The Origins of Dynamic Inventory Modelling under Uncertainty

(The men, their work and connection with the Stanford Studies)

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Abstract

This paper has a double purpose. First, it provides a historical analysis of the developments leading to the first "Stanford Studies", a collection of papers which probably can be considered even today as the single most fundamental reference in the mathematical handling of inventories. From this point of view, the paper is about people and their works. Secondly, we give an insight into the interrelation of mathematics and inventory modelling in a historical context. We will sketch on the one hand the development of the mathematical tools for inventory modelling first of all in statistics, probability theory and stochastic processes but also in game theory and dynamic programming up to the 1950s. Furthermore, we report how inventory problems have motivated the improvement of mathematical disciplines such as Markovian decision theory and optimal control of stochastic systems to provide a new basis of inventory theory in the second half of our century.

1. Introduction

Dates such as 40 years after the first Stanford Studies on inventory or the 10th ISIR Symposium in Budapest are symbolic events which give us the opportunity to analyse the conditions which achieve such exceptional results. Here, we analyse the rise of inventory modelling, restricting ourselves essentially to the aspect of mathematics needed for inventory modelling under uncertainty.

The birth of a new set of mathematical tools is often a result of special circumstances. *Ptolemy* described the planetary orbits as circular movements modified by epicycles. In his time this explanation was adequate. But observations of star positions during the 17th century showed, however, his explanation no longer sufficed. For a theory to *Kepler's* laws *Sir Isaak Newton* had to create a new mathematical tool. With Calculus he was able to explain the planetary motion precisely and much more easily. In the first third of the 20th century, Calculus has been part of the secondary school curriculum and was used to derive the well-known economic order quantity formula (cf. [101, 102]).

Furthermore, new mathematical methods and algorithms have often sprung from practical questions. A well-known example of our century is sequential testing of goods in quality control. Especially if the tested good is destroyed during the test, it is obviously not effective to choose a fixed sample size. It is better to test only as many goods as necessary to give a clear statement about the composition of a batch of these goods.

During World War II, high ranking mathematicians such as *Norbert Wiener* and *Abraham Wald* were confronted by the authorities with such problems.

In connection with anti-aircraft defence *Norbert Wiener* developed a modern prediction and filtering theory of stationary time series. *Abraham Wald* created a statistical decision theory in which the original problem of quality control mentioned above is only a particular case. We will show among other things how this work on statistical decisions created under very special circumstances was also crucial for the development of inventory theory.

Special circumstances led to the great leap in the 1950s in the use of mathematical tools in describing and analysing inventory problems. The increasing practical interest in inventory management after World War II coupled with the development and full acceptance of probability theory as a branch of mathematics have led to such a concentration of combined research efforts of prominent economists and mathematicians (each side having a good knowledge of the other side's science) the result of which is the "Stanford Studies", an unprecedented landmark in the development of inventory theory.

Our idea of writing this paper was prompted by the Balaton Workshop in 1994 (see [19]). Unlike Milne's paper presented at that workshop, our view is directed to persons and ideas more before the Stanford Studies and we have restricted our analysis to the developments regarding sequential decision making on inventories under uncertainty. We may say that we stay in the framework set by two seminal papers which served as starting points to the Stanford Studies: the *Arrow-Harris-Marschak* and the *Dvoretzky-Kiefer-Wolfowitz* papers (see [4, 23]).

This is the reason why we do not deal with several important related fields of stochastic inventory theory. We give just two examples of the areas not analysed here.

We do not write about the area of multi-stage production planning and inventory control opened with *Andrew J. Clark's* path-breaking echelon approach though the roots of these studies are also in the Stanford soil: in the *Clark-Scarf* paper in the second volume of the Stanford Studies (see [6, 20, 8, 109]).

Continuous review inventory modelling is another line not mentioned here. It began with *H.A. Simon's* application of *Wiener's* ideas to control systems in continuous time (see [86], cited in [4]; [103, 104, 82]). The general theory of continuous time controlled stochastic processes developed in the late 1970s led to interesting results on multiproduct inventory and production-inventory systems (see [12, 89, 36, 37]), but definite progress in the field of inventory models with continuous review seems to be more anticipated with Poisson demand or other point processes which are far and away more practicable for certain inventory situation than diffusion processes.

This modelling was initiated by *Martin J. Beckmann* (one of the contributors to the Stanford Studies) and is extending until today (see [9, 18, 90, 22, 38]).

When in the following we draw up the ways which led to the birth of the Stanford Studies we give biographical notes about the most important authors taking account of the role of mathematics in their education and further studies which put them in a position to such a pioneering work.

The overall picture we get from putting these biographies together is extremely interesting in itself: it describes the network of scientists from many different countries, in which the supporting theories and models culminating finally in the Stanford Studies developed.

2. The Golden 1950s in California

Michael C. Lovell put the famous question to the ISIR community: "Why haven't we learned more? (in four or more decades of empirical research on inventory behavior").

He answered: "... we have been trying to do research on the cheap" (see [19] p.41).
In California, in the 1950s, there was a different situation. The Office of Naval Research, The RAND Corporation (a nonprofit research organization under contract with the United States Air Force), the Cowles Commission for Research in Economics, the universities, and other organizations provided intellectual and financial support to logistics projects. Modern inventory theory originates from this atmosphere.

In August 1950 the Second Berkeley Symposium on Mathematical Statistics and Probability was held. The list of participants is a Who's Who in stochastics. It brought together not only the statisticians *Wald, Wolfowitz, Dvoretzky* and *Harris* but also economists like *Marschak* and *Arrow* to present their papers without direct connection to inventory problems but with contents supporting such future studies (see [74]).

At the Logistics Conference of the RAND Corporation, Santa Monica, in the same summer, *Kenneth J. Arrow, Theodore Harris* and *Jacob Marschak* opened the door with the paper "Optimal Inventory Policy" which standing now as a landmark. The mathematicians *Aryeh Dvoretzky, Jack Kiefer* and *Jacob Wolfowitz* followed immediately with their famous papers "The Inventory Problem I, II" under contract with the Office of Naval Research. We will come back to both path breaking contributions in the following sections (see [4], [23]).
After the Third Berkeley Symposium with *Samuel Karlin's* paper "Decision Theory for Pólya Type Distributions", in 1955, two further papers on inventory modelling were published: *Karlin's* "The Structure of Dynamic Programing Models" and "On the Optimal Inventory Equation" written by the RAND Corporation-team *R. Bellman, I. Glicksberg* and *O. Gross* triggering special research leading to *Richard Bellman's* great monograph "Dynamic Programming" (see [75, 47, 48, 11, 10]).

The "Studies in the Mathematical Theory of Inventory and Production" written and edited by the Stanford professors *Arrow, Karlin* and *Herbert Scarf* with contributions by *Martin J. Beckmann, Richard F. Muth*, and *John Gessford* are "one of the best guides to the analysis of inventory and production decisions" - wrote *Alistair Milne*, 36 years after.
The "Studies in Applied Probability and Management Science" with the same editors are natural continuations of this work with parts of Ph.D. dissertations submitted to Stanford University (see [5, 67, 6]).

Finally, we mention the Proceedings of the First Stanford Symposium "Mathematical Methods in the Social Sciences, 1959" (editors: *Arrow, Karlin*, and *Patrick Suppes*) with *Scarfs* famous paper "The Optimality of (S, s) Policies in the Dynamic Inventory Problem", and the volume "Multistage Inventory Models and Techniques" edited by *Scarf, Dorothy Gilford,* and *M. Shelly* with *Scarf's* nice "Survey of Analytic Techniques in Inventory Theory" summarizing some of the results obtained during the last ten years - the golden era of inventory theory (see [7, 78, 80, 79]).

3. Some early foundations of inventory modelling

In the Introduction to the first Stanford studies on inventory, *Arrow* focuses mainly on the optimal control of inventory systems. He characterizes the actual situation as follows:
"most of the individual elements of current inventory models will be found in the earlier

literature (up to, say, 1946), though the form was frequently rather imprecise and little was done to integrate them into a consistent framework" (see [5] p.3).
Such a first framework in the case for control under certainty was given by *Erich Schneider* (b. 1900 in Germany). He yields in [83] a special mathematical model of production over time, where the well-formulated optimization problem with restrictions could be solved by *Schneider* for linear cost and (in an appendix) *Børge Jessen* constructed plausibly a solution for more general cost. In the case of uncertainty the modelling was rather imprecise, for instance in of the paper by *Edward Shaw* (b. 1908), Arrow's predecessor as Executive Head of the Department of Economics at Stanford. *Shaw's* "Elements of a Theory of Inventory" contains an interesting verbal description of a two-period inventory model under risk, without any formula but with figures summarizing the calculations of the optimal operating plan or the expectation interval of weeks 1 and 2. About the demand and cost functions he wrote "It is necessary to discount them, not only for interest but for risk as well" (see [85]). We will try to find the right path describing uncertainty in the history beginning from the roots.

4. Economics and probability

Francis Y. Edgeworth (b. 1845 in Ireland) said before the British Association of Economics in September 1886: "The higher mathematics make two contributions to Social Science: the Calculus of Probabilities, and what we may after Jevons call the Calculation of Utility" (see [28], p. 113). In his books "Mathematical Psychics" and "Metretike" he presented his ideas on an economical calculus containing a mathematical theory of contracts, the generalized utility function and the indifference curves but also his method of measuring probability and utility (see [26, 27]). As Drummond Professor of Economics at Oxford since 1891 he worked successfully in statistics with important contributions to the law of error (especially with the so called Edgeworth expansion) and honourably as the first editor of "The Economic Journal".
John Maynard Keynes (b. 1883 at Cambridge) became editor of the same journal in 1911. He connected economics and probability on a higher level. In his "Treatise on Probability" he asked "what probability we can attribute to our rational beliefs" and developed - under the influence of *Bertrand Russell* - a logical theory of comparative conditional probability "which occupies an intermediate position between the ultimate problems and the applications of the theory" (see [49], p. 18/19). *Keynes* was not only a statistician but a successful financier. With his experience at the India Office, the British Treasury, the National Mutual Life Assurance Co. he could write, as Fellow of King's College, Cambridge, "A Treatise on Money" and his main work "The General Theory of Employment, Interest and Money" creating a new macroeconomic theory (see [50, 51]). An abundance of ideas and economic relationships of his books stimulated several generations of economists for extension of the theory using mathematics.
A simple example is *Keynes'* recognition that there are three motives for holding cash: transactions, precautionary and speculative, and that changes in real interest rates affect inversely the demand for cash. More than 20 years later *J. Tobin* analyzes the speculative motive of investors and develops a theory that explains the behaviour of global decision-making toward risk deriving liquidity preferences of the economy and ensuring *Keynes'* conjecture (see [91]).

About the same time *K. Arrow* used the three motives mentioned above for analyzing inventories (see [5], pp. 3-13). This description of the motivational background for holding inventories is complete: all other classifications of motives in the literature can be reduced to this one.
A microeconomic approach to the consumption-investment problem under unertainty was started by *J. Marschak*. He noticed that the principle of determinateness which was established by *Walras* and *Pareto* and used by himself in his dissertations is not adequately applicable to financial economics.
"The unsatisfactory state of Monetary Theory as compared with general Economics is due to the fact that the principle of determinateness so well established by *Walras* and *Pareto* for the world of perishable consumption goods and labour service has never been applied with much consistency to durable goods and, still less, to claims (securities, loans, cash)." (see [62], p. 312, r. 16 - 21.) He suggested to express preferences for investment by indifference curves in the mean-variance space.
"Just as there are rates of preference between any two yields differing in time or quality, so also there are rates of preference between the mean and the dispersion of yield: we may call these rates safety preferences." [62 b], p. 273, r. 1 - 4. The problem of portfolio selection in the static case is solved for the first time by *H. Markowitz* (see [60, 61, 62, 59]).

The advancement of economic theory in its relation to statistics ad mathematics is the purpose of an international society founded in Cleveland, 29 December 1930. The organizing group consisted of 16 members under them *Ragnar Frisch* (Oslo), *Harold Hotelling* (Stanford), *Karl Menger* (Vienna), *Joseph Schumpeter* (Bonn) and *Norbert Wiener* (MIT). *Irving Fisher* of Yale University (cf. [33]) was elected in absentia the first President of the Econometric Society. *L. von Bortkiewicz* was elected to the Council but he died half a year later. The number of members grew from 16 in 1930 to 1554 in 1951. Among the presidents following *I. Fisher* (1930 - 1935) were *H. Hotelling* (1936 - 1937), *Jacob Marschak* (1946), and *Kenneth Arrow* (1956), who formulated in his presidential address, delivered at Cleveland, Ohio, "almost any realistic decision problem has two fundamental characteristics: it is sequential and it is uncertain. ... An excellent example of a sequential decision problem under uncertainty is that of holding inventories" (see [2], 523/524).
The first issue of the journal of the Economic Society "Econometrica" was published in 1933. This journal was also the cradle of modern inventory theory (see [77, 4, 23, 24]).

F.Y. Edgeworth - President of the Royal Statistical Society in 1912 - introduced his article "Probability" in the Encyclopaedia Britannica as follows: "The mathematical theory of probabilities deals with certain phenomena, which are employed to measure credibility." He noticed a few rows below, that the "philosophical foundations are less clear than the calculation based thereon". At about the same time he used such an unclear basic assumption in a paper on taxation in a regime of monopoly.
Richard von Mises noticed a few years later: "in fact one can scarcely characterize the present state other than probability is not a mathematical discipline". *Kolmogorov* gave a clear foundation of probability in 1933. But for the community of mathematicians probability theory was accepted as an important branch of mathematics only in 1950 (see [30, 29, 68, 52, 44, 74]).

5. Evaluations of holding inventories under uncertainty

Thornton C. Fry (b. 1892), member of the technical staff of Bell Telephone Laboratories, gave a course of lectures at the Massachusetts Institute of Technology in its Department of Electrical Engineering in the 1920s containing the following example:

"A retail chain store, with limited storage facilities, sells on the average 10 boxes of dog-biscuit per week. The usual practice is to stock up Monday morning. How many packages should be adopted as the standard Monday morning stock, in order not to lose more than one sale out of a hundred?" (see [34] p. 229).

Fry calls this "warehouse problem" the simplest of all congestion problems in which demands for service arise from a multiplicity of sources acting more or less independently of one another. The Poisson Law is the fundamental basis upon which most of those problems are solved.
The chance of a demand for j packages is $P(j) = \frac{\lambda^j e^{-\lambda}}{j!}$ with $\lambda = 10$ and the solution of the problem proposed in the example is the least safe stock $S^* = 10$ in accordance with

$$S^* = \min \left\{ n : \sum_{j=n}^{\infty} (j-n)P(j) \leq \frac{\lambda}{100} \right\} . \tag{1}$$

T.C. Fry was more interested in problems of telephone engineering following the work of *A.K. Erlang, T. Engset* and *E.C. Molina.* With respect to the Poisson law he derived the differential equation

$$\frac{dP(j)}{dx} = c[P(j-1) - P(j)] \ , \ \lambda = cx \tag{2}$$

and illustrated the Poisson Law as a limiting case of the Binomial Law with *Bortkiewicz's* data of the number of soldiers killed by the kick of horses (see [16]).

Ladislaus von Bortkiewicz (b. 1868 in St. Peterburg) has studied mathematical statistics under *W. Lexis* at Göttingen and developed further his theory of statistical inference. He wrote two books on the Poisson law and its application (see [16, 17]). At the University of Berlin in 1919 *von Bortkiewicz* was *J. Marschak's* first teacher in Germany, although only for a short time. *Jacob Marschak* continued his study of economics under *Emil Lederer* and *Alfred Weber* in Heidelberg were he presented his dissertation summa cum laude in 1922 and obtained there the honorary doctorate 50 years after. He qualified as a lecturer with a paper on the elasticity of demand and worked at the University of Heidelberg up to April 1933. In exile at Oxford he published with *Lederer* a book on accumulation of capital. He came to the U.S.A. in 1939 (see [60, 61, 64, 65, 66]).

Let us come back to the evaluation of inventory policies. *Fry* tried to justify heuristicly his criterion (1) with a long run argument in [34].
Arrow, Harris, Marschak [4] found an exact approach for a more realistic criterion and an ordering rule of (S, s) type. They chose the present mean value, at time 0, of total future loss:

$$L(y) = L(y \mid s, S) = \sum_{n=0}^{\infty} \alpha^n E^y_{(s,S)}[l(Y_n)], \tag{3}$$

with the one-period “loss” l, the discount factor α, and the mathematical expectation of a probability law which is determined by the initial stock y and the parameter of the ordering rule. It was a long way from criterion (1) to (2). Not only the economical foundation for decisions under uncertainty, but also the theory of stochastic processes had to be developed further.

6. Markov processes

Andrei A. Markov (b. 1856) was a distinguished Russian mathematician working at the University and at the Imperial Academy of Sciences of St. Petersburg. At the beginning of our century he devised a model for dependent trials that was the simplest generalization of independent trial processes which already had been studied by *Abraham de Moivre, Pierre Simon Laplace*, by his teacher *Pafnuty L. Chebyshev*, and others. *Markov* was very interested on mathematical results and showed the law of large numbers and the central limit theorem for his chains. Only in connection with a new edition of his book “Probability Calculus” he gave an application, even performed on the basis of empirical data.
It was a statistical examination of a sequence of 20.000 vowels (v) and consonants (c) $x_1, x_2, \ldots, x_{20.000}$, with $x_n \in \{v, c\}$ in *Aleksandr Pushkins* novel in verse “Yevgeny Onegin”. The “random” transitions between letters of these two types were described by conditional probabilities, e.g. $P(X_{n+1} = v \mid X_n = c, X_{n-1} = v)$, estimated from the data (see [56, 57, 58, 76]).

Yevgeny Sluckij (b. 1880 in Russia) - better known as E. Slutsky - worked as an associated professor for statistics at the Kiev Institute of Business from 1913 to 1919 and knew *Markov's* paper [57] very well. At the same time, *Jacob Marschak* (b. 1898 in Kiev) was a student of mechanical engineering at the Kiev Institute of Technology and was studying mathematical statistics with *Sluckij* who was later researching successfully stochastic processes during his time in Moscow after 1926 (see [88, 64]).

Theodore E. Harris (b. 1919 in Philadelphia) investigated in his doctoral dissertation, presented to the Mathematics Department at Princeton (his teacher was Samuel S. Wilks [105]), a Markov process which originated in the problem of the extinction of families formulated by *Francis Galton* in the 1870s:

“Let $p_0, p_1, p_2, \ldots$ be the respective probabilities that a man has $0, 1, 2, \ldots$ sons, let each son have the same probability for sons of his own and so on. What is the probability that the male line is extinct after r generations ...?” (see [39, 40]).
Let Y_n denote the size of the nth generation,. If there is a progenitor $Y_0 = 1$, then we can model the dependence of successive generations by

$$Y_{n+1} = \sum_{k=1}^{Y_n} X_{kn} \quad , \quad P(X_{kn} = i \mid Y_n \geq k) = p_i, \tag{4}$$

where X_{kn} represents the number of sons of the k-th man of the n-th generation. If the random variables $\{X_{kn}\}$ are stochastically independent, then the process $\{Y_n\}$ from (5) is a Markov process. There is an analogy between the growth of families and nuclear chain reactions. If a neutron collide with an atomic nucleus, then the nucleus may or may not

break up. A nuclear fission generates 2 or 3 neutrons. Such a child (neutron) may as a parent causes a new fission and so on. Now, we can ask the probability that a single neutron (as a progenitor) generates as infinite family. The development of such a family of neutrons might be described by *Harris* and others in the 1940s as a Markov processes a so-called stochastic branching process.
Harris was connected with RAND Corporation for about 20 years, at the beginning at Douglas Aircraft Company, and 1959 - 1966 as chairman of the Department of Mathematics at Santa Monica, California.

The dynamic inventory model with uncertainty in [4] contains two stochastic processes: firstly, the demand process $X_0, X_1, X_2, \ldots$, where X_n denotes the demand over the interval $(n, n+1)$ with a probability distribution of demand $F(x)$ which is independent of n. Denote by Y_n the inventory available at instant n and Q_n the amount ordered at time n with vanishing leadtime. Then, we obtain, secondly, the inventory process defined by the recursion formula

$$Y_{n+1} = \max\,(Y_n + Q_n - X_n, 0). \tag{5}$$

In the case of an ordering rule of (S, s) type with $0 < s < S$, the inventory process (5) is specified

$$Y_{n+1} = \begin{cases} \max\,(S - X_n, 0), & \text{if } Y_n \le s, \\ \max\,(Y_n - X_n; 0), & \text{if } Y_n > s \end{cases} \quad . \tag{6}$$

If the demands in different intervals are stochastically independent, then the inventory process $\{Y_n\}$ from (6) is a Markov process (see [32]) and the mathematical expectation in equation (3) is determined.
From (3) it follows that

$$L(y) = l(y) + \alpha E[L(Y_1)]. \tag{7}$$

Reducing equation (7) to he standard form of the integral equation of renewal theory (see [31], [21]), they got explicitly a solution L and minimizing parameters S^* and s^*.

7. Strategy - statistical decision function - policy

Let us extend the dynamic inventory model with uncertainty in [4]. We have seen in Section 5 that an optimal inventory rule of (S, s) type could be derived minimizing a function of two real variables. This is an example of optimal design. We optimized in the set $\mathfrak{S}$ of ordering rules of (S, s) type. But it seems to be sure that in a more general model with the balance equation (6) an optimal rule need not belong to the restricted set $\mathfrak{S}$.
What kinds of mathematical tools were available in 1950 for building a general inventory model which searches the best ordering rule as it was suggested in the conclusion of [4]? They were game theory with the fundamental term "strategy" and the statistical decision theory with the analogue "statistical decision function", well-known through the famous books "Theory of Games and Economic Behavior" by *John von Neumann* and *Oskar Morgenstern* and "Statistical Decision Functions" by *Abraham Wald* (see [73, 99]).

Wald's two coworkers *A. Dvoretzky* and *J. Wolfowitz* together with *J. Kiefer* developed in their papers "The Inventory problem" (see [23]) a general framework for inventory modelling under uncertainty extending the approach of [4]. They "have shown the way to better economic inventions", *J. Marschak* said at the Boston Meeting of the Econometric Society in December 1951 (see [63]).

A few words about the background of those involved in the story seems to be in order.

Abraham Wald (b. 1902 at Kolozsvár, Hungary - today Cluj in Rumania -) presented his doctoral thesis dealing with a question of axiomatics in geometry at the University of Vienna in 1931. *Karl Menger* advised *Wald* to work also in applied mathematics.

Oskar Morgenstern (b. 1902 at Görlitz, Germany) graduated (Dr. rer. pol.) also at the University of Vienna in 1925. During a Rockefeller Memorial fellowship in Cambridge, Massachusetts, and Paris he wrote a significant essay on economic forecasting. He was Director of the Austrian Institute of Business Cycle Research between 1931 and 1938. *Morgenstern* met *Wald*, appreciated his talents and employed him in his institute. *Wald* worked here on time series analysis. His results were published in a book of the institute's series. At the same time stationary time series have been analyzed by *Slutsky* and *A.N. Kolmogorov* in Moscow, *Herman Wold* in Stockholm and *Norbert Wiener* in connection with anti-aircraft fire control at the Massachusetts Institute of Technology (see [69, 93, 87, 53, 106, 107, 104]).

John von Neumann (b. 1903 in Budapest, Hungary) presented his pioneering paper on game theory to the Mathematical Society of Göttingen in December 1926, also as a recipient of a Rockefeller fellowship. After his great success in mathematical physics, since 1933 as a professor at the Institute for Advanced Study at Princeton University, he returned to game theory. He established contact to economics especially by *O. Morgenstern* initially in connection with a few papers describing mathematical aspects of economic behavior on the different markets as an equilibrium among a large number of agents having conflicting interests. This led later to the famous monograph mentioned above which yields a new mathematical tool for modelling in economics (see [71, 72, 70, 92, 94, 73).
Because of the political situation in Europe *K. Menger, O. Morgenstern* and *A. Wald* left Austria. *Morgenstern* worked at Princeton University (up to 1970), *Wald* went to New York. At Columbia he learned modern statistics from *Harold Hotelling* and by his own research, studying, among others, works of *Jerzy Neyman*.

Wald presented the first of 15 joint papers with *Jacob Wolfowitz* (b. 1910 in Warsaw) to the American Mathematical Society at New York in February 1939. In December of the same year his seminal paper on the theory of statistical estimation and testing hypotheses was published. Extending the Neyman-Pearson theory of testing hypotheses *Wald* formulated a general statistical (decision) problem and discussed Bayes and minimax solutions. *Von Neumann-Morgenstern's* book and the completion of his own book "Sequential Analysis" surrounding the problem of quality control mentioned above were triggering for *Wald* to resume his work on a statistical decision theory in a series of papers culminating in his monograph "Statistical Decision Functions" (see [100, 95 - 99, 25, 108]).
After receiving his Ph.D. in 1942 *J. Wolfowitz* worked as associated director of the statistics research group and from 1946 to 1951 as professor at the new Department of Mathematical Statistics at Columbia University with *A. Wald* as executive officer. In 1951 *Wolfowitz* was called to Cornell University, Ithaca. As professor of mathematics he stayed there until 1970, with a short break as visiting professor of the University of California at Los Angeles,

1952 - 1953. During this time his co-operation intensified with *Aryeh Dvoretzky* (b. 1916 in Ukraine) - on leave of absence from the Hebrew University, Jerusalem, for about 3 years - and with his Ph. D. student *Jack Carl Kiefer* (b. 1924 in Cincinnati) who became an instructor in 1951 and a professor of mathematics at Cornell in 1959.
Let us return to *Arrow's* sentence that a realistic decision problem is sequential and uncertain. *Wald* described the uncertainty by a stochastic process $X = (X_n : n \in \mathbb{N})$ whose distribution is merely known to be an element F of a given class $\mathfrak{F}$ of distributions. In *Wald's* statistical decision problem the experimentation is sequential which means making observations on some of the random variables in the process X in different stages. The control of experimentation together with a terminal decision may be described by a statistical decision function δ. A statistical decision problem may be viewed as a zero sum two person game.
Player 1 is Nature who selects an element F of $\mathfrak{F}$ to be the true distribution of X, and player 2 is the statistician who chooses a decision function δ. The outcome is then given by the risk $r(F, \delta)$ - the expected loss plus the expected cost of experimentation.
In the same way *Dvoretzky, Kiefer* and *Wolfowitz* consider the inventory problem where F is the unknown distribution of the demand process, δ is an ordering policy and $r(F, \delta)$ represents the present value of the total expected loss. If a probability measure μ on the space $\mathfrak{F}$ is given, then an ordering policy δ_μ with the property

$$\int r(F, \delta_\mu) d\mu(F) = \min_\delta \int r(F, \delta) d\mu(F) \tag{8}$$

is called a Bayes solution relative to the given a priori distribution μ. In [23] II a Bayes solution of the inventory problem in the case of finitely many time intervals is constructed with a recursive method as in [99], 4.1.
The first clear presentation of this method of backward induction seems to be published by *K. Arrow, D. Blackwell*, and *Meyer A. Girshick* (b. 1908 in Russia) who presented the paper [3] at the Madison Meeting in September 1948 under the old title "Statistics and the Theory of Games". *David Blackwell* (b. 1919 at Centralia, Illinois), mathematician at Howard University, and the statistician at Stanford *M. Girshick* later wrote together the book "Theory of Games and Statistical Decisions" (see [15]). But the new title and the following sentence is the personal style of *Kenneth J. Arrow* (b. 1921 in New York City):
"It may be remarked that the problem of optimum sequential choice among several actions is closely allied to the economic problem of the rational behavior of an entrepreneur under conditions of uncertainty" (see [3], p. 215).
We should like to emphasize that in [4], section 7:B refered to [3] a program is suggested as it is realized in [23], II.

Arrow's admirable achievement of outstanding results in different fields as optimal economic welfare and optimal inventory policies may be explained at best by his curriculum vitae with his own words:

> "I entered Columbia University for graduate study and received an M.A. in Mathematics in June, 1941, but under the influence of the statistician-economist, *Harold Hotelling*, I changed to the Economics Department for subsequent graduate work. ... The years 1946 - 1949 were spent partly as a graduate student at Columbia University, partly as a research associate of the Cowles Commission for Research in Economics at

the University of Chicago, where I also had the rank of Assistent Professor of Economics in 1948 - 1949. The brilliant intellectual atmosphere of the Cowles Commission, with eager young econometricians and mathematically-inclined economists under the guidance of *Tjalling Koopmans* and *Jacob Marschak*, was a basic formative influence for me as was also the summer of 1948 and subsequent years at the RAND Corporation in the heady days of emerging game theory and mathematical programming" (see [55], p. 207).

Arrow received his Ph.D. from Columbia University in 1951. His doctoral dissertation "Social Choice and Individual Values" (see [1]) was basic for his Nobel Prize for Economics in 1972, cited for "contributions to general economic equilibrium theory and welfare theory". It was a great recognition for the International Society for Inventory Research that he became its first President (from 1982 to 1990) and his continuing support of inventory research is expressed by his being Honorary President since 1990.

8. Functional equations and Markov decision processes

In Section 6 we discussed - following [4] - the functional equation

$$L(y) = l(y) + \alpha E[L(Y_1)] \tag{7}$$

for the expected discounted cost of holding inventories with respect to a given ordering rule. But also in a general model the cost may be divided into two parts: the cost incurred in the first period, and the expected discounted future cost. *Dvoretzky, Kiefer, Wolfowitz* found for the expected discounted total cost f for an inventory system controlled by an optimal policy and initial stock y the functional equation

$$f(y) = \inf_{z \geq y} \left[v(y,z) + \alpha \int_0^z f(z-x)\,d\Phi(x) \right] \tag{9}$$

in the case of "stationarity and independence", that means the random variables of demand in different time intervals are independent and identically distributed with the probability distribution function Φ. The set $\mathfrak{A}$ of admissible ordering policies contains sequences of ordering rules where each rule orders nothing or a positive quantity.

In [23], I, existence and uniqueness of solutions of such functional equations are proved and also an iterative method of solving (9) is given. In practice simple structured ordering policies are prefered to numerical solutions.

Dvoretzky, Kiefer and *Wolfowitz* studied also policies of (S,s) type but only for the static case of one time interval and fixed penalty cost and the usual ordering cost:

$$v(y,z) = p \cdot [1 - \Phi(z)] + cz + \left\{ \begin{array}{lll} 0 & , & \text{if } z = y \\ K & , & \text{if } z < y. \end{array} \right\} . \tag{10}$$

In [23], I 2.5, an example is given where an optimal policy is not of (S,s) type, but for the same cost (10) sufficient optimality conditions for an (S,s) policy may be found in [24] characterizing a suitable distribution function Φ of demand.

The idea was picked up and used for general penalty and holding cost and the dynamic

case by *Samuel Karlin* (b. 1924 in Poland), Ph.D. in mathematics at Princeton University in 1947, teaching at California Institute of Technology, Passadena, 1948 - 1956, Professor at Stanford since 1956. As an expert of game theory he came to inventory research with a paper on the structure of dynamic programming models where he reduced a dynamic model to a static situation. He gave sufficient conditions for establishing that an (S, s) policy is optimal for the dynamic AHM model in the case of Pólya-type demand density functions as a gamma distribution or a truncated normal distribution, and the holding and shortage costs are linear (see [5], chapter 8 and 9, [48]).
The third editor of the famous Stanford studies on inventory is *Herbert E. Scarf* (b. 1930 in Philadelphia) Ph.D. in mathematics at Princeton in 1954. Mathematician at RAND Corporation, 1954 - 1957, Assistant professor of statistics at Stanford, 1957 - 1960. Professor of economics at Yale University since 1963. His conditions of optimality of an (S, s) ordering policy for the dynamic case do not depend explicitly on the demand distribution but on a property of one-period expected cost, the so called K-convexity (see [7]).
We discussed the paper [23] as a direct continuation of [4] for the case of one and of infinitely many time intervals. We have not mentioned the important case of finitely many time intervals, interdependence of demand in the various time periods, time lags in delivery, simultaneous demands for several commodities and so on. *Bellman* made these most advanced results accessible to the audience. He found functional equations like (9) as most important tools for modelling. *Bellman, Glicksberg, Gross* studied a number of such functional equations, not only existence and uniqueness, convergence of the successive approximation, but also the structure of optimal policy (see [11]). Now, this paper is part of the monograph "Dynamic Programming" where *Bellman* revealed the essential mechanisms of deterministic and stochastic multistage decision processes. The foundation of dynamic programming is comparatively simple in the deterministic case. It is more difficult in the stochastic case, in the case of Markov decision processes (see [47, 10]).
The first book on this topic is the outgrowth of *Ronald A. Howard's* Sc.D. thesis submitted to M.I.T. in 1958: "Dynamic Programming and Markov Processes" (see [42, 43]).
A formalized approach is given by *D. Blackwell* for a finite state and action space in 1962 and for general spaces in 1965 for the stationary case (see [13], [14]).
A general non-stationary dynamic modelling which covers the *Dvoretzky-Kiefer-Wolfowitz'* framework including the case where the probability distribution of demand is not completely known is developed by *Karl Hinderer* (see [41]). His monograph was basic for a special direction of mathematical modelling in inventory research in Germany (see [35], [81], [46], [54]).

9. Conclusions

Kenneth J. Arrow wrote as President of the International Society for Inventory Research in a message to members:

> "... The process of inventory accumulation, holding, and decumulation is in itself a significant economic problem ... it manifests elements of some of the deeper concern of all life: the presence of uncertainty, the need for flexibility in facing an uncertain future ..." (see [45] p.1).

In this spirit we tried to sketch the origin of modern inventory modelling under uncertainty.

The key personality of the story ist, *K.J. Arrow* but the roots go a long way back to *F.Y. Edgeworth* who began to connect in his work and life economics and probability remarkably.
We hope that the picture drawn about the development of various branches of mathematics and their applications reflects how, through the work of a number of great scholars, the foundations of modern stochastic inventory theory was put down. Seemingly, rather far areas of mathematics developed through random meetings of people and ideas. In fact, we hope that through the case of stochastic inventory theory we managed to grab a moment of the thousands year long evolution of human knowledge.

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