Transaction costs in finance and inventory research

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Abstract

It is a well-known fact that inventory research has its roots in the monetary theory. In this note we sketch the interaction between inventory theory and finance today. On the one hand we show the development from simple deterministic models to continuous-time stochastic models in modern finance using methods developed in connection with the inventory problem. Hereby we restrict our study to consumption-investment problems in the presence of transaction costs. On the other hand we discuss briefly a few applications of mathematical methods in finance to improve inventory models.

Keywords: Consumption-investment problems; transaction costs; inventory models

In memory of Edward Zabel (1927 - 1999)

1. Introduction

A prominent problem of recent research in modern finance is to extend the well-founded theory of complete financial markets to more realistic (incomplete) markets. That includes pricing and hedging of contingent claims as well as solving consumption-investment problems in the presence of transaction costs (see e.g. [6]).

A few attempts to solve the last problem were made very early. Zabel [32] first examined the impact of transaction costs on optimal consumption and portfolio decisions for a simple dynamic model in discrete time. This happened in 1973 as Black and Scholes [3] published their famous paper of valuing financial instruments of a complete marked in continuous time. Merton [22], [23] and recently Karatzas and Shreve [14] showed that the continuous-time mode of analysis seems to be the preferable method in analyzing basic problems in finance not only for complete markets but also when transaction costs or other marked frictions are introduced.

Discrete-time models dominate inventory research. Holding cash or stocks may be considered as special inventories and a transaction as replenishing. However this obvious analogy is careful to handle. The decision problems, the kinds of costs and the database in finance and inventory research are usually different. Therefore, the methods of analysis in these fields are not the same and develop independently to a great extend. Nevertheless, the interplay of such different methods is important for future research. It may be useful to survey briefly the development of modelling from the deterministic approach given by Ramsey [27] and Hotelling [10] to consumption-investment models under uncertainty with transaction costs. As we shall see, the framework of inventory theory will be used extensively in finance. Reversely, we give finally a preview of applications of mathematical methods in finance to improve inventory models.

2. Consumption-investment problems

„ The natural beginning point for the development of a theory of Finance is the problem of lifetime consumption and portfolio selection for the individual consumer or household unter uncertainty“ ([22], p. 659).
The decision-maker is consumer and investor. He has to decide at each point in time how much of his wealth to consume and how to invest the other part of his wealth in a portfolio of alternative assets. He makes these decisions so as to maximize the expected value of his utility of lifetime consumption.
There is a correspondence between portfolio selection and inventory control. Investor's portfolio may be considered as an inventory (stock) of securities. The portfolio value is depleted or augmented at random as a result of the random fluctuations in the underlying security prices, and inventory is continually depleted by a random demand. Portfolio selection means realign the portfolio proportions, and inventory control leads to reorder stocks for the basic inventory.

The cash balance problem is like a simple consumption-investment problem. In this special case we have only cash in the portfolio, either in the form of cash on hand or as a bank deposit, and a firm as a consumer and investor. Day-to-day inflows and outflows of cash are unpredictable stochastic changes in the cash balance. The goal of the firm is to choose a policy which minimizes expected discounted costs during its lifetime.
Eppen and Fama [8] formulate a discrete cash balance problem with proportional transaction costs as a direct extension of the inventory problem with proportional ordering costs discussed by Bellmann, Glicksberg, Gross [2]. They showed in [8] the optimality of a return-points policy (U, D). That is, if the cash balance is in state $i < U$, the optimal policy is to move it up to state U, while if the cash balance is in state $i > D$, it is optimal to move it down to state D. Inderfurth [11] surveyed further research of cash management.
Phelps [26] investigates one of the simplest stochastic consumption-investment problems. He determines for a risk averse decision-maker the optimal allocation over time of an initial wealth and a given income stream between periodic consumption and investment in a single risky asset by means of dynamic programming.
Consumption-investment problems with transaction costs suppose at least two assets.
Zabel [32] considers a two-period two-asset model. One is a risk-free asset also used for consumption, the other is a risky asset (stock) with a random return. He assumes proportional transaction costs for purchases or sales of stock. The case of fixed cost for purchases or sales of the risky asset has been developed by Mukherjee and Zabel [25]. The results obtained for a special utility function has been extended to all risk averse utility functions by Leland [16] who treats later also option pricing with transaction costs (cf. [17]).
Kamin [13] studies a related portfolio selection model with two risky assets and proportional transaction costs without consumption. In the following sections we will restrict us to develop in detail consumption-investment models with a risk-free and one risky asset.

3. From deterministic to stochastic models

In the 1920s Ramsey [28] asked: „How much of its income should a nation save?“ We formalize this problem using Hotelling's present-value representation of the goal function in [10] as follows:

$$\text{maximize} \quad \int_0^T e^{-\rho t} U[C(t)]dt \tag{1}$$

$$\text{subject to} \quad \dot{W}(t) = P(t) - C(t), W(0) = W_0. \tag{2}$$

Hereby $C(t)$ denotes the consumption rate, $P(t)$ the production rate (income of the community) and $W(t)$ the wealth (capital) at time t. U is a utility function.

From Ramsey's problem (1), (2) we get a deterministic consumption-investment problem if we set

$$P(t) \triangleq rW(t). \tag{3}$$

This means production corresponds to investment in a risk-free asset with rate of yield r. We approximate the derivation of the wealth

$$\begin{aligned} \dot{W}(t) &\approx \frac{\triangle W(t)}{\triangle t} = W(t+1) - W(t) \\ &= rW(t) - [(1+r)W(t) - W(t+1)]. \end{aligned} \tag{4}$$

By discretization we obtain:

$$\text{maximize} \quad \sum_{t=0}^{T} (1+\rho)^{-t} U(C_t) \tag{5}$$

$$\text{subject to} \quad W_{t+1} = (W_t - C_t)(1+r) \tag{6}$$

for prescribed W_0 and W_{T+1}.

Equation (6) may be interpreted as describing the development of wealth in the case of only one risk-free asset.
Samuelson [69] extended the budget equation (6) for a risk-free asset with rate of yield r to an additional risky asset with stochastic return R_t:

$$W_{t+1} = (W_t - C_t)[\alpha_t R_t + (1-\alpha_t)(1+r)], \tag{7}$$

where α_t denotes the fraction of wealth put in the risky asset at times t. Finally, we get *Samuelson's stochastic consumption-investment problem:*

$$\underset{\{C_t,\alpha_t\}}{\text{Max}} \; E \sum_{t=0}^{T} (1+\rho)^{-t} U(C_t) \tag{8}$$

subject to

$$C_t = W_t - \frac{W_{t+1}}{\alpha_t R_t + (1-\alpha_t)(1+r)} \tag{9}$$

W_0 given, W_{T+1} prescribed.

4. From discrete-time to continuous-time models

We return to Ramsey's problem (1), (2) and ask for the behavior of the system in the case of a random influence on production by labor, capital or technical disturbances. That leads to a stochastic control problem whose theory has been began to develop in the 1960s. We will sketch Merton's approach in [20], [21] in formulating and solving a stochastic consumption-investment model in continuous time.

4.1. Price processes

We consider the price change of an asset during a very small time intervall $[t_0, t_0 + h]$. At time $t = t_0 + h$ we have for the price of a risk-free asset

$$X^f(t) = X^f(t_0) \exp{(hr)}, \tag{10}$$

and for the price of a risky asset

$$X(t) = X(t_0) \exp{(hR(t_0))}, \tag{11}$$

where we assume for the rate of return R per unit time:

$$R(t)h = (\mu - \frac{\sigma^2}{2})h + \triangle Y, \tag{12}$$

$$\triangle Y = Y(t) - Y(t_0) = \sigma Z(t)\sqrt{h}, \tag{13}$$

and $Z(t)$ is a process of independent standard normally distributed random variables: $Z(t) \sim \mathcal{N}(0, 1)$.

Passing in the limit to continuous time we obtain as $h \to 0$:

$$dY = \sigma Z(t)\sqrt{dt}, \tag{14}$$

a stochastic differential equation in the sense of Levý [18] as in [20], or

$$dY = \sigma dB \tag{15}$$

a stochastic differential equation in the sense of Itô [12] as in [21].

$B(t)$ is a *Brownian motion*, that is a process of normally distributed random variables: $B(t) \sim \mathcal{N}(0, t)$ with independent increments.

For the price process of a risky asset we find from (11) - (15):

$$dX = X(\mu dt + \sigma dB) \tag{16}$$

$X(t)$ is a *geometric Brownian motion*, that is a process of random variables where the ratio $\frac{X(t_0+h)}{X(t_0)}$ has a lognormal probability distribution with parameters μh and $h\sigma^2$ and is independent of all prices up to time t_0. X is also the basic process in [3].

4.2. The wealth process

We consider the wealth change in the time interval $[t_0, t_0+h]$ when the consumption $C(t_0)h$ is chosen and the proportion $\alpha(t_0)$ of total wealth is invested in the risky asset at time t_0. Then we obtain the discrete-time budget equation (cf. eq. (7)):

$$\begin{aligned} W(t) &= (W(t_0) - C(t_0)h) \cdot \left[\alpha(t_0)\frac{X(t)}{X(t_0)} + (1-\alpha(t_0)\frac{X^f(t)}{X^f(t_0)}\right] \\ &= (W(t_0) - C(t_0)h) \cdot \left[\alpha(t_0)(\exp\ ([\mu - \frac{\sigma^2}{2}]h + \triangle Y) - 1)\right. \\ &\qquad \left. + (1-\alpha(t_0))(\exp\ (hr) - 1) + 1\right]. \end{aligned} \tag{17}$$

The limit as $h \to 0$ leads to the continuous-time budget equation

$$dW(t) = [((\mu - r)\alpha(t) + r)W(t) - C(t)]dt + \sigma\alpha(t)W(t)Z(t)\sqrt{dt}. \tag{18}$$

Using the equivalence of the equations (14) and (15) we find in short form

$$dW = (\mu - r)\alpha W dt + (rW - C)dt + \sigma\alpha W dB. \tag{19}$$

4.3. Merton's consumption-investment problem

After the preparations in the last-sections we are able to formulate a stochastic control problem where $b = b[W(T), T]$ denotes a specified bequest valuation function:

$$\underset{\{\alpha(t), C(t)\}}{\text{Max}}\ E\left(\int_0^T e^{-\rho t}U[C(t)]dt + b[W(T), T]\right) \tag{20}$$

subject to $\quad C(t) \geq 0,\ W(t) > 0,\ W(0) = W_0,$

$$dW(t) = [rW(t) + (\mu - r)\alpha(t)W(t) - C(t)]dt + \sigma\alpha(t)W(t)dB(t). \tag{21}$$

4.4. An example

The stochastic control problem (20), (21) may be solved using the Hamilton-Jacobi-Bellman approach (cf. e.g. [9]). For an infinite horizon $T = \infty, b = 0$, and HARA utility

$$U(c) = \gamma^{-1}c^\gamma\ , \quad 0 < \gamma < 1, \tag{22}$$

we get Merton's optimal policy

$$\alpha^*(t) = \frac{\mu - r}{(1-\gamma)\sigma^2} =: \alpha^*, \qquad C^*(t) = \beta^* W(t) \tag{23}$$

with

$$\beta^* := \frac{\rho}{1-\gamma} - \gamma\left[\frac{(\mu-r)^2}{2\sigma^2(1-\gamma)^2} + \frac{r}{1-\gamma}\right], \tag{24}$$
$$r < \mu < r + (1-\gamma)\sigma^2, \ \rho > \gamma[r + (\mu-r)^2/\sigma^2(1-\gamma)].$$

It is optimal to keep a constant fraction of total wealth in the risky asset and to consume at a rate proportional to total wealth.

5. Inventory processes

We come back to the simple idea to interpret investor's holding in bank and stock expressed in monetary terms as inventories. Let $I_1(t)$ and $I_2(t)$ be the bank inventory and the stock inventory at time t, respectively. Then the wealth at time t is given by

$$W(t) = I_1(t) + I_2(t). \tag{25}$$

We will use these processes to simplify the stochastic consumption-investment problem in Merton's case of no transaction costs and in the case of proportional transaction costs handled by Davis and Norman [7].

5.1. No transaction costs

We consider Merton's model presented in section 4.2 and get

$$(1-\alpha(t))W(t) = I_1(t) \quad , \quad \alpha(t)W(t) = I_2(t). \tag{26}$$

The budget equation (19) may be rewritten

$$\begin{aligned} dW(t) = {} & [(1-\alpha(t))rW(t) - C(t)]dt \\ & + \mu\alpha(t)W(t)dt + \sigma\alpha(t)W(t)dB(t). \end{aligned} \tag{27}$$

Then, we have with (25) and (26) the *evolution equations of inventories*

$$dI_1(t) = [rI_1(t) - C(t)]dt, \tag{28}$$

$$dI_2(t) = I_2(t)[\mu dt + \sigma dB(t)]. \tag{29}$$

Merton's optimal policy (23) induces by (26) „optimal inventories":

$$I_1^*(t) = (1-\alpha^*)W(t), \ I_2^*(t) = \alpha^* W(t). \tag{30}$$

In the (I_1, I_2) plane we obtain the „Merton line“:

$$I_2 = \frac{\alpha^*}{1-\alpha^*} I_1. \tag{31}$$

The investor acts optimally in such a way that the inventory is always on the Merton line.

5.2. Proportional transaction costs

We modify Merton's model supposing that there we charges on all transactions between bank and stock equal to a fixed percentage of the amount transacted. Moreover, we introduce a trading (buying and selling) policy (L_t, N_t), where L_t is the cumulate purchase and N_t the cumulate sale of stock over the interval $[0, t]$. Purchase of dL units of stock requires a payment $(1+\lambda)dL$ from the bank, sale of dN units of stock realizes only $(1-\nu)dN$ in cash.
Therefore, we transfer equations (28), (29) to the evolution equations of the inventories:

$$dI_1(t) = [rI_1(t) - C(t)]dt - (1+\lambda)dL_t + (1-\nu)dN_t, \tag{32}$$

$$dI_2(t) = I_2(t)[\mu dt + \sigma dB(t)] + dL_t - dN_t. \tag{33}$$

Explaining solvency restriction for admissible policies we need the net wealth of the inventory $I = (I_1, I_2) \in \mathbb{R}^2$:

$$\mathcal{W}(I) := I_1 + \min\{(1-\nu)I_2,\ (1+\lambda)I_2\}.$$

The net wealth represents the amount of money in the bank account after performance of the transactions that bring the inventory in the risk asset to zero. Thus, we define

$$\mathcal{R} := \{I \in \mathbb{R}^2 : \mathcal{W}(I) \geq 0\}$$

the solvency region, and

$$\tau := \inf\{t \geq 0\ :\ I(t) \notin \mathring{\mathcal{R}}\}$$

the exit time of the inventory process $I(t)$ of the interior $\mathring{\mathcal{R}}$ of the solvency region $\mathcal{R}$ in the case of bankruptcy.
Davis and Norman's stochastic consumption-investment problem:

$$\underset{\{L_t, N_t, C(t)\}}{\text{Max}}\ E\left(\int_0^\tau e^{-\rho t} U[C(t)]dt\right) \tag{34}$$

subject to $C(t) \geq 0, I(t) \in \mathcal{R}, I(0) = x \in \mathring{\mathcal{R}}$, evolution equations (32), (33).

The problem (34) is a stochastic control problem in which $C(t)$ is a classical stochastic control and L_t, N_t are stochastic singular controls (cf. [9]).

Davis and Norman solved (34) for the HARA utility (22). The optimal policy may be described by a dissection of the solvency region into three wedge-shaped regions

$$\mathcal{R} = \mathcal{R}_b \cup \mathcal{R}_s \cup \mathcal{R}_{nt},$$

where $\mathcal{R}_b$ is the buying region, $\mathcal{R}_s$ the selling region. This means if $x \in \mathcal{R}_b \cup \mathcal{R}_s$, then an optimal behaviour is an instantaneous transaction to the boundary of the no transaction region $\mathcal{R}_{nt}$. If $I(t) \in \overset{\circ}{\mathcal{R}}_{nt}$, then the investor only consumes.

Akian, Menaldi and Sulem [1] investigate the case of more then one risky assets. The value function is obtained as the solution of a variational inequality. This is the same approach as Sulem [31] used to solve an diffusion inventory problem ten years ago.
Further results are surveyed by Cadenillas [6].

6. Conclusion

The interaction between inventory theory and modern finance has been discussed in this note only in one direction. We showed how ideas and methods developed in connection with the classic inventory problem have been used for modelling in finance. Such a job seems to be reasonable and necessary for a broad audience in production and inventory because it is important to take the general shyness of mathematical methods in finance today. In his recent book Korn [15] gives a good introduction for further studies of optimal investment. There are enough researchers who want to apply this tool for their own things. For example, Maccini [19] notices two such open problems: the role of price-setting behavior and inventory movement, and the relationship between interest rates and inventory investment.
Bulinskaya [5] considers inventory control as a part of financial risk management and extend inventory models by capital restrictions. An always actual problem is the oil management. Pricing of commodities and commodity futures (cf. [30], [4], [29], [24]) seems to be an important field of joint research.

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