Conformal Ricci collineations of space–times

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Abstract: We study conformal vector fields on space-times which in addition are compatible with the Ricci tensor (so-called *conformal Ricci collineations*). In the case of Einstein metrics any conformal vector field is automatically a Ricci collineation as well. For Riemannian manifolds, conformal Ricci collineation were called *concircular vector fields* and studied in the relationship with the geometry of geodesic circles. Here we obtain a partial classification of space–times carrying proper conformal Ricci collineations. There are examples which are not Einstein metrics.

Key words: Ricci tensor, conformal gradient field, Einstein spaces, conformal symmetries

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As a well-established concept in General Relativity, *collineations* are symmetry properties of space-times, compare [DS], [Ha]. One can regard them as vector fields preserving certain geometric quantities which are of relevance in General Relativity, like the metric tensor itself, the curvature tensor or the Ricci tensor. Preserving a geometric quantity is usually understood as the vanishing of the Lie derivative of this geometric quantity in direction of this vector field. If X denotes the vector field, we have the *Killing field equation*

$$\mathcal{L}_X g_{ab} = 0$$

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and the equations for *curvature collineations* or *Ricci collineations*

$$\mathcal{L}_X R^a_{bcd} = 0 \quad \text{or} \quad \mathcal{L}_X R_{ab} = 0,$$

respectively, compare [BQ], [CNP], [Fa], [HC], [HRV]. With respect to the Einstein tensor $G_{ab} = R_{ab} - \frac{R}{2} g_{ab}$ or the traceless Ricci tensor $R^{(0)}_{ab} = R_{ab} - \frac{R}{n} g_{ab}$ we can similarly talk about an *Einstein collineation* if $\mathcal{L}_X G_{ab} = 0$ or a *traceless Ricci collineation* if $\mathcal{L}_X R^{(0)}_{ab} = 0$. Similarly, *matter collineations* are considered where the energy-momentum tensor is preserved by the flow in the same manner [HRV].

In the context of a *conformal class of metrics* there is the more general concept that only the conformal class is preserved under the collineation. This is motivated by the idea that all changes are only conformal: Under the flow of X the metric g_{ab} preserves its conformal class, or the Ricci tensor R_{ab} or any other quantity preserves its conformal class (which may be distinct from the conformal class of the metric). In particular, this concerns the case of a *conformal Killing field* characterized by the equation

$$\mathcal{L}_X g_{ab} = 2\sigma g_{ab}$$

where the scalar factor σ is nothing but the divergence of X, up to a constant, compare [SD], [KR4]. In the same context, we define a *conformal Ricci collineation* (CRC) by the combination of the two equations

$$\mathcal{L}_X g_{ab} = 2\sigma g_{ab} \quad \text{and} \quad \mathcal{L}_X R_{ab} = 2\tau g_{ab}.$$

This means that the conformal classes of both the metric and the Ricci tensor are preserved under the flow induced by X. However, we do not require that the Ricci tensor itself is preserved. So our conformal Ricci collineations are more general than the ones considered in [Fa]. In particular for a CRC the flow preserves the eigendirections of the Ricci tensor, not necessarily the Ricci tensor itself, see Lemma 2. This assumption is less restrictive than the classical Ricci collineations.

We call X a *proper conformal Ricci collineation* (PCRC) if X is neither a Killing field nor homothetic, i.e., if σ is not constant. This is a natural generalization of the special case of Einstein metrics satisfying $R_{ab} = \frac{R}{n} g_{ab}$ with a constant scalar curvature R. In this case it is clear that the two equations for a CRC are equivalent and that they reduce to only one equation. Moreover it is well known that for Einstein metrics g_{ab} the equation $\mathcal{L}_X g_{ab} = 2\sigma g_{ab}$ implies

$$\mathcal{L}_{\mathrm{grad}\sigma} g_{ab} = 2\nabla_a \nabla_b \sigma = 2\lambda g_{ab}$$

for a certain scalar function λ. In other words: If an Einstein metric carries a proper conformal Killing field then it carries in addition the conformal gradient field gradσ [Ke2], [KR3]. The classification of all Einstein metrics admitting conformal Killing fields is essentially based on this method [Ka], [KR3]. In the more general case of a metric g_{ab} admitting a PCRC we obtain a similar classification. This is based on the following lemma:

Lemma 1: *For a conformal vector field X on an n-manifold, satisfying $\mathcal{L}_X g = 2\sigma g$, the following conditions (a) and (b) are equivalent:*

(a) $\mathcal{L}_X R_{ab} = 0$ (Ricci collineation)

(b) $\nabla_a \nabla_b \sigma = 0$

Moreover, the following conditions (i) – (v) are equivalent:

(i) $\mathcal{L}_X R_{ab} = 2\tau \cdot g_{ab}$ *for a function* τ (conformal Ricci collineation)

(ii) $\mathcal{L}_X (R_{ab} - \frac{R}{n} g_{ab}) = 0$ (traceless Ricci collination)

(iii) grad(div$X$) *is a conformal vector field*

(iv) $\nabla_a \nabla_b \sigma - \frac{\Delta\sigma}{n} g_{ab} = 0$

(v) X *is* concircular *in the sense of* [Ish], [Ta].

Recall that a vector field X is called *concircular* if the local flow generated by X consists of concircular mappings, i.e., conformal mappings preserving geodesic circles. A transformation of the metric $g \mapsto \bar{g} = \frac{1}{\psi^2} g$ is concircular if and only if $\nabla^2 \psi = \frac{\Delta\psi}{n} g$ (see [Y2], [Ta], [Fe]) or, equivalently, if the two Ricci tensors of g and $\bar{g}$ have the same traceless part, see [KR1]. It seems to be interesting that by Lemma 1 conformal Ricci collineations are precisely the infinitesimal concircular transformations, i.e.

those preserving geodesic circles. Therefore, conformal Ricci collineations could also be called *concircular collineations.* In the slightly different terminology of [Ca], a concircular vector field X is defined as a vector field satisfying $\nabla_Y X = \sigma Y$ for a certain function σ and for any vector Y.

Corollary 1: *If M is compact and if g is positive definite admitting a PCRC then (M, g) is conformally diffeomorphic with the standard sphere.*

This follows from the well known theorem [Ta] that a non-constant solution σ of the equation $\nabla_a\nabla_b\sigma - \frac{\Delta\sigma}{n}g_{ab} = 0$ on a compact Riemannian manifold is possible only for the sphere, up to conformal diffeomorphisms. If, moreover, the scalar curvature is constant, then (M, g) is isometric with the standard sphere, as stated in [YO]. If the two vector fields X and gradσ are parallel everywhere then (M, g) is also isometric with the standard sphere, see Proposition 1.

Proof of Lemma 1: The following equations are well known as part of the integrability condition for the equation $\mathcal{L}_X g = 2\sigma g$, see [Y1;p.160]:

$$\begin{aligned} \mathcal{L}_X R_{ab} &= -(n-2)\nabla_a\nabla_b\sigma - \Delta\sigma \cdot g_{ab} \\ \mathcal{L}_X R &= -2(n-1)\Delta\sigma - 2R\cdot\sigma \ . \end{aligned}$$

Therefore, the implication $(b) \Rightarrow (a)$ is obvious. Furthermore, the trace of $\mathcal{L}_X R_{ab}$ is $-2(n-1)\Delta\sigma$. Consequently, (a) implies $\Delta\sigma = 0$ on the one hand and $\nabla_a\nabla_b\sigma - \frac{\Delta\sigma}{n}g_{ab} = 0$ on the other hand, that is (b).

Another consequence is the equation $(\mathcal{L}_X R_{ab})^\circ = \mathcal{L}_X (R_{ab})^\circ$ where $(\ )^0$ denotes the traceless part of a tensor. This implies $(i) \Leftrightarrow (ii)$. In particular it follows that τ and σ are coupled by the equations

$$\tau = -\frac{n-1}{n}\Delta\sigma, \quad \nabla_a\nabla_b\sigma = -\frac{\tau}{n-1}g_{ab}$$

if (i) is satisfied. $(i) \Leftrightarrow (iv)$ follows from $(\mathcal{L}_X R_{ab})^\circ = -(n-2)(\nabla_a\nabla_b\sigma)^\circ$.

$(iii) \Leftrightarrow (iv)$ is obvious by definition. For $(iv) \Leftrightarrow (v)$ see Theorem 1 in [Ish]. □

Lemma 2: *If X is a conformal Ricci collineation, then the eigendirections of the Ricci tensor are preserved under the flow of the vector field X.*

Proof. For a point $p \in M$ in a neighborhood of p the flow ϕ^t is defined. From the definition of the Lie derivative we obtain:

$$\begin{aligned} \mathcal{L}_X g|_{\phi^s p}(V,W) &= \left.\frac{d}{dt}\right|_{t=s} g|_{\phi^t p}\left(\phi^t_* V, \phi^t_* W\right) \\ &= 2\sigma(\phi^s p) g|_{\phi^s p}\left(\phi^s_* V, \phi^s_* W\right) \end{aligned}$$

It follows that

$$g|_{\phi^s p}(\phi^s_* V, \phi^s_* W) = g_p(V,W)\alpha_s(p)$$

where $\alpha_s(p) := \exp \int_0^s \sigma(\phi^t p)dt$. Now choose an orthonormal basis $e_1, \ldots, e_n$ of eigenvectors for the Ricci tensor in the point p, i.e. we have $\mathrm{Ric}_p(e_i, e_j) = \lambda_i \delta_{ij}$. Here λ_i are the eigenvalues of the Ricci tensor at p.

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=s} \mathrm{Ric}|_{\phi^t p}\left(\phi^t_* e_i, \phi^t_* e_j\right) &= \mathcal{L}_X g|_{\phi^s p}\left(\phi^s_* e_i, \phi^s_* e_j\right) \\ &= 2\tau(\phi^s p) g|_{\phi^s p}\left(\phi^s_* e_i, \phi^s_* e_j\right) \\ &= 2\tau(\phi^s p)\alpha_s(p)\delta_{ij} \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{Ric}|_{\phi^s p}\left(\phi^s_* e_i, \phi^s_* e_j\right) &= \mathrm{Ric}|_p(e_i, e_j) + 2\delta_{ij}\int_0^s \tau(\phi^t p)\alpha_t(p)dt \\ &= \left\{\lambda_i + 2\int_0^s \tau(\phi^t)\alpha_t(p)dt\right\}\delta_{ij} \\ &= \left\{\lambda_i + 2\int_0^s \tau(\phi^t)\alpha_t(p)dt\right\}\alpha_s(p)^{-1} g|_{\phi^s p}\left(\phi^s_* e_i, \phi^s_* e_j\right) \end{aligned}$$

i.e. the flow ϕ^t preserves the eigendirections.

Examples for CRC on space-times:

1. If (M, g) satisfies the Einstein equation with or without cosmological constant then any conformal Killing field is also a CRC in our sense. Conformal Killing fields on semi-Riemannian Einstein spaces (including space-times) were classified in [KR3] after previous work in [Br], [YN], [Half], [Ka], [Ke1], [Ke2].

2. The metric $g_{ab} = -dt^2 + \cos^2(t) g^*_{\alpha\beta}$ carries a PCRC, namely, the gradient field $X = \mathrm{grad} f = \cos(t)\frac{\partial}{\partial t}$ where the function is $f(t) = \sin(t)$. This is independent of the metric $g^*_{\alpha\beta}$ on the 3-dimensional spacelike level. g_{ab} is not an Einstein metric unless $g^*_{\alpha\beta}$ is of constant curvature. In any case, for the function $f(t) = \sin(t)$ one calculates $\nabla^2 f = f'' g = -fg$ and $\nabla^2 f'' = f'''' g = fg$. These are precisely the two equations for a CRC. Here we have $\sigma = f'' = -f,\ \tau = -3f$.

3. The same holds if we replace $\cos(t)$ by $\sin(t), \sinh(t), \cos(t)$ or $\exp(t)$. Up to the choice of additional constants, these possibilities are the only ones where X is a (non-null) gradient and a PCRC which is linearly dependent of grad(div$X$), compare Proposition 1 below. This means that the conformal flow behaves exactly as in the case of a constant curvature metric (de Sitter, anti – de Sitter) except that the levels g^*_{ab} are (more or less) arbitrary.

With regard to the relation between the given conformal vector field X and it divergence, there are essentially two extremal cases:

1. X is parallel to the gradient of the function $\sigma = \frac{1}{4}\mathrm{div}(X)$,

2. X is orthogonal to the gradient of σ.

In the first case there is a local classification as follows:

Proposition 1: *Assume that (M, g) is a space-time admitting a proper conformal Ricci collineation (PCRC) X which is parallel to the gradient of the function $\sigma = \frac{1}{4}\mathrm{div}(X)$. Then on an open and dense subset the metric is a warped product*

$$g_{ab} = \pm dt^2 + \sigma'^2(t) g^*_{\alpha\beta}$$

where $g^*_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ is a metric on a 3-dimensional hypersurface M^*, timelike or spacelike depending on the sign of $\pm dt^2$ and where the warping function $\sigma(t) = \frac{1}{4}\mathrm{div}(X)$ satisfies the equation $\sigma'' + c\sigma = 0$ for a constant $c \neq 0$. Furthermore, $\tau = 3c\sigma$.

If we assume that M is compact and g is positive definite then (M, g) is isometric with the standard sphere. This holds in any dimension.

PROOF: We start with the equations

$$\mathcal{L}_X g_{ab} = 2\sigma g_{ab} \quad \text{and} \quad \mathcal{L}_X R_{ab} = 2\tau g_{ab}$$

where by assumption σ is not constant.
By Lemma 1 we have

$$\mathcal{L}_X R_{ab} = -(n-2)\nabla_a \nabla_b \sigma - \Delta\sigma \cdot g_{ab},$$

$$0 = (\mathcal{L}_X R_{ab})^\circ = -(n-2)(\nabla_a \nabla_b \sigma)^\circ,$$

thus $\nabla_a \nabla_b \sigma = \frac{\Delta\sigma}{n} g_{ab}$, i.e. grad$\sigma$ is conformal. Here n denotes an arbitrary dimension, for space-times we have $n = 4$.

Subcase 1: If $g(\text{grad}\sigma, \text{grad}\sigma) \neq 0$ on an open subset then by a standard lemma [Fi], [KR2] the equation $\nabla_a \nabla_b \sigma = \frac{\Delta\sigma}{n} g_{ab}$ implies that locally the metric is of the form of a warped product

$$g_{ab} = \pm dt^2 + \sigma'^2(t) g^*_{\alpha\beta}$$

where the function σ depends only on the parameter t and where $\sigma' = \frac{d\sigma}{dt}$. Now if X is linearly dependent of gradσ then there is a scalar function $\alpha(t)$ such that $X = \alpha\text{grad}\sigma = \alpha\sigma' \frac{\partial}{\partial t}$. Since X is conformal, this is possible only if

$$2\sigma g(Y,Z) = \mathcal{L}_{\alpha\text{grad}\sigma} g(Y,Z)$$
$$= \alpha \mathcal{L}_{\text{grad}\sigma} g(Y,Z) + (Y\alpha) g(\text{grad}\sigma, Z) + (Z\alpha) g(\text{grad}\sigma, Y)$$
$$= 2\alpha \nabla^2 \sigma(Y,Z) + (Y\alpha)(Z\sigma) + (Z\alpha)(Y\sigma)$$
$$= -\frac{2\alpha\tau}{n-1} g(Y,Z) + (Y\alpha)(Z\sigma) + (Z\alpha)(Y\sigma)$$

for any Y, Z. This is impossible unless α is constant and not zero. Hence we have $X = \frac{1}{c} \cdot \text{grad}\sigma$ for a constant c and, consequently, $\sigma'' = -c\sigma$ and $(n-1)c\sigma = \tau$.

Subcase 2: If gradσ is an isotropic vector field on an open subset then

$$0 = g(\nabla_Y \text{grad}\sigma, \text{grad}\sigma) = \frac{\Delta\sigma}{n} g(Y, \text{grad}\sigma)$$

for any vector field Y. This is possible only if $\Delta\sigma = 0$, hence $\nabla^2\sigma = 0$ and, consequently, $\tau = 0$. This is the case of a Ricci collineation in the classical sense, compare Lemma 1. Note that in this case gradσ is parallel and nowhere vanishing because by assumption σ is not constant. Furthermore, by assumption we have $X = \alpha\mathrm{grad}\sigma$ for a certain function α. By the same argument as above in Subcase 1 it follows that α is constant. But then $2\sigma g_{ab} = \mathcal{L}_X g_{ab} = \alpha\mathcal{L}_{\mathrm{grad}_\sigma} g_{ab} = 2\alpha\nabla_a\nabla_b\sigma = 0$, hence $\sigma = 0$ in contradiction to our assumptions. Hence Subcase 2 cannot occur.

On a compact M the only possibility for σ is a solution of $\sigma'' = -c\sigma$ for a positive c, hence a sine or cosine function. This in turn forces the warped product to be a part of the standard sphere. □

The second case cannot occur for a PCRC according to the following lemma:

Proposition 2: *Assume that (M, g) is a space-time admitting a conformal Ricci collineation (CRC) X which is orthogonal to the gradient of the function $\sigma = \frac{1}{4}\mathrm{div}(X)$. Then either X is a homothetic vector field (i.e., σ is constant) or X is a Ricci collineation in the classical sense, and moreover* gradσ *is a parallel null (or isotropic) vector field. In the latter case X and* gradσ *cannot be parallel because this would contradict Proposition 1.*

PROOF: The Subcase 1 of an isotropic vector field gradσ is the same as in Proposition 1 above. In this case X is necessarily a Ricci collineation in the classical sense, and σ satisfies $\nabla^2\sigma = 0$. Now let us assume that $g(\mathrm{grad}\sigma, \mathrm{grad}\sigma) \neq 0$ on an open subset and that X is orthogonal to gradσ. As in the proof of Proposition 1 we obtain a warped product structure of the metric and

$$\pm 2\sigma(\sigma')^2 = 2\sigma g(\mathrm{grad}\sigma, \mathrm{grad}\sigma) = \mathcal{L}_X g(\mathrm{grad}\sigma, \mathrm{grad}\sigma)$$

$$= 2g(\nabla_{\mathrm{grad}\sigma} X, \mathrm{grad}\sigma) = 2(\sigma')^2 g(\nabla_{\frac{\partial}{\partial t}} X, \tfrac{\partial}{\partial t}) = 0.$$

The last equation follows because on the one hand X and $\frac{\partial}{\partial t}$ are orthogonal, hence $\frac{\partial}{\partial t}\langle X, \frac{\partial}{\partial t}\rangle = 0$, and because on the other hand the t-lines are geodesics. In any case, if $\sigma \neq 0$ on an interval it follows that $\sigma' = 0$, i.e., σ is constant. This means that X is homothetic. □

It seems to be an open question what happens in the mixed case if X and gradσ are neither linearly dependent nor orthogonal. An example seems to be known only in the conformally flat case: The translational vector field on the flat Minkowski space induces a CRC after conformal inversion. This is not a gradient field, and it is not linearly dependent of the gradient of its divergence [KR2].

As far as the global structure is concerned, the most important case seems to be the following:

Corollary 2: *Assume that (M, g) is a space-time admitting a proper conformal Ricci collineation (PCRC) X such that the gradient of $\sigma = \frac{1}{4}\mathrm{div}X$ is timelike and such that X is linearly dependent of* gradσ. *Assume furthermore that for an open t-interval of values of the function σ the t-level of σ is a complete (necessarily spacelike) hypersurface of M. Then $X = c\,\mathrm{grad}\sigma$ for a constant c (thus X is also a gradient field) and (M, g) contains a Lorentzian warped product $g_{ab} = -dt^2 + \sigma'^2(t) g^*_{\alpha\beta}$ where $g^*_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ is a complete Riemannian metric which is independent of t and where the warping function $\sigma(t)$ is defined on a certain interval and satisfies the equation $\sigma'' + k\sigma = 0$ for a constant $k \neq 0$.*

Proof: By Proposition 1 the metric is locally a warped product $g_{ab} = \pm dt^2 + \sigma'^2(t) g^*_{\alpha\beta}$ where $g^*_{\alpha\beta}$ can be chosen as the metric on one of the complete levels. Since the flow of X moves the levels onto one another, their metric can change only by a homothetic factor depending on t which is nothing but the warping function. Then the assertion follows from Proposition 1. □

Depending on the sign of k, the solution can be global in t or not: In the 'elliptic case' $k > 0$ we run into a singularity at each zero of σ' unless the metric on the levels is of constant positive curvature. In the 'hyperbolic case' $k < 0$ there are solutions which are forward complete (if $\sigma'(t) = \sinh t$ or $\sigma'(t) = \exp t$) or complete (if $\sigma'(t) = \cosh t$).

Similar results were formulated in [Ta] for complete Riemannian manifolds, where the condition on the existence of an PCRC is replaced by the assumption that the manifold carries a concircular vector field (or infinitesimal concircular transformation). However, in the Riemannian case emphasis was given to the case $k = 0$ which does not seem to be primarily interesting here.

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