Exercise 16. Let $U$ be the open box $U = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\}$ and let $0 < \lambda_1 \leq \lambda_2 \ldots$ be the eigenvalues of the Dirichlet-Laplace operator (i.e. $Lu = -\Delta u$ with zero boundary conditions). Let

$$N(\lambda) = \text{sup}\{n \in \mathbb{N} : \lambda_n < \lambda\}.$$ 

By explicitly calculating the eigenvalues and eigenfunctions, calculate the limit

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{3/2}}.$$ 

Hint: to find the eigenfunctions use separation of variables.

Exercise 17. A function $u \in H_0^2(U)$ is a weak solution of the biharmonic equation

$$\Delta^2 u = f \text{ in } U$$

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U$$

provided

$$\int_U \Delta u \Delta v \, dx = \int_U f v \, dx \quad \text{for all } v \in H_0^2(U).$$

Given $f \in L^2(U)$ prove that there exists a unique weak solution.

Exercise 18. Let $U \subset \mathbb{R}^n$ be bounded, connected, with $C^1$ boundary $\partial U$ and let $1 \leq p \leq \infty$. Show that there exists a constant $c = c(n, p) > 0$ so that for any $u \in W^{1, p}(U)$

$$\left\| u - \frac{1}{|U|} \int_U u \, dx \right\|_{L^p(U)} \leq c \| Du \|_{L^p(U)}.$$ 

Hint: argue by contradiction and use Rellich-Kondrachov.