gamma$, $m = \dim M$. In the case $M = \mathbb{R}^k$, $n = 2$, (2.3) is proved in [17], Ex. 1.11, the generalization to our situations is straightforward. Note that for $t \ge 0$, (2.2) is a special case of (2.3), for $t < 0$, (2.2) follows by duality.

Moreover, we introduce a right inverse $\mathcal{E}$ of $\mathrm{Tr}_\Gamma$ by $\mathcal{E}h := w$, $h \in H^{\frac{1}{2}}(\Gamma)$, where $w$ solves

$$\left.\begin{aligned} \Delta w &= 0 && \text{in } \Omega, \\ w &= h && \text{on } \Gamma. \end{aligned}\right\}$$

We start by gathering some estimates for $\mathcal{E}h$ and the boundary values of its derivatives. The emphasis is on norms with low index where trace theorems are not available and elliptic estimates have to be obtained by duality arguments. (The existence of the occurring traces is guaranteed as we consider only functions of sufficient smoothness.)

**Lemma 2.1** *(Estimates on $\mathcal{E}$)*
*For any $t \in \mathbb{R}$, there is a constant $C > 0$ such that*

$$||\mathcal{E}h||_{t+\frac{1}{2}}^\Omega + ||\nabla \mathcal{E}h||_{t-1}^\Gamma + \left\|\nabla^2 \mathcal{E}h\right\|_{t-2}^\Gamma \le C||h||_t^\Gamma$$

*for all $h \in H^t(\Gamma) \cap H^{2+}(\Gamma)$.*

**Proof:**
1. At first we show the estimate for $||\mathcal{E}h||_{t+\frac{1}{2}}^\Omega$. For $t \ge \frac{1}{2}$, this is a standard result from elliptic regularity theory. Suppose $t \le -\frac{1}{2}$. Pick $\psi \in H^{-t-\frac{1}{2}}(\Omega)$ arbitrary and define $v$ to be the solution of the BVP

$$\left.\begin{aligned} \Delta v &= \psi && \text{in } \Omega, \\ v &= 0 && \text{on } \Gamma \end{aligned}\right\}.$$

By standard results,

$$||v||_{-t+\frac{3}{2}}^\Omega \le C||\psi||_{-t-\frac{1}{2}}^\Omega$$

with $C$ independent of $\psi$. So

$$\begin{aligned}\left|\int_\Omega w\psi\,dx\right| &= \left|\int_\Omega w\Delta v\,dx\right| = \left|\int_\Gamma h\partial_{\nu^0} v\,d\Gamma\right| \\ &\le C\|h\|_t^\Gamma \|\partial_{\nu^0} v\|_{-t}^\Gamma \le \|h\|_t^\Gamma \|\psi\|_{-t-\frac{1}{2}}^\Omega .\end{aligned}$$

This proves

$$\|\mathcal{E}h\|_{t+\frac{1}{2}}^\Omega \le C\|h\|_t^\Gamma$$

for $t \le -\frac{1}{2}$, and the same estimate for $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ follows by interpolation.

2. To show the estimate on $\mathrm{Tr}_\Gamma \nabla \mathcal{E}h$, it is sufficient to show

$$\|\partial_{\nu^0} w\|_{-t-1}^\Gamma \le C\|h\|_{-t}^\Gamma \tag{2.4}$$

because the gradient at the boundary can be represented in terms of normal and tangential derivatives. Eq. (2.4) can be proved using a similar argument as before or the fact that $[h \mapsto \partial_{\nu^0} w]$ is given by a pseudodifferential operator of order 1.

3. From the previous results and the well-known identity

$$\partial_n^2 w = -\Delta_\Gamma \mathrm{Tr}_\Gamma w + \mathrm{Tr}_\Gamma \Delta w - \kappa \partial_n w \tag{2.5}$$

where $-\Delta_\Gamma$ is the Laplace-Beltrami operator on $\Gamma$ with respect to the standard metric and $\kappa$ is the mean curvature of $\Gamma$ one obtains the estimate

$$\left\|\partial_n^2 \mathcal{E}h\right\|_{t-2}^\Gamma \le C\|h\|_t^\Gamma .$$

By decomposition with respect to tangential and normal components, the estimate on $\mathrm{Tr}_\Gamma \nabla^2 \mathcal{E}h$ follows from this and another application of the previous results. ∎

Fix $\sigma > \frac{N-1}{2}$ and choose $s \ge \sigma + 4$ integer. (The restriction to integer values for $s$ is just for the sake of simplicity.) Let $\mathcal{U}$ be a small open neighborhood of the identity in $H^{s+1}(\Gamma, \mathbb{R}^N)$ which will be shrunken in the sequel whenever necessary without further mentioning. For $\phi \in \mathcal{U}$, set

$$\Phi := \mathcal{E}(\phi - \mathrm{Id}_\Gamma) + \mathrm{Id}_\Omega$$

(with $\mathcal{E}$ acting componentwise on $\mathbb{R}^N$-valued functions) and note that $\Phi \in C^3(\overline{\Omega}, \mathbb{R}^N) \cap \mathrm{Diff}(\Omega, \Phi(\Omega))$ due to the Sobolev embedding theorems and the smallness of $\mathcal{U}$.

Now set $\Omega_\Phi = \Phi(\Omega)$ and let $e_i$, $i = 1, \dots, N$ denote the standard unit vector fields on $\Omega_\Phi \subset \mathbb{R}^N$. On this domain, consider the augmented BVP

$$\left.\begin{aligned}-\Delta U^i + \partial_i P + \lambda_1^i &= 0 && \text{in } \Omega_\Phi, \\ \partial_i U^i &= 0 && \text{in } \Omega_\Phi, \\ (\partial_i U^j + \partial_j U^i + \lambda_2^{ij})\nu_j &= \mathrm{div}_{\Gamma_\Phi}((\gamma \circ \Phi^{-1})\nabla_{\Gamma_\Phi} x^i) && \text{on } \Gamma_\Phi, \\ \textstyle\int_{\Omega_\Phi} U^i\,dx &= 0, \\ \textstyle\int_{\Omega_\Phi} (\partial_i U^j - \partial_j U^i)\,dx &= 0,\end{aligned}\right\} \tag{2.6}$$

$i, j = 1, \dots, N$, for (sufficiently smooth) functions $U^i$, $P$, and real numbers $\lambda_1^i$, $\lambda_2^{ij}$ with $\lambda_2^{ij} = -\lambda_2^{ji}$, where $\Gamma_\Phi := \partial\Omega_\Phi$, $\mathrm{div}_{\Gamma_\Phi}$ and $\nabla_{\Gamma_\Phi}$ denote the divergence and gradient with respect to the Riemannian metric induced on $\Gamma_\Phi$ from $\mathbb{R}^N$, $x^i$ denotes the $i$-th component of the embedding $\Gamma_\Phi \hookrightarrow \mathbb{R}^N$, and $\nu$ is the outer unit normal on $\Gamma_\Phi$.

Let $\boldsymbol{g}$ denote the pull-back of the standard Riemannian metric from $\Omega_\Phi$ to $\Omega$ by $\Phi$ and $\tilde{\boldsymbol{g}}$ the metric induced by $\boldsymbol{g}$ on $\Gamma$. In cartesian coordinates on $\Omega$ and coordinates from a (regular) local parametrization $(\xi^i)$, $i = 1, \ldots, N$, on $\Gamma$, these metrics have representations

$$g_{ij} = \partial_i \Phi^k \partial_j \Phi^k, \quad \tilde{g}_{\alpha\beta} = \partial_\alpha \xi^i g_{ij} \partial_\beta \xi^j.$$

Furthermore, we set $G = (g_{ij})$, $g = \det G$, $g^{ij} = (G^{-1})_{ij}$, and introduce analogous notation for $\tilde{\boldsymbol{g}}$. Moreover, let

$$a^i_k := \partial_k (\Phi^{-1})^i \circ \Phi = ((D\Phi)^{-1})_{ik}.$$

On $\Omega_\Phi$ we consider the contravariant tensor fields $U := U^i e_i$, $\Lambda_{(1)} := \lambda^i_1 e_i$, and $\Lambda_{(2)} := \lambda^{ij}_2 e_i \otimes e_j$, where $\otimes$ denotes the tensor product. Finally, by $u := \Phi^* U$ and $p := \Phi^* P$ we denote the pull-backs of $U$ and $P$ to $\Omega$ by $\Phi$. The cartesian coordinates of $u$ will be denoted by $u^i$.

**Lemma 2.2** *(Transformation)*
*With the notation introduced above,* (2.6) *is equivalent to*

$$\left.\begin{array}{rcll} -\nabla^k \nabla_k u^i + \nabla^i p + \lambda^k_1 a^i_k & = & 0 & \textit{in } \Omega, \\ \nabla_i u^i & = & 0 & \textit{in } \Omega, \\ (\nabla^i u^j + \nabla^j u^i - g^{ij} p + \lambda^{kl}_2 a^i_k a^j_l) n_j & = & \frac{1}{\sqrt{g}} a^i_k \partial_\alpha (\sqrt{\tilde{g}} \gamma \tilde{g}^{\alpha\beta} \partial_\beta \Phi^k) & \textit{on } \Gamma, \\ \int_\Omega \sqrt{g} \partial_k \Phi^i u^k \, dx & = & 0, & \\ \int_\Omega \sqrt{g} \partial_k \Phi^i \partial_l \Phi^j (\nabla^k u^l - \nabla^l u^k) \, dx & = & 0, & \end{array}\right\} \qquad (2.7)$$

*where* $i, j = 1, \ldots, N$, $\nabla_k$ *denotes the covariant differentiation operator (in cartesian coordinates), and* $n$ *denotes the conormal vector field on* $\Gamma$, *both with respect to* $\boldsymbol{g}$.

In (2.7) and in the sequel, unless otherwise stated, the usual conventions of "index raising" and "index lowering" are applied with respect to $\boldsymbol{g}$, i.e. $\nabla^k := g^{kj} \nabla_j$, $n_i := g_{ij} n^j$ etc.
**Proof:** Without reference to coordinates, (2.6) can be written as

$$\left.\begin{array}{rcll} -\mathrm{Tr}_{12} \sharp_1 \nabla\nabla U + \sharp \nabla P + \Lambda_{(1)} & = & 0 & \text{in } \Omega_\Phi, \\ \mathrm{Tr} \nabla U & = & 0 & \text{in } \Omega_\Phi, \\ \mathrm{Tr}_{23}((\mathrm{Sym}(\sharp \nabla U) - \sharp \mathrm{Id} P + \Lambda_{(2)}) \otimes \flat \nu) & = & w^i e_i & \text{on } \Gamma_\Phi, \\ \int_{\Omega_\Phi} \mathrm{Tr}(U \otimes \flat e_i) \, dx & = & 0, & i = 1, \ldots, N, \\ \int_{\Omega_\Phi} \mathrm{Tr}\mathrm{Tr}_{13}(\mathrm{Antisym}(\sharp \nabla U) \otimes \flat e_i \otimes \flat e_j) \, dx & = & 0, & i, j = 1, \ldots, N, \end{array}\right\}$$

where

$$w^i := \mathrm{div}_{\Gamma_\Phi}((\gamma \circ \Phi^{-1}) \nabla_{\Gamma_\Phi} x^i),$$

$\nabla$, $\sharp$, and $\flat$ denote the operations of covariant differentiation, "index raising" and "index lowering" with respect to the standard metric on $\Omega_\Phi$, and Tr, Sym, and Antisym denote trace, symmetrization and antisymmetrization. (Where necessary, indices at the operators are used to determine to which of the tensor indices the operations refer.)

To the first three equations, $\Phi^*$ is applied while in the last two we use the integral transformation formula

$$\int_{\Omega_\Phi} \varphi \, dx = \int_\Omega \sqrt{g} \Phi^* \varphi \, dx.$$

All occurring operations commute with $\Phi^*$ in the sense that for tensor fields on $\Omega$, the operations $\nabla$, $\sharp$, and $\flat$ are to be understood with respect to $\boldsymbol{g}$ and $\Phi^*\mathrm{div}_{\Gamma_\Phi}\Phi_*$ and $\Phi^*\nabla_{\Gamma_\Phi}\Phi_*$ are the divergence and gradient on $\Gamma$ with respect to $\tilde{\boldsymbol{g}}$. Using this and

$$\Phi^*\nu = n, \quad (\Phi^* e_i)^k = a_i^k, \quad g_{jk} a_i^k = \partial_j \Phi^i,$$

one straightforwardly obtains (2.7). ■

All geometric quantities occurring in (2.7) will be considered as functions of $\phi$ (although the argument will be suppressed in the notation). Additionally, we recall the definition of the Christoffel symbols

$$\Gamma_{ij}^k = \frac{1}{2} g^{km} (\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}), \tag{2.8}$$

and define

$$\begin{aligned} \omega_{\boldsymbol{g}} &:= \frac{\sqrt{\tilde{g}}}{\sqrt{\tilde{g}(\mathrm{Id})}}, \\ f^k &:= \frac{1}{\sqrt{\tilde{g}}} \partial_\alpha (\sqrt{\tilde{g}} \gamma \tilde{g}^{\alpha\beta} \partial_\beta \Phi^k), \end{aligned}$$

and $f := (f^1, \ldots, f^N)$. Note that $\omega_{\boldsymbol{g}} d\Gamma$ is the measure on $\Gamma$ induced by $\boldsymbol{g}$.

**Lemma 2.3** *(Geometric quantities)*
*For $i, j, k = 1, \ldots, N$, $\alpha, \beta = 1 \ldots N-1$ and $\mathcal{U}$ sufficiently small we have*

$$\begin{aligned} g_{ij}, g^{ij}, a_j^i, \sqrt{g} &\in C^\infty(\mathcal{U}, H^{s+\frac{1}{2}}(\Omega)), \\ \tilde{g}_{\alpha\beta}, \tilde{g}^{\alpha\beta}, \tilde{g}^{\pm\frac{1}{2}}, \omega_{\boldsymbol{g}}, n_j &\in C^\infty(\mathcal{U}, H^s(\Gamma)), \\ \Gamma_{ij}^k &\in C^\infty(\mathcal{U}, H^{s-\frac{1}{2}}(\Omega)), \\ f^k &\in C^\infty(\mathcal{U}, H^{s-1}(\Gamma)). \end{aligned}$$

**Proof:** The spaces $H^{t+\frac{1}{2}}(\Omega)$ and $H^t(\Gamma)$ are Banach algebras for $t > \frac{N-1}{2}$. This fact (for various $t$) is the basis for the asserted results. To begin, the statement on $g_{ij}$ and $\tilde{g}_{\alpha\beta}$ is immediate from this. Analogous results hold for $g$, $\tilde{g}$, $G$ and $\tilde{G}$. For any Banach algebra, the inversion is a smooth operation on the set of its invertible elements. Application of this result to $G$, $D\Phi$, and $\tilde{G}$ in the Banach algebras $H^{s+\frac{1}{2}}(\Omega, \mathcal{L}(\mathbb{R}^N))$ and $H^s(\Gamma, \mathcal{L}(\mathbb{R}^N))$, respectively, yields the statement on $g^{ij}$, $a_j^i$, and $\tilde{g}^{\alpha\beta}$. The statement on $\Gamma_{ij}^k$ follows now by (2.8). The statement on $\sqrt{g}$, $\tilde{g}^{\frac{1}{2}}$, and $\omega_{\boldsymbol{g}}$ is implied by the fact that the superposition operator generated by the square root is smooth on the positive cones of $H^{s+\frac{1}{2}}(\Omega)$ and $H^s(\Gamma)$. The statement on $\tilde{g}^{-\frac{1}{2}}$ follows again from the smoothness of the inversion. From this, one gets the statement on $f^k$. Finally, the statement on $n_j$ is obtained from the identity

$$n^j = \frac{\sqrt{g}}{\omega_{\boldsymbol{g}}} g^{ij} \nu_i^0.$$

■

To describe the dependence of the solution of (2.7) on $\phi$, we introduce the spaces

$$\begin{aligned} V &:= \left\{(c^{ij}) \mid i,j=1,\ldots,N, c^{ij}\in\mathbb{R}, c^{ij}=-c^{ji}\right\},\\ X_t &:= H^{t+\frac{1}{2}}(\Omega,\mathbb{R}^N)\times H^{t-\frac{1}{2}}(\Omega)\times(\mathbb{R}^N\times V),\\ Y_t &:= H^{t-\frac{3}{2}}(\Omega,\mathbb{R}^N)\times H^{t-\frac{1}{2}}(\Omega)\times H^{t-1}(\Gamma,\mathbb{R}^N)\times\mathbb{R}^N\times V \end{aligned}$$

for $t \geq \frac{1}{2}$ and the operator $L : \mathcal{U} \longrightarrow \mathcal{L}(X_s, Y_s)$ by

$$L(\phi)(u,p,\lambda) := \begin{pmatrix} -\nabla^k\nabla_k u^i + \nabla^i p + \lambda_1^k a_k^i \\ \nabla_i u^i \\ (\nabla^i u^j + \nabla^j u^i - g^{ij}p + \lambda_2^{kl} a_k^i a_l^j) n_j \\ \int_\Omega \sqrt{g}\partial_k \Phi^i u^k \, dx \\ \int_\Omega \sqrt{g}\partial_k \Phi^i \partial_l \Phi^j (\nabla^k u^l - \nabla^l u^k)\, dx \end{pmatrix}^T .$$

We will denote the canonical projection of $X_t$ onto its $i$-th component by $\Pi_i$ and by $E_3$ the operator in $\mathcal{L}(H^{t-1}(\Gamma,\mathbb{R}^N), Y_t)$ mapping $H$ to $E_3 H := (0,0,H,0,0)$. For Banach spaces $E$ and $F$, let $\mathcal{L}_{is}(E,F)$ denote the set of continuous isomorphisms from $E$ to $F$ with the topology inherited from $\mathcal{L}(E,F)$.

**Lemma 2.4** *(Solvability of* (2.7) *and dependence on* $\phi$*)*
*For sufficiently small* $\mathcal{U}$*,*

$$L \in C^\infty(\mathcal{U}, \mathcal{L}_{is}(X_s, Y_s)).$$

**Proof:** Due to the Banach algebra property of the occurring Sobolev spaces and Lemma 2.3, one immediately gets $L \in C^\infty(\mathcal{U}, \mathcal{L}(X_s, Y_s))$. As $\mathcal{L}_{is}(X_s, Y_s)$ is open in $\mathcal{L}(X_s, Y_s)$, it remains to show that $L(\mathrm{Id})$ is an isomorphism which is a regularity result for the (augmented) Stokes equations with Neumann boundary conditions. It is proved in [8], Lemma 2(i). ■

For $\phi \in \mathcal{U}$, we introduce the linear operators $D\Phi$ on $H^{s+\frac{1}{2}}(\Omega,\mathbb{R}^N)$, $H^{s-\frac{3}{2}}(\Omega,\mathbb{R}^N)$, and $H^{s-1}(\Gamma,\mathbb{R}^N)$, $S(\phi)$ on $X_s$ and $T(\phi)$ on $Y_s$ by

$$\begin{aligned} (D\Phi z)^i &= \partial_j \Phi^i z^j,\\ S(\phi)(u,p,\lambda) &:= (D\Phi u, p, \lambda),\\ T(\phi)(F,K,H,M_1,M_2) &:= (D\Phi F, K, D\Phi H, M_1, M_2). \end{aligned}$$

By the same arguments as above, one shows

$$\begin{aligned} [\phi \mapsto D\Phi] &\in C^\infty(\mathcal{U}, \mathcal{L}_{is}(H^{s+\frac{1}{2}}(\Omega,\mathbb{R}^N))) \cap C^\infty(\mathcal{U}, \mathcal{L}_{is}(H^{s-\frac{3}{2}}(\Omega,\mathbb{R}^N)))\\ &\quad \cap C^\infty(\mathcal{U}, \mathcal{L}_{is}(H^{s-1}(\Gamma,\mathbb{R}^N))),\\ [\phi \mapsto S(\phi)] &\in C^\infty(\mathcal{U}, \mathcal{L}_{is}(X_s)),\\ [\phi \mapsto T(\phi)] &\in C^\infty(\mathcal{U}, \mathcal{L}_{is}(Y_s)). \end{aligned}$$

Moreover, we introduce

$$\begin{aligned} A &\in C^\infty(\mathcal{U}, \mathcal{L}(H^{s-1}(\Gamma,\mathbb{R}^N), H^s(\Gamma,\mathbb{R}^N))), & A(\phi) &:= \mathrm{Tr}_\Gamma \Pi_1 L(\phi)^{-1} E_3,\\ \mathcal{F} &\in C^\infty(\mathcal{U}, H^s(\Gamma,\mathbb{R}^N)), & \mathcal{F}(\phi) &:= D\Phi A(\phi)(D\Phi)^{-1} f(\phi),\\ \tilde{\nu} &\in C^\infty(\mathcal{U}, H^{s-1}(\Gamma,\mathbb{R}^N)), & \tilde{\nu}(\phi) &:= D\Phi n. \end{aligned}$$

We remark that $\widetilde{\nu} = \nu \circ \phi$.

With this notation, we introduce the initial value problem

$$\left.\begin{array}{rcl} \dot{\phi} & = & \mathcal{F}(\phi), \\ \phi(0) & = & \phi_0 \end{array}\right\} \tag{2.9}$$

with $\phi_0 \in \mathcal{U}$. With a slight abuse of notation, we will denote by $\phi$ both elements of $\mathcal{U}$ and functions of time valued in $\mathcal{U}$; the interpretation should be clear from the context. For our present purposes, we regard $\mathcal{F}$ as a function of $\phi$ only and consider $\gamma$ to be fixed. Define

$$\begin{array}{rcl} \Phi_0 & := & \mathcal{E}(\phi_0 - \mathrm{Id}_\Gamma) + \mathrm{Id}_\Omega, \\ \Phi(t) & := & \mathcal{E}(\phi(t) - \mathrm{Id}_\Gamma) + \mathrm{Id}_\Omega. \end{array}$$

Using (3.11) which will be proved below, it is not hard to check that (2.9) is equivalent to (1.1), (1.2) with data

$$\begin{array}{rcl} \Omega(0) & := & \Phi_0(\Omega), \\ \gamma_0 & := & \gamma \circ \phi_0^{-1} \end{array}$$

and solution

$$\begin{array}{rcl} \Omega(t) & = & (\Phi(t))(\Omega), \\ \widetilde{u}(\cdot,t) & = & \Phi(t)_* \Pi_1 L(\phi(t))^{-1} E_3 (D\Phi(t))^{-1} f(\phi(t)), \\ \widetilde{p}(\cdot,t) & = & (\Pi_2 L(\phi(t))^{-1} E_3 (D\Phi(t))^{-1} f(\phi(t))) \circ \Phi(t)^{-1}, \\ \widetilde{\gamma}(\cdot,t) & = & \gamma \circ \phi(t)^{-1} \end{array}$$

for $t \geq 0$.

# 3 Linearization

Our next aim is to study the linearization of (2.9). As a preparation for this, we give estimates for the (transformed) Neumann BVPs both for the Laplacian and for the Stokes operator, i.e. for problem (2.7). (The former has to be treated because it will occur as an auxiliary problem in the discussion of the Stokes equations, cf. e.g. [6].) The emphasis is on estimates in weak norms and on the uniformity of all estimates with respect to $\phi \in \mathcal{U}$.

The Laplace-Beltrami operators on $(\Omega, \boldsymbol{g})$ and $(\Gamma, \widetilde{\boldsymbol{g}})$ will be denoted by $\Delta_{\boldsymbol{g}}$ and $\Delta_{\widetilde{\boldsymbol{g}}}$, respectively. Moreover, we set $\partial_n := n^i \mathrm{Tr} \partial_i$.

**Lemma 3.1** *(The Neumann problem for the Laplacian)*

*For $\phi \in \mathcal{U}$, $(F, H) \in H^{s-\frac{3}{2}}(\Omega) \times H^{s-1}(\Gamma)$, the (augmented) BVP*

$$\left.\begin{array}{rcll} \Delta_{\boldsymbol{g}} v + \mu & = & F & \text{in } \Omega, \\ \partial_n v & = & H & \text{on } \Gamma, \\ \int_\Omega \sqrt{g} v \, dx & = & 0, & \\ \mu & = & \dfrac{\int_\Omega \sqrt{g} F \, dx - \int_\Gamma \omega_{\boldsymbol{g}} H \, d\Gamma}{\int_\Omega \sqrt{g} \, dx} & \end{array}\right\}$$

*is uniquely solvable, and for $t \in [-1, \sigma+2]$ there is a constant $C = C_t$ independent of $\phi$ such that*

$$\|v\|_t^\Gamma + \|v\|_{t+\frac{1}{2}}^\Omega + \|\nabla v\|_{t-1}^\Gamma + \|\nabla v\|_{t-\frac{1}{2}}^\Omega \le C\left(\|F\|_{t-\frac{3}{2}}^\Omega + \|H\|_{t-1}^\Gamma\right).$$

**Remark:** For $t$ large enough, the estimates of various terms on the left follow by trivial estimates and the trace theorem from the ones for $\|v\|_{t+\frac{1}{2}}^\Omega$.

**Proof:**

Step 1: We prove the estimates for $\|v\|_t^\Gamma$ and $\|v\|_{t+\frac{1}{2}}^\Omega$. For $t \ge \frac{1}{2}$, the result is standard and follows from a discussion of the weak formulation, elliptic regularity theory, and interpolation. (To see the validity of the regularity results in the case of general $\boldsymbol{g}$, it is also possible to apply a perturbation argument as in Lemma 2.4.)

It remains to treat the case $t < \frac{1}{2}$. For arbitrary $\psi \in H^{-t}(\Gamma)$, consider the problem

$$\left.\begin{array}{rcll} \Delta_{\boldsymbol{g}} w + \eta &=& 0 & \text{in } \Omega, \\ \partial_n w &=& \frac{1}{\omega_{\boldsymbol{g}}}\psi & \text{on } \Gamma, \\ \int_\Omega \sqrt{g} w\, dx &=& 0, & \\ \eta &=& -\dfrac{\int_\Gamma \psi\, d\Gamma}{\int_\Omega \sqrt{g}\, dx}. & \end{array}\right\}$$

As $-t > -\frac{1}{2}$, we have

$$\|w\|_{-t+1}^\Gamma + \|w\|_{-t+\frac{3}{2}}^\Omega \le C\|\psi\|_{-t}^\Gamma.$$

By Green's formula,

$$\begin{aligned} \int_\Gamma v\psi\, d\Gamma &= \int_\Gamma \omega_{\boldsymbol{g}} v \partial_n w\, d\Gamma = \int_\Gamma \omega_{\boldsymbol{g}} \partial_n v w\, d\Gamma + \int_\Omega \sqrt{g}(\Delta_{\boldsymbol{g}} w\, v - w\Delta_{\boldsymbol{g}} v)\, dx \\ &= \int_\Gamma \omega_{\boldsymbol{g}} H w\, d\Gamma - \int_\Omega \sqrt{g} w F\, dx, \end{aligned}$$

hence

$$\left|\int_\Gamma v\psi\, d\Gamma\right| \le C\left(\|H\|_{t-1}^\Gamma \|w\|_{-t+1}^\Gamma + \|F\|_{t-\frac{3}{2}}^\Omega \|w\|_{-t+\frac{3}{2}}^\Omega\right)$$

which implies

$$\|v\|_t^\Gamma \le C\left(\|F\|_{t-\frac{3}{2}}^\Omega + \|H\|_{t-1}^\Gamma\right).$$

Analogously, for arbitrary $\chi \in H^{-t-\frac{1}{2}}(\Omega)$ consider the problem

$$\left.\begin{array}{rcll} \Delta_{\boldsymbol{g}} w + \eta &=& \frac{1}{\sqrt{g}}\chi & \text{in } \Omega, \\ \partial_n w &=& 0 & \text{on } \Gamma, \\ \int_\Omega \sqrt{g} w\, dx &=& 0, & \\ \eta &=& \dfrac{\int_\Omega \chi\, dx}{\int_\Omega \sqrt{g}\, dx}. & \end{array}\right\}$$

The solution satisfies an estimate

$$\|w\|_{-t+1}^\Gamma + \|w\|_{-t+\frac{3}{2}}^\Omega \le C\|\psi\|_{-t-\frac{1}{2}}^\Omega.$$

Using Green's formula as above, we find

$$\int_\Omega v\chi\,dx = \int_\Omega \sqrt{g} w F\,dx - \int_\Gamma \omega_{\boldsymbol{g}} H w\,d\Gamma$$

which implies

$$\|v\|^\Omega_{t+\frac{1}{2}} \le C\left(\|F\|^\Omega_{t-\frac{3}{2}} + \|H\|^\Gamma_{t-1}\right).$$

Step 2: Due to (2.1), we have

$$\|\nabla v\|^\Omega_{t-\frac{1}{2}} \le C\left(\|v\|^\Omega_{t+\frac{1}{2}} + \|v\|^\Gamma_t\right)$$

for $t \le -\frac{1}{2}$, and the same estimate is trivially true for $t \ge \frac{1}{2}$. The estimate for $\|\nabla v\|^\Omega_{t-\frac{1}{2}}$ follows from this by the results of Step 1 and interpolation. Finally, due to the decomposition of $\nabla v$ at the boundary with respect to the normal and tangential components, one has

$$\|\nabla v\|^\Gamma_{t-1} \le C\left(\|v\|^\Gamma_t + \|H\|^\Gamma_{t-1}\right).$$

This implies the estimate for $\|\nabla v\|^\Gamma_{t-1}$. ■

**Lemma 3.2** *(The Neumann problem for the Stokes equations)*
*For $\phi \in \mathcal{U}$, $(F, K, H, M_1, M_2) \in Y_s$, set*

$$(u, p, \lambda) := L(\phi)^{-1}(F, K, H, M_1, M_2).$$

*Then, for $t \in [-\frac{1}{2}, \sigma + 2]$ there is a constant $C = C_t$ independent of $\phi$ such that*

$$\begin{aligned} &\|u\|^\Gamma_t + \|u\|^\Omega_{t+\frac{1}{2}} + \|\nabla u\|^\Gamma_{t-1} + \|p\|^\Gamma_{t-1} + \|p\|^\Omega_{t-\frac{1}{2}} \\ \le\;& C\left(\|F\|^\Omega_{t-\frac{3}{2}} + \|K\|^\Omega_{t-\frac{1}{2}} + \|\nabla K\|^\Omega_{t-\frac{3}{2}} + \|K\|^\Gamma_{t-1} + \|H\|^\Gamma_{t-1} + |M_1| + |M_2|\right). \end{aligned}$$

**Remark:** As in the previous lemma, various terms, both on the left and on the right, can be omitted due to trivial estimates and the trace theorem if $t$ is large enough.

**Proof:**

Step 1: Suppose $t \ge \frac{3}{2}$. For $\boldsymbol{g}$ euclidean, the result is proved in [8], Lemma 2(i); the result for general $\boldsymbol{g}$ follows again by a perturbation argument.

Step 2: We first consider the case $t \in \left[\frac{1}{2}, \frac{3}{2}\right)$, $K = 0$ and prove the estimates for $\|u\|^\Gamma_t$, $\|u\|^\Omega_{t+\frac{1}{2}}$, and $\|p\|^\Omega_{t-\frac{1}{2}}$.

The equation $L(\phi)(u, p, \lambda) = (F, 0, H, M_1, M_2)$ implies its weak formulation

$$L_w(\phi)(u, p, \lambda) = ([v \mapsto \int_\Omega \sqrt{g} F^i v_i\,dx + \int_\Gamma \omega_{\boldsymbol{g}} H^i v_i\,d\Gamma], 0, 0, 0), \tag{3.1}$$

where

$$L_w(\phi):\ X_{\frac{1}{2}} \longrightarrow H^1(\Omega, \mathbb{R}^N)' \times H^0(\Omega) \times \mathbb{R}^N \times V$$

is given by

$$L_w(\phi)(u, p, \lambda) := \begin{pmatrix} [v \mapsto \int_\Omega \sqrt{g}(\nabla^i u^j + \nabla^j u^i)(\nabla_i v_j + \nabla_j v_i)\,dx + \int_\Omega \sqrt{g} p \nabla_i v^i\,dx \\ +\lambda_1^i \int_\Omega \sqrt{g}\partial_j \Phi^i v^j\,dx \\ +\lambda_2^{ij} \int_\Omega \sqrt{g}\partial_k \Phi^i \partial_l \Phi^j (\nabla^k v^l - \nabla^l v^k)\,dx] \\ \nabla_i u^i \\ \int_\Omega \sqrt{g}\partial_k \Phi^i u^k\,dx \\ \int_\Omega \sqrt{g}\partial_k \Phi^i \partial_l \Phi^j (\nabla^k u^l - \nabla^l u^k)\,dx \end{pmatrix}^T.$$

Arguments parallel to those in the proof of Lemma 2.4 show that

$$L_w \in C^\infty\Big(\mathcal{U}, \mathcal{L}(X_{\frac{1}{2}}, H^1(\Omega,\mathbb{R}^N)' \times H^0(\Omega) \times \mathbb{R}^N \times V)\Big)$$

and

$$[\phi \mapsto [f \mapsto [v \mapsto \int_\Gamma \omega_{\boldsymbol{g}} f^i v_i \, d\Gamma]]] \in C^\infty\Big(\mathcal{U}, \mathcal{L}(H^{-\frac{1}{2}}(\Gamma,\mathbb{R}^N), H^1(\Omega,\mathbb{R}^N)')\Big). \tag{3.2}$$

It is shown in [8], Lemma 1(i), that

$$L_w(\mathrm{Id}) \in \mathcal{L}_{is}(X_{\frac{1}{2}}, H^1(\Omega,\mathbb{R}^N)' \times H^0(\Omega) \times \mathbb{R}^N \times V)).$$

Hence

$$L_w \in C^\infty\Big(\mathcal{U}, \mathcal{L}_{is}(X_{\frac{1}{2}}, H^1(\Omega,\mathbb{R}^N)' \times H^0(\Omega) \times \mathbb{R}^N \times V)\Big).$$

Together with (3.2), this implies our estimate for $t = \frac{1}{2}$. For $t \in (\frac{1}{2}, \frac{3}{2})$, the result follows by interpolation.

Step 3: Suppose $t < \frac{1}{2}$ and $K = 0$, pick $\psi \in H^{s-1}(\Gamma,\mathbb{R}^N)$ arbitrary and set

$$(v, q, \mu) := L(\phi)^{-1} E_3\left(\tfrac{1}{\omega_{\boldsymbol{g}}} g^{ij}\psi^j\right).$$

Then, due to the regularity results above,

$$||v||^\Omega_{-t+\frac{3}{2}} + ||q||^\Omega_{-t+\frac{1}{2}} + |\mu_1| + |\mu_2| \le C||\psi||^\Gamma_{-t}. \tag{3.3}$$

By the Green formula for the Stokes operator (cf. [8], Eq. (2.3))

$$\begin{aligned}
\int_\Gamma \psi^j u^j \, d\Gamma &= \int_\Gamma \omega_{\boldsymbol{g}} u_i \frac{1}{\omega_{\boldsymbol{g}}} g^{ij}\psi^j \, d\Gamma \\
&= \int_\Gamma \omega_{\boldsymbol{g}} u_i (\nabla^i v^j + \nabla^j v^i - g^{ij} p + \mu_2^{kl} a_k^i a_l^j) n_j \, d\Gamma \\
&= \int_\Omega \sqrt{g}(-\nabla^j \nabla_j u^i + \nabla^i p) v_i \, dx \\
&\quad + \int_\Gamma \omega_{\boldsymbol{g}} v_i (\nabla^i u^j + \nabla^j u^i - g^{ij} q + \mu_2^{kl} a_k^i a_l^j) n_j \, d\Gamma \\
&\quad - \int_\Omega \sqrt{g}(-\nabla^j \nabla_j v^i + \nabla^i q) u_i \, dx \\
&= \int_\Omega \sqrt{g}(F^i - a_k^i \lambda_1^k) v_i \, dx \\
&\quad + \int_\Gamma \omega_{\boldsymbol{g}} (H^i + (\mu_2^{kl} - \lambda_2^{kl}) a_k^i a_l^j n_j) v_i \, d\Gamma + \int_\Omega \sqrt{g}\mu_1^k a_k^i u_i \, dx.
\end{aligned}$$

Using

$$\begin{aligned}
\int_\Omega \sqrt{g}\lambda_1^k a_k^i v_i \, dx &= \lambda_1^k \int_\Omega \sqrt{g}\partial_i \Phi^k v^i \, dx = 0, \\
\int_\Omega \sqrt{g}\mu_1^k a_k^i v_i \, dx &= \mu_1^k M_1^k,
\end{aligned}$$

and the Stokes theorem in the form

$$\int_\Gamma \omega_{\boldsymbol{g}} \lambda_2^{kl} a_k^i a_l^j n_j v_i \, d\Gamma = \lambda_2^{kl} \int_\Omega \sqrt{g} \partial_i \Phi^k \partial_j \Phi^l (\nabla^i v^j - \nabla^j v^i) \, dx = 0,$$

$$\int_\Gamma \omega_{\boldsymbol{g}} \mu_2^{kl} a_k^i a_l^j n_j v_i \, d\Gamma = \mu_2^{kl} M_2^{kl}$$

we obtain

$$\left| \int_\Gamma \psi^j u^j \, d\Gamma \right| \leq C \left( ||F||^\Omega_{t-\frac{3}{2}} ||v||^\Omega_{-t+\frac{3}{2}} + ||H||^\Gamma_{t-1} ||v||^\Gamma_{-t+1} + |\mu_1||M_1| + |\mu_2||M_2| \right)$$

$$\leq C \left( ||F||^\Omega_{t-\frac{3}{2}} + ||H||^\Gamma_{t-1} + |M_1| + |M_2| \right) ||\psi||^\Gamma_{-t},$$

where the trace theorem and (3.3) have been used in the last inequality. This proves the desired estimate on $||u||^\Gamma_t$.

Analogously, pick $\chi \in H^{s-\frac{3}{2}}(\Omega, \mathbb{R}^N)$ and set

$$(v, q, \mu) := L(\phi)^{-1} \left( \tfrac{1}{\sqrt{g}} g^{ij} \chi^j, 0, 0, 0, 0 \right).$$

Due to the regularity results above,

$$||v||^\Omega_{-t+\frac{3}{2}} + ||q||^\Omega_{-t+\frac{1}{2}} + |\mu_1| + |\mu_2| \leq C ||\chi||^\Gamma_{-t-\frac{1}{2}}. \tag{3.4}$$

By the Green formula,

$$\int_\Omega u^i \chi^i \, dx = \int_\Omega \sqrt{g} F^i v_i \, dx + \int_\Gamma \omega_{\boldsymbol{g}} H^i v_i \, d\Gamma + \mu_1^k M_1^k + \mu_2^k M_2^k$$

and thus

$$\left| \int_\Omega u^i \chi^i \, dx \right| \leq C \left( ||F||^\Omega_{t-\frac{3}{2}} ||v||^\Omega_{-t+\frac{3}{2}} + ||H||^\Gamma_{t-1} ||v||^\Gamma_{-t+1} + |\mu_1||M_1| + |\mu_2||M_2| \right).$$

Together with (3.4) and the trace theorem, this yields the desired estimate for $||u||^\Omega_{t+\frac{1}{2}}$.

Finally, pick $\eta \in H^{s-\frac{1}{2}}(\Omega)$ and set

$$(v, q, \mu) := L(\phi)^{-1} \left( 0, \tfrac{1}{\sqrt{g}} \eta, 0, 0, 0 \right).$$

Due to the regularity results of Step 1,

$$||v||^\Omega_2 + ||q||^\Omega_1 + |\mu_1| + |\mu_2| \leq C ||\eta||^\Omega_1. \tag{3.5}$$

By the Green formula,

$$\int_\Omega p\eta \, dx = -\int_\Omega \sqrt{g} (\partial_i \eta u^i + F^i v_i) \, dx - \int_\Gamma \omega_{\boldsymbol{g}} H^i v_i \, d\Gamma - \mu_1^k M_1^k - \mu_2^k M_2^k$$

and thus

$$\left| \int_\Omega p\eta \, dx \right| \leq C \left( ||u||^\Omega_0 ||\eta||^\Omega_1 + ||F||^\Omega_{-2} ||v||^\Omega_2 + ||H||^\Gamma_{-\frac{3}{2}} ||v||^\Gamma_{\frac{3}{2}} + |\mu_1||M_1| + |\mu_2||M_2| \right).$$

Together with (3.5), the previously given estimate for $\|u\|_0^\Omega$, and the trace theorem, this yields

$$\|p\|_{-1}^\Omega \le C\left(\|F\|_{-2}^\Omega + \|H\|_{-\frac{3}{2}}^\Gamma + |M_1| + |M_2|\right).$$

The desired estimates for $\|p\|_t^\Omega$ follow from this and the estimates of Step 1 by interpolation.

Step 4: We consider the case $(F, K, H, M_1, M_2) = (0, K, 0, 0, 0)$. Define $\psi$ to be the solution of the Neumann BVP

$$\left.\begin{array}{rcll} \Delta_{\boldsymbol{g}}\psi &=& K & \text{in } \Omega, \\ \partial_n \psi &=& \frac{\int_\Omega \sqrt{g}K\,dx}{\int_\Gamma \omega_{\boldsymbol{g}}\,d\Gamma} & \text{on } \Gamma, \\ \int_\Omega \sqrt{g}\psi\,dx &=& 0. & \end{array}\right\}$$

Hence

$$\nabla^k\nabla_k\nabla^i\psi = (\nabla^k\nabla^i - \nabla^i\nabla^k)\nabla_k\psi + \nabla^i K = R^j{}_k{}^{ki}\nabla_j\psi + \nabla^i K = R^{ij}\nabla_j\psi + \nabla^i K, \quad (3.6)$$

where $R^j{}_l{}^{ki}$ and $R^{ij}$ denote the coordinates of the Riemann and Ricci tensors of $\boldsymbol{g}$, respectively. We extend the outer normal vector $n$ differentiably to a neighborhood of $\Gamma$ and keep the same notation for the extension. There we have

$$\nabla^i\nabla^j\psi n_j = \nabla^i(\nabla^j\psi n_j) - \nabla^j\psi\nabla^i n_j.$$

On $\Gamma$, the first term on the right is a $\boldsymbol{g}$-normal vector because $\nabla^j\psi n_j = \partial_n\psi$ is constant along $\Gamma$. Hence, introducing the orthogonal projection $P^{\boldsymbol{g}}$ onto the tangent bundle with respect to $\boldsymbol{g}$, given by

$$(P^{\boldsymbol{g}}v)^j = \partial_\alpha\phi^i\tilde{g}^{\alpha\beta}\partial_\beta\phi^j v_i,$$

and using the transformed version of (2.5), we find on $\Gamma$

$$\begin{aligned} \nabla^i\nabla^j\psi n_j &= n^i n_k\nabla^k\nabla^j\psi n_j - \left(P^{\boldsymbol{g}}(\nabla^j\psi\nabla^{\boldsymbol{g}}n_j)\right)^i \\ &= n^i(\mathrm{Tr}_\Gamma\Delta_{\boldsymbol{g}}\psi - \Delta_{\tilde{\boldsymbol{g}}}\mathrm{Tr}_\Gamma\psi - \kappa\partial_n\psi) - \left(P^{\boldsymbol{g}}(\nabla^j\psi\nabla^{\boldsymbol{g}}n_j)\right)^i, \qquad (3.7) \end{aligned}$$

where $\kappa$ is the mean curvature of $\Gamma$ with respect to $\boldsymbol{g}$.

Thus, setting $\overline{u} := u - \nabla^{\boldsymbol{g}}\psi$ we obtain

$$\begin{aligned} L(\phi)(\overline{u}, p, \lambda) &= (0, K, 0, 0, 0) - L(\phi)(\nabla^{\boldsymbol{g}}\psi, 0, 0) \\ &= \begin{pmatrix} R^{ij}\nabla_j\psi + \nabla^i K \\ 0 \\ 2\left(-n^i(\mathrm{Tr}_\Gamma K - \Delta_{\tilde{\boldsymbol{g}}}\mathrm{Tr}_\Gamma\psi - \kappa\partial_n\psi) + \left(P^{\boldsymbol{g}}(\nabla^j\psi\nabla^{\boldsymbol{g}}n_j)\right)^i\right) \\ \int_\Omega \sqrt{g}\partial_k\Phi^i\nabla^k\psi\,dx \\ 0 \end{pmatrix}^T. \end{aligned}$$

Due to Lemma 3.1, we have

$$\begin{aligned} &\left\|R^{ij}\nabla_j\psi + \nabla^i K\right\|_{t-\frac{3}{2}}^\Omega + \left\|-n^i(\mathrm{Tr}_\Gamma K - \Delta_{\tilde{\boldsymbol{g}}}\mathrm{Tr}_\Gamma\psi - \kappa\partial_n\psi) + \left(P^{\boldsymbol{g}}(\nabla^j\psi\nabla^{\boldsymbol{g}}n_j)\right)^i\right\|_{t-1}^\Gamma \\ &+ \left|\int_\Omega \sqrt{g}\partial_k\Phi^i\nabla^k\psi\,dx\right| \\ \le\ & C\left(\|\nabla\psi\|_{t-\frac{3}{2}}^\Omega + \|\nabla\psi\|_{t-1}^\Gamma + \|\psi\|_{t+1}^\Gamma + \|\nabla K\|_{t-\frac{3}{2}}^\Omega + \|K\|_{t-1}^\Gamma\right) \\ \le\ & C\left(\|K\|_{t-\frac{1}{2}}^\Omega + \|\nabla K\|_{t-\frac{3}{2}}^\Omega + \|K\|_{t-1}^\Gamma\right). \end{aligned}$$

To estimate $\overline{u}$ and $p$, the results of the previous steps are applicable, and the asserted estimates for $u$ follow from $u = \overline{u} + \nabla^{g}\psi$ by another application of Lemma 3.1.

Step 5: It remains to show the estimates for $||\nabla u||^{\Gamma}_{t-1}$ and $||p||^{\Gamma}_{t-1}$. Fix $x \in \Gamma$ and choose local coordinates in an $\mathbb{R}^N$-neighborhood $M$ of $x$ such that $\partial_1, \ldots \partial_{N-1}$ are tangential derivatives on $\Gamma \cap M$ and $\partial_N$ is normal to $\Gamma \cap M$. Differing from our usual notation, during this step of the proof, $u^i$, $T^i$, $g_{ij}$, $n_i$ etc. will denote components with respect to these local coordinates. Moreover, without loss of generality we demand $n_i = \delta_i^N$ and $g_{ij} = \delta_{ij}$ at $x$. It is sufficient to show (with obvious notation)

$$\sum_{i=1}^{N} ||\nabla_N u^i||^{\Gamma\cap M}_{t-1} + ||p||^{\Gamma\cap M}_{t-1} \leq C\left(||u||^{\Gamma}_{t} + ||H||^{\Gamma}_{t-1} + ||K||^{\Gamma}_{t-1}\right) \tag{3.8}$$

because obviously $\left\|\partial_j u^i\right\|^{\Gamma\cap M}_{t-1} \leq C||u||^{\Gamma}_{t}$, $j = 1, \ldots, N-1$. Note that on $\Gamma$

$$\begin{aligned}
(g^{iN}\nabla_N u^j + g^{jN}\nabla_N u^i - g^{ij}p)n_j &= H^i - \sum_{k=1}^{N-1}(g^{ik}\nabla_k u^j + g^{jk}\nabla_k u^i)n_j, \\
\nabla_N u^N &= K - \sum_{k=1}^{N-1}\nabla_k u^k.
\end{aligned}$$

These $N+1$ linear equations constitute a system for $(\nabla_N u^1, \ldots, \nabla_N u^N, p)$ whose coefficient matrix at $x$ is

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 2 & -1 \\ & & & 1 & 0 \end{pmatrix},$$

which is invertible. Using arguments similar to those in the proof of Lemma 2.3 and estimates for the right-hand side

$$\begin{aligned}
&\left\|H^i - \sum_{k=1}^{N-1}(g^{ik}\nabla_k u^j + g^{jk}\nabla_k u^i)n_j\right\|^{\Gamma\cap M}_{t-1} + \left\|K - \sum_{k=1}^{N-1}\nabla_k u^k\right\|^{\Gamma\cap M}_{t-1} \\
\leq\ & C\left(||u||^{\Gamma}_{t} + ||H||^{\Gamma}_{t-1} + ||K||^{\Gamma}_{t-1}\right)
\end{aligned}$$

we get (3.8). ■

Using the weak formulation in Step 2 of the proof we determine the auxiliary parameters $\lambda_1^k$ and $\lambda_2^{kl}$.

**Lemma 3.3** *(Auxiliary parameters)*

*For $\phi \in \mathcal{U}$, let $(u, p, \lambda)$ be the solution of*

$$L(\phi)(u, p, \lambda) = E_3 H.$$

*Then*

$$\lambda_1^k = \frac{\int_\Gamma \omega_{\boldsymbol{g}} H_i a_k^i \, d\Gamma}{\int_\Omega \sqrt{g}\, dx}, \tag{3.9}$$

$$\lambda_2^{kl} = \frac{1}{2\int_\Omega \sqrt{g}\, dx}\left(\int_\Gamma \omega_{\boldsymbol{g}} H_i(\phi^l a_k^i - \phi^k a_l^i)\, d\Gamma - \int_\Omega \sqrt{g}\left(\lambda_1^k \Phi^l - \lambda_1^l \Phi^k\right)\, dx\right). \tag{3.10}$$

*In particular,*

$$\Pi_3 L(\phi)^{-1}(D\Phi)^{-1} f(\phi) = 0. \tag{3.11}$$

**Remark:** The last statement shows that the auxiliary parameters in (2.7) vanish, i.e. its solution $(u, p)$ is indeed a solution to the Stokes equations without unnatural forcing terms.

**Proof:** From (3.1) we get

$$\begin{aligned}\int_\Omega \sqrt{g}(\nabla^i u^j + \nabla^j u^i)(\nabla_i v_j + \nabla_j v_i)\,dx + \int_\Omega \sqrt{g} p \nabla_i v^i\,dx \\ +\lambda_1^i \int_\Omega \sqrt{g}\partial_j \Phi^i v^j\,dx + \lambda_2^{ij}\int_\Omega \sqrt{g}\partial_k\Phi^i\partial_l\Phi^j(\nabla^k v^l - \nabla^l v^k)\,dx = \int_\Gamma \omega_{\boldsymbol{g}} H^i v_i\,d\Gamma\end{aligned}$$

for all $v \in H^1(\Omega, \mathbb{R}^N)$. For $k = 1, \ldots, N$, setting $v^i = a_k^i$ yields $\nabla_j v^i = 0$ and hence (3.9). Similarly, for $k, l = 1, \ldots, N$, setting $v^i = \Phi^l a_k^i - \Phi^k a_l^i$ yields $\nabla_i v^i = 0$, $\nabla_j v_i = -\nabla_i v_j$, $\nabla_j v_i - \nabla_i v_j = 2\partial_i \Phi^k \partial_j \Phi^l$, and one finds (3.10). The last statement follows from setting (in local coordinates)

$$H^i := \frac{a_r^i}{\sqrt{\tilde{g}}}\partial_\alpha\left(\sqrt{\tilde{g}}\gamma\tilde{g}^{\alpha\beta}\partial_\beta\Phi^r\right)$$

and calculating the integrals

$$\begin{aligned}\int_\Gamma \omega_{\boldsymbol{g}} H_i a_k^i\,d\Gamma &= \int_\Gamma \frac{1}{\sqrt{\tilde{g}(\mathrm{Id})}}\partial_\alpha\left(\sqrt{\tilde{g}}\gamma\tilde{g}^{\alpha\beta}\partial_\beta\Phi^k\right)\,d\Gamma = 0,\\ \int_\Gamma \omega_{\boldsymbol{g}} H_i(\phi^l a_k^i - \phi^k a_l^i)\,d\Gamma &= \int_\Gamma \frac{1}{\sqrt{\tilde{g}(\mathrm{Id})}}\phi^l\partial_\alpha\left(\sqrt{\tilde{g}}\gamma\tilde{g}^{\alpha\beta}\partial_\beta\Phi^k\right)\,d\Gamma \\ &\quad -\int_\Gamma \frac{1}{\sqrt{\tilde{g}(\mathrm{Id})}}\phi^k\partial_\alpha\left(\sqrt{\tilde{g}}\gamma\tilde{g}^{\alpha\beta}\partial_\beta\Phi^l\right)\,d\Gamma = 0,\end{aligned}$$

where integration by parts on $\Gamma$ has been used in the first line. ∎

**Lemma 3.4** *(Linearization of $f$)*
*We have*

$$(D\Phi)^{-1} f'(\phi)[h] = \gamma\Delta_{\tilde{\boldsymbol{g}}}(h^i\tilde{\nu}^i)n + R_1(\phi)h,$$

*where $R_1(\phi)$ is a first order differential operator whose coefficients are smooth functions of $\phi$ and its derivatives up to order 3.*

**Proof:** Suppressing the argument $\phi$ in the geometric quantities, we have due to a well-known result in the theory of (hyper)surfaces (e.g. [5], Theorem 2.5.1)

$$f = \tilde{g}^{\alpha\beta}\partial_\alpha\gamma\partial_\beta\phi + \gamma\kappa\tilde{\nu} = \tilde{g}^{\alpha\beta}\partial_\alpha\gamma\partial_\beta\phi + \gamma\tilde{g}^{\alpha\beta}\partial_{\alpha\beta}\phi^k\tilde{\nu}^k\tilde{\nu},$$

where $\kappa \circ \phi^{-1}$ is the $((N-1)$-fold) mean curvature of $\Gamma_\Phi$. So

$$(D\Phi)^{-1} f'[h] = \gamma\tilde{g}^{\alpha\beta}\partial_{\alpha\beta}h^k\tilde{\nu}^k n + R_2(\phi)h,$$

where $R_2(\phi)$ is a first order differential operator whose coefficients are smooth functions of $\phi$ and its derivatives up to order 2. On the other hand,

$$\begin{aligned}\gamma\Delta_{\tilde{\boldsymbol{g}}}(h^i\tilde{\nu}^i)n &= \frac{\gamma}{\sqrt{\tilde{g}}}\partial_\alpha(\sqrt{\tilde{g}}\tilde{g}^{\alpha\beta}(\partial_\beta h^k\tilde{\nu}^k + h^k\partial_\beta\tilde{\nu}^k))n \\ &= \gamma\tilde{g}^{\alpha\beta}\partial_{\alpha\beta}h^k\tilde{\nu}^k n + \gamma\tilde{g}^{\alpha\beta}\partial_\alpha(h^k\partial_\beta\tilde{\nu}^k)n + \frac{\gamma}{\sqrt{\tilde{g}}}\partial_\alpha(\sqrt{\tilde{g}}\tilde{g}^{\alpha\beta})\partial_\beta(h^k\tilde{\nu}^k)n.\end{aligned}$$

This proves the lemma. ■

We set

$$\widetilde{A}(\phi) := D\Phi A(\phi)(D\Phi)^{-1} = \mathrm{Tr}_\Gamma \Pi_1 \widetilde{L}(\phi)^{-1} E_3,$$

where

$$\widetilde{L}(\phi) := T(\phi) L(\phi) S(\phi)^{-1}$$

corresponds to the augmented BVP (2.6) transformed to $\Omega$ componentwise, i.e. without regard of the tensor field character of $U$, $\Lambda_{(i)}$ etc. We have $\widetilde{L} \in C^\infty(\mathcal{U}, \mathcal{L}_{is}(X_s, Y_s))$ and (cf. [8], p. 321)

$$\widetilde{L}(\phi)(u,p,\lambda) = \begin{pmatrix} -a_j^k \partial_k (a_j^l \partial_l u^i) + a_i^k \partial_k p + \lambda_1^i \\ a_i^k \partial_k u^i \\ (\mathrm{Tr}_\Gamma (a_i^k \partial_k u^j + a_j^k \partial_k u^i - p\delta_{ij}) + \lambda_2^{ij}) \widetilde{\nu}^j \\ \int_\Omega \sqrt{g} u^i \, dx \\ \int_\Omega \sqrt{g} (a_i^k \partial_k u^j - a_j^k \partial_k u^i) \, dx \end{pmatrix}^T .$$

Now, by Lemma 3.4

$$\begin{aligned} \mathcal{F}'[h] &= \widetilde{A}'[h] f + \widetilde{A} f'[h] = R_3 h + R_4 h + D\Phi A(\gamma \Delta_{\widetilde{g}} (h^i \widetilde{\nu}^i) n) \\ &= R_3 h + R_4 h + D\Phi A(\gamma \Delta_{\widetilde{g}} (k^i n_i) n), \end{aligned}$$

where $k$ is defined by $k^m := a_i^m h^i$,

$$\begin{aligned} R_3 h &:= \widetilde{A}'[h] f, \\ R_4 h &:= \widetilde{A} R_1 h. \end{aligned}$$

Then

$$\begin{aligned} (\mathcal{F}'[h], h)_{H^0(\Gamma, \mathbb{R}^N)} &= \int_\Gamma (\mathcal{F}'[h])^i h^i \, d\Gamma \\ &= R_5 + \int_\Gamma A^i (\gamma \Delta_{\widetilde{g}} (k^j n_j) n) k_i \, d\Gamma \\ &= R_5 + \int_\Gamma A^i (\gamma \Delta_{\widetilde{g}} l \, n)(l n_i + (P^g k)_i) \, d\Gamma \\ &= R_5 + R_6 + \int_\Gamma A^i (\gamma \Delta_{\widetilde{g}} l \, n) l n_i \, d\Gamma, \end{aligned}$$

where $l = k^j n_j$ and

$$\begin{aligned} R_5 &:= (R_3 h + R_4 h, h)_{H^0(\Gamma, \mathbb{R}^N)}, \\ R_6 &:= \int_\Gamma (P^g A(\gamma \Delta_{\widetilde{g}} l \, n))^i k_i \, d\Gamma. \end{aligned}$$

Thus

$$\begin{aligned} (\mathcal{F}'[h], h)_{H^0(\Gamma, \mathbb{R}^N)} &= \sum_{m=5}^{7} R_m + \int_\Gamma n_i A^i (\Delta_{\widetilde{g}} (\gamma l) n) l \, d\Gamma = \sum_{m=5}^{8} R_m - \frac{1}{2} \int_\Gamma A_0 (\gamma l) l \, d\Gamma \\ &= \sum_{m=5}^{8} R_m - \frac{1}{2} \int_\Gamma \omega_g A_0 \left( \chi \frac{l}{\omega_g} \right) \frac{l}{\omega_g} \, d\Gamma, \end{aligned}$$

where

$$\begin{aligned} R_7 &:= \int_\Gamma n_i A^i([\gamma, \Delta_{\tilde{g}}]ln)l\, d\Gamma, \\ [\gamma, \Delta_{\tilde{g}}]l &:= \gamma \Delta_{\tilde{g}} l - \Delta_{\tilde{g}}(\gamma l), \\ R_8 &:= \int_\Gamma (n_i A^i(\Delta_{\tilde{g}}(\gamma l)n) + \frac{1}{2} A_0(\gamma l))l\, d\Gamma, \\ \chi &:= \omega_{\boldsymbol{g}} \gamma. \end{aligned}$$

To define $A_0 = A_0(\phi)$, consider the Laplace-Beltrami operator $\Delta_{\boldsymbol{g}}$ on $(\Omega, \boldsymbol{g})$ and note that

$$(\Delta_{\boldsymbol{g}}, \mathrm{Tr}_\Gamma) \in C^\infty(\mathcal{U}, \mathcal{L}_{is}(H^{s+\frac{1}{2}}(\Omega), H^s(\Gamma))),$$

which can be proved in analogy to the corresponding results on (2.7) above. We set

$$A_0(\phi) := \partial_n (\Delta_{\boldsymbol{g}}, \mathrm{Tr}_\Gamma)^{-1}(0, \cdot),$$

i.e. $A_0(\phi)$ is the Dirichlet-Neumann operator on the boundary of the Riemannian manifold $(\Omega, \boldsymbol{g})$.

The corresponding second Green formula implies the symmetry property

$$\int_\Gamma \omega_{\boldsymbol{g}} A_0(\psi)\eta\, d\Gamma = \int_\Gamma \omega_{\boldsymbol{g}} A_0(\eta)\psi\, d\Gamma \quad \forall \psi, \eta \in H^s(\Gamma).$$

Consequently,

$$(\mathcal{F}'[h], h)_{H^0(\Gamma, \mathbb{R}^N)} = \sum_{m=5}^{8} R_m - \frac{1}{2} \int_\Gamma \omega_{\boldsymbol{g}} \chi A_0(\psi)\psi\, d\Gamma \tag{3.12}$$

with $\psi := \frac{l}{\omega_{\boldsymbol{g}}} = \frac{1}{\omega_{\boldsymbol{g}}} a^i_m n_i = \partial_j \phi^m n^j h^m$.

To estimate $R_6$ and $R_8$ we need an additional result which may loosely be stated as follows: Up to lower order terms, $A$ has diagonal structure with respect to the decomposition in tangential and normal components:

**Lemma 3.5** *("Diagonal structure" of $A$)*
*There are constants $C$ such that*

**(i)**

$$||P^{\boldsymbol{g}} A(hn)||^\Gamma_t \le C||h||^\Gamma_{t-2}, \quad t \in \{0, 2\}$$

*for $\phi \in \mathcal{U}$, $h \in H^{s-1}(\Gamma)$,*

**(ii)**

$$\left\|n_i A^i(w)\right\|^\Gamma_0 \le C||w||^\Gamma_{-2}$$

*for $\phi \in \mathcal{U}$, $w \in P^{\boldsymbol{g}}(H^{s-1}(\Gamma, \mathbb{R}^N))$.*

**Proof:** Fix $\phi \in \mathcal{U}$, $h \in H^{s-1}(\Gamma)$ and set

$$(u, p, \lambda) := L(\phi)^{-1} E_3(hn).$$

Then, by (3.9), (3.10),

$$|\lambda_1| + |\lambda_2| \le C\|h\|_{-2}^{\Gamma}. \tag{3.13}$$

There is a function $d \in H^s(\Omega)$ such that $\mathrm{Tr}_\Gamma d = 0$ and $\partial_j d = n_j$ on $\Gamma$. (Near $\Gamma$, $d$ can be chosen to be the distance function from $\Gamma$ with respect to the metric $\boldsymbol{g}$. The statement about its smoothness and the existence on the whole of $\Omega$ follow then from [7], Lemma 14.16, together with straightforward transformation and cutoff arguments.) We extend $n$ to $\Omega$ by setting $n_j = \partial_j d$.

Set

$$\overline{h} := \frac{1}{\int_\Gamma \omega_{\boldsymbol{g}}\, d\Gamma} \int_\Gamma \omega_{\boldsymbol{g}} h \, d\Gamma$$

and consider the (augmented) Neumann BVP for the Laplace-Beltrami operator of $\boldsymbol{g}$

$$\left.\begin{array}{rcll} \Delta_{\boldsymbol{g}} \psi &=& 0 & \text{in } \Omega, \\ 2\partial_n \psi &=& h - \overline{h} & \text{on } \Gamma, \\ \int_\Omega \sqrt{g}\psi\, dx &=& 0. & \end{array}\right\}$$

By Lemma 3.1,

$$\|\psi\|_{\frac{3}{2}}^{\Omega} \le C\left\|h - \overline{h}\right\|_0^{\Gamma} \le C\|h\|_0^{\Gamma}, \tag{3.14}$$

$$\|\psi\|_{-\frac{1}{2}}^{\Omega} + \|\psi\|_{-1}^{\Gamma} \le C\|h\|_{-2}^{\Gamma} \tag{3.15}$$

with $C$ independent of $\phi \in \mathcal{U}$.

Now set

$$\begin{array}{rcl} v^i &:=& \psi n^i - d\nabla^i \psi, \\ q &:=& -2\nabla^j \psi n_j. \end{array}$$

Note that

$$P^{\boldsymbol{g}} \mathrm{Tr}_\Gamma v = 0$$

and (cf. (3.6))

$$\nabla^j \nabla_j \nabla^i \psi = (\nabla^j \nabla^i - \nabla^i \nabla^j)\nabla_j \psi = R^{k\;\;ji}_{\;\;j}\nabla_k \psi = R^{ik}\nabla_k \psi.$$

Now

$$\begin{array}{rcl} -\nabla^j \nabla_j v^i + \nabla^i q &=& -\Delta_{\boldsymbol{g}} \psi n^i - 2\nabla^j \psi \nabla_j n^i - \psi \nabla^j \nabla_j n^i \\ && +\nabla^j n_j \nabla^i \psi + 2n^j \nabla_j \nabla^i \psi + d\nabla^j \nabla_j \nabla^i \psi - 2\nabla^i \nabla_j \psi n^j - 2\nabla_j \psi \nabla^i n_j \\ &=& -2\nabla^j \psi(\nabla_j n^i + \nabla^i n_j) - \psi \nabla^j \nabla_j n^i + \nabla^j n_j \nabla^i \psi + dR^{ik}\nabla_k \psi, \end{array}$$

$$\begin{array}{rcl} \nabla_i v^i &=& \nabla_i \psi n^i + \psi \nabla_i n^i - n_i \nabla^i \psi - d\Delta_{\boldsymbol{g}} \psi = \psi \nabla_i n^i, \\ \nabla^i v^j + \nabla^j v^i - g^{ij} q &=& \nabla^i \psi n^j + \nabla^j \psi n^i + \psi(\nabla^i n^j + \nabla^j n^i) \\ && -n^i \nabla^j \psi - n^j \nabla^i \psi - 2d\nabla^i \nabla^j \psi + 2g^{ij}\nabla^k \psi n_k, \end{array}$$

$$
\begin{aligned}
\mathrm{Tr}_\Gamma(\nabla^i v^j + \nabla^j v^i - g^{ij} q) n_j &= \mathrm{Tr}_\Gamma(\psi(\nabla^i n^j + \nabla^j n^i)) n_j + (h - \overline{h}) n^i, \\
\int_\Omega \sqrt{g}\partial_k \Phi^i v^k \, dx &= 2\int_\Omega \sqrt{g}\partial_k \Phi^i \psi n^k \, dx, \\
\int_\Omega \sqrt{g}\partial_k \Phi^i \partial_l \Phi^j (\nabla^k v^l - \nabla^l v^k) \, dx &= \int_\Gamma \omega_{\boldsymbol{g}} \partial_k \Phi^i \partial_l \Phi^j (n^j v^i - n^i v^j) \, d\Gamma = 0.
\end{aligned}
$$

Set $(w, s) := (v, q) - (u, p)$ and note that

$$P^{\boldsymbol{g}} A(hn) = P^{\boldsymbol{g}} \mathrm{Tr}_\Gamma u = -P^{\boldsymbol{g}} \mathrm{Tr}_\Gamma w$$

and

$$L(w, s, 0) = L(v, q, 0) - E_3(hn) + L(0, 0, \lambda).$$

The calculations above and (3.13), (3.14) imply

$$
\begin{aligned}
\|L(0, 0, \lambda)\|_{Y_2} &\leq C\|h\|_{-2}^{\Gamma}, \\
\|L(v, q, 0) - E_3(hn)\|_{Y_2} &\leq C\|h\|_0^{\Gamma}.
\end{aligned}
$$

Thus, by Lemma 3.2 and the trace theorem,

$$\|P^{\boldsymbol{g}} A(hn)\|_2^{\Gamma} = \|P^{\boldsymbol{g}} \mathrm{Tr}_\Gamma w\|_2^{\Gamma} \leq C\|(w, s, 0)\|_{X_2} \leq C\|h\|_0^{\Gamma}.$$

Similarly, using Lemma 3.2, (2.1), and (3.15) one gets

$$\|P^{\boldsymbol{g}} A(hn)\|_0^{\Gamma} \leq C\|\mathrm{Tr}_\Gamma w\|_0^{\Gamma} \leq C\|h\|_{-2}^{\Gamma}.$$

Hence (i) is proved.

To show (ii), note at first that $A$ is symmetric with respect to the $L^2(\Gamma, \mathbb{R}^N)$-scalar product induced by $\boldsymbol{g}$ on $\Gamma$, i.e.

$$\int_\Gamma \omega_{\boldsymbol{g}} \varphi_i A^i(\chi) \, d\Gamma = \int_\Gamma \omega_{\boldsymbol{g}} A^i(\varphi) \chi_i \, d\Gamma. \tag{3.16}$$

This fact is a consequence of the Green formula for the Stokes operator with respect to $\boldsymbol{g}$, cf. the proof of Lemma 3.2.

Now fix $w \in P^{\boldsymbol{g}}(H^{s-1}(\Gamma, \mathbb{R}^N))$ and pick $\varphi \in H^{s-1}(\Gamma)$. Using (3.16), the orthogonality of $P^{\boldsymbol{g}}$ and (i) we get

$$
\begin{aligned}
\left|\int_\Gamma \varphi n_i A^i(w) \, d\Gamma\right| &= \left|\int_\Gamma \omega_{\boldsymbol{g}} A^i\left(\widetilde{g}^{-\frac{1}{2}} \varphi n\right) (P^{\boldsymbol{g}} w)_i \, d\Gamma\right| \\
&= \left|\int_\Gamma \omega_{\boldsymbol{g}} \left(P^{\boldsymbol{g}} A\left(\widetilde{g}^{-\frac{1}{2}} \varphi n\right)\right)^i (P^{\boldsymbol{g}} w)_i \, d\Gamma\right| \\
&\leq C\left\|P^{\boldsymbol{g}} A\left(\widetilde{g}^{-\frac{1}{2}} \varphi n\right)\right\|_2^{\Gamma} \|w\|_{-2}^{\Gamma} \leq C\|\varphi\|_0^{\Gamma} \|w\|_{-2}^{\Gamma}.
\end{aligned}
$$

This implies (ii). ∎

The following estimate plays the key role in the analysis of (2.9).

**Lemma 3.6** *($L^2$-a priori estimate for $\mathcal{F}'$)*
*There is a constant $C$ such that*

$$(\mathcal{F}'(\phi)[h], h)_{H^0(\Gamma,\mathbb{R}^N)} \leq C\|h\|_0^{\Gamma\,2}$$

*for all $h \in H^{s+1}(\Gamma, \mathbb{R}^N)$ and $\phi \in \mathcal{U}$.*

**Proof:** We will subsequently give the necessary estimates for all terms occurring in (3.12).

Suppressing the argument $\phi$ again, we have

$$\widetilde{A}'[h]f = -\mathrm{Tr}_\Gamma \Pi_1 \widetilde{L}^{-1}\widetilde{L}'[h]\widetilde{L}^{-1}E_3 f = -\mathrm{Tr}_\Gamma \Pi_1 S L^{-1}T^{-1}\widetilde{L}'[h]z,$$

where $z := SL^{-1}T^{-1}E_3 f$. Note that

$$a_j^{k\,\prime}(\phi)[h] = \widetilde{a}_j^{k\,\prime}(\phi)[\mathcal{E}h],$$

where $\widetilde{a}_j^{k\,\prime}(\phi)$ is a first order differential operator whose coefficients are smooth functions of the first derivatives of $\Phi$. Corresponding results hold for $\widetilde{\nu}'(\phi)[h]$ and $g'(\phi)[h]$. Consequently, using (2.1), (2.2), Lemma 2.1, and the fact that $z$ is uniformly bounded in $X_s$ with respect to $\phi$ we find

$$\|\widetilde{L}'[h]z\|_{Y_t} \leq C\|h\|_t^\Gamma, \qquad t \in [\tfrac{3}{2}, s]$$

$$\left\|\Pi_1\widetilde{L}'[h]z\right\|_{-2}^{\Omega} + \left\|\Pi_2\widetilde{L}'[h]z\right\|_{-1}^{\Omega} + \left\|\nabla\Pi_2\widetilde{L}'[h]z\right\|_{-2}^{\Omega} + \left\|\Pi_2\widetilde{L}'[h]z\right\|_{-\frac{1}{2}}^{\Gamma}$$
$$+\left\|\Pi_3\widetilde{L}'[h]z\right\|_{-\frac{1}{2}}^{\Gamma} + |\Pi_4\widetilde{L}'[h]z| + |\Pi_5\widetilde{L}'[h]z| \leq C\|h\|_{-\frac{1}{2}}^{\Gamma};$$

$$\left\|\Pi_2\widetilde{L}'[h]z\right\|_0^{\Omega} \leq C\|h\|_{\frac{1}{2}}^{\Gamma}.$$

Interpolation yields

$$\left\|\Pi_1\widetilde{L}'[h]z\right\|_{t-\frac{3}{2}}^{\Omega} + \left\|\Pi_2\widetilde{L}'[h]z\right\|_{t-\frac{1}{2}}^{\Omega} + \left\|\nabla\Pi_2\widetilde{L}'[h]z\right\|_{t-\frac{3}{2}}^{\Omega} + \left\|\Pi_2\widetilde{L}'[h]z\right\|_{t}^{\Gamma}$$
$$+\left\|\Pi_3\widetilde{L}'[h]z\right\|_{t}^{\Gamma} + |\Pi_4\widetilde{L}'[h]z| + |\Pi_5\widetilde{L}'[h]z| \leq C\|h\|_{t}^{\Gamma} \tag{3.17}$$

for $t \in [-\frac{1}{2}, \sigma+2]$. In particular, Lemma 3.2 yields

$$\|R_3 h\|_0^\Gamma = \left\|\widetilde{A}'[h]f\right\|_0^\Gamma \leq C\|h\|_0^\Gamma. \tag{3.18}$$

Lemmas 3.2 and 3.4 yield

$$\|R_4 h\|_0^\Gamma = \left\|\Pi_1 S L^{-1}T^{-1}E_3 R_1 h\right\|_0^\Gamma \leq C\|h\|_0^\Gamma. \tag{3.19}$$

Lemma 3.5 (i) with $t = 0$ implies

$$|R_6| \leq C\left\|P^{g}A(\gamma\Delta_{\widetilde{g}}l\, n)\right\|_0^\Gamma \|k\|_0^\Gamma \leq C\left\|\gamma\Delta_{\widetilde{g}}l\right\|_{-2}^\Gamma \|k\|_0^\Gamma \leq C\|h\|_0^{\Gamma\,2}. \tag{3.20}$$

The commutator $[\gamma, \Delta_{\bar{g}}]$ is a first order differential operator in $\Gamma$ whose coefficients are smooth functions of $\phi$ and its first derivatives. Hence, by Lemma 3.2,

$$|R_7| \leq C\big\|A([\gamma, \Delta_{\bar{g}}]ln)\big\|_0^\Gamma \|l\|_0^\Gamma \leq C\big\|[\gamma, \Delta_{\bar{g}}]l\big\|_{-1}^\Gamma \|l\|_0^\Gamma \leq C\|h\|_0^{\Gamma^2}. \tag{3.21}$$

To estimate $R_8$, consider the BVP

$$\left.\begin{array}{rcll} \Delta_g \Psi & = & 0 & \text{in } \Omega, \\ \Psi & = & \frac{1}{2}\gamma l & \text{on } \Gamma. \end{array}\right\}$$

It follows from standard results that this problem is uniquely solvable, and dual estimates as in the proofs of Lemma 2.1 and (3.15) together with an interpolation argument show that there are constants $C$ independent of $\phi \in \mathcal{U}$ such that

$$\|\Psi\|_{\frac{1}{2}}^\Omega + \|\partial_n \Psi\|_{-1}^\Gamma \leq C\|\gamma l\|_0^\Gamma \leq C\|h\|_0^\Gamma. \tag{3.22}$$

With $w := \nabla^{\boldsymbol{g}}\Psi$, we have in $\Omega$ (cf. (3.6))

$$\begin{array}{rcl} \nabla^j \nabla_j w^i & = & R^{ki}\nabla_k \Psi, \\ \nabla_i w^i & = & 0, \end{array}$$

and on $\Gamma$ (cf. (3.7))

$$\begin{array}{rcl} (\nabla^i w^j + \nabla^j w^i)n_j & = & 2\nabla^i \nabla^j \Psi n_j = 2\nabla^k \nabla^j \Psi n_j n_k n^i + 2P^{\boldsymbol{g}}(\nabla^{\boldsymbol{g}}(\nabla^j \Psi n_j))^i \\ & = & -\Delta_{\bar{g}}(\gamma l)n^i - \kappa \partial_n \Psi n^i + 2P^{\boldsymbol{g}}(\nabla^{\boldsymbol{g}}(\nabla^j \Psi n_j))^i. \end{array}$$

Setting $(u, p, \lambda) := L^{-1}E_3(\Delta_{\bar{g}}(\gamma l)n)$ we obtain

$$n_i A^i(\Delta_{\bar{g}}(\gamma l)n) + \frac{1}{2}A_0(\gamma l) = n_i \mathrm{Tr}_\Gamma(u^i + w^i)$$

and

$$(u + w, p, \lambda) = L^{-1}\begin{pmatrix} R^{ki}\nabla_k \Psi \\ 0 \\ -\kappa \partial_n \Psi n^i + 2P^{\boldsymbol{g}}(\nabla^{\boldsymbol{g}}(\nabla^j \Psi n_j))^i \\ \int_\Omega \sqrt{g}\partial_k \Phi^i \nabla^k \Psi \, dx \\ 0 \end{pmatrix}.$$

Using (2.1), Lemmas 3.2 and 3.5 (ii), and (3.22) one finds from this

$$|R_8| \leq C\big\|n_i(u^i + w^i)\big\|_0^\Gamma \|l\|_0^\Gamma \leq C\|h\|_0^{\Gamma^2}. \tag{3.23}$$

To estimate the last term in (3.12), we choose an extension of $\chi$ which will be denoted by the same symbol such that $\chi \in C^2(\overline{\Omega})$ and $\chi \geq 0$. We set $u := (\Delta_{\boldsymbol{g}}, \mathrm{Tr}_\Gamma)^{-1}(0, \psi)$ and note that, by dual estimates as above,

$$\|u\|_0^\Omega \leq C\|\psi\|_{-\frac{1}{2}}^\Gamma \leq C\|h\|_0^\Gamma$$

with $C$ independent of $\phi \in \mathcal{U}$. Then

$$\int_\Gamma \omega_{\boldsymbol{g}}\chi A_0(\psi)\psi \, d\Gamma = \int_\Gamma \omega_{\boldsymbol{g}}\partial_n u \chi \psi \, d\Gamma = \int_\Omega \sqrt{g}\nabla^i u \nabla_i(u\chi) \, dx = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= \int_\Omega \sqrt{g}\chi\nabla^i u \nabla_i u\,dx = \int_\Omega \sqrt{g}\chi \boldsymbol{g}(\nabla^{\boldsymbol{g}} u, \nabla^{\boldsymbol{g}} u)\,dx \geq 0\\
I_2 &= \int_\Omega \sqrt{g} u \nabla^i u \nabla_i \chi\,dx = \frac{1}{2}\int_\Omega \sqrt{g}\nabla^i(u^2)\nabla_i\chi\,dx\\
&= -\frac{1}{2}\int_\Omega \sqrt{g} u^2 \Delta_{\boldsymbol{g}}\chi\,dx + \frac{1}{2}\int_\Gamma \omega_{\boldsymbol{g}}\psi^2\partial_n\chi\,d\Gamma\\
&\geq -C\|u\|_0^{\Omega\,2} - C\|\psi\|_0^{\Gamma\,2} \geq -C\|h\|_0^{\Gamma\,2}.
\end{aligned}$$

Together with (3.18)–(3.21) and (3.23), this proves the lemma. ■

# 4 Chain rule and main result

To derive estimates like the one in Lemma 3.6 for stronger norms, we will apply a chain rule based on the invariance of our problem under reparametrizations. For a precise formulation, we recall that the right hand side of $(2.9)_1$ depends not only on $\phi$ but also on the surface tension coefficient $\gamma$, i.e.

$$\mathcal{F}(\phi) =: \mathcal{G}(\phi, \gamma),$$

where

$$\mathcal{G} \in C^\infty\left(\mathcal{U} \times H^s(\Gamma), H^s(\Gamma, \mathbb{R}^N)\right)$$

and the dependence on the second argument is linear.

We recall that

$$\mathcal{G}(\phi, \gamma) = U|_{\Gamma_\Phi} \circ \phi,$$

where $(U, P, \lambda)$ solves (2.6). Let $\psi \in \mathrm{Diff}^\infty(\Gamma)$, $\phi\circ\psi \in \mathcal{U}$ and set $\Phi_\psi := \mathrm{Id}_\Omega + \mathcal{E}(\phi\circ\psi - \mathrm{Id}_\Gamma)$. Then, because of $\Phi(\Omega) = \Phi_\psi(\Omega)$ and $\gamma \circ \psi \circ (\phi \circ \psi)^{-1} = \gamma \circ \phi^{-1}$,

$$\mathcal{G}(\phi \circ \psi, \gamma \circ \psi) = U|_{\Gamma_\Phi} \circ \phi \circ \psi = \mathcal{G}(\phi, \gamma) \circ \psi. \tag{4.1}$$

Let $D_1, \ldots, D_N$ be $N$ smooth vector fields on $\Gamma$, identified with first order differential operators, such that

$$\mathrm{span}\{D_1, \ldots, D_N\} = T_x\Gamma \quad \forall x \in \Gamma. \tag{4.2}$$

For example, one can choose $D_i$ as orthogonal projection of $\Gamma \times \{e_i\}$ on $T\Gamma$. Then, for all $n \in \mathbb{N}$, the scalar product $(\,\cdot\,,\cdot)_n$ defined by

$$(u, v)_n := \sum_{|\alpha|\leq n} (D^\alpha u, D^\alpha v)_{H^0(\Gamma)}, \quad D^\alpha := D_1^{\alpha_1} \ldots D_N^{\alpha_n}, \quad |\alpha| := \alpha_1 + \ldots \alpha_N,$$

generates a norm on $H^n(\Gamma)$ which is equivalent to the usual one. This follows (via localization) from (4.2) and the compactness of $\Gamma$.

Each $D_i$ generates a one-parameter group of smooth diffeomorphisms $t \mapsto \psi_t^i$. For any $\phi \in \mathcal{U} \cap H^{s+2}(\Gamma)$, there is a small interval $I$ around 0 such that the mapping $t \mapsto (\phi \circ \psi_t^i, \gamma \circ \psi_t^i)$ is in $C^1(I, \mathcal{U} \times H^s(\Gamma))$. Setting $\psi := \psi_t^i$ in (4.1) and differentiating with respect to $t$ at $t = 0$ yields

$$D_i\mathcal{G}(\phi, \gamma) = \mathcal{G}'(\phi, \gamma)[(D_i\phi, D_i\gamma)]$$

for all $\phi \in \mathcal{U} \cap H^{s+2}(\Gamma, \mathbb{R}^N)$. Note that this formula can be interpreted as a generalized chain rule for the nonlocal operator $\mathcal{G}$ as it links the spatial derivatives of $\mathcal{G}(\phi, \gamma)$ to those of $\phi$ and $\gamma$. More generally,

$$\begin{aligned} D^\alpha \mathcal{G}(\phi, \gamma) &= \mathcal{G}'(\phi, \gamma)\,[(D^\alpha \phi, D^\alpha \gamma)] \\ &\quad + \sum_{k=2}^{|\alpha|} \sum_{\alpha_1 + \ldots + \alpha_k = \alpha} C_{\alpha_1 \ldots \alpha_k} \mathcal{G}^{(k)}(\phi, \gamma)\,[(D^{\alpha_1}\phi, D^{\alpha_1}\gamma), \ldots, (D^{\alpha_k}\phi, D^{\alpha_k}\gamma)] \end{aligned} \tag{4.3}$$

for $\phi \in \mathcal{U} \cap H^{s+|\alpha|+1}(\Gamma, \mathbb{R}^N)$, where the second sum is to be taken over all additive decompositions of $\alpha$ with nonzero multiindices $\alpha_l$. In particular, $\mathcal{F}$ is a smooth map from $\mathcal{U} \cap H^{s+|\alpha|+1}(\Gamma, \mathbb{R}^N)$ to $H^{s+|\alpha|}(\Gamma, \mathbb{R}^N)$. For a proof of this (in a completely parallel context) we refer to [14], Lemma 6. As $\mathcal{G}$ is linear in the second argument,we easily find

$$\begin{aligned} &\mathcal{G}^{(k)}(\phi, \gamma)\,[(D^{\alpha_1}\phi, D^{\alpha_1}\gamma), \ldots, (D^{\alpha_k}\phi, D^{\alpha_k}\gamma)] = \mathcal{F}^{(k)}(\phi)[D^{\alpha_1}\phi, \ldots, D^{\alpha_k}\phi] \\ &+ \sum_{j=1}^{k} \left(\partial_\phi^{k-1}\mathcal{G}\right)(\phi, D^{\alpha_j}\gamma)[D^{\alpha_1}\phi, \ldots, D^{\alpha_{j-1}}\phi, D^{\alpha_{j+1}}\phi, \ldots, D^{\alpha_k}\phi], \end{aligned} \tag{4.4}$$

where $\partial_\phi^l \mathcal{G}$ denotes the $l$-th Frechet derivative of $\mathcal{G}$ with respect to the first argument. The following lemma gives the crucial estimate for such derivatives (with general smooth $\gamma$). It shows that the terms in (4.3) that contain higher Frechet derivatives are "of lower order" in the nonlinear estimates.

**Lemma 4.1** *(Estimates on higher Frechet derivatives)*
*For $k \geq 2$, $\phi \in \mathcal{U} \cap H^{s+|\alpha|+1}(\Gamma, \mathbb{R}^N)$, $\gamma \in C^\infty(\Gamma)$, and $(\alpha_1 \ldots \alpha_k)$ as in (4.3), there is a constant $C = C_{k,\mathcal{U},\gamma}$ such that*

$$\left\| \mathcal{F}^{(k)}(\phi)[D^{\alpha_1}\phi, \ldots, D^{\alpha_k}\phi] \right\|_0^\Gamma \leq C \|\phi\|_{|\alpha|}^\Gamma.$$

**Proof:** We have $\mathcal{F}(\phi) = u$ where $(u, p, \lambda) \in X_s$ solves

$$\widetilde{L}(\phi)(u, p, \lambda) = E_3 f(\phi).$$

(Note that the meaning of $u$ here differs from the one in (2.7), Lemma 2.2 etc.) Due to (3.11), we have

$$\begin{aligned} \lambda &= \Pi_3 \widetilde{L}(\phi)^{-1} E_3 f(\phi) = \Pi_3 S(\phi) L(\phi)^{-1} T(\phi)^{-1} E_3 f(\phi) \\ &= \Pi_3 L(\phi)^{-1} E_3 (D\Phi)^{-1} f(\phi) = 0. \end{aligned}$$

As above, we consider $u$ and $p$ as (smooth) functions of $\phi \in \mathcal{U}$ and introduce the shortened notation

$$(u, p)^{(j)}(\phi)[h_1 \ldots, h_j] := (u^{(j)}(\phi)[h_1 \ldots, h_j], p^{(j)}(\phi)[h_1 \ldots, h_j], 0)$$

for their Frechet derivatives. By differentiation one obtains for $k \in \mathbb{N}$

$$\widetilde{L}(\phi)(u, p)^{(k)}(\phi)[D^{\alpha_1}\phi, \ldots, D^{\alpha_k}\phi]$$

$$
\begin{aligned}
&= \quad E_3 f^{(k)}(\phi)[D^{\alpha_1}\phi, \dots, D^{\alpha_k}\phi] \\
&\quad -\sum_{l=1}^{k}\sum_{\pi\in \mathsf{S}_k} \widetilde{L}^{(l)}(\phi)[D^{\alpha_{\pi(1)}}\phi, \dots, D^{\alpha_{\pi(l)}}\phi](u,p)^{(k-l)}(\phi)[D^{\alpha_{\pi(l+1)}}\phi, \dots, D^{\alpha_{\pi(k)}}\phi] \\
&=: \quad (F_k, K_k, H_k, M_{1,k}, M_{2,k}), \qquad (4.5)
\end{aligned}
$$

where $\mathsf{S}_k$ is the set of the permutations of $\{1,\dots,k\}$. For the sake of brevity we introduce the aggregated norm (cf. Lemma 3.2)

$$
|||(u,p)|||_t := ||u||_t^\Gamma + ||u||_{t+\frac{1}{2}}^\Omega + ||\nabla u||_{t-1}^\Gamma + ||p||_{t-1}^\Gamma + ||p||_{t-\frac{1}{2}}^\Omega .
$$

Setting $k=1$ in (4.5) and using (3.17) and Lemmas 3.4 and 3.2 yields

$$
|||(u,p)'(\phi)[h]|||_t \le C||h||_{t+1}^\Gamma, \qquad t\in[-\tfrac{1}{2},\sigma+2]. \qquad (4.6)
$$

We are going to prove the estimate

$$
|||(u,p)^{(j)}(\phi)[D^{\alpha_1}\phi,\dots,D^{\alpha_j}\phi]|||_t \le C||\phi||^\Gamma_{t+\sum_{i=1}^j|\alpha_i|} \qquad (4.7)
$$

for $t\in[0,\sigma+2]$ and arbitrary nonzero multiindices $\alpha_i$ by induction over $j\ge 2$. (4.7) clearly implies the assertion of the lemma.

Let $k\ge 2$ and assume (4.7) holds for all $j$ with $2\le j<k$. Lemma 3.2, (4.5), and $\widetilde{L}(\phi)^{-1} = S(\phi)L(\phi)^{-1}T(\phi)^{-1}$ imply

$$
\begin{aligned}
&|||(u,p)^{(k)}(\phi)[D^{\alpha_1}\phi,\dots,D^{\alpha_k}\phi]\,|||_t \\
\le\ & C\left(||F_k||^\Omega_{t-\frac{3}{2}} + ||K_k||^\Omega_{t-\frac{1}{2}} + ||\nabla K_k||^\Omega_{t-\frac{3}{2}} + ||K_k||^\Gamma_{t-1} + ||H_k||^\Gamma_{t-1} + |M_{1,k}| + |M_{2,k}|\right). \qquad (4.8)
\end{aligned}
$$

We will prove (4.7) for $j=k$ by separately estimating the terms on the right.

1. The Frechet derivative $f^{(k)}(\phi)[D^{\alpha_1}\phi,\dots,D^{\alpha_k}\phi]$ is a finite sum of terms of the form

$$
\mathcal{T}_1 := \mathrm{Tr}_\Gamma a(\nabla\Phi,\nabla^2\Phi)\prod_{j=1}^{k}\partial^{\nu_j}\mathcal{E}D^{\alpha_j}\phi
$$

where $a$ is smooth, $1\le|\nu_j|\le 2$, $\sum_{j=1}^k|\nu_j| = k+1$. We set

$$
\beta_j := |\alpha_j|+|\nu_j|, \quad I:=\{j\,|\,\beta_j>3\}, \quad m:=\sharp I.
$$

and assume without loss of generality that $\beta_k$ is maximal.

1.1. If $m\le 1$ we estimate, using (2.2), Lemma 2.1, and the fact that $|\alpha_k|\le|\alpha|-1$

$$
||\mathcal{T}_1||^\Gamma_{t-1} \le C||\partial^{\nu_k}\mathcal{E}D^{\alpha_k}\phi||^\Gamma_{t-1} \le C||\phi||^\Gamma_{t+|\alpha|}.
$$

1.2. If $m\ge 2$ we set

$$
b:=\sum_{j\in I}\beta_j, \quad \lambda_j := \frac{(\beta_j-3)^+}{b-3m}, \quad \tau:=(t-1)^+, \quad s_j := (1-\lambda_j)(\sigma+1)+\lambda_j\tau.
$$

Then $0 \leq \tau \leq \sigma+1$, $\tau \leq s_j$, $\sum_{j=1}^k s_j = \tau+(k-1)(\sigma+1) > \tau+(k-1)\frac{N-1}{2}$, and by (2.3) and Lemma 2.1,

$$||\mathcal{T}_1||_{t-1}^\Gamma \leq ||\mathcal{T}_1||_\tau^\Gamma \leq C\prod_{j\in I} ||\partial^{\nu_j}\mathcal{E}D^{\alpha_j}\phi||_{s_j}^\Gamma \leq C\prod_{j\in I}||\phi||_{s_j+\beta_j}^\Gamma.$$

Furthermore, because of $b \leq |\alpha|+m+1$ and $m\geq 2$,

$$s_j+\beta_j = (1-\lambda_j)(\sigma+4)+\lambda_j(\tau+b-3m+3) \leq (1-\lambda_j)(\sigma+4)+\lambda_j(\tau+|\alpha|)$$

and hence, by the logarithmic convexity of $t\mapsto ||\phi||_t^\Gamma$,

$$||\phi||_{s_j+\beta_j}^\Gamma \leq C{||\phi||_{\sigma+4}^\Gamma}^{1-\lambda_j}{||\phi||_{\tau+|\alpha|}^\Gamma}^{\lambda_j}.$$

Due to $\tau\leq t$ and $\sum_{j\in I}\lambda_j = 1$, this implies

$$||\mathcal{T}_1||_{t-1}^\Gamma \leq C||\phi||_{t+|\alpha|}^\Gamma.$$

2. The term $F_k$ and the first spatial derivatives of $K_k$ can be written as a finite sum of terms of the form

$$\mathcal{T}_2 := a(\nabla\Phi)\prod_{j=1}^{l}\partial^{\nu_j}\mathcal{E}D^{\alpha_{\pi(j)}}\phi\,\partial^{\nu_{l+1}}u^{(k-l)}(\phi)\left[D^{\alpha_{\pi(l+1)}}\phi,\ldots,D^{\alpha_{\pi(k)}}\phi\right]$$

with $1\leq l\leq k$, $1\leq|\nu_j|\leq 2$, $\sum_{j=1}^{l+1}|\nu_j| = l+2$, and

$$\mathcal{T}_3 := a(\nabla\Phi)\prod_{j=1}^{l}\partial^{\nu_j}\mathcal{E}D^{\alpha_{\pi(j)}}\phi\,\partial^{\nu_{l+1}}p^{(k-l)}(\phi)\left[D^{\alpha_{\pi(l+1)}}\phi,\ldots,D^{\alpha_{\pi(k)}}\phi\right]$$

with $1\leq l\leq k$, $|\nu_j|=1$.

At first we consider $\mathcal{T}_2$. We set

$$\alpha' := \sum_{j=l+1}^{k}\alpha_{\pi(j)},\quad \beta_j := \begin{cases} |\nu_j|+|\alpha_{\pi(j)}|, & j\leq l,\\ |\nu_{l+1}|+|\alpha'|+1, & j=l+1,\end{cases}\qquad I := \{j\,|\,\beta_j>4\},\ m := \sharp I.$$

Moreover, without loss of generality we assume $\beta_j\leq\beta_l$ for $j\leq l$ and introduce the shortened notation

$$u^{(k-l)} := u^{(k-l)}(\phi)\left[D^{\alpha_{\pi(l+1)}}\phi,\ldots,D^{\alpha_{\pi(k)}}\phi\right].$$

2.1. If $m=0$ or $I=\{l\}$ one straightforwardly obtains from (2.2), Lemma 2.1, and (4.6) or (4.7), respectively,

$$\begin{aligned} ||\mathcal{T}_2||_{t-\frac{3}{2}}^\Omega &\leq C\prod_{j=1}^{l-1}||\partial^{\nu_j}\mathcal{E}D^{\alpha_{\pi(j)}}\phi||_{\sigma+\frac{1}{2}}^\Omega ||\partial^{\nu_l}\mathcal{E}D^{\alpha_{\pi(l)}}\phi||_{t-\frac{3}{2}}^\Omega\left\|\partial^{\nu_{l+1}}u^{(k-l)}\right\|_{\sigma+\frac{1}{2}}^\Omega\\ &\leq C{||\phi||_{\sigma+4}^\Gamma}^{l-1}||\phi||_{t+|\alpha|}^\Gamma\left\|u^{(k-l)}\right\|_{\sigma+|\nu_{l+1}|+\frac{1}{2}}^\Omega\\ &\leq C||\phi||_{t+|\alpha|}^\Gamma||\phi||_{\sigma+\beta_{l+1}}^\Gamma \leq C||\phi||_{t+|\alpha|}^\Gamma.\end{aligned}$$

2.2 If $I = \{l+1\}$ one has by (2.2)

$$\|\mathcal{T}_2\|^{\Omega}_{t-\frac{3}{2}} \le C\left\|\partial^{\nu_{l+1}} u^{(k-l)}\right\|^{\Omega}_{t-\frac{3}{2}}.$$

2.2.1. Suppose $k-l=1$. Then, for $t \ge \frac{3}{2}$,

$$\|\partial^{\nu_{l+1}} u'(\phi)\,[D^{\alpha_{\pi(k)}}\phi]\|^{\Omega}_{t-\frac{3}{2}} \le C\|u'(\phi)\,[D^{\alpha_{\pi(k)}}\phi]\|^{\Omega}_{t+\frac{1}{2}} \le C\|\phi\|^{\Gamma}_{t+|\alpha|}$$

because of $|\alpha_{\pi(k)}| \le |\alpha|-1$ and (4.6). Moreover, using (2.1),

$$\begin{aligned}\|\partial^{\nu_{l+1}} u'(\phi)\,[D^{\alpha_{\pi(k)}}\phi]\|^{\Omega}_{-2} \quad &\le \quad C\Big(\|u'(\phi)\,[D^{\alpha_{\pi(k)}}\phi]\|^{\Omega}_{0} + \|\nabla u'(\phi)\,[D^{\alpha_{\pi(k)}}\phi]\|^{\Gamma}_{-\frac{3}{2}} \\ &+ \|u'(\phi)\,[D^{\alpha_{\pi(k)}}\phi]\|^{\Gamma}_{-\frac{1}{2}}\Big) \le C\|\phi\|^{\Gamma}_{-\frac{1}{2}+|\alpha|}.\end{aligned}$$

Interpolation yields

$$\|\partial^{\nu_{l+1}} u'(\phi)\,[D^{\alpha_{\pi(k)}}\phi]\|^{\Omega}_{t-\frac{3}{2}} \le C\|\phi\|^{\Gamma}_{t+|\alpha|}$$

in the general case.

2.2.2. Suppose $k-l \ge 2$. For $t \ge \frac{3}{2}$ we have, by the induction assumption,

$$\left\|\partial^{\nu_{l+1}} u^{(k-l)}\right\|^{\Omega}_{t-\frac{3}{2}} \le C\left\|u^{(k-l)}\right\|^{\Omega}_{t+\frac{1}{2}} \le C\|\phi\|^{\Gamma}_{t+|\alpha'|} \le C\|\phi\|^{\Gamma}_{t+|\alpha|}.$$

For $t \in (\frac{1}{2}, \frac{3}{2})$ we have, using the induction assumption and $|\alpha'| \le |\alpha|-1$,

$$\left\|\partial^{\nu_{l+1}} u^{(k-l)}\right\|^{\Omega}_{t-\frac{3}{2}} \le \left\|\partial^{\nu_{l+1}} u^{(k-l)}\right\|^{\Omega}_{0} \le C\|\phi\|^{\Gamma}_{\frac{3}{2}+|\alpha'|} \le C\|\phi\|^{\Gamma}_{t+|\alpha|}.$$

For $t \le \frac{1}{2}$ we have, using additionally (2.1),

$$\left\|\partial^{\nu_{l+1}} u^{(k-l)}\right\|^{\Omega}_{t-\frac{3}{2}} \le C\left(\left\|\nabla u^{(k-l)}\right\|^{\Omega}_{t-\frac{1}{2}} + \left\|\nabla u^{(k-l)}\right\|^{\Gamma}_{t-1}\right)$$

and

$$\left\|\nabla u^{(k-l)}\right\|^{\Omega}_{t-\frac{1}{2}} \le \left\|\nabla u^{(k-l)}\right\|^{\Omega}_{0} \le C\left\|u^{(k-l)}\right\|^{\Omega}_{1} \le C\|\phi\|^{\Gamma}_{1+|\alpha'|} \le C\|\phi\|^{\Gamma}_{t+|\alpha|}.$$

Together with the induction assumption, these inequalities imply

$$\left\|\partial^{\nu_{l+1}} u^{(k-l)}\right\|^{\Omega}_{t-\frac{3}{2}} \le C\|\phi\|^{\Gamma}_{t+|\alpha|}.$$

2.3. If $m \ge 2$ we set

$$b = \sum_{j\in I}\beta_j, \quad \lambda_j := \frac{(\beta_j-4)^+}{b-4m}, \quad \tau := (t-2)^+, \quad s_j := (1-\lambda_j)\sigma + \lambda_j\tau.$$

Then $0 \le \tau \le \sigma$, $\tau \le s_j \le \sigma$, and $\sum_{j=1}^{l+1}(s_j+\frac{1}{2}) = \tau + \frac{1}{2} + l(\sigma+\frac{1}{2}) > \tau + \frac{1}{2} + l\frac{N}{2}$.

2.3.1. If $l+1 \in I$ then by (2.3), Lemma 2.1, and the induction assumption

$$\begin{aligned} \|\mathcal{T}_2\|^{\Omega}_{t-\frac{3}{2}} &\leq \|\mathcal{T}_2\|^{\Omega}_{\tau+\frac{1}{2}} \leq C \prod_{j\in I\setminus\{l+1\}} \|\partial^{\nu_j}\mathcal{E}D^{\alpha_j}\phi\|^{\Omega}_{s_j+\frac{1}{2}} \left\|u^{(k-l)}\right\|^{\Omega}_{s_{l+1}+|\nu_{l+1}|+\frac{1}{2}} \\ &\leq C\prod_{j\in I} \|\phi\|^{\Gamma}_{s_j+\beta_j}. \end{aligned}$$

As in 1., we find from $b \leq |\alpha| + m + 2$ and $m \geq 2$

$$s_j + \beta_j \leq (1-\lambda_j)(\sigma+4) + \lambda_j(\tau + |\alpha|),$$

and hence

$$\|\mathcal{T}_2\|^{\Omega}_{t-\frac{3}{2}} \leq C\prod_{j\in I} \|\phi\|^{\Gamma}_{s_j+\beta_j} \leq C\|\phi\|^{\Gamma}_{t+|\alpha|}. \tag{4.9}$$

2.3.2. If $l+1 \notin I$, we apply (2.3), estimate, as in 2.1.,

$$\left\|\partial^{\nu_{l+1}}u^{(k-l)}\right\|^{\Omega}_{\sigma+\frac{1}{2}} \leq C,$$

and also get (4.9).

3. The estimates for $\|\mathcal{T}_3\|^{\Omega}_{t-\frac{3}{2}}$ can be obtained in a way completely parallel to 2. Note, however, that one has to set $\beta_{l+1} := |\alpha'| + 3$ here.

4. The term $K_k$ can be written as a finite sum of terms of the form

$$\mathcal{T}_4 := a(\nabla\Phi)\prod_{j=1}^{l} \partial^{\nu_j}\mathcal{E}D^{\alpha_{\pi(j)}}\phi\partial^{\nu_{l+1}}u^{(k-l)}(\phi)\left[D^{\alpha_{\pi(l+1)}}\phi,\ldots,D^{\alpha_{\pi(k)}}\phi\right]$$

with $1 \leq l \leq k$, $|\nu_j| = 1$. The estimates on $\|\mathcal{T}_4\|^{\Omega}_{t-\frac{1}{2}}$ are parallel again, with $\beta_j$ as in 2. and $I := \{j \mid \beta_j > 3\}$. If $m \geq 2$ one has to set

$$\lambda_j := \frac{(\beta_j - 3)^+}{b - 3m}, \quad \tau := (t-1)^+, \quad s_j := (1-\lambda_j)(\sigma+1) + \lambda_j\tau$$

and to use $b \leq |\alpha| + m + 1$.

5. Additionally to the term discussed in 1., $H_k$ contains terms of the forms

$$\mathcal{T}_5 := \mathrm{Tr}_\Gamma a(\nabla\Phi)\prod_{j=1}^{l} \partial^{\nu_j}\mathcal{E}D^{\alpha_{\pi(j)}}\phi\,\partial^{\nu_{l+1}}u^{(k-l)}(\phi)\left[D^{\alpha_{\pi(l+1)}}\phi,\ldots,D^{\alpha_{\pi(k)}}\phi\right]$$

and

$$\mathcal{T}_6 := \mathrm{Tr}_\Gamma a(\nabla\Phi)\prod_{j=1}^{l} \partial^{\nu_j}\mathcal{E}D^{\alpha_{\pi(j)}}\phi\,p^{(k-l)}(\phi)\left[D^{\alpha_{\pi(l+1)}}\phi,\ldots,D^{\alpha_{\pi(k)}}\phi\right]$$

with $1 \leq l \leq k$, $|\nu_j| = 1$. The estimates on $\|\mathcal{T}_5\|^{\Gamma}_{t-1}$ are completely parallel to 4.

6. The estimates on $\|\mathcal{T}_6\|^{\Gamma}_{t-1}$ are also parallel to 4., one has to set $\beta_{l+1} := |\alpha'| + 2$.

7. To estimate the integrals that occur from the terms $M_{1,k}$ and $M_{2,k}$, one uses

$$\left|\int_\Omega v\,dx\right| \le C\|v\|^\Omega_{t-\frac{1}{2}}$$

and proceeds as in 4., as the integrands have the form discussed there (or are simpler). ■

From now on, only function spaces on $\Gamma$ will play a role, and to simplify our notation, we will write $\|\cdot\|_t$ instead of $\|\cdot\|^\Gamma_t$.

Consider the "nonlinear commutator" $[D^\alpha, \mathcal{F}]$ defined on $\mathcal{U} \cap H^{s+\alpha+1}(\Gamma, \mathbb{R}^N)$ by

$$[D^\alpha, \mathcal{F}](\phi) := D^\alpha \mathcal{F}(\phi) - \mathcal{F}'(\phi)[D^\alpha \phi].$$

Equations (4.3) and (4.4) and Lemma 4.1 (with $\gamma$ replaced by its partial derivatives) imply

$$\|[D^\alpha, \mathcal{F}](\phi)\|_0 \le C_\alpha \|\phi\|_{|\alpha|}, \qquad \phi \in \mathcal{U} \cap H^{s+|\alpha|+1}(\Gamma, \mathbb{R}^N). \tag{4.10}$$

This implies the following a priori estimate:

**Lemma 4.2** *( $H^r$ - a priori estimate for $\mathcal{F}$)*
*Let $r \ge s+1$ be integer. Then*

$$(\mathcal{F}(\phi), \phi)_r \le C_r\left(1 + \|\phi\|_r^2\right), \qquad \phi \in \mathcal{U} \cap H^{r+1}(\Gamma, \mathbb{R}^N).$$

**Proof:** It is sufficient to show the estimate for smooth $\phi \in \mathcal{U}$. For such $\phi$, we have from Lemma 3.6, the definition of $(\cdot,\cdot)_r$, and (4.10)

$$\begin{aligned}(\mathcal{F}(\phi), \phi)_r &= \sum_{|\alpha|\le r} (D^\alpha \mathcal{F}(\phi), D^\alpha\phi)_0 = (\mathcal{F}(\phi), \phi)_0 + \sum_{1\le|\alpha|\le r} (D^\alpha \mathcal{F}(\phi), D^\alpha\phi)_0 \\ &= (\mathcal{F}(\phi), \phi)_0 + \sum_{1\le|\alpha|\le r} (\mathcal{F}'(\phi)[D^\alpha\phi], D^\alpha\phi)_0 + \sum_{1\le|\alpha|\le r} ([D^\alpha, \mathcal{F}](\phi), D^\alpha\phi)_0 \\ &\le C_r\left(1 + \|\phi\|_r^2\right).\end{aligned}$$

■

We are ready now to prove our main result. Both its formulation and its proof are oriented at [11]. In particular, we use "norm compression" to simultaneously control two different norms in the Sobolev scale $H^t(\Gamma, \mathbb{R}^N)$. Differing from the treatment of the Euler equations in [11], we cannot conclude strong continuity of the solution in time from strong right continuity because our problem is not time reversible. Instead, we use a nonlinear interpolation argument in our Sobolev scale to show continuous dependence on the initial data. Combined with a compactness argument, this yields the strong continuity result.

We introduce the following notation: Let $r \ge r_0 := s+1$ be integer and set $\mathcal{V} := \mathcal{U} - \mathrm{Id}$, where we assume that $\mathcal{V}$ is a ball of radius $\delta > 0$ around 0 in $H^{r_0}(\Gamma, \mathbb{R}^N)$. Moreover, for $T > 0$ and $X$ an open set in a reflexive Banach space we denote by $C_w([0,T], X)$ the space of functions which are continuous from $[0,T]$ to $X$ with respect to its weak topology. Analogously, $C^1_w([0,T], X)$ denotes the set of weakly differentiable functions from $[0,T]$ to $X$ whose derivative is in $C_w([0,T], X)$.

Setting $\psi := \phi - \mathrm{Id}$, instead of (2.9) we consider the equivalent problem

$$\left.\begin{aligned}\dot\psi &= \mathcal{F}(\psi + \mathrm{Id}), \\ \psi(0) &= \psi_0 := \phi_0 - \mathrm{Id}.\end{aligned}\right\} \tag{4.11}$$

**Theorem 4.3** *(Existence, uniqueness, and regularity of solutions to* (4.11)*)*

**(i)** *For any* $\psi_0 \in \mathcal{V} \cap H^r(\Gamma, \mathbb{R}^N)$ *with* $\|\psi_0\|_{r_0} \leq K < \delta$ *there is a* $T = T(K, r)$ *such that* (4.11) *has a unique solution*

$$\psi = \Psi(\cdot, \psi_0) \in C\left([0, T], \mathcal{V} \cap H^r(\Gamma, \mathbb{R}^N)\right) \cap C^1\left([0, T], H^{r-1}(\Gamma, \mathbb{R}^N)\right).$$

**(ii)** *For any* $r \geq r_0$, $K \in (0, \delta)$, *and* $t \in [0, T(K, r)]$, *the mappings* $\Psi(t, \cdot)$ *are continuous from* $B_0\left(K, H^{r_0}(\Gamma, \mathbb{R}^N)\right) \cap H^r(\Gamma, \mathbb{R}^N)$ *to* $H^r(\Gamma, \mathbb{R}^N)$, *uniformly with respect to* $t$.

**(iii)** *Suppose* $\psi \in C([0, T], \mathcal{V})$ *is a solution to* (4.11) *with* $\psi_0 \in H^r(\Gamma, \mathbb{R}^N)$. *Then* $\psi \in C\left([0, T], H^r(\Gamma, \mathbb{R}^N)\right)$.

**Proof:**

1. We show that for $\psi_0 \in \mathcal{V} \cap H^r(\Gamma, \mathbb{R}^N)$ with $\|\psi_0\|_{r_0} \leq K < \delta$ there is a $T = T(K, r)$ such that (4.11) has a solution

$$\psi \in C_w\left([0, T], \mathcal{V} \cap H^r(\Gamma, \mathbb{R}^N)\right) \cap C_w^1\left([0, T], H^{r-1}(\Gamma, \mathbb{R}^N)\right).$$

By standard arguments, we find that $[\psi \mapsto \mathcal{F}(\psi + \mathrm{Id})]$ is weakly sequentially continuous from $\mathcal{V} \cap H^r(\Gamma, \mathbb{R}^N)$ to $H^{r-1}(\Gamma, \mathbb{R}^N)$ (cf. [14], Lemma 7). There is a $\mu > 0$ such that $K + 3\mu < \delta$. For fixed positive $\varepsilon \leq 1$, we define on $H^r(\Gamma, \mathbb{R}^N)$ the equivalent scalar product $(\cdot, \cdot)_{r,\varepsilon}$ by

$$(u, v)_{r,\varepsilon} := \varepsilon^2 (u, v)_r + (u, v)_{r_0}.$$

The corresponding norm will be denoted by $\|\cdot\|_{r,\varepsilon}$. By Lemma 4.2, $\psi \in \mathcal{V} \cap H^{r+1}(\Gamma, \mathbb{R}^N)$ implies

$$\begin{aligned}(\mathcal{F}(\psi + \mathrm{Id}), \psi)_r &= (\mathcal{F}(\psi + \mathrm{Id}), \psi + \mathrm{Id})_r - (\mathcal{F}(\psi + \mathrm{Id}), \mathrm{Id})_r \\ &\leq C_r(1 + \|\psi\|_r^2) + C_r\, \|\mathcal{F}(\psi + \mathrm{Id})\|_{r-1}\, \|\mathrm{Id}\|_{r+1} \leq C_r(1 + \|\psi\|_r^2)\end{aligned}$$

and analogously

$$(\mathcal{F}(\psi + \mathrm{Id}), \psi)_{r_0} \leq C(1 + \|\psi\|_{r_0}^2).$$

Consequently,

$$\begin{aligned}(\mathcal{F}(\psi + \mathrm{Id}), \psi)_{r,\varepsilon} &= \varepsilon^2 (\mathcal{F}(\psi + \mathrm{Id}), \psi)_r + (\mathcal{F}(\psi + \mathrm{Id}), \psi)_{r_0} \\ &\leq \varepsilon^2 C_r \left(1 + \|\psi\|_r^2\right) + C\left(1 + \|\psi\|_{r_0}^2\right) \leq C_r^* \left(1 + \|\psi\|_{r,\varepsilon}^2\right). \end{aligned} \tag{4.12}$$

We set $\varepsilon := \min\{1, \frac{\mu}{\|\psi_0\|_r}\}$ so that $\|\psi_0\|_{r,\varepsilon} \leq K + \mu$. Moreover, we set $\beta(\rho) := 2C_r^*(1 + \rho)$ and choose $T = T(K, r) > 0$ small enough that any solution $\rho(\cdot)$ of the ODE $\rho' = \beta(\rho)$ with $\rho(0) \leq (K + 2\mu)^2$ satisfies $\rho(t) < \delta^2$ for $t \in [0, T]$. Now the assertion follows as in Theorem A in [11] with $V := H^{r+1}(\Gamma, \mathbb{R}^N)$, $H := \left(H^r(\Gamma, \mathbb{R}^N), \|\cdot\|_{r,\varepsilon}\right)$, $X := H^{r-1}(\Gamma, \mathbb{R}^N)$. Note that we need a slight modification due to the local character of our considerations: A simple ODE argument shows that (4.12) implies $\left\|\psi^{(n)}(t)\right\|_{r,\varepsilon} < \delta$ for all $t \in [0, T]$ and all Galerkin approximations $\psi^{(n)}$, and due to the choice of $T$ we also get $\|\psi(t)\|_{r,\varepsilon} < \delta$, $t \in [0, T]$, for their limit $\psi$. This implies both $\psi^{(n)}(t), \psi(t) \in \mathcal{V}$ and, together with the definition of $\varepsilon$,

$$\|\psi(t)\|_r^2 \leq \delta^2 \varepsilon^{-2} \leq C(1 + \|\psi_0\|_r^2), \quad t \in [0, T]. \tag{4.13}$$

2. Let $\psi, \widetilde{\psi} \in C_w([0,T], \mathcal{V}) \cap C_w^1\left([0,T], H^{r_0-1}(\Gamma,\mathbb{R}^N)\right)$ be two solutions of (4.11) with initial values $\psi_0$ and $\widetilde{\psi}_0$, respectively. Then $\psi, \widetilde{\psi} \in C^1\left([0,T], H^0(\Gamma,\mathbb{R}^N)\right)$, and by Lemma 3.6

$$\begin{aligned}\frac{d}{dt}\left(\left\|\psi-\widetilde{\psi}\right\|_0^2\right) &= 2(\mathcal{F}(\psi+\mathrm{Id})-\mathcal{F}(\widetilde{\psi}+\mathrm{Id}), \psi-\widetilde{\psi})_0 \\ &= 2\int_0^1 (\mathcal{F}'(\widetilde{\psi}+\tau(\psi-\widetilde{\psi})+\mathrm{Id})[\psi-\widetilde{\psi}], \psi-\widetilde{\psi})_0\, d\tau \le C\left\|\psi-\widetilde{\psi}\right\|_0^2.\end{aligned}$$

This implies

$$\left\|\psi(t)-\widetilde{\psi}(t)\right\|_0 \le C\left\|\psi_0-\widetilde{\psi}_0\right\|_0, \quad t\in[0,T], \tag{4.14}$$

and hence uniqueness of the solution to (4.11) constructed in 1.

3. For fixed $t \in [0,T]$ we consider the nonlinear operator $\Psi(t,\cdot)$ assigning to any initial value $\psi_0 \in \mathcal{V}$ the value of the corresponding solution of (4.11) at time $t$. Taking into account that the Sobolev spaces $H^r(\Gamma,\mathbb{R}^N)$, $r\in\mathbb{R}$, form a real interpolation scale with $p=2$ and using (4.13) (with $r$ replaced by $r+1$) and (4.14), (ii) follows from the application of the nonlinear interpolation result given in [3], Proposition A.1 and Remark A.2, to $\Psi(t,\cdot)$.

4. Suppose $\psi \in C([0,T],\mathcal{V})$ is a solution to (4.11) with $\psi_0 \in H^r(\Gamma,\mathbb{R}^N)$. We show that then

$$\psi \in C_w\left([0,T], \mathcal{V}\cap H^r(\Gamma,\mathbb{R}^N)\right) \cap C_w^1\left([0,T], H^{r-1}(\Gamma,\mathbb{R}^N)\right).$$

Set

$$T^* := \sup\left\{t\in[0,T] \;\middle|\; \psi|_{[0,t]} \in C_w\left([0,t], \mathcal{V}\cap H^r(\Gamma,\mathbb{R}^N)\right)\cap C_w^1\left([0,t], H^{r-1}(\Gamma,\mathbb{R}^N)\right)\right\}.$$

We will show $T^*=T$. From 1. and 2., we have $T^*>0$. Assume $T^*<T$. There is a constant $K$ such that $\|\psi(t)\|_{r_0} \le K$ for $t\in[0,T^*]$. Choose now $T_1 \in [0,T^*)$ such that $T^*-T_1 < T(K,r)$ where $T(K,r)$ is given by 1. Due to 1. and the translational invariance of $(4.11)_1$ in time, the initial value problem

$$\begin{aligned}\dot{\widetilde{\psi}} &= \mathcal{F}(\widetilde{\psi}+\mathrm{Id}), \\ \widetilde{\psi}(T_1) &= \psi(T_1)\end{aligned}$$

has a solution $\widetilde{\psi}$ on $[T_1,T_2]$, where $T_2 := \min\{T, T_1+T(K,r)\}$ and

$$\widetilde{\psi} = \psi|_{[T_1,T_2]} \in C_w\left([T_1,T_2], \mathcal{V}\cap H^r(\Gamma,\mathbb{R}^N)\right)\cap C_w^1\left([T_1,T_2], H^{r-1}(\Gamma,\mathbb{R}^N)\right)$$

due to 2. This contradicts the definition of $T^*$ because of $T_2 > T^*$.

5. To complete the proof of (i) and (iii), it remains to show that if

$$\psi \in C_w\left([0,T], \mathcal{V}\cap H^r(\Gamma,\mathbb{R}^N)\right)\cap C_w^1\left([0,T], H^{r-1}(\Gamma,\mathbb{R}^N)\right)$$

is a solution of (4.11) with $\|\psi(t)\|_{r_0} \le K < \delta$, $t\in[0,T]$, then actually

$$\psi \in C\left([0,T], \mathcal{V}\cap H^r(\Gamma,\mathbb{R}^N)\right)\cap C^1\left([0,T], H^{r-1}(\Gamma,\mathbb{R}^N)\right).$$

This will be done by showing that for any $T_1\in[0,T]$,

$$\psi|_{[T_1,T_2]} \in C\left([T_1,T_2], \mathcal{V}\cap H^r(\Gamma,\mathbb{R}^N)\right)\cap C^1\left([T_1,T_2], H^{r-1}(\Gamma,\mathbb{R}^N)\right), \tag{4.15}$$

where $T_2 := \min\{T, T_1 + T(K, r)\}$. Approximate $\psi(T_1)$ by $\psi_1^{(n)} \in H^{r+1}(\Gamma, \mathbb{R}^N)$ so that $\psi_1^{(n)} \to \psi(T_1)$ in $H^r(\Gamma, \mathbb{R}^N)$ and $\left\|\psi_1^{(n)}\right\|_{r_0} \leq K$. By 1., 2., and 4., $\Psi(\cdot, \psi_1^{(n)})$ exists on $[0, T_2 - T_1]$, and

$$\begin{aligned} \Psi(\cdot, \psi_1^{(n)})|_{[0,T_2-T_1]} &\in C_w\left([0, T_2 - T_1], H^{r+1}(\Gamma, \mathbb{R}^N)\right) \cap C_w^1\left([0, T_2 - T_1], H^r(\Gamma, \mathbb{R}^N)\right) \\ &\subset C\left([0, T_2 - T_1], H^r(\Gamma, \mathbb{R}^N)\right) \cap C^1\left([0, T_2 - T_1], H^{r-1}(\Gamma, \mathbb{R}^N)\right). \end{aligned}$$

Moreover, for $t \in [T_1, T_2]$, by 3. and the translational invariance in time,

$$\psi(t) = \Psi(t - T_1, \psi(T_1)) = \lim_{n\to\infty} \Psi(t - T_1, \psi_1^{(n)})$$

in $H^r(\Gamma, \mathbb{R}^N)$, uniformly in $t \in [T_1, T_2]$. As the uniform limit of continuous functions is continuous, this implies (4.15). ■

**Remarks:**

1. Due to the smoothness of $\mathcal{F}$, a solution $\psi \in C\left([0, T], \mathcal{V} \cap H^r(\Gamma, \mathbb{R}^N)\right)$ automatically satisfies
$$\psi \in \bigcap_{k=0}^{r-r_0+1} C^k\left([0, T], H^{r-k}(\Gamma, \mathbb{R}^N)\right).$$
2. Part (iii) shows that no smoothness is lost during the evolution as long as a solution exists, in particular, solutions having smooth initial data are smooth in space and time. Moreover, the existence time ensured by (i) is actually independent of $r$.

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# References


[1] Antanovskii, L.K.: Analyticity of a free boundary in plane quasi-steady flow of a liquid form subject to variable surface tension, in: *Proc. of Conference: The Navier-Stokes Equations II: Theory and Numerical Methods, Oberwolfach 1991,* Springer 1993, 1–16

[2] Antanovskii, L.K.: Creeping thermocapillary motion of a two-dimensional deformable bubble: existence theorem and numerical simulation, *Eur. J. Mech., B/Fluids* **11** (1992) 6 741–758

[3] Beyer, K., Günther, M.: On the Cauchy problem for a capillary drop, Part I: Irrotational motion, *Math. Meth. Appl. Sci.* **21** (1998) 1149–1183

[4] Cummings, L.J., Howison, S.D., King, J.R.: Conserved quantities in Stokes flow with free surfaces, *Phys. Fluids* **9** (1997) 3 477–480

[5] Dierkes, U., Hildebrandt, S., Küster, A., Wohlrab, O.: *Minimal surfaces I,* Springer 1992

[6] Giga, Y.: Analyticity of the semigroup generated by the Stokes operator in $L_r$ spaces, *Math. Z.* **178** (1981) 297–329

[7] GILBARG, D., TRUDINGER, N.S.: *Elliptic partial differential equations of second order,* second edition, Springer 1983

[8] GÜNTHER, M., PROKERT, G.: Existence results for the quasistationary motion of a capillary liquid drop, *Zeitschrift für Analysis und ihre Anwendungen* **16** (1997) 2 311–348

[9] HOPPER, R.W.: Plane Stokes flow driven by capillarity on a free surface, *J. Fluid Mech.* **213** (1990) 349–375

[10] HOWISON, S.D., RICHARDSON, S.: Cusp development in free boundaries, and two-dimensional viscous flows. EUROPEAN JOURN. APPL. MATH. **6** (1995) 441–454

[11] KATO, T., LAI, C.Y.: Nonlinear evolution equations and the Euler flow, *Journ. of Funct. Anal.* **56** (1984) 15–28

[12] LADYSHENSKAYA, O.A.: *The mathematical theory of viscous incompressible flow*, Gordon and Breach 1963

[13] PROKERT, G.: On the existence of solutions in plane quasistationary Stokes flow driven by surface tension, *Euro. Journ. of Appl. Math.* **6** (1995) 539–558

[14] PROKERT, G.: Existence results for Hele-Shaw flow driven by surface tension, *European J. Appl. Math.* **9** (1998) 2 195–221

[15] PROKERT, G.: On evolution equations for moving domains, *Zeitschrift für Analysis und ihre Anwendungen* **18** (1999) 1 67-95

[16] PROKERT, G.: Hyperbolic evolution equations for moving boundary problems, *European J. Appl. Math.* **10** (1999) 607–622

[17] SAINT RAYMOND, X.: *Elementary introduction to the theory of pseudodifferential operators,* CRC Press, 1991

[18] SOLONNIKOV, V.A.: On quasistationary approximation in the problem of motion of a capillary drop, in: Escher, J., Simonett, G. (Eds.): *Topics in nonlinear analysis*, Birkhäuser 1999, 643–671