CONFORMAL EINSTEIN SPACES IN N-DIMENSIONS

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ABSTRACT. This paper presents a set of necessary and sufficient conditions for a n-dimensional semi-Riemannian manifold to be conformal to an Einstein space. We extend results due to C.N. Kozameh, E.T. Newman and K.P. Tod who solved the problem in the four dimensional Lorentz case for manifolds with non-degenerate Weyl tensor, i.e. for space-times with $J \neq 0$. In particular, in n-dimension we will find tensorial conditions if the Weyl tensor operates injectively on the alternating two-forms. Moreover, in the four dimensional Riemannian case we can always decide whether a manifold is locally conformal to an Einstein space.

1. INTRODUCTION

Already in the 1920's conformal transformations of Einstein spaces have been studied by Brinkmann [3], [4]. The aim of the present paper is to find necessary and sufficient tensorial conditions for manifolds being conformally related to Einstein spaces. Conformal Einstein spaces are of particular interest in General Relativity and Quantum Gravity. But also from the mathematical viewpoint these manifolds are very important, since a conformal class of such manifolds which we will consider is represented (up to scaling) by exactly one Einstein space.

In 1985, Kozameh, Newman and Tod [11] found a set of two independent conditions being necessary and sufficient for a conformal Einstein space-time with $J \neq 0$. To solve the problem in n-dimension for manifolds with any signature we will modify the ideas of Kozameh, Newman and Tod. One of their ideas is to generalize the problem and to ask: Under which tensorial conditions is a manifold conformally related to a C-space, i.e. a space with harmonic Weyl tensor? This question is a much more easier one than the one above, since it is linear in the gradient of the conformal factor. With the solution of the linear problem it is possible to give necessary and sufficient conditions for a conformal Einstein space.

We consider smooth semi-Riemannian manifolds (M^n, g) of dimension $n \geq 4$, because in three dimension every Einstein space is conformally flat and thus the problem is trivially solved by the Weyl-Schouten theorem [15, Thm. C.9]. A manifold having a Weyl tensor with vanishing divergence is called *C-space* or *space with harmonic Weyl tensor*. A semi-Riemannian manifold (M^n, g) is conformal to such a C-space if and only if there exists a smooth function $\phi : M \to \mathbb{R}$ satisfying

$$0 = (div_4 W)(X, Y, Z) + (3 - n) W(X, Y, Z, grad_g \phi) \, .$$

In section 3 we show that if we consider W as endomorphism of $\Lambda^2(M)$:

$$W : \Lambda^2(M) \to \Lambda^2(M) \, , \; \omega \mapsto W[\omega]$$

and if this endomorphism is injective in every point of M, i.e. $(\det W)(p) \neq 0$ for all $p \in M$, then the gradient of ϕ can be written as:

$$grad_g \phi = \frac{1}{(n-3)(n-1)} \sum_{i,k=1}^{n} \epsilon_i \epsilon_k W^{-1}[(div_4 W)_{E_i}](E_i, E_k) \, E_k \, ,$$

where W^{-1} is the inverse map of $W \in End(\Lambda^2(M))$. Let $\mathbb{T}$ be the vector field given by the right hand side of this Equation, then $\mathbb{T}$ can be defined for all manifolds with $\det W \neq 0$. Since every

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Einstein space is a C-space, we use in section 4 this special vector field to give a tensorial condition for a conformal Einstein space. Let F_V be the $(0,2)$ tensor field given by

$$F_V(X,Y) := \langle \nabla_X V, Y\rangle + \langle X, V\rangle \langle Y, V\rangle - \frac{1}{n}\left[div(V) + \langle V, V\rangle\right]\langle X, Y\rangle \ ,$$

then a semi-Riemannian manifold with $(\det W)(p) \neq 0$ for all p is locally conformally related to an Einstein space if and only if

$$Ric^\circ + (n-2)F_{\mathbb{T}} = 0\,,$$

where Ric° denotes the traceless Ricci tensor. In section 5 we extend these results to the four dimensional Riemannian case. If we consider Riemannian four manifolds with $W \neq 0$, then we can use the identity $W_{abci}W^{abcj} = \delta_i^j |W^2|$ to derive $\mathbb{T}$, since $W \neq 0$ implies $|W^2| = \frac{1}{4}W_{abcd}W^{abcd} \neq 0$. That means for every point p with $W_{|p} \neq 0$, the vector field $\mathbb{T}$ is given by:

$$\mathbb{T} = \frac{2}{|W^2|}\sum_{i,k=1}^{4} W[(div_4 W)_{E_i}](E_i, E_k)\, E_k\,.$$

Moreover, we will show that a connected Einstein manifold is conformally flat or there is no open subset with $W = 0$. Thus, the problem to find necessary and sufficient tensorial conditions for a conformal Einstein space is solved in the four dimensional Riemannian case.

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2. Preliminaries

Let (M^n, g) be a (smooth) semi-Riemannian manifold of dimension $n \geq 4$, i.e. g is a non-degenerate inner product. We denote by ∇ the Levi-Civita connection and by R the Riemannian curvature tensor:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,.$$

Mostly, we consider R as $(0,4)$ tensor and use $\langle X, Y\rangle$ instead of $g(X,Y)$. The Ricci tensor Ric is given by $Ric(X,Y) = trace\{V \mapsto R(V,X)Y\}$ and the normalized scalar curvature by $S = \frac{1}{n(n-1)}trace(Ric)$. Using the Kulkarni-Nomizu product:

$$(g \star h)(X,Y,Z,T) := g(X,T)h(Y,Z) + g(Y,Z)h(X,T) - g(X,Z)h(Y,T) - g(Y,T)h(X,Z)$$

we obtain the Weyl tensor W and the Schouten tensor $\mathfrak{h}$ (cf. [15, Cor. B.8]):

$$W := R - g\star\mathfrak{h} \quad , \qquad \mathfrak{h} := \frac{1}{n-2}\left(Ric - \frac{n}{2}Sg\right)\,.$$

Let $\mathfrak{X}(M)$ be the set of all vector fields and $\mathfrak{T}^r_s(M)$ be the set of all (r,s) tensor fields of M. There exist two differential operators for vector fields:

$$div : \mathfrak{X}(M) \to C^\infty(M)\,,\ V \mapsto \mathfrak{C}(\nabla V)\,,$$

$$rot : \mathfrak{X}(M) \to \mathfrak{T}^0_2(M)\,,\ V \mapsto dV^\#\,,$$

where $\mathfrak{C}$ is the natural contraction, $V^\#$ is the corresponding one-form to V (i.e. $V^\#(X) = \langle V, X\rangle$ for all X) and d is the exterior derivative of alternating forms. It is easy to verify (cf. [17, Exercise 3.18]):

$$rot(V)(X,Y) = \langle \nabla_X V, Y\rangle - \langle X, \nabla_Y V\rangle\ . \tag{1}$$

One can generalize the divergence operator on tensor fields:

$$div_r : \mathfrak{T}^0_s(M) \to \mathfrak{T}^0_{s-1}\,,\ A \mapsto \mathfrak{C}_{r(s+1)}(\nabla A)\,,$$

where $\mathfrak{C}_{rs}$ is the metric (r,s) contraction and ∇A is the $(0,s+1)$ tensor field given by:

$$(\nabla A)(X_1,...,X_s,V) := (\nabla_V A)(X_1,...,X_s)\,.$$

Using a frame field $E_1, ..., E_n$, i.e. $\langle E_i, E_j \rangle = \epsilon_i \delta_{ij}$, $\epsilon_i = \pm 1$, we obtain:

$$(div_r A)(X_1, ..., X_{s-1}) = \sum_{i=1}^{n} \epsilon_i (\nabla_{E_i} A)(X_1, ..., X_{r-1}, E_i, X_r, ..., X_{s-1}) \,. \tag{2}$$

Moreover, there is a generalization of the rotation operator on tensor fields:

$$rot : \mathfrak{T}_2^0(M) \to \mathfrak{T}_3^0(M) \,,\ A \mapsto rot(A) \,,$$

where $rot(A)$ is given by

$$rot(A)(X, Y, Z) := (\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z) \,.$$

With the help of these differential operators the second Bianchi identity now supplies the following relations between the Riemannian curvature tensor and the Ricci tensor, as well as between the Weyl and the Schouten tensor (cf. [2, (16.3)]):

$$div_4 R = rot(Ric) \quad , \qquad div_4 W = (n-3) rot(\mathfrak{h}) \,. \tag{3}$$

Two semi-Riemannian manifolds (M, g) and (N, h) are called *conformally equivalent* if there exist a diffeomorphism $f : M \to N$ and a smooth function $\psi : M \to (0, \infty)$ satisfying $f^* h = \psi^{-2} g$. Since this diffeomorphism f is an isometry from $(M, \overline{g} := \psi^{-2} g)$ to (N, h), we consider conformal transformations of the type: $(M, g) \to (M, \overline{g} := \psi^{-2} g)$. The corresponding symbols for $(M, \overline{g})$ will be denoted by $\overline{\nabla}, \overline{R}, \overline{W}, ...$ Now let $(M, g) \to (M, \overline{g} := \psi^{-2} g)$ be such a conformal transformation with $\psi = e^{\phi}$ and $\phi : M \to \mathbb{R}$ smooth, then the Levi-Civita connections and the Weyl tensors satisfy the relations (cf. [12, Lemma A.1] and [15, Prop. C.4]):

$$\overline{\nabla}_X Y = \nabla_X Y - [(X\phi) Y + (Y\phi) X - \langle X, Y \rangle \, grad_g \phi] \,, \tag{4}$$

$$\overline{W} = \psi^{-2} W \,. \tag{5}$$

3. Spaces with harmonic Weyl tensor

Definition 1. A semi-Riemannian manifold (M^n, g) of dimension $n \geq 4$ is called *C-space* or *space with harmonic Weyl tensor* (cf. [2, (16.D)]) if the divergence of the Weyl tensor vanishes:

$$div_4 W = 0 \,.$$

Lemma 1. *Let $(M, g) \to (M, \overline{g} := \psi^{-2} g)$ with $\psi = e^{\phi}$ be a conformal transformation. The divergences of the Weyl tensors satisfy for all $X, Y, Z \in \mathfrak{X}(M)$:*

$$(\overline{div_4} \overline{W})(X, Y, Z) = (div_4 W)(X, Y, Z) + (3 - n) W(X, Y, Z, grad_g \phi) \,.$$

Proof. Use the formulae (2), (5) and (4) and the first Bianchi identity. □

Now we want to find tensorial conditions for manifolds being conformally related to C-spaces. First of all, we define the Weyl tensor as an operator on $(0, 2)$ tensor fields.

Definition 2. Let $A \in \mathfrak{T}_4^0(M)$ have the same symmetries like the Riemannian curvature tensor. A is an endomorphism of $\mathfrak{T}_2^0(M)$ in the following way:

$$A : \mathfrak{T}_2^0(M) \to \mathfrak{T}_2^0(M) \,,\ b \mapsto A[b] := \mathfrak{C}_{34} \mathfrak{C}_{16} (A \otimes b) \,,$$

where $A \otimes b$ is the $(0, 6)$ tensor field given by $(A \otimes b)(X, Y, Z, T, U, V) := A(X, Y, Z, T)\, b(U, V)$. Using a frame field $E_1, ..., E_n$, we obtain:

$$A[b](X, Y) = \sum_{i,j} \epsilon_i \epsilon_j A(E_i, X, Y, E_j) b(E_j, E_i) \,.$$

When $\Lambda^2(M)$ denotes the set of all skew-symmetric $(0, 2)$ tensor fields, the first Bianchi identity implies for all $\omega \in \Lambda^2(M)$:

$$A[\omega](X, Y) = \frac{1}{2} \sum_{i,j} \epsilon_i \epsilon_j A(E_i, E_j, X, Y) \omega(E_i, E_j) \,,$$

and thus A is in particular an endomorphism of $\Lambda^2(M)$.
For vector fields X and Y let $X \wedge Y$ be the two-form with

$$(X \wedge Y)(U,V) := \det \begin{pmatrix} \langle X,U\rangle & \langle X,V\rangle \\ \langle Y,U\rangle & \langle Y,V\rangle \end{pmatrix} ,$$

then the operation above satisfies:

$$A[X \wedge Y](Z,T) = A(X,Y,Z,T) \,. \tag{6}$$

Definition 3. Let T be a vector field. Then we call the $(0,3)$ tensor C_T given by

$$C_T(X,Y,Z) := (div_4 W)(X,Y,Z) + (3-n)W(X,Y,Z,T)$$

conformal C-space integrability tensor.

Definition 4. We consider W as an endomorphism of $\Lambda^2(M)$. If the smooth function $\det W : M \to \mathbb{R}$ is different from zero for all $p \in M$, the Weyl tensor W is called *non-degenerate*.

If W is non-degenerate, there exists an endomorphism W^{-1} of $\Lambda^2(M)$ satisfying $W^{-1} \circ W = Id_{\Lambda^2(M)}$. Moreover, from Equation (5) we can conclude that non-degeneracy is preserved under conformal transformations.

Proposition 1. *Let (M^n, g) be a semi-Riemannian manifold with non-degenerate Weyl tensor, then the following holds:*
a) Every vector field T satisfying the relation

$$C_T = 0$$

is locally a gradient, i.e. every point has a neighbourhood U and a smooth function $\phi : U \subset M \to \mathbb{R}$ with $grad_g \phi = T_{|U}$.
b) There is at most one vector field T with $C_T = 0$.

Proof. a) We show that $C_T = 0$ implies $rot(T) = dT^{\#} = 0$: Taking the divergence with respect to the third argument of the $(0,3)$ tensor C_T, we obtain using a frame field $E_1, ..., E_n$:

$$\begin{aligned} 0 &= div_3(C_T)(X,Y) \\ &= div_3(\mathfrak{C}_{45}(\nabla W + (3-n)W \otimes T^{\#}))(X,Y) \\ &= (div_3 div_4 W)(X,Y) + (3-n)\sum \epsilon_j \mathfrak{C}_{45}[(\nabla_{E_j} W) \otimes T^{\#} + W \otimes (\nabla_{E_j} T^{\#})](X,Y,E_j) \,. \end{aligned}$$

$div_3(div_4 W) = 0$ is a consequence of (3) and $div_3(div_4 R) = 0$, and thus we conclude:

$$\begin{aligned} 0 &= \sum\sum \epsilon_i \epsilon_j [(\nabla_{E_j} W)(X,Y,E_j,E_i)T^{\#}(E_i) + W(X,Y,E_j,E_i)(\nabla_{E_j} T^{\#})(E_i)] \\ &= -(div_4 W)(X,Y,T) + \tfrac{1}{2}\sum\sum \epsilon_i \epsilon_j W(X,Y,E_j,E_i)[(\nabla_{E_j} T^{\#})(E_i) - (\nabla_{E_i} T^{\#})(E_j)] \\ &= (3-n)W(X,Y,T,T) + W[dT^{\#}] \,. \end{aligned}$$

Since $W \in End(\Lambda^2(M))$ is injective, the claim follows from $W[dT^{\#}] = 0$ and the Poincaré Lemma.
b) We have to prove that the homogeneous problem:

$$0 = W(X,Y,Z,T) \quad \text{for all} \quad X,Y,Z \in \mathfrak{X}(M)$$

has only the trivial solution. Using the operation of W on $\Lambda^2(M)$, this is equivalent to:

$$0 = W[Z \wedge T] \quad \text{for all} \quad Z \in \mathfrak{X}(M) \,.$$

Since W is injective, we conclude from the last Equation:

$$0 = Z \wedge T \quad \text{for all} \quad Z \,,$$

and thus we obtain $T = 0$. □

The last proposition implies that for manifolds with non-degenerate Weyl tensor there is (up to scaling) at most one way to transform a manifold conformal into a C-space. The question is now: How can we express the gradient of the conformal factor? Since the covariant derivative preserves the symmetries of a tensor, we can define the divergence of the Weyl tensor with respect to a vector field as alternating two-form:

$$(div_4 W)_Z \in \Lambda^2(M)\,, \quad \text{where} \quad (div_4 W)_Z(X,Y) := (div_4 W)(X,Y,Z)\,.$$

Let (M^n, g) be conformal to a C-Space. Then there exists a function $\phi : M \to \mathbb{R}$ satisfying for all $X, Y, Z \in \mathfrak{X}(M)$

$$0 = (div_4 W)(X,Y,Z) + (3-n)W(X,Y,Z,grad_g\phi)\,.$$

This Equation is under consideration of formula (6) equivalent to:

$$0 = (div_4 W)_Z + (3-n)W[Z \wedge grad_g\phi] \quad \text{for all} \quad Z\,.$$

If W is non-degenerate, we obtain for all $Z \in \mathfrak{X}(M)$

$$Z \wedge grad_g\phi = \frac{1}{n-3}W^{-1}[(div_4 W)_Z]\,.$$

Use a frame field $E_1, ..., E_n$ and choose $Z = E_i$, then the last Equation implies for all $k \neq i$:

$$\epsilon_i E_k\phi = \frac{1}{n-3}W^{-1}[(div_4 W)_{E_i}](E_i, E_k)\,.$$

After multiplication by ϵ_i the left hand side is independent of i and so the right hand side has to be equal for all $i \neq k$:

$$E_k\phi = \frac{1}{(n-1)(n-3)}\sum_{i=1}^{n}\epsilon_i W^{-1}[(div_4 W)_{E_i}](E_i, E_k)$$

Definition 5. Let (M^n, g) be a manifold with non-degenerate Weyl tensor. When $E_1, ..., E_n$ is a frame field, the following vector field can be defined on M:

$$\mathbb{T} := \frac{1}{(n-1)(n-3)}\sum_{i,k=1}^{n}\epsilon_i\epsilon_k W^{-1}[(div_4 W)_{E_i}](E_i, E_k)\,E_k\,.$$

From the calculation above respectively from Proposition 1 we can see that $\mathbb{T}$ is the only possible solution of $C_T = 0$, and thus we can formulate the following theorem:

Theorem 1. *A semi-Riemannian manifold (M^n, g) having a non-degenerate Weyl tensor is locally conformally related to a C-space if and only if*

$$C_{\mathbb{T}} = 0\,.$$

Proof. The calculation above supplies the necessity of condition $C_{\mathbb{T}} = 0$.
Proposition 1 implies that $\mathbb{T}$ is locally a gradient, and so we obtain the claim from Lemma 1. □

4. Conformal Einstein Spaces

Definition 6. A semi-Riemannian manifold (M, g) of dimension $n \geq 3$ is called *Einstein space* if the traceless Ricci tensor

$$Ric^\circ := Ric - (n-1)Sg$$

vanishes. In this case, S is constant.

We consider, because of the argument in the introduction, manifolds of dimension $n \geq 4$. Let $(M, g) \to (M, \overline{g} := \psi^{-2}g)$ with $\psi = e^\phi$ be a conformal transformation. The Ricci tensor has the following transformation behaviour (cf. [12, Lemma A.1]):

$$\overline{Ric} = Ric + (n-2)[\nabla^2\phi + d\phi \otimes d\phi] + [\triangle\phi - (n-2)\,\langle grad_g\phi, grad_g\phi\rangle]\,g\,, \tag{7}$$

where $\nabla^2\phi$ is the Hessian of ϕ (i.e. $\nabla^2\phi(X,Y) = \langle\nabla_X grad_g\phi, Y\rangle$) and $(d\phi\otimes d\phi)(X,Y) = (X\phi)(Y\phi)$. If $(M, \overline{g})$ is an Einstein space, we conclude taking the trace of the last Equation for the Ricci tensor of (M, g) the condition:

$$0 = Ric - (n-1)Sg + (n-2)[\nabla^2\phi + d\phi \otimes d\phi] - \frac{n-2}{n}\,[\triangle\phi + \langle grad_g\phi, grad_g\phi\rangle]\,g\,. \tag{8}$$

Definition 7. Let V be a vector field, then the traceless $(0,2)$ tensor field F_V given by

$$F_V(X,Y) := \langle \nabla_X V, Y\rangle + \langle X, V\rangle \langle Y, V\rangle - \frac{1}{n}\left[div(V) + \langle V, V\rangle\right] \langle X, Y\rangle$$

is called *Schwarzian tensor* (cf. [18, Introduction]).

Remark 1. Compare formula (1), then the Poincaré Lemma implies: The $(0,2)$ tensor F_V is symmetric, if and only if V is locally a gradient.

With the help of Definition 7 Equation (8) can be written as follows:

(9) $$0 = Ric^\circ + (n-2)F_{grad_g \phi}\,,$$

i.e. a manifold is conformal to an Einstein space if and only if there exists a function $\phi : M \to \mathbb{R}$ that satisfies (9). In [4], Brinkmann considered solutions of the form: $F_{grad_g \phi} = 0$ which is equivalent to $\nabla^2 e^\phi = \frac{\triangle e^\phi}{n} g$. In this case, also (M,g) is an Einstein manifold. In [18], Osgood and Stowe call a conformal diffeomorphism $f : (M,g) \to (\overline{M}, \overline{g} = e^{-2\phi} g)$ between Riemannian manifolds a *Möbius transformation,* if the gradient of ϕ satisfies $F_{grad_g \phi} = 0$.

Proposition 2. *Every Einstein space has harmonic Weyl tensor, i.e. it is a C-space.*

Proof. Let (M,g) be an Einstein space: $Ric = (n-1)Sg$ and $S = const$. Then the Schouten tensor is given by: $\mathfrak{h} = \frac{1}{n-2}\left[(n-1)Sg - \frac{n}{2}Sg\right] = \frac{1}{2}Sg$. Since S is constant, the Schouten tensor is parallel, and thus we obtain from (3): $div_4 W = 0$. □

In section 3 we showed that if the Weyl tensor is non-degenerate, there is only one possible gradient of the conformal factor, and thus we can formulate a necessary and sufficient tensorial condition for a manifold to be conformal to an Einstein space:

Theorem 2. *A semi-Riemannian manifold with non-degenerate Weyl tensor is locally conformally related to an Einstein space if and only if the vector field $\mathbb{T}$ given in Definition 5 satisfies:*

(10) $$Ric^\circ + (n-2)F_{\mathbb{T}} = 0\,.$$

Proof. Let (10) be satisfied. Since Ric° is symmetric, $F_{\mathbb{T}}$ is symmetric and thus $\mathbb{T}$ is locally a gradient. From Equation (9) we conclude that the manifold is locally conformal to an Einstein space.
Conversely: Let (M,g) be a manifold which has non-degenerate Weyl tensor and is conformal to an Einstein space $(M, \overline{g} = e^{-2\phi}g)$. Because of Proposition 1, there is (up to scaling) only one way to transform (M,g) conformal into a C-space. Since $(M,\overline{g})$ is a C-space (Proposition 2), we obtain $\mathbb{T} = grad_g \phi$. Equation (9) now supplies the claim. □

Theorem 3. *A simply connected semi-Riemannian manifold having a non-degenerate Weyl tensor is conformally related to an Einstein space if and only if:*

$$Ric^\circ + (n-2)F_{\mathbb{T}} = 0\,.$$

Proof. If (M,g) is a conformal Einstein space, the condition follows from the last theorem.
Conversely: Since M is simply connected and $F_{\mathbb{T}}$ has to be symmetric, it follows from the Poincaré Lemma that $\mathbb{T}$ is (globally) a gradient of a smooth function $\phi : M \to \mathbb{R}$. Thus, (M,g) is (globally) conformally related to an Einstein space $(M, \overline{g} = e^{-2\phi}g)$. □

Remark 2. Let $(M^n, g) \to (M, \overline{g} := e^{-2\phi})$ be a conformal transformation, so that $(M,\overline{g})$ is an Einstein space. Proposition 2 implies:

$$0 = C_{grad_g \phi} = div_4 W + (3-n)\mathfrak{C}_{45}[W \otimes d\phi]\,.$$

Take the divergence with respect to the first argument of this tensor, we obtain:

$$0 = div_1 div_4 W + (3-n)W[\nabla^2\phi] + (3-n)(n-3)W[d\phi \otimes d\phi]\,.$$

Since $(M,\overline{g})$ is an Einstein space, we conclude from Equation (8) and $W[\lambda g] = 0$ (for any $\lambda : M \to \mathbb{R}$):

$$0 = div_1 div_4 W + \frac{n-3}{n-2}W[Ric] + (3-n)(n-4)W[d\phi \otimes d\phi]\,.$$

In four dimension this expression does not depend on the conformal factor, and the symmetric $(0,2)$ tensor field

$$B := div_1 div_4 W + \frac{1}{2} W[Ric]\,,$$

called *Bach tensor* (cf. [1]), has to vanish identically for every four dimensional conformal Einstein space. However, the vanishing of the Bach tensor B is not sufficient for a four manifold being locally conformally related to an Einstein space (cf. [20]). Kozameh, Newman and Tod [11] found an additional condition to the vanishing of the Bach tensor for space times with $J \neq 0$, so that these two conditions are necessary and sufficient for a conformal Einstein space. We can verify that $J \neq 0$ is equivalent to the non-degeneracy of the Weyl tensor. Moreover, the result of Kozameh, Newman and Tod [11] can be extended to four manifolds of any signature:
A four manifold with non-degenerate Weyl tensor is locally conformally related to an Einstein space if and only if

$$C_{\mathbb{T}} = 0 \qquad and \qquad B = 0\,.$$

To prove that these two conditions are sufficient, the following is to be needed: If the Weyl tensor is non-degenerate, every symmetric traceless $(0,2)$ tensor k satisfying $W[k] = 0$ and $W^*[k] = 0$ has to vanish identically, where W^* denotes the dual Weyl tensor, i.e. $W^* = * \circ W$ and $*$ is the Hodge star. This claim can be seen by using the Hodge duality on four manifolds.
Furthermore, in four dimensions it is possible to define the vector field $\mathbb{T}$ without calculating W^{-1}. Kozameh, Newman and Tod [11] used special identities of the Weyl tensor. With the help of the Einstein summation convention two of these identities are given as follows:

$$W_{kb}{}^{cd}\, W_{cd}{}^{ef}\, W_{ef}{}^{lb} = \frac{1}{4}\delta^l_k W^3$$

and

$$W^*_{kb}{}^{cd}\, W_{cd}{}^{ef}\, W_{ef}{}^{lb} = (-1)^{ind(M,g)}\frac{1}{4}\delta^l_k (W^*)^3\,,$$

where $ind(M,g)$ is the number of negative eigenvalues of g and W^3 as well as $(W^*)^3$ are given by $W^3 = W_{abcd}\, W^{cd}{}_{ef}\, W^{abef}$ and $(W^*)^3 = W^*_{ab}{}^{cd}\, W^*_{cd}{}^{ef}\, W^*_{ef}{}^{ab}$. If the Weyl tensor is non-degenerate, we can verify that W^3 or $(W^*)^3$ is different from zero, so that one of these identities can be used to derive the vector field $\mathbb{T}$. In order to see the calculation of $\mathbb{T}$ with such identities compare also next section.

Example 1. **a)** Spaces with degenerate Weyl tensor are
- self-dual manifolds (i.e. $W^- = 0$, cf. [5]) and
- conformally flat manifolds.
b) Examples of manifolds with non-degenerate Weyl tensor are
- the Schwarzschild exterior space-time (cf. [17, §13]) and
- the Riemannian product $(M^{2n}, g) = (M_1 \times M_2, g_1 \oplus g_2)$ $(n \geq 2)$ of two space forms (M_1^n, g_1) and (M_2^n, g_2) with constant sectional curvatures K_1 and K_2, so that K_1 is different from $-K_2$, i.e. (M,g) is not conformally flat.
c) A manifold with degenerate Weyl tensor on a hypersurface is the Reissner-Nordström solution (cf. [7, (5.5)]): Let

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

with $e > m$ be the line element on $M := \mathbb{R} \times \mathbb{R}^{>0} \times_r S^2$. Then (M, ds^2) is a four dimensional Lorentz manifold. On the hypersurface $N := \left\{p \in M\ ,\ r(p) = \frac{e^2}{m}\right\}$ the Weyl tensor vanishes, but on $M - N$ the Weyl tensor is non-degenerate.

Remark 3. The second example in b) provides C-spaces which are not conformal to Einstein spaces: Since the Weyl tensor is parallel, (M,g) is a C-space, but (M,g) is an Einstein space if and only if $K_1 = K_2$. Hence (M,g) is for $K_1 \neq \pm K_2$ a C-space which is not conformally related to an Einstein space, because from Proposition 1 we know that every conformal transformation of (M,g) into a C-space $(M,\overline{g})$ has to be a scaling.

When the Weyl tensor is non-degenerate, we have found necessary and sufficient conditions for a manifold to be conformal to an Einstein space. But what happens when the Weyl tensor is degenerate on the manifold or on a subset of it? Wünsch [21] found necessary and sufficient conditions for space-times of Petrov type III, but the method he used seems not to be extendible to dimensions other than four. If the Weyl tensor is degenerate on a subset $U \subset M$, it is very difficult to express solutions of $C_{\mathbb{T}} = 0$ in known terms. Moreover, if $d(p) := \dim\ker W_{|p}$ is greater than or equal to $n-1$ for all $p \in U$, the vector field $\mathbb{T}$ is generally not unique in U. In particular, if d varies on a subset the case is very tricky, because the number of restrictions changes.

However, if the set $(\det W)^{-1}(0)$ contains no interior points, the results given in this section can be extended. Let (M,g) be a manifold such that the interior of $N := (\det W)^{-1}(0)$ is empty, then (M,g) is locally conformally related to an Einstein space if and only if $(M-N, g_{|M-N})$ is locally conformally Einstein and the vector field $\mathbb{T}$, defined on $M-N$, is extendible to a vector field on M. This claim follows from continuous reasons, since the extension of $\mathbb{T}$ on M, if there is any, is unique, and this uniqueness is all what we need for the Proof.

5. The four dimensional Riemannian case

We consider four dimensional Riemannian manifolds. The Riemannian metric induces a positive definite inner product on the alternating two-forms. Since W is self-adjoint with respect to this product, W is a diagonalizable endomorphism of $\Lambda^2(M)$. That means

$$W = A^T \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_6 \end{pmatrix} A \, : \, \Lambda^2(M) \to \Lambda^2(M)\,,$$

where $\lambda_1, ..., \lambda_6 : M \to \mathbb{R}$ are smooth functions and $A^T A = Id_{\Lambda^2(M)}$. Denote by $|W^2|$ the trace of the endomorphism $W \circ W$, then we obtain $|W^2| = \lambda_1^2 + ... + \lambda_6^2$ and thus, if W is not zero in $p \in M$, $|W^2|$ is different from zero in p. In four dimensions we have for the Weyl tensor the following relation (cf. [5, Eq. (31)]):

$$W_{abck}W^{abcl} = |W^2|\delta_k^l\,,$$

which in our notation can be written as follows:

$$\sum_{i=1}^{4} W[W[E_i \wedge X]](E_i, Y) = \frac{1}{2}|W^2|\,\langle X, Y\rangle\ . \tag{11}$$

Let (M,g) be conformal to an Einstein space $(M,\overline{g} = e^{-2\phi}g)$. From Proposition 2 we know that $(M,\overline{g})$ is a C-space, and thus $C_{grad_g\phi}$ vanishes identically:

$$0 = (div_4 W)_Z - W[Z \wedge grad_g\phi] \quad \text{for all} \quad Z\,.$$

Apply W to the last Equation, then we conclude using (11) for all $X \in \mathfrak{X}(M)$:

$$0 = \sum_i W[(div_4 W)_{E_i}](E_i, X) - \frac{1}{2}|W^2|(X\phi)\,,$$

i.e. if $|W^2|$ is different from zero, we obtain for the gradient of ϕ:

$$grad_g\phi = \frac{2}{|W^2|}\sum_{i,k} W[(div_4 W)_{E_i}](E_i, E_k)\, E_k\,.$$

Definition 8. Let (M,g) be a Riemannian manifold of dimension four. Then

$$M_1 := \{p \in M\,,\, |W^2|(p) \neq 0\}$$

is an open subset of M, and the following vector field can be defined on M_1:

$$\mathbb{T} := \frac{2}{|W^2|} \sum_{i,k=1}^{4} W[(div_4 W)_{E_i}](E_i, E_k)\, E_k \,.$$

Moreover, denote by M_2 the interior of $M - M_1$.

With the help of (11), it is possible to define the vector field $\mathbb{T}$ even if the Weyl tensor is degenerate, since non-degeneracy means that for all $i = 1, ..., 6$ the eigenvalue function λ_i has no zeros in M_1. The uniqueness of the vector field $\mathbb{T}$ follows from the special structure of the Weyl tensor in four dimension. If $(M_1, g_{|M_1})$ has non-degenerate Weyl tensor and is locally conformally related to a C-space, then the vector fields $\mathbb{T}$ given in Definition 5 and 8 are equal.

Lemma 2. *Let (M, g) be a four dimensional Riemannian manifold.*
a) If $M_2 \neq \emptyset$, then $(M_2, g_{|M_2})$ is conformally flat, in particular $(M_2, g_{|M_2})$ is locally conformally related to an Einstein space.
b) If $M_1 \neq \emptyset$, then $(M_1, g_{|M_1})$ is locally conformally related to an Einstein space if and only if the vector field $\mathbb{T}$ given in Definition 8 satisfies

$$Ric^\circ + 2F_{\mathbb{T}} = 0\,.$$

Proof. a) This follows from the Weyl-Schouten theorem (cf. [15, Thm C.9]), since $W_{|M_2} = 0$.
b) Let $(M_1, g_{|M_1})$ be conformal to an Einstein space $(M_1, e^{-2\phi} g_{|M_1})$. Since this Einstein space is a C-space, we obtain from the calculation above $\mathbb{T} = grad_g \phi$. Equation (9) now implies the claim.
Conversely: Since $F_{\mathbb{T}}$ has to be symmetric, $\mathbb{T}$ is locally a gradient. Thus, $(M_1, g_{|M_1})$ is locally conformal to an Einstein space. □

Proposition 3. *If (M, g) is a connected (Riemannian) Einstein space, then M_1 or M_2 is empty.*

Proof. We can see from Theorem 5.26 in Besse [2] that in geodesic normal coordinates the metric g is real analytic. For this reason also the Weyl tensor W is in these coordinates real analytic and thus, if the Weyl tensor vanishes on an open subset U of M, the Weyl tensor has to vanish identically in M. □

Remark 4. The last Proposition and formula (5) imply that if a connected Riemannian manifold with non-empty M_2 is locally conformal to an Einstein space, then this manifold is conformally flat.

Theorem 4. *Let (M, g) be a connected Riemannian manifold of dimension four, so that there is a point $p \in M$ with $W_{|p} \neq 0$, i.e. in particular (M, g) is not conformally flat. Then (M, g) is locally conformally related to an Einstein space if and only if the following two conditions hold:*

(i) On M_1 the vector field $\mathbb{T}$ given in Definition 8 satisfies:

$$0 = Ric^\circ + 2F_{\mathbb{T}}\,.$$

(ii) M_2 is empty and if $M - M_1$ is not empty, then $\mathbb{T}$ is extendible to a vector field of M.

If in addition M is simply connected and (i) as well as (ii) are satisfied, then (M, g) is conformal to an Einstein space.

Proof. Let (M, g) be locally conformal to an Einstein space. Then Lemma 2 implies (i). From Proposition 3 and Equation (5) it is clear that M_2 has to be empty. If $M - M_1$ is not empty, let p be any point of $M - M_1$. Then there is a neighbourhood U of p and a smooth function $\phi : U \to \mathbb{R}$, so that $(U, e^{-2\phi} g_{|U})$ is an Einstein space. Since the gradient of ϕ corresponds to the vector field $\mathbb{T}$ on $U - U \cap (M - M_1)$, $\mathbb{T}$ is uniquely extendible on $U \cup M_1$. Thus, we obtain the second condition.
Conversely: Let (i) and (ii) be satisfied. Then there exists a unique vector field $\widehat{T} \in \mathfrak{X}(M)$ which is equal to $\mathbb{T}$ on M_1. Because of continuous reasons and (i), $\widehat{T}$ satisfies on M:

$$0 = Ric^\circ + 2F_{\widehat{T}}\,.$$

Since $F_{\widehat{T}}$ has to be symmetric, $\widehat{T}$ is locally a gradient, and we obtain the claim from (9).
If in addition M is simply connected, then $\widehat{T}$ is globally a gradient, i.e. there exists a smooth function $\phi : M \to \mathbb{R}$ satisfying $grad_g \phi = \widehat{T}$. Thus, $(M, e^{-2\phi} g)$ is an Einstein space. □

This theorem includes also self-dual manifolds, i.e. four manifolds being half conformally flat ($W^- = 0$, cf. [5]). A generalization of the last theorem to the four dimensional Lorentz case seems to be very difficult, because if W is different from zero, we can generally not follow that $|W^2|$ or a other trace-invariant of the Weyl tensor is different from zero. Moreover, the result used in the Proof of Proposition 3 is given in Besse [2] only in the Riemannian case.

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