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Mathematics 3

2. Exercise Sheet

Exercise 1

1. Let $V$ be an $n-1$ dimensional subspace of $\mathbb{R}^n$. Show that
   \[ |V|_e = 0. \]

2. Let $A$ be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e. $A \in \mathbb{R}^{n \times n}$. Assume that $\text{rang}(A) < n$. Show that
   \[ |A(E)|_e = 0 \text{ for all } E \subset \mathbb{R}^n. \]

Solution of Exercise 1

1. Without loss of generality we can rotate $V$ and by this assume $V = \mathbb{R}^{n-1} \times \{0\}$. Now overlap a given rational point $q \in V \cap \mathbb{Q}^n$ by a rectangle $R = [-1,1]^{n-1} \times [-\varepsilon, \varepsilon]$. Since $\varepsilon > 0$ is arbitrary, we have $|R|_e = 0$ and because of that the measure is invariant under translations we also know that $|R + q_k^{(n)}| = 0$ for the sequence of all rational points $q_k^{(n)} \in \mathbb{Q}^n$ (precisely, all vectors $q_k^{(n)} = (q_k^1, \ldots, q_k^n) \in \mathbb{Q}^n$). Using the $\sigma$–additivity of the Lebesgue measure and the fact that $\mathbb{Q}^n$ is countable we can conclude that $|V|_e \leq \bigcup_{k=1}^\infty |R + q_k^{(n)}|_e \leq \sum_{k=1}^\infty |R|_e = 0$ and so $|V|_e = 0$.

2. Since the image under $A$ has dimension less or equal to $n - 1$, we conclude from 1. the result for the case $\text{rang}(A) = n - 1$. If $\text{rang}(A) = n - k$ ($1 \leq k \leq n - 1$) use the same procedure with a rectangle of the form $R = [-1,1]^{n-k} \times [-\varepsilon, \varepsilon]^k$. If $k = n$ then $A = 0$ and so of course $|A(E)|_e = |\{0\}|_e = 0$ for all $E$. (Note that $\text{det}(A) = 0$ whenever $\text{rang}(A) < n$ and so the formula $|A(E)|_e = \text{det}(A)|E|_e$ given in the lecture even holds for singular matrices.)

Exercise 2

Show that a set $E \subset \mathbb{R}^n$ is Lebesgue measurable if and only if it satisfies the Caratheodory criterium:
   \[ |F|_e = |F \cap E|_e + |F \setminus E|_e \text{ for all } F \subset \mathbb{R}^n. \]

Solution of Exercise 2

Assume $E$ satisfies the Caratheodory criterium. Let $\varepsilon > 0$ be given. As shown in the lecture there is $U \supset E$ open such that $|U|_e < |E|_e + \varepsilon$. Now we apply the Caratheodory criterium with $F = U$ and recall that $U \cap E = E$ hence $|U|_e = |E|_e + |U \setminus E|_e$ which is equivalent to
   \[ |U \setminus E|_e = |U|_e - |E|_e < \varepsilon. \]

Thus $E$ is measurable.

Let now $E$ be measurable. Given any set $F \subset \mathbb{R}^n$ as shown in the lecture there is $A \supset F$ measurable with $|F|_e = |A|_e$. Since $A \cap E$ and $A \setminus E$ are disjoint we have $|A|_e = |A \cap E|_e + |A \setminus E|_e$. Now we combine it with the monotonicity ($F \cap E \subset A \cap E$, $F \setminus E \subset A \setminus E$) to get
   \[ |F|_e = |A|_e = |A \cap E|_e + |A \setminus E|_e \geq |F \cap E|_e + |F \setminus E|_e. \]

The reverse inequality just follows from $F \subset (F \cap E) \cup (F \setminus E)$. 

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Exercise 3

Give an example of a measurable subset $E \subset [0, 1]$ (i.e. $|E|_e \leq 1$) with

$$|\partial E|_e > 0.$$ 

Note: It is possible to construct $E$ such that $|E|_e = 0$ and $|\partial E|_e = 1$.

Solution of Exercise 3

Since $\mathbb{Q}$ is countable, we have $|\mathbb{Q} \cap [0, 1]|_e = 0$. Because of $\partial \mathbb{Q} = \mathbb{R}$ we see that $\partial (\mathbb{Q} \cap [0, 1]) = [0, 1]$ and so $|\partial (\mathbb{Q} \cap [0, 1])|_e = 1 > 0$.

Exercise 4

Show that in fact Corollary 3.6 (i) holds for general sets (not necessarily measurable): Let $A_i$ be a monoton increasing sequence of subsets of $\mathbb{R}^n$, i.e. for all $i$ let $A_i \subset A_{i+1}$ then

$$\lim_{i \to \infty} |A_i|_e = |\bigcup_{i=1}^{\infty} A_i|_e.$$  

Hint: To each $A_i$ choose an appropriate measurable set $E_i$ with $A_i \subset E_i$ and $|A_i|_e = |E_i|_e$. Try to apply Corollary (3.6) (i) to the sequence $E_i$.

Solution of Exercise 4

Choose $E_i \supset A_i$ measurable with $|A_i| = |E_i|$ and a measurable set $\tilde{F}$ which covers $\bigcup_{i=1}^{\infty} A_i$ and fulfills $|\bigcup_{i=1}^{\infty} A_i| = |\tilde{F}|$. Now define $F_i = \tilde{F} \cap (\bigcap_{j \geq i} E_j) \supset A_i$. Then $|A_i| \leq |F_i| \leq |E_i| = |A_i|$ and so $|F_i| = |E_i|$. In the same way we see that

$$|\bigcup_{i=1}^{\infty} A_i| \leq |\bigcup_{i=1}^{\infty} F_i| = |\bigcup_{i=1}^{\infty} (\tilde{F} \cap (\bigcap_{j \geq i} E_j))| = |\tilde{F} \cap (\bigcup_{i=1}^{\infty} \bigcap_{j \geq i} E_j)| \leq |\tilde{F}| = |\bigcup_{i=1}^{\infty} A_i|$$

and so $|\bigcup_{i=1}^{\infty} F_i| = |\bigcup_{i=1}^{\infty} A_i|$. We know the statement for measurable sets and therefore ($F_i$ are measurable) we get the result by

$$\lim_{i \to \infty} |A_i| = \lim_{i \to \infty} |F_i| = |\bigcup_{i=1}^{\infty} F_i| = |\bigcup_{i=1}^{\infty} A_i|$$

Exercise 5 (extra exercise)

Let $E_1 := \mathcal{N}$ denote the non-measurable subset of $[0, 1]$ presented in the exercise session and set $E_2 = [0, 1] \setminus E_1$.

1. Show that $|E_2|_e = 1$. (Prove it by contradiction. You may use that every measurable subset of $\mathcal{N}$ has measure 0.)

2. Conclude that there are two sets $A_1, A_2$ with $A_1 \cap A_2 = \emptyset$ but

$$|A_1|_e + |A_2|_e \neq |A_1 \cup A_2|_e.$$ 

Solution of Exercise 5

1. If not, then $|E_2|_e = 1 - \varepsilon$ for some $\varepsilon > 0$. Now we know that we can find some measurable set $F \subset [0, 1]$ s.t. $E_2 \subset F$ and $|F|_e = |E_2|_e$ and in fact $|F|_e = 1 - \varepsilon$, i.e. $|[0, 1] \setminus F|_e = \varepsilon$. But now we have $[0, 1] \setminus F \subset E_1$ and by this $|[0, 1] \setminus F|_e = 0$ in contradiction to $\varepsilon > 0$.

2. We know that all measure zero sets are measurable. By this, we can conclude that $|\mathcal{N}|_e > 0$, lets say $|\mathcal{N}|_e = \varepsilon$. But then

$$|\mathcal{N}|_e + |[0, 1] \setminus \mathcal{N}|_e = \varepsilon + 1 > 1 = |\mathcal{N} \cup ([0, 1] \setminus \mathcal{N})|. $$