Exercise 10. Let $U \subset \mathbb{R}^n$ be open and bounded and $u \in H^1(U)$ a weak solution of
\[- \sum_{ij} \partial_j (a^{ij} \partial_i u) = 0 \quad \text{in } U,
\]
where the coefficients $a^{ij} \in L^\infty(U)$ are uniformly elliptic. Prove that the weak maximum principle holds:
\[\sup_U u \leq \sup_{\partial U} u^+\]
Hint: use $v := (u - \sup_{\partial U} u^+)^+$ as a test function in the definition of weak solution. Why is $v$ a valid test function?

Exercise 11. Let $U = (0, \pi)$ and consider the Dirichlet problem
\[
Lu = f \quad \text{in } U, \\
u = 0 \quad \text{on } \partial U,
\]
where $Lu = -\frac{d^2}{dx^2} u - u$ and $f \in L^2(U)$. Under what conditions on $f$ does there exist a weak solution $u$?

Exercise 12. Let $(a, b) \subset \mathbb{R}$ and $f \in H^1_0(a, b)$. Show that
\[
\int_a^b |f(x)|^2 \, dx \leq \left( \frac{b-a}{\pi} \right)^2 \int_a^b |f'(x)|^2 \, dx.
\]
Show also that the constant $\left( \frac{b-a}{\pi} \right)^2$ is the best possible, i.e. there exists no constant $c < \left( \frac{b-a}{2} \right)^2$ so that the inequality would hold with $c$ for all $f \in H^1_0(a, b)$. Hint: one possibility is to use Fourier series.

Exercise 13. Show by example, that if $u \in L^1_{loc}(\mathbb{R}^n)$ and, for some bounded open subset $U \subset \mathbb{R}^n$, $\|D^h u\|_{L^1(U)} \leq C$ for all $|h| > 0$, it does not necessarily follow that $u \in W^{1,1}(U)$.
Recall that $D^h u$ denotes the difference quotient of $u$.
Exercise 14. Let $U \subset \mathbb{R}^n$ be open and bounded. Let $\lambda$ be the Rayleigh-quotient, defined as

$$\lambda := \inf \left\{ \frac{\|\nabla u\|^2_{L^2(U)}}{\|u\|^2_{L^2(U)}} : u \in H^1_0(U) \setminus \{0\} \right\}.$$ 

(i) Show that there exists $w \in H^1_0(U)$ with $\|w\|_{L^2(U)} = 1$ and $\|\nabla w\|^2_{L^2(U)} = \lambda$.

Hint: Take a sequence $u_j \in H^1_0(U)$ such that $\|u_j\|_{L^2(U)} = 1$ and $\|\nabla u_j\|^2_{L^2(U)} \to \lambda$. Show that there exists a subsequence such that $u_j \to w$ in $H^1_0(U)$ and $u_j \to w$ in $L^2(U)$.

(ii) Show that $w$ satisfies

$$\int_U \nabla w \cdot \nabla v \, dx = \lambda \int_U wv \, dx \quad \forall v \in H^1_0(U).$$

Hint: test the definition of $\lambda$ with $w + tv$ for small $t \in \mathbb{R}$.

(iii) Deduce that $\lambda$ is the lowest eigenvalue of the Dirichlet-Laplace operator (i.e. $Lu = -\Delta u$ with zero boundary conditions).

Exercise 15. Let $U \subset \mathbb{R}^n$ be open and bounded. Given $f \in L^2(U)$ consider the Dirichlet problem

$$-\Delta u = f \quad \text{in } U$$

$$u = 0 \quad \text{on } \partial U.$$ 

Let $\{\psi_1, \ldots, \psi_m\}$ be an orthonormal set of functions in $H^1_0(U)$. Construct an approximate solution by setting $u_m = \sum_{k=1}^m c_k \psi_k$, where the coefficients $c_k \in \mathbb{R}$ are to satisfy

$$\int_U \nabla u_m \cdot \nabla \psi_k \, dx = \int_U f \psi_k \, dx \quad k = 1, \ldots, m.$$

Show that these conditions on $\{c_k\}_{k=1}^m$ can be written as an algebraic system of $m$ linear equations for $m$ unknowns, and prove that this system has a unique solution. Prove furthermore, that the unique solution $u_m$ satisfies an estimate of the form

$$\|u_m\|_{H^1(U)} \leq C,$$

where the constant $C$ depends on $f$ but is independent of $m$ and $\{\psi_i\}$. 

4