Mathematics 3

4. Exercise Sheet

Exercise 1

Show that the Lebesgue measure of the triangle with edges at \((0, 0)\), \((a, 0)\) and \((0, b)\) \((a, b > 0)\) is given by \(\frac{ab}{2}\). (Keep in mind that by definition of the Lebesgue measure you can just measure rectangles!)

Exercise 2

Define

\[
 f : \mathbb{R}^n \to \mathbb{R} \\
 f(x) = \begin{cases} 
 |x|^{-\beta} & |x| > 1 \\
 0 & \text{else}
\end{cases}
\]

Show that \(f\) is measurable and determine all \(\beta \in \mathbb{R}\) for which \(\int_{\mathbb{R}^n} f(x)dx < \infty\).

Exercise 3

1. Let \(\{E_i\}_{i=1}^N\) be given measurable sets in \(\mathbb{R}^n\). Show that there are disjoint measurable sets \(\{E_j^*\}_{j=1}^{N^*}\) s.t.

\[
\begin{align*}
(i) & \quad \bigcup_{i=1}^N E_i = \bigcup_{j=1}^{N^*} E_j^* \\
(ii) & \quad E_i = \bigcup_{E_j^* \subset E_i} E_j^*
\end{align*}
\]

Show further that \(N^* = 2^N - 1\). \textit{Hint:} Do this by induction in \(N\).

2. Let \(f(x) = \sum_{i=1}^N a_i 1_{E_i}\) be a given measurable simple function. Show that one may write \(f = \sum_{j=1}^{N^*} a_j^* 1_{E_j^*}\) with disjoint sets \(E_j^*\) and \(a_j^* = \sum_{i: E_j^* \subset E_i} a_i\).

\textit{Note:} This exercise shows that we can always wlog assume that the \(E_i\) are disjoint and by this simplify proofs that use simple functions.

Exercise 4

1. Let \(v \in \mathbb{R}^n\) and \(f : \mathbb{R}^n \to \mathbb{R}\) be measurable. Define \(f_v(x) := f(x - v)\). Show that the integral over the hole space is invariant under translation, i.e.

\[
\int_{\mathbb{R}^n} f_v(x)dx = \int_{\mathbb{R}^n} f(z)dz.
\]
2. Let \( \lambda_1, \ldots, \lambda_n > 0 \) and \( E \subset \mathbb{R}^n \) be measurable. Define \( f_\lambda(x) := f(\lambda_1 x_1, \ldots, \lambda_n x_n) \). Show that the integral behaves the correct way under rescaling, that is,

\[
\int_E f_\lambda(x) \, dx = \lambda_1^{-1} \cdots \lambda_n^{-1} \int_{\lambda E} f(z) \, dz
\]

where we set \( \lambda E = \{(\lambda_1 x_1, \ldots, \lambda_n x_n) : x \in E\} \).

Hint: Use ex. 2 from sheet 1 to show that \( |R(E, f_\lambda)| = \lambda_1^{-1} \cdots \lambda_n^{-1} |R(\lambda E, f)| \) and the analogous statement for \( R(\mathbb{R}^n, f_v) \).