**Exercise 1 (3 Points).** Let $H$ be a Hilbert space and $T \in B(H)$ a self-adjoint operator. Show, that $T$ is positive, i.e. $\langle Tx| x \rangle \geq 0$ for any $x \in H$, if and only if there is $A \in B(H)$ self-adjoint such that $T = A^2$.

**Exercise 2 (3 Points).** Let $H$ be a Hilbert space, 

$$f(t) = \sum_{n=0}^{\infty} \alpha_n t^n$$

a power series with radius of convergence $r > 0$ and $T \in B(H)$ self-adjoint with $\|T\| < r$. Show, that

$$\sum_{n=0}^{\infty} \alpha_n T^n$$

is convergent and its value is $f(T)$.

**Exercise 3 (3 Points).** For any bounded sequence $(\lambda_n) \in \ell^\infty$ define the multiplication operator

$$M_{(\lambda_n)} : \ell^2 \rightarrow \ell^2$$

$$(x_n) \mapsto (\lambda_n x_n).$$

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $(\lambda_n) \in \ell^\infty$ real-valued. Show that $f \left( M_{(\lambda_n)} \right) = M_{(f(\lambda_n))}$. 

Hint: Consider polynomial functions $f$ first.

**Exercise 4 (3 Points).** Let $H$ be a Hilbert space, $T \in B(H)$ self-adjoint and $\lambda \in \sigma(T)$ an isolated point of the spectrum, i.e. there is $\epsilon > 0$ such that $(\lambda - \epsilon, \lambda + \epsilon) \cap \sigma(T) = \{\lambda\}$. Show, that the characteristic function

$$\mathbb{1}_{\lambda} : \sigma(T) \rightarrow \mathbb{R}$$

$$t \mapsto \begin{cases} 1 & t = \lambda \\ 0 & \text{otherwise} \end{cases}$$

is continuous and that $\mathbb{1}_{\lambda}(T)$ is a non-zero orthoprojection. Show, that $\lambda$ is an eigenvalue of $T$. 

The written solutions to these exercises should be handed in immediately before the lecture on Monday the 04.06.2018.