Homework accompanying the lecture

Mathematics 4 for Physicists

Worksheet 7

The base field for all the vector spaces on this worksheet are the complex numbers.

Exercise 1 (2 Points). Let $V$ be a Banach space and $T \in B(V)$ be continuously invertible. Use the Neumann series to show, that there is $\epsilon > 0$ such that $S$ is invertible provided $\|T - S\| \leq \epsilon$.

Hint: $S = T(I - T^{-1}(T - S))$.

Exercise 2 (3 Points). Let $T : l^2 \to l^2$ be the left shift operator 

$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Show that any $\lambda \in \mathbb{C}$ fulfilling $|\lambda| < 1$ is an eigenvalue of $T$.

Determine the spectrum of $T$ and of the right shift operator $T^*$.

Exercise 3 (3 Points). Let $(\lambda_n) \subset \mathbb{C}$ be a bounded sequence. Determine the spectrum of the operator

$$T : l^2 \to l^2$$

$$(x_n) \mapsto (\lambda_n x_n).$$

Exercise 4 (4 Points). Let $H$ be a Hilbert space and $T \in B(H)$ self-adjoint. Suppose $T$ is a positive operator, i.e. $\langle x| Tx \rangle \geq 0$ for any $x \in H$.

Show, that for any $\epsilon > 0$, the operator $T + \epsilon$ is bounded below, $\ker(T + \epsilon) = \{0\}$ and $\text{im}(T + \epsilon)$ is dense in $H$.

Show, that $\sigma(T) \subset \{\lambda \in \mathbb{R} | \lambda \geq 0\}$.

The written solutions to these exercises should be handed in immediately before the lecture on Monday the 28.05.2018.