Asymptotically Euclidean ends of Ricci flat manifolds, and conformal inversions

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Abstract: We prove that a Ricci flat end of a Riemannian manifold is asymptotically Euclidean if it is obtained from a smooth metric by a conformal inversion. A number of consequences are discussed.

Key Words: conformal compactification, asymptotically flat, Ricci curvature, scalar curvature, twistor spinors, holonomy

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Introduction and Results.

It is a basic feature of conformal geometry that the Euclidean space $\mathbb{E}^n$ admits a conformal compactification which is the standard sphere S^n. Vice versa, if one starts with the standard sphere (S^n, g) then, by stereographic projection, there is a conformal factor $\varphi\colon S^n \to \mathbb{R}$ with exactly one zero $p \in S^n$ such that $(S^n \setminus \{p\}, \frac{1}{\varphi^2} g)$ is isometric with the Euclidean space. A more sophisticated example of this type of a conformal inversion is a complete metric on an even–dimensional Euclidean space constructed in [KR2] which carries a twistor spinor with exactly one zero point p. If we send that point p to infinity by a suitable conformal factor we obtain a Ricci flat metric which is asymptotically flat at the end corresponding to the point p. In dimension 4 this is a version of the *Eguchi–Hanson metric,* cf. [KR1]. We will describe this metric in more detail in the following Examples. In each of these two cases, the metric and its asymptotic behaviour *at infinity* comes from a smooth metric around a particular point p after a type of a conformal inversion. So in some sense geometric properties *in the large* can be translated into purely local properties, and vice versa.

In more generality and in this context, one can ask two natural questions as follows:

1. *Suppose (M, g) is a Riemannian manifold with one end. Can one find a compactification of this end by one additional point ∞ and by a conformal factor φ with one zero at ∞ ?*
2. *Vice versa, suppose (M, g) is a Riemannian manifold and suppose $\varphi\colon M \to \mathbb{R}$ is a function with an isolated zero at p. What can be said about the asymptotic behaviour of $(M \setminus \{p\}, \frac{1}{\varphi^2} g)$ in a neighborhood of this end?*

Examples: 1. Let g be the standard flat metric on $\mathbb{R}^n$, and let $\overline{g} = \frac{1}{\varphi^2} g$.

(a) For $\varphi(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$ we obtain $\mathrm{Ric}_{\overline{g}} = 0$. In fact, $\overline{g}$ is again Euclidean. This transformation is nothing but the standard conformal inversion $x \mapsto \|x\|^{-2} \cdot x$.

(b) For $\varphi(x_1, \ldots x_n) = x_1^4 + \ldots + x_n^4$ we obtain

$$\mathrm{Ric}_{\overline{g}|x} = \varphi^{-1}\Big[12(n-2)A + \Big(12(x_1^2 + \ldots + x_n^2) - 16(n-1)\varphi^{-1}(x_1^6 + \ldots + x_n^6)\Big)E\Big].$$

where $A = ((\delta_{ij} x_i^2))_{i,j}$ and where E is the identity matrix. Depending on the direction $x = (x_1, \ldots, x_n)$, this expression tends to infinity for $x \to 0$. Along the diagonal $x_1 = \ldots = x_n$ we obtain $\mathrm{Ric}_{\overline{g}|x} = (x_1)^{-2} E$, up to a non-zero constant factor. Nevertheless, the eigenvalues of the Ricci tensor of $\overline{g}$ tend to zero. The same happens for any function with a zero of order $k > 2$ at the origin. In this case $\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = (k-2)||x||^{-2}$ for $||x|| \to 0$, up to a bounded function. This indicates that one has to be careful with the order of the zero.

2. On the n–dimensional sphere S^n we fix a point p. In terms of geodesic polar coordinates $(r, \theta) \in (0, \pi) \times S^{n-1}$ around p for the function $\varphi(r, \theta) = 1 - \cos r$ the manifold $(S^n \setminus \{p\}, \varphi^{-2} g)$ is isometric to Euclidean space, the corresponding transformation is the stereographic projection.

3. Here we describe explicitely the metric we already mentioned above: We consider the complete metric on the $2m$–dimensional Euclidean space $\mathbb{R}^{2m}$ of the form

$$g = dr^2 + r^2(1 + r^{2m}) h_{(1+r^{2m})^{-1}}(y) \tag{1}$$

in polar coordinates $(r, y) \in \mathbb{R}^+ \times S^{2m-1}$, the metric h_t on the sphere S^{2m-1} is the *Berger metric* or the so-called *canonical variation* of the standard metric. This means that we take the standard metric orthogonal to the direction of the canonical S^1–action on S^{2m-1}, and in the direction of this action we multiply the metric by the positive number $t = (1 + r^{2m})^{-1}$. As we show in [KR2] the function $\varphi(r, y) = r^2\sqrt{1 + r^{2n}}$ satisfies the following: The manifold $(\mathbb{R}^{2m} \setminus \{0\}, \overline{g} = \frac{1}{\varphi^2} g)$ is in inverted polar coordinates $(\rho, y) = (1/r, y)$ of the form

$$\overline{g} = \frac{1}{(1 + \frac{1}{\rho^{2m}})^{1-\frac{1}{m}}} d\rho^2 + (1 + \rho^{2n})^{\frac{1}{m}} h_{\left(1 - \frac{1}{1+\rho^{2m}}\right)}.$$

The holonomy group is the special unitary group $\mathrm{SU}(m)$, in particular it is a Ricci flat Kähler metric which is locally irreducible. In addition the metric is asymptotically Euclidean of order $2m$. In dimension 4 this is a form of the Eguchi–Hanson metric, in higher dimensions it occurs in the work of Calabi [Ca], where it is used for obtaining complete metrics with holonomy $\mathrm{SU}(m)$ on a complex line bundle over the $(m-1)$–dimensional complex projective space.

4. With the help of the metric g on $\mathbb{R}^{2m}$ given in Equation 1 and the fuction ϕ and a covering argument one can show that there are Ricci flat manifolds with an arbitrary number of asymptotically euclidean ends which are obtained from a smooth metric by a conformal inversion:

The metric g as well as the function ϕ is defined on $\mathbb{R}^{2m}$. Now $M := \mathbb{R}^{2m} - \{(x_1, \ldots, x_{2m-2}, 0, 1)\,;\, x_j \in \mathbb{R}\}$ is diffeomorphic to $\mathbb{R}^{2m-2} \times \mathbb{R}^+ \times S^1$. Hence on the covering $\tilde{M} := \mathbb{R}^{2m-2} \times \mathbb{R}^+ \times (-1, k); k \in \mathbb{Z}^+$ resp. $\tilde{M} := \mathbb{R}^{2m-2} \times \mathbb{R}^+ \times \mathbb{R}$ we obtain a metric $\tilde{g}$ and a function $\tilde{\phi}$ with the following properties: The function $\tilde{\phi}$ has k resp. ∞ zeros corresponding to k resp. ∞ asymptotically euclidean ends of the conformally equivalent and Ricci flat metric $\tilde{\phi}^{-2}\tilde{g}$ on $\tilde{M} - \tilde{\phi}^{-1}(0)$.

Concerning Question 1, an answer was recently given by M.Herzlich in [He], under suitable assumptions on the decay of the Weyl curvature. The most essential assumption here is the asymptotical flatness at the end.

In this paper we are going to given an answer to Question 2 as follows:

Theorem: *Let (M, g) be a Riemannian manifold together with a non-constant function $\varphi: M \to \mathbb{R}$ which has at least one zero. Assume that the metric $\overline{g} = \frac{1}{\varphi^2} g$ is Ricci flat outside the zero set of φ.*

Then the zero set of φ consists of isolated points, and $(M \setminus \varphi^{-1}(0), \overline{g})$ is asymptotically Euclidean of order 2 at each end corresponding to a point in $\varphi^{-1}(0)$ in suitable coordinates. These coordinates are inverted normal coordinates *around the corresponding zero of φ, after a conformal change of the metric g with a nonzero conformal factor.*

Moreover, the metric $\overline{g}$ is either flat or locally irreducible. The function φ is unique (up to scaling), unless g is locally conformally flat.

We call a coordinate system $z = (z_1, \ldots, z_n)$ with $\rho^2 = \sum_k z_k^2 > \rho_1$ for some $\rho_1 > 0$ *asymptotically Euclidean of order* τ for the metric $\overline{g}$ if the metric coefficients $\overline{g}_{ij}$ with respect to these coordinates satisfy:

$$\overline{g}_{ij} = \delta_{ij} + O(\rho^{-\tau}); \frac{\partial}{\partial z_k} \overline{g}_{ij} = O(\rho^{-\tau-1}), \frac{\partial^2}{\partial z_k \partial z_l} \overline{g}_{ij} = O(\rho^{-\tau-2})$$

for $\rho^2 = \sum_k z_k^2 \to \infty$. A shorthand notation for this asymptotic behaviour is

$$\overline{g}_{ij} = \delta_{ij} + O''(\rho^{-\tau}).$$

A Riemannian manifold N is called *asymptotically Euclidean* of order τ if the complement of a compact subset carries asymptotically Euclidean coordinates of order τ.

Before giving the proof in the next section, we want to formulate a few corollaries which illustrate the theorem.

Corollary 1: *If in addition to the assumptions in the theorem either the manifold M is compact or the Riemannian manifold $(\overline{M}, \overline{g}) = (M \setminus \varphi^{-1}(0), \overline{g})$ is complete then (M, g) is conformally diffeomorphic with the standard sphere.*

We can also apply these results to manifolds carrying twistor spinors with zeros. A spinor field ψ on a Riemannian spin manifold is called *twistor spinor,* if it satisfies the *twistor equation*

$$\nabla_X \psi + \frac{1}{n} X \cdot D\psi = 0$$

for every vector field X. Here $\nabla_X \psi$ is the spinor derivative of the spinor field ψ in direction of the vector field X. D denotes the *Dirac operator*, the dot $\cdot$ denotes the Clifford multiplication and $n = \dim M$ is the dimension of the manifold, cf. [BFGK, ch.1.4]. If the twistor spinor ψ has a nontrivial set of zeros then the *length* of ψ, the function $\varphi = \langle \psi, \psi \rangle$ satisfies the assumption of our Theorem, cf. [BFGK, ch.2.3]. Here $\langle \, , \, \rangle$ denotes the induced hermitian inner product on the spinor bundle. Hence we obtain as

Corollary 2 ([KR3, Thm.1.2]): *Let (M, g) be a simply–connected Riemannian spin manifold carrying a twistor spinor ψ with a non–empty set Z_ψ of zeros and assume that the metric is not conformally flat. Then every twistor spinor on (M, g) vanishes exactly at Z_ψ. For the dimension N of the space of twistor spinors and the holonomy* Hol *of the conformally equivalent and Ricci flat metric $(\overline{M}, \overline{g})$ one of the following holds:*

a) $n = 2m, m \geq 2, \mathrm{Hol} = \mathrm{SU}(m)$ *and* $N = 2$.

b) $n = 4m, m \geq 2, \mathrm{Hol} = \mathrm{Sp}(m)$ *and* $N = m + 1$.

c) $n = 8, \mathrm{Hol} = \mathrm{Spin}(7)$ *and* $N = 1$.

d) $n = 7, \mathrm{Hol} = G_2$ *and* $N = 1$.

The metric we described in Example 3 (cf. [KR2]) gives an example for case a) for every m. The covering argument given in Example 4 shows that there is for any $k \in \mathbb{Z}^+ \cup \{\infty\}$ a Riemannian spin manifold which is not conformally flat and which carries a twistor spinor with exactly k zeros. In addition we obtain from Corollary 1 the following Corollary due to Lichnerowicz whose first proof used the solution of the Yamabe problem:

Corollary 3 (Lichnerowicz [Li;Thm.7]): *A compact Riemannian spin manifold carrying a non–trivial twistor spinor with zero is conformally equivalent to the standard sphere.*

Proofs.

Proof of the theorem: We start with the following well known formula for the behaviour of the Ricci tensor under conformal changes $\overline{g} = \frac{1}{\varphi^2} g$:

$$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = \varphi^{-2}\Big((n-2)\cdot\varphi\cdot\nabla^2\varphi + \Big[\varphi\cdot\Delta\varphi - (n-1)\cdot||\nabla\varphi||^2\Big]\cdot g\Big), \qquad (2)$$

see [Be; 1J]. With respect to the metric g, ∇ and ∇^2 denote the *gradient* and the *Hessian,* respectively. Δ denotes the *Laplacian,* i.e. the (positive) trace of the Hessian. By assumption we have $\mathrm{Ric}_{\overline{g}} = 0$. For the trace this implies the equation

$$\frac{1}{n(n-1)} S_g = \frac{||\nabla\varphi||^2}{\varphi^2} - \frac{2}{n}\frac{\Delta\varphi}{\varphi}$$

where $S = S_g$ denotes the scalar curvature of g. Now let us consider an arbitrary trajectory of the gradient $\nabla\varphi$, parametrized by arc length t. Then we have

$$\nabla\varphi = \varphi'(t)\frac{\partial}{\partial t}, \quad \varphi'(t) = \frac{d\varphi(t)}{dt}.$$

Along this trajectory we assume $\varphi(0) = 0$ and $\varphi(t) > 0$ for $0 < t < \varepsilon$, $\varepsilon > 0$ being a constant.

<u>Claim 1:</u> $\Delta\varphi(0) = n\varphi''(0), \varphi'(0) = 0$.

Proof: From the first equation above we obtain

$$\frac{2}{n}\Delta\varphi = -\frac{S}{n(n-1)}\varphi + \frac{\varphi'^2}{\varphi}.$$

Since S is bounded, this implies $\varphi'(0) = 0$ and

$$\frac{2}{n}\Delta\varphi(0) = \lim_{t\to 0}\frac{\varphi'^2(t)}{\varphi(t)} = \lim_{t\to 0}\frac{2\varphi'(t)\varphi''(t)}{\varphi'(t)} = \lim_{t\to 0} 2\varphi''(t) = 2\varphi''(0).$$

In particular it follows that $\nabla^2\varphi(0) = \varphi''(0)\cdot g$.

<u>Claim 2:</u> Without loss of generality we may assume that $\lim_{t\to 0}\dfrac{n\varphi''(t) - \Delta\varphi(t)}{\varphi(t)} = 0$.

Proof: If $()^0$ denotes the traceless part of a $(0,2)$-tensor, then the first formula implies

$$(\nabla^2\varphi)^0 = -(n-2)\varphi\mathrm{Ric}^0.$$

Hence the limit in Claim 2 vanishes if at that point Ric^0 vanishes. Fortunately, there is a conformal change of the metric g such that in the new metric $\mathrm{Ric}^0 = 0$ holds at one point. This can be seen in normal coordinates around that point, see [KR3, Lemma 3.2].

Claim 3: $(\Delta\varphi)'(0) = \varphi'''(0) = 0$. If moreover $\varphi''(0) = 0$ then all derivatives of $\Delta\varphi$ and all derivatives of φ vanish at $t = 0$.

PROOF: From the formulae above we obtain

$$\frac{1}{n-1}S(0) = \lim_{t\to 0}\frac{n\varphi'^2(t) - 2\varphi(t)\Delta\varphi(t)}{\varphi^2(t)}$$

$$= \lim_{t\to 0}\frac{2n\varphi'(t)\varphi''(t) - 2\varphi'(t)\Delta\varphi(t) - 2\varphi(t)(\Delta\varphi)'(t)}{2\varphi(t)\varphi'(t)}$$

$$= \lim_{t\to 0}\frac{n\varphi''(t) - \Delta\varphi(t)}{\varphi(t)} - \lim_{t\to 0}\frac{(\Delta\varphi)'(t)}{\varphi'(t)} = -\lim_{t\to 0}\frac{(\Delta\varphi)'(t)}{\varphi'(t)}.$$

This implies $(\Delta\varphi)'(0) = 0$, similarly for the higher derivatives of $\Delta\varphi$. We also have from Claim 2 that

$$\frac{1}{n(n-1)}S(0) = \lim_{t\to 0}\frac{\varphi'^2 - 2\varphi\varphi''}{\varphi^2} = \lim_{t\to 0}\frac{2\varphi'\varphi'' - 2\varphi'\varphi'' - 2\varphi\varphi'''}{2\varphi\varphi'} = -\lim_{t\to 0}\frac{\varphi'''}{\varphi'}.$$

This implies $\varphi'''(0) = 0$ and the rest of Claim 3.

Claim 4: $\varphi''(0) > 0$.

PROOF: Without loss of generality, we can use Claims 1 to 4 above, in particular $\varphi'(0) = 0$. We set $\varphi(t) = \exp(y(t))$, necessarily with $\lim_{t\to 0} y(t) = -\infty, \lim_{t\to 0} y'(t) = +\infty$. From $\varphi' = y'\varphi, \varphi'' = (y'^2 + y'')\varphi$ and from Claim 2 we obtain

$$\frac{1}{n(n-1)}S(t) = \frac{\varphi'^2 - 2\varphi\varphi''}{\varphi^2} + \frac{2\varphi\varphi'' - \frac{2}{n}\varphi\Delta\varphi}{\varphi^2}$$

$$= -(y'^2 + 2y'') + o(t).$$

By scaling of the metric g we may assume that $-n(n-1) < S(0) < n(n-1)$ and hence

$$-1 < -(y'^2 + 2y'') < 1$$

in a certain interval $(0, \varepsilon)$. This is a Riccati equation for y', and a Riccati comparison argument (see [Ka;1.6]) implies that the solution satisfies

$$y'_{-1} < y' < y'_1 \quad \text{where} \quad (y'_{\pm 1})^2 + 2y''_{\pm 1} = \pm 1.$$

Regarding the pole of the solution at $t = 0$, the comparison solutions $y_{\pm 1}$ are

$$y'_{-1}(t) = \cot(\tfrac{t}{2}) \quad \text{and} \quad y'_1(t) = \coth(\tfrac{t}{2}).$$

It follows that

$$y_{-1}(t) = 2\log(\sin(\tfrac{t}{2})) + C_{-1} \quad \text{and} \quad y_1(t) = 2\log(\sinh(\tfrac{t}{2})) + C_1$$

with constants $C_{-1} \leq C_1$. This implies

$$\exp(C_{-1})\sin^2(\tfrac{t}{2}) = \exp(y_{-1}(t)) < \varphi(t) < \exp(y_1(t)) = \exp(C_1)\sinh^2(\tfrac{t}{2})).$$

In any case $\varphi''(t)$ is strictly positive for $0 \leq t < \varepsilon$, bounded away from 0. Compare the analogous comparison solution $y_0(t) = 2\log t + C$ for $(y_0')^2 + 2y_0'' = 0$ leading to $\varphi_0(t) = \exp(C)t^2$, the well known solution in the flat case, see the introduction above.

Claim 5: The zeros of φ are isolated. This follows from the fact that each zero is also a critical point, together with $\nabla^2\varphi(0) = \varphi''(0) \cdot g$, a positive definite Hessian.

Claim 6: If $x \in U \mapsto f(x) \in \mathbb{R}$ is a smooth function defined on an open neigborhood U of 0 in $\mathbb{R}^n$ with asymptotic behaviour

$$f(x) = 1 + O(r^2),$$

where $r^2 = \sum_j x_j^2$ for $r \to 0$ and if $x = \frac{z}{\|z\|^2}; \rho^2 = \sum_j z_j^2 = r^{-2}$ then we obtain the following asymptotic behaviour

$$F(z) := f(x(z)) = 1 + O''(\rho^{-2})$$

for $\rho \to \infty$. This is the shorthand notation for the following equations:

$$F(z) = 1 + O(\rho^{-2}), \frac{\partial F}{\partial z_j} = O(\rho^{-3}), \frac{\partial^2 F}{\partial z_i \partial z_j} = O(\rho^{-4})$$

PROOF: This one concludes from the chain rule, since

$$\frac{\partial}{\partial z_i} = r^2 \frac{\partial}{\partial x_i} - 2x_i \sum_k x_k \frac{\partial}{\partial x_k}.$$

Claim 7: The metric $\overline{g}$ is asymptotically Euclidean of order 2 at each end.

PROOF: Assume that $x = (x_1, \ldots, x_n)$ are geodesic normal coordinates around $p \in \varphi^{-1}(0)$ with respect to a metric g satisfying Claim 3 and Claim 4 above. Hence the corresponding metric coefficients $g_{ij}(x) = g(\partial/\partial x_i, \partial/\partial x_j)$ satisfy

$$g_{ij} = \delta_{ij} + O(r^2),$$

with $r^2 = \sum_k x_k^2$. Then the metric coefficients $h_{ij} = r^{-4}g(\partial/\partial z_i, \partial/\partial z_j)$ of the conformally equivalent metric $h = r^{-4}g$ with respect to the *inverted normal coordinates* $z = (z_1, \ldots, z_n); z_i = r^{-2}x_i$ and $\rho^2 = \sum_k z_k^2 = r^{-2}$ satisfy:

$$h_{ij} = \delta_{ij} + O''(\rho^{-2}). \tag{3}$$

This is well known, cf. [LP, ch.6], [KR3, ch.2]. As in Claim 6 it again follows from the chain rule.

From Claim 3 we know that $\Delta\varphi(0) > 0$, by scaling we may assume $\Delta\varphi(0) = n$. Then we conclude from Claim 3 and Claim 4 that

$$\varphi(x) = r^2(1 + O(r^2)),$$

which implies by Claim 6 that in inverted normal coordinates $z : r^4\varphi^{-2}(z) = 1 + O''(\rho^{-2})$. This together with Equation (2) finally shows that the coefficients $\overline{g}_{ij}$ of the metric $\overline{g} = \varphi^{-2}g = (r^4\varphi^{-2})(r^{-4}g)$ are the following, with respect to the inverted normal coordinates:

$$\overline{g}_{ij} = (1 + O''(\rho^{-2}))(\delta_{ij} + O''(\rho^{-2})) = \delta_{ij} + O''(\rho^{-2}).$$

Claim 8: The metric $\overline{g}$ is either flat or locally irreducible.

PROOF: (Cf. [KR3, Proof of Thm.1.2]) Claim 7 implies that for sufficiently large $\rho_1 > 0$ all principal curvatures κ_j of the hypersurfaces $S_\rho := \{z|\rho^2 = \sum_k z_k^2\}$ for $\rho > \rho_1$ satisfy $\kappa_j > (2\rho)^{-1}$. This shows that all geodesics γ which start in the direction of growing ρ, i.e. $\overline{g}(\gamma'(0), \partial_\rho) > 0$ are defined for all positive real numbers and $\lim_{t\to\infty} \rho(\gamma(t)) = \infty$. In geodesic normal coordinates the Ricci–flat metric $\overline{g}$ is analytic (cf. [Be, ch.5F]), hence the Riemann curvature tensor does not vanish on an open set unless the metric itself is flat.

Now we assume that the curvature tensor does not vanish on an open subset and we assume that the metric $\overline{g}$ is locally reducible. Then we can choose a geodesic $\gamma : [0, \infty) \to M \setminus \{p\}$ with $\lim_{t\to\infty} \rho(\gamma(t)) = \infty$ which in an open neigborhood U of $\gamma(0)$ lies in the factor U_1 of the Riemannian product $U = U_1 \times U_2$ and such that the Riemann curvature tensor R_2 at $\gamma(0)$ of the factor U_2 does not vanish. Hence we can choose analytic parallel vector fields $X(t), Y(t)$ along γ tangential to U_2 which span a tangent plane with non–zero sectional curvature $K(X, Y) = K_2(X, Y)$, where K_2 is the sectional curvature of the factor U_2. Hence the function $t \mapsto K(X, Y)$ is a non–zero constant k for small t. Then the analyticity implies that $\lim_{t\to\infty} K(X(t), Y(t)) = k$. But the asymptotic behaviour of the coordinate system z implies that $K(X(t), Y(t)) = O(\rho^{-4}(\gamma(t))) = 0$, a contradiction.

Claim 9: The function φ is uniquely determined up to multiplication with a constant, unless g is locally conformally flat.

PROOF: If the functions φ_1 and φ_2 both satisfy the assumptions of the Theorem then outside the discrete set (see Claim 3) $\varphi_1^{-1}(0) \cup \varphi_2^{-1}(0)$ the metrics $\overline{g}_i = \varphi_i^{-2}g$ are both Ricci flat. Hence the metrics $\overline{g}_1$ and $u^{-2}\overline{g}_1$ with $u := \varphi_1\varphi_2^{-1}$ are both Ricci flat.

Then Equation (1) implies the equations $\nabla^2 u = \frac{\Delta u}{n}\overline{g}_1$ and $2f\Delta u = n \cdot \overline{g}_1(\nabla u, \nabla u)$. Here the Hessian ∇^2, the Laplacian Δ and the gradient ∇ have to be taken with respect to the metric $\overline{g}_1$. If t denotes the arc length on the trajectories of ∇u and $u' = \frac{du}{dt}$ then the first equation implies that $\overline{g}_1$ is a warped product metric $dt^2 + u'(t)^2 g_*$, the second equation implies that $2uu'' = u'^2$. Up to a shift of the parameter and the choice of constants we have $u'(t) = t$. The metric g_* must be Einstein with scalar curvature $n(n-1)$. Then $\overline{g}_2$ is of the form $dt^2 + g_*$, i.e. it is locally reducible in contradiction to Claim 7.

PROOF OF COROLLARY 1: If M is compact, then the zero set $\varphi^{-1}(0)$ of φ is a finite set $\{p_1, \ldots, p_m\}$. Then the manifold $(\overline{M}, \overline{g}) = (M - Z_\psi, \varphi^{-2}g)$ is a complete, Ricci flat manifold which is locally irreducible and has m ends. Then we conclude from the *splitting theorem* due to Cheeger–Gromoll, that $(\overline{M}, \overline{g})$ can have only one end, i.e. $m = 1$.

Since the end of $(\overline{M}, \overline{g})$ is asymptotically Euclidean of order 2 it follows from the *volume comparison theorem* due to Bishop in a formulation of Gromov that $(\overline{M}, \overline{g})$ is isometric to the Euclidean space (cf. [KR2; Lemma 2.3]). Hence (M, g) is conformally equivalent to the standard sphere since it is a conformal one–point compactification of the Euclidean space.

If $(\overline{M}, \overline{g})$ is complete then one concludes as above that it is isometric to Euclidean space.

Proof of Corollary 2: As remarked above the length $\varphi = \langle \psi, \psi \rangle$ of a twistor spinor ψ with zero satisfies the assumption of our Theorem. Since φ is uniquely determined (up to scaling) all twistor spinors have the same set of zeros. Under the conformal equivalence the twistor spinors on (M, g) correspond to the parallel spinors on $(\overline{M}, \overline{g})$. Hence the list in Corollary 3 coincides with the list of possible holonomies of irreducible manifolds carrying parallel spinors, which is given in [Wa].

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