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Cohomology Operations from S^1-Cobordisms in Floer Homology

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Abstract

In this work, Floer homology is considered as a relative Morse theory for the symplectic action functional on the loop space of a symplectic manifold (M, ω). It is assumed that M is closed and the cohomology classes $\{\omega\}, c_1(TM) \in H^2(M)$ vanish on $\pi_2(M)$. In Floer homology the relative gradient flow for the Hamiltonian action functional is analysed by means of a class of nonlinear Fredholm operators for maps of the infinite standard cylinder $\mathbb{R} \times S^1$ into M. The operators involved are elliptic partial differential operators of Cauchy-Riemann type. The boundary conditions are given by contractible non-degenerate 1-periodic solutions of a fixed Hamiltonian equation.

It is shown that the analytical concept of Floer homology for these nonlinear Cauchy-Riemann operators can be generalized from the standard cylinder to arbitrary Riemann surfaces which are either closed or have ends which are endowed with the standard cylindrical structure and carry a specified orientation. A nonlinear Fredholm analysis is used to define algebraic operations on the Floer homology groups. In our work the equivalent description as a cohomology theory is chosen. Under the specified conditions these groups are known to be canonically isomorphic to the singular homology of M as a graded $\mathbb{Z}_2$-vector space. The cohomological operations associated to Riemann surfaces give rise to further algebraic structures on this graded vector space.

The main result is that the Floer cohomology $HF^*(M, \mathbb{Z}_2)$ carries the structure of a $\mathbb{Z}$-graded associative and commutative algebra over $\mathbb{Z}_2$ with unit and with a non-degenerate symmetric bilinear form.

The operation $Z(\Sigma)$ on Floer cohomology associated to a Riemann surface Σ is defined by counting solutions of nonlinear Cauchy-Riemann type partial differential equations under generic conditions. It is proven that the operator $Z(\Sigma)$ is uniquely determined by the topological type of the model surface Σ, that is, the oriented homeomorphisms class of the compactified surface. Here, the oriented cylindrical ends are compactified as $(0,1] \times S^1$ and $[-1,0) \times S^1$, respectively. Thus, every such surface can be viewed as an oriented S^1-cobordism. It turns out that our theory leads to a functor Z which assigns to each such cobordism Σ with $a+b$ ends of specified orientation a multi-linear operator $Z(\Sigma)\colon \bigotimes^a HF^* \to \bigotimes^b HF^*$. It is proven that this functor Z satisfies the axioms of a topological field theory in dimension $1+1$ in the sense of Atiyah, [3]. This is equivalent to describing the vector space HF^* as an algebra over $\mathbb{Z}_2$ with unit and non-degenerate bilinear form compatible with the multiplication.

Altogether, our work establishes a multiplicative structure on Floer homology. The multiplication is associated to the topological type of a model surface with oriented boundary given by the standard compact 2-disk where two open disks are removed from the interior. The so-called the pair-of-pants product is a multiplicative operation of degree n on the Floer cohomology HF^*, if $\dim M = 2n$. The unit element is associated to the standard disk and has the degree n. The non-degenerate bilinear form is represented by an annulus with boundary circles oriented in opposite direction. The latter bilinear form on HF^* of degree 0 corresponds to the Poincaré duality under the canonical vector space isomorphism $HF^*(M, \mathbb{Z}_2) \cong H_{n-*}(M, \mathbb{Z}_2)$. With respect to this isomorphism, it is legitimate to denote the operators $Z(\Sigma)$ as cohomology operations.

The emphasis of this work lies on the construction of the cohomology operations based on the analysis of the nonlinear Fredholm operators in Floer homology theory.

Zusammenfassung

In dieser Arbeit wird Floer-Homologie in der Version einer relativen Morse-Theorie für das Wirkungsfunktional auf dem Schleifenraum einer symplektischen Mannigfaltigkeit (M, ω) behandelt. Es wird angenommen, daß die Mannigfaltigkeit M geschlossen ist und die Cohomologieklassen $\{\omega\}, c_1(TM) \in H^2(M)$ auf $\pi_2(M)$ verschwinden. In der Floer-Homologietheorie wird der relative Gradientenfluß für das Hamiltonsche Wirkungsfunktional mit Hilfe einer Klasse nichtlinearer Fredholm-Operatoren für Abbildungen des unendlich langen Standardzylinders $\mathbb{R} \times S^1$ in M definiert. Diese Operatoren sind elliptische partielle Differentialoperatoren von dem Typ eines quasilinearen Cauchy-Riemann-Operators. Die Randbedingung ist durch zusammenziehbare nichtdegenerierte 1-periodische Lösungen einer Hamilton-Gleichung gegeben.

Es wird gezeigt, daß das analytische Konzept der Floer-Homologietheorie für diese nichtlinearen Cauchy-Riemann-Operatoren von dem Standardzylinder auf Riemannsche Flächen verallgemeinert werden kann, die entweder geschlossen sind, oder deren Enden die Struktur des Standardzylinders tragen und jeweils mit einer festen Orientierung versehen sind. Diese erweiterte nichtlineare Fredholm-Analysis im Rahmen der Floer-Homologietheorie führt zu algebraischen Operationen auf den Floer-Homologiegruppen. In der vorliegenden Arbeit wird die äquivalente Beschreibung als eine Cohomologie-Theorie gewählt. Es ist bekannt, daß diese Cohomologie-Gruppen unter den gegebenen Bedingungen zur singulären Homologie der Mannigfaltigkeit M in kanonischer Weise isomorph sind, und zwar als graduierter Vektorraum über $\mathbb{Z}_2$. Die Operationen, die den Riemannschen Flächen zugeordnet werden, führen zusätzliche algebraische Strukturen auf diesem graduierten Vektorraum ein.

Das Hauptresultat der vorliegenden Arbeit besagt, daß die Floer-Cohomologie $HF^*(M, \mathbb{Z}_2)$ die Struktur einer $\mathbb{Z}$-graduierten assoziativen und kommutativen Algebra über $\mathbb{Z}_2$ mit Eins und mit einer nichtdegenerierten symmetrischen Bilinearform trägt.

Die einer Riemannschen Fläche Σ zugeordnete Operation $Z(\Sigma)$ auf der Floer-Cohomologie wird durch Zählen von Lösungen einer nichtlinearen partiellen Differentialgleichung von Cauchy-Riemann-Typ unter generischen Bedingungen definiert. Es wird bewiesen, daß die den Riemannschen Flächen zugeordneten Operatoren auf der Floer-Cohomologie durch den topologischen Typ der Fläche eindeutig bestimmt sind. Dieser Typ bezeichnet die Homöomorphieklasse der jeweiligen kompaktifizierten Fläche, wobei die orientierten

zylindrischen Enden als $(0, 1] \times S^1$ bzw. $[-1, 0) \times S^1$ kompaktifiziert werden. Auf diese Weise wird eine solche Fläche als orientierter S^1-Cobordismus betrachtet. Es stellt sich heraus, daß die vorliegende Erweiterung der Floer-Homologietheorie einen Funktor Z beschreibt, der jedem solchen Cobordismus Σ mit $a + b$ Enden entsprechender Orientierung einen multilinearen Operator $Z(\Sigma)\colon \bigotimes^a HF^* \to \bigotimes^b HF^*$ zuordnet. Es wird gezeigt, daß dieser Funktor Z die Axiome einer topologischen Feldtheorie der Dimension $1 + 1$ gemäß [3] erfüllt. Dies ist zu einer Darstellung von HF^* als Algebra über $\mathbb{Z}_2$ mit Eins und nichtdegenerierter Bilinearform äquivalent. Es wird also in dieser Arbeit eine multiplikative Struktur in der Floer-(Co)-Homologie hergeleitet. Diese Produkt-Struktur wird dem topologischen Typ einer Fläche mit orientiertem Rand zugeordnet, die durch die kompakte Standard-Kreisscheibe gegeben ist, aus der zwei offene disjunkte Kreisscheiben entfernt worden sind. In der Floer-Cohomologie HF^* ist dieses sogenannte Hosen-Produkt eine Multiplikation vom Grad n, wobei $\dim M = 2n$ gilt. Das neutrale Element folgt aus dem Typ der Standard-Kreisscheibe und hat den Grad n. Die nichtdegenerierte Bilinearform ist durch einen kompakten Zylinder gegeben, dessen Randkurven entgegengesetzt orientiert sind. Diese Bilinearform auf HF^* entspricht unter dem kanonischen Vektorraum-Isomorphismus $HF^*(M, \mathbb{Z}_2) \cong H_{n-*}(M, \mathbb{Z}_2)$ der Poincaré-Dualität. Aufgrund dieser natürlich auftretenden Darstellung der Poincaré-Dualität werden an der gebräuchlichen Bezeichnung Floer-Homologie festgehalten und dennoch die Operatoren $Z(\Sigma)$ als Cohomologie-Operationen bezeichnet.

Der Schwerpunkt der vorliegenden Arbeit liegt in der Konstruktion der Cohomologie-Operationen mit Hilfe einer erweiterten Analysis für die nichtlinearen Fredholm-Operatoren der Floer-Homologietheorie.

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Contents

CHAPTER 1

Introduction

The aim of this work is to construct a functor Z which assigns to every oriented S^1-cobordism an operator acting on Floer homology. It gives a homology theory which is derived from a relative version of Morse theory on the loop space of a symplectic manifold (M,ω). The term relative Morse theory refers to a generalized Morse theory in which merely those trajectories of the gradient flow of a functional are considered, which have bounded flow energy. In particular, no global gradient flow is required. Indeed, in our case of a Morse theory for the symplectic action functional there is no global gradient flow, and the bounded trajectories are constructed as solutions of elliptic boundary value problems for Cauchy-Riemann type equations.

1.1 The Floer Homology Groups

Let (M,ω) be a closed symplectic manifold of dimension $2n$. We associate to every smooth Hamiltonian $H\colon M\to\mathbb{R}$ the Hamiltonian vector field X_H. Floer homology is generated by the 1-periodic trajectories of this vector field on M. In general, we have to allow perturbations of H as a non-autonomous Hamiltonian $H\in C^\infty(S^1\times M,\mathbb{R})$. The associated vector field X_{H_t} is defined by

$$\omega(X_{H_t}(x),\cdot\,)=-dH_t(x),\quad (t,x)\in S^1\times M\,.$$

The periodic orbits are smooth solutions of the Hamiltonian equation

$$\dot x(t)=X_{H_t}(x(t)),\quad x(0)=x(1)\,. \tag{1.1}$$

Floer homology deals with the case where these periodic orbits are assumed to be non-degenerate. Recall that every orbit gives rise to a fixed point for the time-1 map of the flow $\psi\colon\mathbb{R}\times M\to M$ of X_{H_t},

$$\dot\psi_t(x)=X_{H_t}(\psi_t(x)),\quad \psi_0=\mathrm{id}_M,$$

and the orbit $x(t)=\psi_t(x(0))$ is called **non-degenerate** if 1 is not an eigenvalue of the linearization of the time-1 map,

$$1\notin\sigma\Big(T\psi_1(x(0))\Big)\,.$$

Moreover, it is sufficient to concentrate on the contractible periodic solutions of (1.1), because Floer homology turns out to be invariant under continuation and (1.1) admits no non-contractible solutions if H is time-independent with $\|H\|_{C^2}$ small enough. In particular, the components of the free loop space

$\Omega(M)$ consisting of homotopically non-trivial loops will have no relevance for Floer homology theory. We denote the set of contractible solutions of (1.1) by

$$\begin{aligned}\mathcal{P}_1(H)&=\{x\in C^\infty(S^1,M)\,|\,x\text{ is contractible and solves (1.1)}\,\}\\ &\subset\Omega^\circ(M)=\{\text{ contractible smooth loops}\,\}\,.\end{aligned}$$

Fixing any point $p_o\in M$ we obtain, in view of the canonical embedding $M\hookrightarrow\Omega^\circ(M)$,

$$\pi_1(\Omega^\circ(M),p_o)=\pi_2(M,p_o)\,.$$

The second homotopy group of M plays a decisive role within Floer homology theory. The influence of this abelian group is characterized by the following group homomorphisms. We identify $\pi_2(M)$ with its image in $H_2(M,\mathbb{Z})$ under the Hurewicz homomorphism. The homomorphism $\phi_\omega\colon\pi_2(M)\to\mathbb{R}$ is defined by evaluation of the cohomology class $\{\omega\}\in H^2(M,\mathbb{R})$,

$$\phi_\omega(\{u\})=\int_{S^2}u^*\omega,\quad u\in C^0(S^2,M)\,,$$

and analogously

$$\phi_c\colon\pi_2(M)\to\mathbb{Z},\quad \phi_c(A)=\langle c_1,A\rangle\,.$$

Here, $c_1=c_1(TM)$ is the first Chern class of the tangent bundle of M, whose isomorphism class as a complex vector bundle (TM,J) is canonically defined by the symplectic structure. Namely, we can choose any almost complex structure $J\in\mathrm{End}(TM)$ on M, $J^2=-\,\mathrm{id}$, which is compatible with ω, in the sense that

$$\omega\circ(\mathrm{id}\times J)=\langle\cdot,\cdot\rangle_J$$

is a Riemannian metric. The space of ω-**compatible almost complex structures** is contractible.

The abelian group $\pi_2(M)/\ker\phi_\omega$ describes the ambiguity of the Hamiltonian **action functional**

$$\mathcal{A}_H\colon\Omega^\circ(M)\to\mathbb{R}/\operatorname{im}(\phi_\omega),\quad \mathcal{A}_H(x)=\int_{D^2}\bar x^*\omega-\int_{S^1}H_t(x(t))dt\,.$$

Here, $\bar x\colon D^2\to M$ is any extension of $x\colon S^1\to M$ to the standard disk.

Floer homology can be viewed as a variational method in the sense that the contractible solutions of (1.1) are exactly the critical points of $\mathcal{A}_H$. Moreover, the action functional is well-defined as a real-valued functional on the covering of $\Omega^\circ(M)$ with $\pi_2(M)/\ker\phi_\omega$ as group of covering transformations. In particular, it is real-valued in the relative sense that it assigns the relative action $\Phi([u])=\big(\mathcal{A}_H(y)-\mathcal{A}_H(x)\big)_{[u]}$ to every homotopy class of continuous paths $u\colon[0,1]\to\Omega^\circ(M)$ with $u(0)=x$, $u(1)=y$. The **relative gradient flow** for $\mathcal{A}_H$ is well-defined in terms of the space of bounded energy trajectories,

$$\begin{aligned}\mathcal{M}(J,H)=\{\,u\in C^\infty(\mathbb{R}\times S^1,M)\,|\,\overline\partial_{J,H}(u)=0,\\ \iint_{\mathbb{R}\times S^1}|\partial_s u|_J^2\,ds dt<\infty\,\}\,.\end{aligned}$$

Here, $\overline{\partial}_{J,H}$ is the partial differential operator defined for maps from the infinite standard cylinder $\mathbb{R} \times S^1$ to M as

$$\begin{aligned}\overline{\partial}_{J,H}(u) &= \partial_s u + J(t,u)\partial_t u + \nabla H(t,u) \\ &= \partial_s u + J(t,u)\big[\partial_t u - X_{H_t}(u)\big] .\end{aligned}$$

For sake of regularity, in general, J has to be allowed to depend explicitly on $t \in S^1$. The space of connecting trajectories for $x, y \in \mathcal{P}_1(H)$ is defined by

$$\mathcal{M}_{x,y}(J,H) = \{\, u \in C^\infty(\mathbb{R} \times S^1, M) \mid \overline{\partial}_{J,H}(u) = 0,\ u(-\infty) = x,\ u(\infty) = y \,\},$$

where $u(\pm\infty) = x$ is meant as a short hand notation for the uniform limit in $C^\infty(S^1, M)$ with respect to $t \in S^1$ as $s \to \pm\infty$. The Hamiltonian function H is called **regular** if all contractible 1-periodic solutions of (1.1), $x \in \mathcal{P}_1(H)$, are non-degenerate. In this case, $\mathcal{P}_1(H) \subset \mathrm{Fix}(\psi_1)$ is a finite set because M is compact, and we obtain the decomposition

$$\mathcal{M}(J,H) = \bigcup_{x,y \in \mathcal{P}_1(H)} \mathcal{M}_{x,y}(J,H) .$$

The first step in Floer theory is to show the existence of a generic set of so-called regular J such that all components of $\mathcal{M}(J,H)$ as a subspace of $C^\infty(\mathbb{R} \times S^1, M)$ are finite dimensional manifolds[1]. For each component not only the relative action is well-defined but also a relative Morse index given by the component's finite dimension. This fact is related to the homomorphism ϕ_c as follows. Given two connecting trajectories $u, u' \in \mathcal{M}_{x,y}$, the relative homotopy class with respect to the boundary condition,

$$([u] - [u'])_{\mathrm{rel}\{\pm\infty\} \times S^1} = A \in H_2(M, \mathbb{Z}),$$

is well-defined and

$$\dim_{\mathrm{loc}} u - \dim_{\mathrm{loc}} u' = 2c_1(A) .$$

The relative index is realized by the Conley-Zehnder index which induces a grading on the set of periodic solutions,

$$\mu_{CZ} \colon \mathcal{P}_1(H) \to \mathbb{Z}/2N\mathbb{Z},$$

where the minimal Chern number N is defined by $\phi_c(\pi_2(M)) = N\mathbb{Z} \subset \mathbb{Z}$. In the analysis of Floer theory it turns out that the influence of $\pi_2(M)$ is closely related to Gromov's theory of pseudo-holomorphic spheres[2]. These are solutions of the quasi-linear elliptic Cauchy-Riemann operator $\overline{\partial}_J$,

$$u \in C^\infty(S^2, M), \quad \overline{\partial}_J(u) = Tu + J(u) \circ Tu \circ j = 0 .$$

Here, J is assumed to be t-independent and j is a complex structure on S^2. Note that solutions $\overline{\partial}_J(u) = 0$ with $[u] \in \ker \phi_\omega$ necessarily are constant, because for J-holomorphic curves $\int_{S^2} u^*\omega$ measures the area with respect to $\langle \cdot, \cdot \rangle_J$.

[1] The subtle point concerning the explicit t-dependence of J is located here.
[2] In contrast to Gromov's theory our surfaces are not assumed to be compact.

In the present work the results about the cohomology operations induced from S^1-cobordisms will be proven under the assumption that $\phi_\omega = 0$ and $\phi_c = 0$. Thus we rule out the influence of $\pi_2(M)$. However, in the end, we will indicate the generalization for the case of monotone symplectic manifolds, that is $\phi_\omega = \lambda \cdot \phi_c$ for some non-negative constant $\lambda \geqslant 0$. This larger class of symplectic manifolds, including for example $\mathbb{C}P^n$ with the Fubini-Study metric, admits more possibilities of "new" multiplicative structures on Floer homology arising from the functor Z to be constructed.

Let us continue with the exposition of the classical Floer theory under the assumption that ω and c_1 vanish on $\pi_2(M)$. After establishing the transversality result stating that the trajectory spaces $\mathcal{M}_{x,y}(J,H)$ are finite dimensional manifolds of dimension

$$\dim \mathcal{M}_{x,y}(J,H) = \mu_{CZ}(y) - \mu_{CZ}(x),$$

one obtains the so-called moduli spaces of unparametrized trajectories

$$\widehat{\mathcal{M}}_{x,y}(J,H) = \mathcal{M}_{x,y}(J,H)/\mathbb{R}$$

by taking the quotient with respect to the shifting action of $\mathbb{R}$,

$$\mathbb{R} \times \mathcal{M}_{x,y} \to \mathcal{M}_{x,y}, \quad u * \tau(s,t) = u(s+\tau, t) .$$

The action is free precisely if $x \neq y$. One proves by means of the a priori energy estimate

$$\Phi(u) = \int_{-\infty}^{\infty} \int_{S^1} |\partial_s u|_J^2 \, ds dt = \mathcal{A}_H(y) - \mathcal{A}_H(x)$$

that the 0-dimensional moduli spaces, in which case the relative index is given by

$$\mu_{CZ}(y) - \mu_{CZ}(x) = 1,$$

are compact and therefore finite. The compactness follows by an analysis which is analogous to the compactness result for the moduli spaces of pseudo-holomorphic curves[3]. Here, the so-called bubbling-off analysis is simplified by the fact that non-trivial J-holomorphic spheres cannot occur due to $\phi_\omega = 0$.

In our work, we will consider a cohomology theory rather than a homology theory. A priori, this is of no relevance because there is no canonical preference between positive or negative relative gradient flow, what reflects the inherent Poincaré duality for the closed symplectic manifold[4]. Here, we have opted for the defining Cauchy-Riemann equation with the additional Hamiltonian term such that it geometrically describes the positive relative gradient flow for the action $\mathcal{A}_H$. Its L^2-gradient with respect to the ω-compatible structure J is computed as

$$\mathrm{grad}_{J,L^2} \mathcal{A}_H = -J\big(\tfrac{\partial}{\partial t} - X_{H_t}\big) .$$

Both, the action $\mathcal{A}_H$ and the index μ_{CZ} are strictly increasing along non-trivial "flow lines". Reversing the sign of the Hamiltonian function corresponds to

[3] Cf. [38] and [31].
[4] Compare Section 5.2 in [50].

reversing the orientation of the periodic solutions, $x^{-1}(t) = x(-t)$. This is reflected by the index as

$$\mu_{CZ}(x^{-1}) = -\mu_{CZ}(x),$$

see [48] and [21][5]. This reversal of the orientation of S^1 will also be described by the cobordism functor Z in Section 5.5. It is represented by the operations identified with the annuli with boundary circles oriented in opposite direction. It is proven that this corresponds to the Poincaré duality involution.

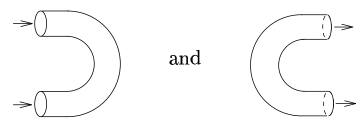

Figure 1.1: Reversal of orientation

Opting for a formulation as a cohomology theory, we define the underlying cochain groups as

$$C^k(H) = \mathrm{span}_{\mathbb{Z}_2}\Big(\{\, x \in \mathcal{P}_1(H) \mid \mu_{CZ}(x) = k \,\}\Big),$$

that is, $\mathbb{Z}_2$-vector spaces generated by the 1-periodic solutions of index k. Counting the connecting unparametrized trajectories of relative index 1 modulo 2,

$$n(x,y) = \#\widehat{\mathcal{M}}_{x,y}(J,H) \mod 2,$$

defines an operation $\delta = \delta(J,H)$ of degree 1,

$$\delta\colon C^k(H) \to C^{k+1}(H), \quad \delta\, x = \sum_{\mu(y)=\mu(x)+1} n(x,y)y,$$

for all $k \in \mathbb{Z}$. Note that the operation is first defined on the basis elements $x \in \mathcal{P}_1(H)$ and then extended to $C^k(H)$ as a $\mathbb{Z}_2$-linear map.

1.1.1 Theorem (Floer) *$(C^*(H), \delta(J,H))$ is a cochain complex*

$$\ldots \to C^{k-1}(H) \xrightarrow{\delta^{k-1}} C^k(H) \xrightarrow{\delta^k} C^{k+1}(H) \to \ldots,$$

that is, $\delta^{k+1} \circ \delta^k = 0$ for all $k \in \mathbb{Z}$.

The Floer (co-)homology groups associated to the regular pair (J,H) are denoted by

$$H^k(J,H) = \ker \delta^k / \operatorname{im} \delta^{k-1},$$

where the grading by $k \in \mathbb{Z}$ is also indicated as $H^*(J,H) = \big(H^k(J,H)\big)_{k\in\mathbb{Z}}$.

[5] There is a sign difference in the definition of the index due to the choice of the sign of J! In this work, the sign convention from [21] is chosen.

The important observation is that **Conley's Continuation Principle** can be applied to this homology theory with respect to the set $\mathcal{I}$ of all regular pairs (J,H). The cohomology H^* together with $\mathcal{I}$ form a so-called connected simple system, that is, there exist **canonical isomorphisms**

$$\Phi^*_{\beta\alpha}\colon H^*(J^\alpha, H^\alpha) \xrightarrow{\cong} H^*(J^\beta, H^\beta)$$

for all $\alpha, \beta \in \mathcal{I}$ such that

$$\Phi^*_{\gamma\beta} \circ \Phi^*_{\beta\alpha} = \Phi^*_{\gamma\alpha}, \quad \Phi^*_{\alpha\alpha} = \mathrm{id}\,. \tag{1.2}$$

This allows the definition of the (co-)homology groups which are independent of the chosen regular pair (J,H),

$$HF^k(M,\mathbb{Z}_2) = \{\, (x_\alpha) \in \prod_{\alpha\in\mathcal{I}} H^k(J^\alpha, H^\alpha) \mid x_\beta = \Phi^*_{\beta\alpha} x_\alpha \text{ f.a. } \alpha,\beta \in \mathcal{I} \,\}\,.$$

In Chapter 5 this $\mathbb{Z}$-graded vector space over $\mathbb{Z}_2$ will be shortly denoted by $\mathcal{A}^* = HF^*(M,\mathbb{Z}_2)$.

1.1.2 Theorem (Floer) *The (co-)homology groups $HF^*(M,\mathbb{Z}_2)$ are canonically isomorphic to the standard homology $H^{sing}_*(M,\mathbb{Z}_2)$. More precisely, there are canonical vector space isomorphisms*

$$HF^k(M,\mathbb{Z}_2) \cong H^{sing}_{n-k}(M,\mathbb{Z}_2)\,.$$

This follows from the above Continuation Principle by considering a time-independent Morse function $H \in C^\infty(M,\mathbb{R})$. Obviously, $\mathrm{Crit}\, H \subset \mathcal{P}_1(H)$ in view of the canonical embedding $M \hookrightarrow \Omega^\circ(M)$. If $\|H\|_{C^2}$ is small enough, then H is regular as a Hamiltonian and there are no other 1-periodic solutions of (1.1) than the stationary critical points. The Morse index and the Conley-Zehnder index are related by

$$\mu_{CZ}(x) = n - \mu_{\mathrm{Morse}}(x), \quad x \in \mathrm{Crit}\, H\,.$$

Moreover if J is also chosen t-independent, the gradient flow lines for the negative gradient flow on M defined by the Morse function H on M are t-independent trajectories of the relative positive gradient flow for the associated Hamiltonian action,

$$\{\, \gamma \in C^\infty(\mathbb{R}, M) \mid \gamma' + \nabla H(\gamma) = 0,\, \gamma(-\infty) = x,\, \gamma(\infty) = y \,\} \subset \mathcal{M}_{x,y}(J,H)\,.$$

Here, γ' denotes the s-derivative. For $\|H\|_{C^2}$ small enough, cf. [48], Morse-Smale transversality which is given for a generic t-dependent J, implies regularity of (J,H) as far as solutions of relative index 1 are concerned. Therefore, the cochain complex $\big(C^*(H), \delta(J,H)\big)$ is identical with the Thom-Smale-Witten complex $\big(C^{\mathrm{Morse}}_*, \partial^{\mathrm{Morse}}\big)$ associated to the negative gradient flow of the Morse function H and the Riemannian metric induced by J. The latter chain complex is known to induce the classical homology of M, see e.g. [50].

To put it in a nutshell, the graded vector space $HF^*(M,\mathbb{Z}_2)$ consisting of the (co-)homology groups associated to the symplectic manifold (M,ω) is completely identified as the standard homology of M. This has also been carried out for coefficients in $\mathbb{Z}$, that is for the module $H_*(M,\mathbb{Z})$, in [21].

Let us now consider some analytical details from the construction of the canonical isomorphisms $\Phi^*_{\beta\alpha}$, because our cobordism functor Z will be a generalization of exactly this feature. Given two regular pairs $(J^\alpha,H^\alpha),(J^\beta,H^\beta)\in\mathcal{I}$ one considers a homotopy of pairs with the homotopy parameter $s\in\mathbb{R}$,

$$(J^{\beta\alpha}(s),H^{\beta\alpha}(s))=\begin{cases}(J^\alpha,H^\alpha), & s\leqslant -T,\\ (J^\beta,H^\beta), & s\geqslant T,\end{cases}$$

for some $T>0$. This homotopy exists because the space of ω-compatible almost complex structures is contractible. Identifying the homotopy parameter s with the "flow" parameter s the associated Cauchy-Riemann type equation

$$\partial_s u+J^{\beta\alpha}(s,t,u)\partial_t u+\nabla H_t^{\beta\alpha}(s,u)=0 \tag{1.3}$$

leads to the solution spaces

$$\widetilde{\mathcal{M}}_{x_\alpha,x_\beta}(J^{\beta\alpha},H^{\beta\alpha})=\{\,u\in C^\infty(\mathbb{R}\times S^1,M)\mid u \text{ satisfies (1.3)},\\ u(-\infty)=x_\alpha,\ u(\infty)=x_\beta\,\}$$

for $x_\alpha\in\mathcal{P}_1(H^\alpha)$, $x_\beta\in\mathcal{P}_1(H^\beta)$. The cornerstones for the analysis of these solution spaces to be verified are again:

1. the transversality result which states that there exists a generic homotopy $(J^{\beta\alpha},H^{\beta\alpha})$ such that all solution spaces are component-wise finite dimensional manifolds,

2. an a priori energy estimate stating that the "flow" energy $\iint|\partial_s u|_J^2 ds dt$ can be uniformly estimated by a constant only depending on the relative action $\mathcal{A}_{H^\beta}(x_\beta)-\mathcal{A}_{H^\alpha}(x_\alpha)$ and the chosen homotopy $(J^{\beta\alpha},H^{\beta\alpha})$, and

3. the index formula $\dim_{\mathrm{loc}} u=\mu_{CZ}(x_\beta)-\mu_{CZ}(x_\alpha)$.

The energy estimate provides the compactness result that the solutions spaces $\widetilde{\mathcal{M}}_{x_\alpha,x_\beta}$ are compact in dimension 0. Note that here is no shifting symmetry to consider. Therefore the number

$$\langle x_\alpha,x_\beta\rangle=\#\widetilde{\mathcal{M}}_{x_\alpha,x_\beta}\mod 2$$

is defined for relative index 0. Analogously to the δ-operator one defines the $\mathbb{Z}_2$-linear operator

$$\Phi_{\beta\alpha}x_\alpha=\sum_{\mu(x_\alpha)=\mu(x_\beta)}\langle x_\alpha,x_\beta\rangle x_\beta$$

on the level of cochain groups. The crucial point is that outside the compact subset $[-T,T]\times S^1$ of the cylinder, equation (1.3) describes the usual relative

gradient flow for $\mathcal{A}_{H^\alpha}$ and respectively $\mathcal{A}_{H^\beta}$. This is the key for the proof of the identity

$$\delta^\beta\circ\Phi_{\beta\alpha}=\Phi_{\beta\alpha}\circ\delta^\alpha\,. \tag{1.4}$$

Consequently, $\Phi_{\beta\alpha}$ descends to an operator on the level of cohomology

$$\Phi^*_{\beta\alpha}\colon H^*(\alpha)\to H^*(\beta)\,.$$

An important property of the entire theory is the **homotopy invariance** with respect to parameters which leave the asymptotic boundary conditions fixed. Choosing different pairs $(J_\lambda^{\beta\alpha},H_\lambda^{\beta\alpha})$, $\lambda=0,1$, of homotopies for the same regular pairs $\alpha,\beta\in\mathcal{I}$, amounts to the same operator on the level of cohomology,

$$\Phi^*(J_0^{\beta\alpha},H_0^{\beta\alpha})=\Phi^*(J_1^{\beta\alpha},H_1^{\beta\alpha})\,.$$

This follows from the fact that one can find a homotopy of homotopy pairs, $\lambda\mapsto(J_\lambda^{\beta\alpha},H_\lambda^{\beta\alpha})$, $\lambda\in[0,1]$. This λ-homotopy induces an algebraic homotopy operator Ψ, that is an operator $\Psi\colon C^*(\alpha_0)\to C^{*-1}(\alpha_1)$ of degree -1 such that

$$\Phi^*_{\beta\alpha}(\lambda=0)+\Phi^*_{\beta\alpha}(\lambda=1)=\delta^\beta\circ\Psi+\Psi\circ\delta^\alpha\,.$$

This principle of homotopy invariance is the core of the proof that the cobordism functor Z will lead to a topological theory. This means that already the topological type of the model surface Σ generalizing the standard cylinder characterizes uniquely the associated cohomology operation $Z(\Sigma)$.

Essential Elements for the Theory

Let us sum up the characteristic features of Floer homology theory. The central object for the definition of the solution spaces is the quasi-linear Cauchy-Riemann operator $\overline{\partial}_J=\partial_s+J\partial_t$. Over the cylindrical ends $\mathbb{R}\times S^1\setminus([-T,T]\times S^1)$ we add the 0-order term ∇H associated to the given Hamiltonian function such that, there, the solutions of the resulting PDE describe the relative L^2-gradient flow for the symplectic action,

$$\partial_s u=-J\big(\partial_t u-X_{H_t}(u)\big)\,.$$

Using the regularity of the Hamiltonian H we obtain appropriate boundary conditions for the elliptic partial differential equation $\overline{\partial}_{J,H}(u)=0$ for mappings $u\colon\mathbb{R}\times S^1\to M$ by

$$u(-\infty)=x,\quad u(\infty)=y,\quad x,y\in\mathcal{P}_1(H)\,.$$

Namely, the non-degeneracy of the periodic solutions x,y together with the ellipticity of $\overline{\partial}_J$ imply that the linearization D_u of $\overline{\partial}_{J,H}$ at a solution u is a Fredholm operator.

Next, it is necessary to prove the existence of regular pairs (J,H) such that the solution spaces $\overline{\partial}_{J,H}^{-1}(0)$ with finite flow energy are component-wise finite dimensional manifolds. The computation of the Fredholm index of the linearizations D_u provides the formula for the local dimension.

Finally, an a priori energy estimate like $\iint |\partial_s u|_J^2 ds dt = \mathcal{A}_H(y) - \mathcal{A}_H(x) + \text{const}$ for solutions u has to be established in order to prove compactness of the appropriate solution spaces in dimension 0. Again, the crucial element for this estimate is the relative gradient flow outside a compact subset of the domain $\mathbb{R} \times S^1$.

1.1.1 The Extension for Generalized Model Surfaces

The analysis introduced above can be generalized for arbitrary model surfaces with cylindrical ends. Let Σ be any connected Riemann surface with cylindrical ends. This particularly includes closed surfaces. By cylindrical ends we mean the complement of some compact subset Σ_0 endowed with infinite cylindrical coordinates ψ_k. Topologically, this surface with ν cylindrical ends can be considered as the interior of a compact surface $\overline{\Sigma}$ with boundary. Let every boundary component be supplied, in addition, with its own orientation. Thus, using the cylindrical coordinates, we distinguish 'entrances', $\epsilon_k = -1$, from exits, $\epsilon_k = +1$,

$$\psi_k \colon Z^{\epsilon_k} \hookrightarrow \Sigma, \quad k = 1, \dots, \nu,$$
$$Z^+ = (0, \infty) \times S^1,\ Z^- = (-\infty, 0) \times S^1,$$

such that $\psi_k(Z^{\epsilon_k}) \cap \psi_l(Z^{\epsilon_l}) = \emptyset$ for $k \neq l$ and $\Sigma_0 = \Sigma \setminus \bigcup_k \psi_k(Z^{\epsilon_k})$. The complex structure j on Σ is chosen to agree with the standard structure $\psi_k^* i$ on the cylindrical ends induced from $(\mathbb{R} \times S^1, i)$. Now, assign with each cylindrical end a regular pair $(J^k, H^k) \in \mathcal{I}$ such that $J^k, H^k \in C^\infty(Z^{\epsilon_k} \times M, \mathbb{R})$ are independent of s for $|s| \geqslant T$ for some $T > 0$ and $H^k(s, \cdot)$ vanishes for $|s| < \epsilon$. Let J be an extension of $(J^k)_{k=1,\dots,\nu}$ as ω-compatible almost complex structure to the compact complement Σ_0, i.e. $J \in C^\infty(\Sigma \times M)$. Compared to the definition of the above canonical isomorphisms $\Phi^*_{\beta\alpha}$ these preparations allow the generalization of the underlying solution spaces.

Given $x_i \in \mathcal{P}_1(H^i)$, $i = 1, \dots, \nu$, the following problem is well-defined. Consider smooth maps $u \colon \Sigma \to M$ satisfying the conditions:

(a) u is J-holomorphic on the compact complement Σ_0, that is,

$$Tu(z) + J(z, u(z))Tu(z)j(z) = 0 \quad \text{for all } z \in \Sigma_0,$$

(b) the cylindrical restrictions $u_i = u \circ \psi_i$ satisfy

$$\partial_s u_i + J^i(s,t,u_i)\partial_t u_i + \nabla H^i(s,t,u_i) = 0,$$
$$\lim_{\epsilon_i s \to \infty} u_i(s) = x_i \quad \text{in } C^\infty(S^1, M),$$

(c) and an extension $\hat{u}$ to the closed surface $\hat{\Sigma} = \Sigma \cup \bigcup_{i=1}^{\nu}(\epsilon_i D^2)$, which is obtained from Σ by gluing disks with appropriate orientation into the cylindrical ends, represents a fixed class, $[\hat{u}] = A \in H_2(M, \mathbb{Z})/\mathcal{S}$.

Here, $\mathcal{S} \subset H_2(M, \mathbb{Z})$ is the subgroup of spherical classes, that is, the image of $\pi_2(M)$ under the Hurewicz homomorphism.

The same analytical program as in the classical Floer theory for the cylindrical domain has to be carried out for these generalized solutions. First, we deal again with a Cauchy-Riemann problem. The boundary conditions are described by fixing 1-periodic orbits as asymptotic solutions for the cylindrical ends. Conditions (a) and (b) allow to prove the Fredholm property of the linearization. This will be the part of the linear theory, in particular the computation of the index of the linearized problem. Then, the transversality result has to be proven. It states that there exists a generic extension of the structures (J^i) on the cylindrical ends to $J \in C^\infty(\Sigma \times M)$ such that the linearization D_u of (a) and (b) at any solution u is onto. For such a generic extension J all solution spaces are component-wise finite-dimensional manifolds and the local dimension is given then by the Fredholm index of the linearization D_u. Condition (c) has to be fixed in order to control the index and the generalized "flow" energy in relation with ϕ_ω and ϕ_c. The index formula which provides the dimension of the solution spaces will be computed as

$$\dim_{\text{loc}} u = \sum_{i=1}^{\nu} \epsilon_i \mu_{CZ}(x_i) + n(2 - 2g - \nu) + 2c_1(A)\,.$$

This holds for solutions u of (a), (b) and (c) under the assumption that $\phi_c = 0$ so that $c_1(A)$ is well-defined. Here the integer g is the genus of the capped connected surface $\hat{\Sigma}$ and the indices of the 1-periodic solutions are counted with the sign ϵ_i according to the orientation of the cylindrical end.

Analogously, the fact that we again deal with the relative gradient flow for the Hamiltonian actions associated to (H^i) outside some compact subset Σ_0 in the interior of Σ, gives rise to an a priori energy estimate,

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|Tu\|^2_{J,L^2(\Sigma_0)} + \sum_{i=1}^{\nu} \iint_{Z^{\epsilon_i}} |\partial_s u_i|^2_{J^i} ds dt \\ &\leqslant \sum_{i=1}^{\nu} \epsilon_i \mathcal{A}_{H^i}(x_i) + \omega(A) + c\,.\end{aligned}$$

Note that the L^2-norm of the gradient Tu is uniquely determined by j and J. The constant $c \geqslant 0$ is independent of u, A and (x_i). It stems from the deviation from the gradient flow equation on the compact subset of Σ where either H^i are explicitly depending on the variable s or on Σ_0 where we do not have any fixed conformal coordinates (s, t). Again $\omega(A)$ and the Hamiltonian actions in the formula are well-defined if we assume that $\phi_\omega = 0$. This energy estimate provides the key for the compactness result in dimensions 0 and 1.

Summing up, Floer theory is generalized to Riemann surfaces with cylindrical ends such that all essential results can be obtained: transversality, the Fredholm property and the compactness result. These analytical results allow the definition of algebraic operations associated to Σ by counting the finitely many zero dimensional solutions. It is important to point out that the case of closed model surfaces is included. However, note that here, no symmetry group action is taken into consideration. Instead, we consider parametrized curves where eventual symmetries are excluded by perturbing the structure J which may explicitly depend on $z \in \Sigma$.

1.2 Cohomology Operations from S^1-cobordisms

Let us fix $A \in H_2(M,\mathbb{Z})/\mathcal{S}$, a connected model surface $(\Sigma, (\psi_i))$, regular pairs $(J^i, H^i)_{i=1,\dots,\nu}$, 1-periodic solutions $x_i \in \mathcal{P}_1(H^i)$ and a generic extension J as above. Then, counting modulo 2 the 0-dimensional solutions of (a), (b) and (c) leads to the number $\langle x_1, \dots, x_\nu; A\rangle_\Sigma$. More precisely, we assume without loss of generality that the cylindrical ends of Σ are oriented such that $\epsilon_i = -1$ for $i = 1, \dots, a$ and $\epsilon_i = +1$ for $i = a+1, \dots, \nu$, say $\nu = a + b$. Then for fixed $x_i \in \mathcal{P}_1(H^i)$, $i = 1, \dots, a$, we define

$$\mathcal{O}(x_1 \otimes \ldots \otimes x_a) = \sum_{\substack{y_i \in \mathcal{P}_1(H^{i+a}),\\ i=1,\dots,b}} \langle x_1, \dots, x_a, y_1, \dots, y_b; A\rangle_\Sigma \, y_1 \otimes \ldots \otimes y_b \,.$$

This way, a $\mathbb{Z}_2$-linear operator is defined on the tensor spaces

$$\mathcal{O}\colon C^*(H^1) \otimes \ldots \otimes C^*(H^a) \to C^*(H^{a+1}) \otimes \ldots \otimes C^*(H^{a+b})$$

having degree $\deg \mathcal{O} = n(a + b + 2g - 2) - 2c_1(A)$. The main theorem states that, exactly like for the canonical isomorphisms $\Phi^*_{\beta\alpha}$, the operator $\mathcal{O}$ commutes on the cochain level with the induced coboundary operator δ for the tensor cochain complexes. It thus descends to the level of cohomology and, due to the Künneth formula, defines an operator between the tensor complexes of the cohomology groups associated to (J^i, H^i). Moreover, the operator also commutes on the level of cohomology with the canonical isomorphisms $\Phi^*_{\beta\alpha}$ from (1.2). Altogether, $\mathcal{O}$ provides a cohomology operation

$$Z(\Sigma)\colon \bigotimes^a HF^*(M,\mathbb{Z}_2) \to \bigotimes^b HF^*(M,\mathbb{Z}_2)\,.$$

Now, the crucial question is whether this operator depends on the assumed structures on Σ, that is, on J, the conformal structure j, the cylindrical coordinates (ψ_i), etc. To show that it does not depend on these structures we apply the homotopy principle as in the classical Floer theory. We show that homotopies of the parameters J and H on some compact subset in the interior of the model surface $\mathbb{R} \times S^1$ define an algebraic homotopy operator Ψ proving that the homological operation is invariant. The same can be proven for the generalized model surfaces. Since any two conformal structures with fixed restriction to the cylindrical ends are homotopic, it follows that the operation $Z(\Sigma)$ depends at the most on the differentiable surface Σ and the cylindrical coordinates (ψ_i). Moreover, given two connected model surfaces of the same topological type (a, b, g), that is, the number of entrances, of exits and the genus are identical, we can always find a diffeomorphism between the surfaces which

extends the identity on the cylindrical ends with respect to the cylindrical coordinates. In particular, any permutation of the equally oriented cylindrical ends of a connected model surface leaves the operator $Z(\Sigma)$ invariant. Thus, the map Z which assigns to every model surface Σ a cohomology operation leads to a purely combinatorial concept as far as the surface Σ is concerned.

1.2.1 The Cobordism Functor Z

An important property which has to be analyzed for the map Z is the behaviour with respect to gluing model surfaces along their cylindrical ends. This has already been used for proving the fact that $Z(\Sigma)$ commutes with the canonical isomorphisms $\Phi^*_{\beta\alpha}$ which are exactly the operators $Z([0,1] \times S^1)$.

Already for the standard Floer theory dealing with the unparametrized connecting trajectories a gluing operation for solutions has to be considered. Compared to splitting of trajectories occuring in the compactness analysis, gluing describes the reverse process. Splitting appears in the generalized framework if the underlying model surface is deformed on the region of a properly embedded annulus by stretching its cylindrical coordinates $[-R, R] \times S^1$, $R \to \infty$. This corresponds to a homotopy of the conformal structure j associated with these cylindrical coordinates. The homotopy invariance of the operator $Z(\Sigma)$ together with a uniform energy estimate for this stretching process provide a decomposition result. If a given model surface Σ is decomposed by a finite number of properly embedded pairwise disjoint circles, the associated operator $Z(\Sigma)$ is identical to the composition of the operators associated to the decomposed parts of Σ. For the converse, gluing model surfaces along specified boundary circles with opposite orientation corresponds to the composition of the associated operators.

The following aspect is crucial for the decomposition result. A priori, it is not obvious that the stretching process leads to split solutions which asymptotically cohere by a contractible loop. It is essential for the entire theory to prove that the cohomology operation $Z(\Sigma)$ associated to the model surface Σ and a fixed class $A \in H_2(M)/\mathcal{S}$, is always the trivial 0-operation if $A \notin \mathcal{S}$. This means, that only the solutions of spherical type are relevant for the cohomology theory. Already here, the crucial role of $\pi_2(M)$ becomes apparent although we assumed $\phi_\omega, \phi_c = 0$.

In view of the (de-)composition result it is possible to describe the map Z in terms of the axioms of a topological field theory[6]. Associate to every oriented closed 1-manifold, that is a finite number of 1-spheres, a tensor product of the Floer cohomology vector space $HF^*(M,\mathbb{Z}_2)$ and its dual space $\overline{HF}^*$ such that there is a factor for every component, and reversal of orientation corresponds to taking the dual. Then Z assigns to every compact oriented surface Σ with oriented boundary, consisting a entrances and b exits, a vector in the tensor space,

$$Z(\Sigma) \in \bigotimes^a \overline{HF}^* \otimes \bigotimes^b HF^*\,.$$

[6]cf. Atiyah, [3] and [4].

This map Z is a functor in the trivial sense that $Z(\Sigma)$ is invariant under diffeomorphisms preserving the orientation of the surface and its boundaries. This functor Z satisfies the axiom of associativity, namely that a composition of the oriented S^1-cobordism leads to composition of the associated operators. It is non-trivial in the sense that Z associates the identity operator to the cylinder $S \times [0,1]$ where S is a closed 1-manifold.

Moreover, this functor Z is consistent with the structure of $HF^*(M, \mathbb{Z}_2)$ as a $\mathbb{Z}$-graded vector space over $\mathbb{Z}_2$. Using the grading with negative sign for the dual vector space

$$\overline{HF}^k = \mathrm{Hom}(HF^{-k}, \mathbb{Z}_2) = HF_{-k},$$

that is, for the Floer homology groups, the tensor spaces inherit a natural grading. The degree of the operator $Z(\Sigma)$ associated to a surface with a entrances, b exits, k components and g the sum of the genera of all components is

$$\deg Z(\Sigma) = n(a + b + 2g - 2k)\,.$$

This is obviously compatible with the composition rule. Observe, for example, that the annuli with opposite oriented ends provide an involution $HF^k \xrightarrow{\cong} HF_{-k}$ (see also Figure 1.1).

It is, however, an immediate consequence of the axioms that a 1 + 1-dimensional topological field theory is uniquely determined by the following set of basic elements. Every oriented compact surface with oriented boundary can be decomposed into the pair-of-pants with two entrances and one exit, the

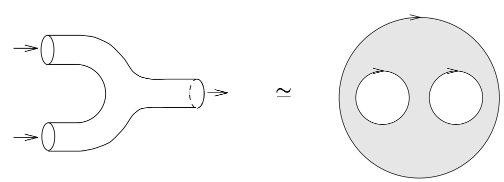

Figure 1.2: The pair of pants.

annuli with two entrances and respectively two exits and the disk with one exit. Therefore, the cobordism functor Z is equivalently determined by the composition rule and the operators associated to these basic elements. The entire theory for the map Z describes Floer (co-)homology $HF^*(M, \mathbb{Z}_2)$ as a graded commutative algebra over $\mathbb{Z}_2$ with unit and a non-degenerate symmetric bilinear form β such that the latter is compatible with the multiplication m,

$$\beta(a, m(b,c)) = \beta(m(a,b), c)\,.$$

The present work on the functor Z will finish at this abstract stage. We already discussed above the argument from the classical Floer theory proving that the non-degenerate bilinear form which is associated to the annulus with

two entrances corresponds to the Poincaré duality under the canonical isomorphism $HF^*(M, \mathbb{Z}_2) \cong H^{\mathrm{sing}}_{n-*}(M, \mathbb{Z}_2)$ as $\mathbb{Z}_2$-vector spaces. The identification of the multiplicative structure given by the pair-of-pants, for example as the cup product, however, is beyond the scope of this exposition. The present work primarily deals with the construction of the cobordism functor Z, and it will be shown in [49] that the multiplicative structure associated to the pair of pants is isomorphic to a deformed cup product. The analysis requires a detailed and concrete treatment of a suitably chosen model surface for the pair of pants.

Here, the construction is only carried out under the conditions that $\phi_\omega = \phi_c = 0$. It is possible to generalize the functor Z for the case of weakly monotone symplectic manifolds. This requires a more refined transversality and compactness analysis dealing with the presence of pseudo-holomorphic spheres. Originally, Theorems 1.1.1 and 1.1.2 were already formulated for the monotone case. It means that the homomorphisms ϕ_ω and ϕ_c are proportional by a positive constant $\lambda \geqslant 0$, $\phi_\omega = \lambda\phi_c$. This generalization implies that the index for 1-periodic solutions is only well-defined as a cyclic integer in $\mathbb{Z}/2N\mathbb{Z}$ where $\phi_c(\pi_2(M)) = N\mathbb{Z}$. Moreover, the energy estimate for connecting orbits is no longer given uniformly for all solutions with fixed ends but it is coupled to the local dimension of the solution's component by the monotonicity condition. It is uniform for connecting orbits representing a fixed homology class in $H_2(M, \mathbb{Z})$ relative to the fixed ends. The monotonicity condition guarantees that bubbling-off of J-holomorphic spheres cannot occur for families of connecting orbits of local dimension 0 and 1. Therefore, compactness of the respective moduli spaces still holds. Floer (co-)homology is well-defined as a $\mathbb{Z}_{2N}$-graded $\mathbb{Z}_2$-vector space and the general version of Theorem 1.1.2 states that

$$HF^k(M, \mathbb{Z}_2) \cong \bigoplus_{l \equiv k \mod 2N} H^l(M, \mathbb{Z}_2)$$

for all $k \in \mathbb{Z}/2N\mathbb{Z}$. In [29], this result is generalized for the case of weakly monotone symplectic manifolds. This requires the choice of coefficients for HF^* in the completion of the group ring of $\pi_2(M)/(\ker\phi_\omega \cap \ker\phi_c)$.

Such generalizations are also possible for the cobordism functor Z. Considering the multiplicative structure associated to the pair of pants one can hope that this ring structure on HF^* detects differences between HF^* and the cohomology ring $H^*_{\mathrm{sing}}(M, \mathbb{Z}_2)$ whereas both algebras are canonically isomorphic as $\mathbb{Z}_2$-vector spaces. It turns out that HF^* with the pair-of-pants product is related to the quantum cohomology of (M, ω), cf. [38]. Namely the difference between HF^* and the standard cohomology ring of M is generated by the presence of J-holomorphic spheres[7]. Indeed, we shall prove in [49] that counting pair-of-pants solutions, which define the multiplication m, amounts to counting J-holomorphic spheres under the condition of three additional incidence relations. Namely three prescribed points on the spherical domain have to be mapped into given homology cycles which are represented by unstable manifolds of critical points for the negative gradient flow for Morse functions on M.

[7] For the deformation of the cup product see also [57], [45], [38], [43].

1.3 Overview

In this theory, algebraic operators in a cohomology theory are explicitly defined by studying solutions of a special type of elliptic partial differential equations. In our work, we first have to develop the functional analytic set-up for a class of nonlinear Cauchy-Riemann type equations. Next, the analytical part involves a theory of Fredholm operators within a suitable framework of function spaces over noncompact domains. The required geometric operations for the Riemann surfaces which topologically represent S^1-cobordisms have to be carried out within the nonlinear Fredholm analysis for the solution spaces. Finally, we have to interpret these geometric operations in algebraic terms which are relevant for Floer homology.

The second chapter treats the definition of a model surface. In order to be able to apply the analysis of Floer theory, the model surface has to be supplied with several additional structures although it turns out in the end that only the topological data play a role. We will introduce the coordinates of the standard cylinder on the ends of Σ, such that the induced conformal structure and measure for defining the suitable function spaces are fixed. Then the generalized Cauchy-Riemann operator will be defined and its linearization is computed. Finally we compile the local elliptic estimates used in the sequel.

The entire third chapter is devoted to the linear Fredholm theory. The important point is that, in contrast to Gromov's original theory, the known elliptic estimate for the Cauchy-Riemann operator is not sufficient for providing the essential Fredholm property for noncompact model surfaces with cylindrical ends. It has to be combined with the asymptotic behaviour of the linearized operator on the cylindrical ends. A problematic feature is the fact that, working with Sobolev spaces $W^{1,p}$ on a 2-dimensional domain, we need L^p-estimates over the cylindrical ends with $p > 2$. The required analysis is worked out in detail. Furthermore, the index cannot be computed anymore directly in terms of a spectral flow like for the standard infinite cylinder[8]. We will derive it from a known index formula for closed surfaces, namely the formula of Riemann-Roch,

$$\operatorname{ind}\overline{\partial} = 2n(1-g) + 2c_1(u^*TM)[\Sigma]$$

This is the real index of the linearized Cauchy-Riemann operator at a solution of $\overline{\partial}_J(u) = 0$ for a closed Riemann surface. The general index formula follows by means of the index additivity with respect to gluing of model surfaces. Thus, it suffices to compute the index explicitly for a trivial bundle over the standard disk. We remark that it is not necessary to relate the general formula back to that of Riemann-Roch. In fact, the index additivity which will be proven is sufficient to deduce the original index formula for closed surfaces. Once again, we point out that, in contrast to Gromov's original theory, this theory deals with noncompact surfaces.

The fourth chapter contains all necessary nonlinear results for the solution spaces. At first we have to show that these solution spaces can be obtained

[8]Compare equation (4.6) in [48].

as vanishing sets for a suitable nonlinear Cauchy-Riemann operator viewed as a smooth section in a Banach space bundle over a Banach manifold of maps. The appropriate transversality analysis proves that we find a suitable set of generic regular extensions of (J^i, H^i) over the interior compact complement where we lose hold on the cylindrical coordinates. Regularity means that all linearizations of the nonlinear Cauchy-Riemann operator at solutions are onto such that the solution sets are smooth manifolds, component-wise of finite dimension. Then, we will prove the necessary compactness properties by the standard (non-)bubbling-off analysis based on an a priori L^2-energy estimate. It is here where the simplifying assumption of $\omega_{|\pi_2(M)} = 0$ is made. Finally we construct in detail the gluing operation for the solution spaces. This will be needed not only for proving that the operator $\mathcal{O}$ on chain level commutes with the coboundary operator δ. It is also used for the general decomposition result showing that the operator Z on the level of cohomology has the crucial associativity property.

In the fifth chapter, we finally carry out the construction of the cobordism functor Z and draw the conclusions from the analytical preparations before. The operator $Z(\Sigma)$ is defined by counting the number of 0-dimensional solutions in $\mathbb{Z}_2$, the results about the 1-dimensional solution spaces account for proving that $Z(\Sigma)$ acts on the Floer cohomology groups. Then we develop step by step the invariance of $\Sigma \to Z(\Sigma)$ under changes of uncanonical parameters like for instance the conformal structure on Σ. This is proven in terms of a homotopy principle and it turns out that finally $Z(\Sigma)$ only depends on the topological class of the S^1-cobordism Σ. In Theorem 5.4.11, we obtain a functor Z satisfying the axioms of a $1+1$-dimensional topological field theory as they were set up in [3]. In Corollary 5.4.12 we compute the generic $\mathbb{Z}_2$-number of parametrized J-holomorphic tori which equals in $\mathbb{Z}_2$ the Euler characteristic of M. Independently from this direct computation, the number of tori follows immediately from the axiomatic presentation in 5.4.11. Thus, Corollary 5.4.12 serves as a verification of consistency. Finally, we prove that the Floer (co-)homology is a graded algebra over $\mathbb{Z}_2$ with an associative and commutative multiplication which is associated with the pair-of-pants surface (Theorem 5.5.1).

CHAPTER 2

Foundations

This chapter lays down the foundational material for the analysis of the spaces of solutions. In Chapter 4 these spaces of solutions of a perturbed Cauchy-Riemann type equation are analyzed in detail. Here we develop the notion of a model surface, the Banach manifold setup for the analysis, and we construct the nonlinear elliptic operator the zero sets of which will be precisely the mappings of the model surface we have to count. We also compute its linearization and compile the required fundamental elliptic estimates.

2.1 The Model Surface and the Manifold of Maps

The crucial point of the setup for the solution spaces of mappings from surfaces with S^1-boundaries to the symplectic manifold (M,ω) is the proper definition of the model surface. Since, for homological reasons, we want to impose asymptotic relative gradient flows for the action functional, we have to consider model surfaces Σ with cylindrical ends of the type $[0,\pm\infty)\times S^1\subset\mathbb{C}/i\mathbb{Z}$. This explicit structure is important for the following two reasons.

1. The Fredholm analysis upon which the construction of the solution spaces is based, crucially relies on the correct definition of the Banach spaces involved. Those are chosen to be Sobolev spaces, which, on the cylindrical ends, are of the type $H^{k,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$. Hence, we have to fix this concrete Lebesgue measure on Σ.
2. Since we want to consider solutions of a generalized Cauchy-Riemann type PDE which is meant to describe the relative gradient flow of symplectic action on the ends of Σ, we need the conformal structure i of $\mathbb{R}\times S^1\subset\mathbb{C}/i\mathbb{Z}$ fixed on the cylindrical ends.

We define a model surface as follows.

2.1.1 Definition *Let $\overline{\Sigma}$ be a connected, compact, oriented differentiable surface with oriented boundary. We denote the interior by Σ°. Let the boundary components of $\partial\Sigma=\overline{\Sigma}\setminus\Sigma^\circ$ be enumerated and labeled by $i=1,\dots,\nu$, and the orientation of each component indicated by $\epsilon_i\in\{\pm1\}$. Let $\overline{\Sigma}$ be provided with a smooth embedding*

$$\psi_i\colon Z^{\epsilon_i}\hookrightarrow\Sigma^\circ,\quad i=1,\dots,\nu$$

for each component i where $Z^+=[0,\infty)\times S^1\subset\mathbb{R}\times S^1=\mathbb{C}/i\mathbb{Z}$ and $Z^-=(-\infty,0]\times S^1\subset\mathbb{R}\times S^1$ such that

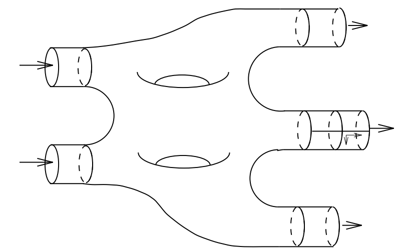

Figure 2.1: Model surface of type (a,b,g)

(i) $\psi_i(Z^{\epsilon_i})\cap\psi_j(Z^{\epsilon_j})=\emptyset$ *for $i\neq j$ and*

(ii) $\psi_i(s)=\psi(s,\cdot)$ *converges in $C^0(S^1,\overline{\Sigma})$ to an orientation preserving homeomorphism $S^1\overset{\approx}{\to}\partial\Sigma_i$ onto the i-th boundary component as $\epsilon_i s\to\infty$.*

*The collection $\Sigma=(\overline{\Sigma},(\psi_i)_{i=1,\dots,\nu})$ is called a **model surface of type** $(\mathbf{a},\mathbf{b},\mathbf{g})$, where $a=\#\{\epsilon_i=-1\mid i=1,\dots,\nu\}$ denotes the number of inward oriented ends, $b=\nu-a$ the number of exits and g is the genus of the capped topological surface $\hat{\Sigma}$ obtained from $\overline{\Sigma}$ by gluing disks into the cylindrical ends. Without loss of generality we assume that they are numbered as*

$$\epsilon_i=-1\quad\textit{for}\quad 1\leqslant i\leqslant a\quad\textit{and}\quad\epsilon_i=+1\quad\textit{for}\quad a+1\leqslant i\leqslant\nu\,.$$

Note that only the differentiable structure of Σ° is used. We will use the following notations,

$$Z_T^+=[T,\infty)\times S^1,\quad Z_T^-=(-\infty,-T]\times S^1,$$

and

$$\Sigma_Z=\bigcup_{i=1}^{\nu}\psi_i(Z^{\epsilon_i}),\quad\Sigma_T=\overline{\Sigma\setminus\left(\bigcup_{i=1}^{\nu}\psi_i(Z_T^{\epsilon_i})\right)}$$

for $T\geqslant0$. Note that we use the notation Σ for the topological set $\overline{\Sigma}$ equipped with additional structure as the cylindrical coordinates (ψ_i). Below we will introduce a differentiable structure on $\overline{\Sigma}$ which is closely related to these coordinates and extends the originally given structure on Σ°.

Our first aim is to construct a suitable space of mappings $u\colon\overline{\Sigma}\to M$ with $u_{|\partial\Sigma}\subset C^\infty(S^1,M)$ in a fixed set of contractible loops such that it is endowed with the structure of a separable Banach manifold modeled on a Sobolev space. This Banach manifold is meant to contain the solution spaces in question as the regular zero locus of a Fredholm map. The subtle point is to find a well-defined Sobolev space concept for the model surface Σ. Here we rely on the concept of manifolds of maps as it was described by H. Eliasson in [13].

The strategy for the construction will be as follows. First, we define the Sobolev spaces of sections $H^{1,p}_\Sigma(\xi)$ and $L^p_\Sigma(\xi)$ for a suitable class of smooth vector bundles $\xi \to \overline{\Sigma}$. Then, we construct the space $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ as a Banach manifold of mappings

$$u\colon \overline{\Sigma} \to M, \quad (u \circ \psi_i)(s,\cdot) \overset{C^0(S^1)}{\longrightarrow} x_i \in C^0(S^1, M) \quad \text{for } \epsilon_i s \to \infty,$$

such that it is modeled on $H^{1,p}_\Sigma(\mathbb{R}^{2n})$. In order to provide a unique Sobolev space structure, we restrict to a special class of smooth functions as a skeleton.

2.1.2 Definition *We say that a mapping $\overline{\Sigma} \to N$ into a smooth manifold or a function $f\colon \overline{\Sigma} \to \mathbb{R}$ is **$\mathbf{C}^{\mathbf{k}}_\Sigma$-differentiable** on Σ if $f \in C^k(\Sigma^\circ, \mathbb{R})$ and its restriction to the cylindrical ends allows a smooth extension as*

$$(f \circ \psi_i)\Big(\frac{\epsilon_i s}{\sqrt{1-s^2}}, t\Big) = \varphi_{i|[0,1)}(s,t), \quad \varphi_i \in C^k([0,1] \times S^1)\,.$$

We denote these subsets of smooth maps by $C^k_\Sigma(N)$ and $C^k_\Sigma(\mathbb{R})$ and C^k_Σ-maps are also called Σ-smooth. Moreover, given a ν-tuple of smooth loops $x_1, \dots, x_\nu \in C^\infty(S^1, M)$, we define the curve space

$$C^\infty_{x_1,\dots,x_\nu}(\Sigma, M) = \{\, h \in C^\infty_\Sigma(M) \,|\, \varphi_i(1,\cdot) = x_i,\ i = 1, \dots, \nu \,\}\,.$$

This class of differentiable maps on Σ has the following property.

2.1.3 Lemma *For each $f \in C^1_\Sigma(\mathbb{R})$ there is a constant $c(f) > 0$, such that each $f_i = f \circ \psi_i : Z^{\epsilon_i} \to \mathbb{R}$ satisfies*

$$\Big|\frac{\partial}{\partial s} f_i(s,t)\Big| \leqslant \frac{c(f)}{(1+s^2)^{\frac{3}{2}}} \quad \textit{for all } (s,t) \in Z^{\epsilon_i}\ .$$

PROOF. Observe that from $f_i(s,t) = \varphi_i(\frac{\epsilon_i s}{\sqrt{1+s^2}}, t)$, $\varphi_i \in C^1([0,1])$, we obtain $f_i' = \varphi_i' \cdot A$ with

$$A = \begin{pmatrix} -\frac{\epsilon_i}{(1+s^2)^{\frac{3}{2}}} & 0 \\ 0 & 1 \end{pmatrix},$$

such that $c(f) = \sup_{i=1,\dots,\nu} \|\varphi_i\|_{C^1([0,1])}$ provides the estimate. ∎

In view of the definition, we consider each $f \in C^k_\Sigma$ as defined on the compact surface $\overline{\Sigma}$. However, note that the former differentiable structure of Σ is only used on the interior Σ°

2.1.4 Corollary *If $f \in C^2_\Sigma(\mathbb{R})$ vanishes on the boundary, $f_{|\partial\overline{\Sigma}} \equiv 0$, the associated functions in cylindrical coordinates $f_i = f \circ \psi_i$, $i = 1, \dots, \nu$ satisfy $f_i \in H^{1,p}(Z^{\epsilon_i}, \mathbb{R})$ for all $p \geqslant 1$.*

Here $H^{k,p}(Z^\pm, \mathbb{R})$ denotes the standard Sobolev space with respect to the Lebesgue measure of $Z^\pm \subset Z = \mathbb{R} \times S^1 = \mathbb{C}/i\mathbb{Z}$.

PROOF. First, Lemma 2.1.3 obviously implies $|\frac{\partial}{\partial s} f_i| \in \bigcap_{p\geqslant 1} L^p$. We obtain the estimate

$$\begin{aligned} |f_i(s,t) - f_i(s_0,t)| &= \Big|\int_{s_0}^s \frac{\partial}{\partial s} f_i(\tau,t) d\tau\Big| \leqslant \Big|\int_{s_0}^s \frac{c(f)}{(1+\tau^2)^{\frac{3}{2}}} d\tau\Big| \\ &\leqslant c(f)\,\Big|\int_{s_0}^s \frac{d\tau}{\tau^3}\Big| = c(f)\,\big|s^{-2} - {s_0}^{-2}\big|\ . \end{aligned}$$

Thus, $s_0 \to \pm\infty$ together with $f_i(\pm\infty, t) = 0$ for all $t \in S^1$ imply

$$|f_i(s,t)| \leqslant c(f)\,|s|^{-2} \ \text{ for all } t \in S^1$$

and therefore $f_i \in \bigcap_{p\geqslant 1} L^p$. The same argument for $g_i = \frac{\partial}{\partial t} f_i$ where $\frac{\partial}{\partial t} f \in C^1_\Sigma(\mathbb{R})$, $\frac{\partial}{\partial t} f_{|\partial\Sigma} \equiv 0$ finally leads to $\nabla f \in \bigcap_{p\geqslant 1} L^p$. ∎

From these computations we conclude that this notion of differentiability with respect to the boundary of Σ is strong enough to provide C^∞_Σ as a good skeleton for the definition of Sobolev spaces on Σ. It is yet weak enough to contain all functions $f \in C^\infty(\Sigma^\circ, \mathbb{R})$ with any exponential decrease. This is the crucial part for the construction of the solution spaces for which we want to find a suitable analytic framework.

We now proceed to define the Sobolev space structures which will be needed on pull-back bundles u^*TM for maps $u\colon \overline{\Sigma} \to M$ with $u_{|\partial\Sigma} \subset C^\infty(S^1, M)$ component-wise contractible. Since for the Floer theory it is sufficient to concentrate on contractible periodic solutions, we are allowed to merely consider vector bundles over $\overline{\Sigma}$, which are trivial when restricted to the cylindrical ends.

2.1.5 Definition *We denote by $\mathrm{Vec}^*_{C^\infty}(\Sigma)$ the set of vector bundles over $\overline{\Sigma}$ which are Σ-smooth with trivial restrictions to $\partial\Sigma$. Given $\xi \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$, let $(\phi_i)_{i=1,\dots,\nu}$ be any Σ-smooth trivialization of $\xi_{|\Sigma_Z}$, where $\Sigma_Z = \bigcup_{i=1,\dots,\nu} \psi_i(Z^{\epsilon_i})$, we define the spaces of sections*

$$\begin{aligned} H^{1,p}_\Sigma(\xi) &= \{\, s \in H^{1,p}_{loc}(\Sigma^\circ, \xi) \,|\, \phi_i(s \circ \psi_i) \in H^{1,p}(Z^{\epsilon_i}, \mathbb{R}^n) \quad \text{f.a. } i = 1, \dots, \nu \,\}\,, \\ L^p_\Sigma(\xi) &= \{\, s \in L^p_{loc}(\Sigma^\circ, \xi) \,|\, \phi_i(s \circ \psi_i) \in L^{1,p}(Z^{\epsilon_i}, \mathbb{R}^n) \quad \text{f.a. } i = 1, \dots, \nu \,\} \end{aligned}$$

for $p > 2$. These spaces are considered as topological vector spaces with a Banach space topology induced by $(\phi_i)_{i=1,\dots,\nu}$.

If $p > 2$, we know from Sobolev's embedding theorem that $H^{1,p}_{\mathrm{loc}}(\Sigma^\circ) \subset C^0(\Sigma^\circ)$. Any change of trivialization is Σ-smooth, $\tilde\phi \circ \phi^{-1} \in C^\infty_\Sigma(\mathcal{GL}(\mathbb{R}^n))$, thus $\tilde\phi \circ \phi^{-1} \in \bigcap_{p\geqslant 1} H^{1,p}_\Sigma$ due to Corollary 2.1.4. Hence, the definitions of $H^{1,p}_\Sigma$ and L^p_Σ are independent of the cylindrical trivialization ϕ. Moreover, the Banach space topology is uniquely determined, because any two norms induced by $(\phi_i)_{i=1,\dots,\nu}$ and a finite open covering of $\overline{\Sigma \setminus \Sigma_Z}$ are equivalent, as well as for different ϕ, $\tilde\phi$.

Our aim is now to complete the curve space $C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ to a Banach manifold which is modeled on $H^{1,p}_\Sigma(\Sigma \times \mathbb{R}^{2n})$ so that we are provided with a suitable framework for the Fredholm analysis. In order to define the local structure we will consider an exponential map on TM. We first assume M to be a compact smooth manifold endowed with a smooth Riemannian metric. (Later we will deal with the symplectic structure (M,ω).) Let exp be any exponential map defined by a spray, see e.g. [34, 14]. For example, we can consider the unique exponential map associated to the Levi-Civita connection on M, i.e. given by the geodesic spray. Let $\mathcal{D} \subset TM$ denote the associated injectivity neighbourhood of the zero section in the tangent bundle $\tau\colon TM \to M$, such that the map

$$\mathcal{D} \xrightarrow{\approx} V(\triangle) \subset M \times M, \qquad v \mapsto (\tau(v), \exp(v)),$$

describes a diffeomorphism onto a neighbourhood of the diagonal $\triangle$ within $M \times M$. We denote the space of smooth contractible loops in M by

$$\Omega^o(M) = \{\, s \in C^\infty(S^1, M) \,|\, x \text{ contractible} \,\}.$$

We already noticed that, for each $h \in C^\infty_\Sigma(M)$ with $h_{|\partial\Sigma} \subset \Omega^o(M)$,

$$h^*TM = \{\, (z,\xi) \in \overline{\Sigma} \times TM \,|\, h(z) = \tau(\xi) \,\} \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$$

is a Σ-smooth vector bundle over $\overline{\Sigma}$ with trivial ends.

2.1.6 Definition *Given a family $(x_i)_{i=1,\dots,\nu} \subset \Omega^o(M)$, we define*

$$\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) = \\ \{\, \exp\circ v \in C^0(\overline{\Sigma}, M) \,|\, v \in H^{1,p}_\Sigma(h^*\mathcal{D}),\ h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M) \,\}\ .$$

Here, we note that $H^{1,p}_\Sigma(h^*\mathcal{D}) = \{\, v \in H^{1,p}_\Sigma(h^*TM) \,|\, v(z) \in \mathcal{D}$ f.a. $z \in \overline{\Sigma} \,\}$ is an open subset because $H^{1,p}_\Sigma(h^*TM) \hookrightarrow C^0(\overline{\Sigma}, h^*TM)$ embeds continuously. We often use the alternative notation

$$\exp_h v = \exp\circ v \subset C^0(\overline{\Sigma}, M), \qquad (\exp_h v)(z) = \exp_{h(z)} v(z)\,.$$

For a fixed $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$, the map

$$\Phi_h\colon h^*\mathcal{D} \to \overline{\Sigma} \times M, \quad (z,\xi) \mapsto (z, \exp_{h(z)} \xi),$$

is an embedding onto an open neighbourhood of the graph of h. We define

$$\begin{aligned} &\mathcal{V}_h = H^{1,p}_\Sigma(h^*\mathcal{D}) \subset H^{1,p}_\Sigma(h^*TM), \\ &\mathcal{U}_h = \{\, g \in C^0(\overline{\Sigma}, M) \,\big|\, \operatorname{graph} g \subset \Phi_h(h^*\mathcal{D}),\ \Phi_h^{-1}\circ(\mathrm{id}, g) \in \mathcal{V}_h \,\}, \\ &\Psi_h\colon \mathcal{U}_h \to \mathcal{V}_h, \quad g \mapsto \Phi_h^{-1}\circ(\mathrm{id}, g)\,. \end{aligned}$$

Thus, we construct a covering of

$$\mathcal{P}(x_1,\dots,x_\nu) = \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) = \bigcup_{h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)} \mathcal{U}_h\,.$$

2.1.7 Theorem *$\mathcal{P}(x_1,\dots,x_\nu)$ is endowed with the differentiable structure of an infinite dimensional separable Banach manifold modeled on $H^{1,p}_\Sigma(\Sigma \times \mathbb{R}^m)$, with $m = \dim M$. The family*

$$(\mathcal{U}_h, \Psi_h)_{h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)}, \qquad \Psi_h^{-1}(s) = \exp_h s \quad \text{for } s \in H^{1,p}_\Sigma(h^*\mathcal{D})\,,$$

represents an associated atlas of charts. Moreover, the differentiable structure is independent of the Riemannian metric g on M.

Note that any countable subset $X \subset C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ which lies dense within $C^0_{x_1,\dots,x_\nu}(\Sigma, M) = \{\, h \in C^0(\overline{\Sigma}, M) \,|\, h \circ \phi_i(1,\cdot) = x_i,\ i = 1,\dots,\nu \,\}$ provides a countable sub-atlas for the manifold $\mathcal{P}(x_1,\dots,x_\nu)$.

In this section we want to confine the exposition to the definitions and constructions of the elements of the analytic framework. We already have set up the Banach manifold of maps which will contain the central solution spaces as zero loci of smooth sections in vector bundles over $\mathcal{P}(x_1,\dots,x_\nu)$. In order to present a transparent and compact foundation for this analytical framework, we postpone the rather technical proofs of Theorem 2.1.7 and the following statements to the appendix. They all deal with special Banach manifold charts and local trivializations of Banach space bundles, which are defined along the lines of Eliasson's description of manifolds of maps, see [13].

2.1.8 Corollary *Any smooth map $f \in C^\infty(M,N)$ between finite dimensional manifolds M and N induces a well-defined smooth map $\mathfrak{S}(f)$ between the associated Banach manifolds,*

$$\mathfrak{S}(f)\colon \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) \to \mathcal{P}^{1,p}_{f(x_1),\dots,f(x_\nu)}(\Sigma, N), \quad \gamma \mapsto f\circ\gamma$$

with $f(x_i) = f \circ x_i \in \Omega^o(N)$, $i = 1,\dots,\nu$.

2.2 Vector Bundles over Manifolds of Maps

Next, we have to introduce further objects in relation with the Banach manifold, which is meant to contain the solution spaces in question as zero loci of smooth Fredholm maps. Those maps will actually be smooth sections in vector bundles over $\mathcal{P}(x_1,\dots,x_\nu)$. In order to describe those vector bundles we require some more constructions.

2.2.1 Theorem *The vector spaces $H^{1,p}_\Sigma(g^*TM)$ and $L^p_\Sigma(g^*TM)$ are well-defined for every $g \in \mathcal{P}(x_1,\dots,x_\nu)$. Moreover,*

$$H^{1,p}_\Sigma(\mathcal{P}(x_1,\dots,x_\nu)^*TM) = \bigcup_{g \in \mathcal{P}(x_1,\dots,x_\nu)} H^{1,p}_\Sigma(g^*TM)$$

and

$$L^p_\Sigma(\mathcal{P}(x_1,\dots,x_\nu)^*TM) = \bigcup_{g \in \mathcal{P}(x_1,\dots,x_\nu)} L^p_\Sigma(g^*TM)$$

are smooth vector bundles over $\mathcal{P}(x_1,\dots,x_\nu)=\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M)$ *for all smooth loops* $x_1,\dots,x_\nu\in\Omega^o(M)$*. There is a natural identification*

$$T\mathcal{P}(x_1,\dots,x_\nu)=H^{1,p}_\Sigma(\mathcal{P}(x_1,\dots,x_\nu)^*TM)\,.$$

We now add further structures both to the model surface $\Sigma=(\overline{\Sigma},(\psi_i))$ and the target manifold M. We already noticed that the cylindrical coordinates over the ends of Σ were explicitly used to specify the measure on Σ, so that we deal with the suitable Sobolev spaces. Now, we endow the model surface Σ with a conformal structure j, which on the ends is already determined by

$$j(\psi_i(s,t))\circ T\psi_i(s,t)=T\psi_i(s,t)\circ i,\quad (s,t)\in Z^{\epsilon_i},\quad i=1,\dots,\nu,$$

where i is the standard structure from $Z^{\epsilon_i}\subset\mathbb{R}\times S^1=\mathbb{C}/i\mathbb{Z}$. We note that an extension j of $\psi_*(i)=T\psi_i\circ i\circ T\psi_i^{-1}$ is not unique but always exists. For example, the space of complex structures on a 2-dimensional real vector space V is parametrized as

$$\begin{aligned}\mathcal{H}(V)&=\{\,J\in\mathbb{R}^{2\times2}\,|\,J^2=-\operatorname{Id}\,\}\\ &\cong\left\{\begin{pmatrix}z & x+y\\ y-x & z\end{pmatrix}\,\Big|\,x^2-y^2-z^2=1,\ x,y,z\in\mathbb{R}\right\}.\end{aligned}$$

Thus, an orientation of Σ corresponds to a compatible choice of one of the components of the bundle $\mathcal{H}(T\Sigma)\to\Sigma$. This component is fibre-wise contractible. Hence the space of all extensions of $\psi_*(i)$ is contractible. This will be an important fact later for the principle of homotopy invariance.

To sum up, we consider our model surface Σ as a Riemann surface with cylindrical ends. Now, let the target manifold M be a closed symplectic manifold (M,ω) of dimension $2n$.

2.2.2 Definition *A* $(1,1)$*-Tensor* $J\in C^\infty(\operatorname{End}(TM))$ *is called* ω***-compatible*** *almost complex structure, if*

1. $J^2=-\operatorname{Id}_{TM}$ *and*
2. $\omega\circ(\operatorname{Id}\times J)=g_J$ *is a Riemannian metric on* TM.

As a simple consequence we note that such a J is orthogonal and anti-symmetric with respect to $g_J=\langle\cdot,\cdot\rangle$, i.e. $\langle Jv,Jw\rangle=\langle v,w\rangle$. According to the sign convention we obtain for the Hamiltonian vector field X_H associated to a Hamiltonian function $H\in C^\infty(M,\mathbb{R})$:

$$\omega(X_H,\cdot)=-dH,\quad X_H=J\nabla H\,.$$

In view of the required transversality results for our solution spaces which are still to be defined, we allow the almost complex structure J to depend on points in $\overline{\Sigma}$.

2.2.3 Definition *Using the projection* $\operatorname{pr}\colon\overline{\Sigma}\times M\to M$*, we define*

$$\begin{aligned}\mathcal{J}=\{\,J\in C^\infty_\Sigma(\operatorname{pr}^*\operatorname{End}(TM))\,|\,&J^2=-\operatorname{Id}\\ &\omega\circ(\operatorname{Id}\times J(z,\cdot))\ \text{Riem. metric on } TM \text{ f.a. } z\in\overline{\Sigma}\,\}\,.\end{aligned}$$

We further introduce the bundles L *and* X^J *over* $\overline{\Sigma}\times M$ *by*

$$\begin{aligned}L&=T^*\overline{\Sigma}\otimes TM, & L_{z,m}&=\operatorname{Hom}(T_z\overline{\Sigma},T_mM),\\ X^J&=T^{0,1}\overline{\Sigma}\otimes_J TM\subset L, & X^J_{z,m}&=\{\,\phi\in L_{z,m}\,|\,\phi\circ j(z)=-J(z,m)\circ\phi\,\}\,.\end{aligned}$$

for $J\in\mathcal{J}$*. Finally, we define the linear map between the vector spaces of sections*

$$\begin{aligned}&\Lambda_J\colon C^\infty_\Sigma(L)\to C^\infty_\Sigma(X^J),\quad \phi\mapsto\phi+J\circ\phi\circ j,\\ &\Lambda_J(\phi)(z,m)=\phi(z,m)+J(z,m)\circ\phi(z,m)\circ j(z)\in\operatorname{Hom}(T_z\overline{\Sigma},T_mM).\end{aligned}$$

The bundle X^J consists of the (j,J)-antiholomorphic homomorphisms. It is a rank-$2n$-bundle as one can see as follows. Using local conformal coordinates at $z\in\Sigma$, $z=s+it$, with $\frac{\partial}{\partial s},\frac{\partial}{\partial t}$ a basis of $T_z\Sigma$ such that $j(z)\frac{\partial}{\partial s}=\frac{\partial}{\partial t}$, we easily observe that

$$\lambda(z)\colon X_{(z,m)}\to T_mM,\quad \phi\mapsto\phi\cdot\tfrac{\partial}{\partial s},$$

is an isomorphism. We use the notation $A\cdot v$ for the linear map A applied to the vector v. The inverse map in local coordinates $z=s+it$ is given by

$$\begin{aligned}&\lambda(z)^{-1}\cdot v=d\bar z\otimes_J v=\phi_v,\\ &\phi_v(a\tfrac{\partial}{\partial s}+b\tfrac{\partial}{\partial t})=av-bJ(z,m)v,\ v\in T_mM\,.\end{aligned}$$

Moreover, the mapping $\Lambda\colon\mathcal{J}\to\operatorname{End}\big(C^\infty_\Sigma(L)\big)$ is affine.

We now introduce two further Banach space bundles which are needed for the analysis.

2.2.4 Definition *For any* $u\in C^0(\overline{\Sigma},M)$ *we consider* $\bar u=(\operatorname{id},u)$*, the graph embedding* $\bar u\colon\overline{\Sigma}\hookrightarrow\overline{\Sigma}\times M$*. Pulling back provides us with the bundles over* $\overline{\Sigma}$*,* $\bar u^*F$*,* $\bar u^*X^J\to\overline{\Sigma}$ *denoted by* u^*F *and* u^*X^J*. We define*

$$\begin{aligned}L^p_\Sigma(\mathcal{P}(x_1,\dots,x_\nu)^*L)&=\bigcup_{u\in\mathcal{P}(x_1,\dots,x_\nu)}\{u\}\times L^p_\Sigma(u^*L)\qquad\text{and}\\ L^p_\Sigma(\mathcal{P}(x_1,\dots,x_\nu)^*X^J)&=\bigcup_{u\in\mathcal{P}(x_1,\dots,x_\nu)}\{u\}\times L^p_\Sigma(u^*X^J)\,.\end{aligned}$$

We obtain the following result

2.2.5 Theorem *The bundles* $L^p_\Sigma(\mathcal{P}^*E)$*,* $E=L,X^J$*, are well-defined extensions of* $L^p_\Sigma(C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)^*E)$ *and they form smooth Banach space bundles over the Banach manifold* $\mathcal{P}(x_1,\dots,x_\nu)$ *for any* $x_1,\dots,x_\nu\in\Omega^o(M)$*. The linear map* Λ_J *for* $J\in\mathcal{J}$ *induces a smooth bundle homomorphism*

$$\Lambda_J\colon L^p_\Sigma(\mathcal{P}^*L)\to L^p_\Sigma(\mathcal{P}^*X^J)\,.$$

Since we will have to carry out several explicit computations in terms of local coordinates, we describe the local trivializations of the smooth bundle $L^p_\Sigma(\mathcal{P}^*X^J)$ already here, the complete proofs are located in the appendix. The technical point of this concept of bundles over mapping manifolds is that we have to use connections and covariant derivatives on the bundle TM. In order to keep these computations simple it is convenient to consider the unique Levi-Civita connection on M associated to a Riemannian metric, for instance $g_J = \omega \circ (\mathrm{Id} \times J)$. However, for the bundle with fibers $\Omega^{0,1}(u^*TM) \subset L^p_\Sigma(\mathcal{P}^*X^J)$ of complex anti-linear morphisms along maps $u \in \mathcal{P}(x_1, \dots, x_\nu)$, we prefer to work with Hermitian connections with respect to J, such that complex antilinearity is preserved. A Hermitian connection ∇^J is a connection with

$$\nabla^J J = 0 .$$

However, only Kähler manifolds M admit both Levi-Civita and Hermitian connections. In this situation, we decide for the latter, thus allowing torsion terms in the linearization of the crucial Cauchy-Riemann type operator. At this point we differ from the expositions by Salamon and Zehnder, in [48].

Let us consider the pull-back bundle

$$\tilde{\tau} \colon \mathrm{pr}^* TM \to \overline{\Sigma} \times M$$

equipped with a Hermitian connection map $K^J \colon T(TM) \to TM$ which is associated to an ω-compatible almost complex $J \in \mathcal{J}$. We note that K^J inducing a Hermitian connection ∇^J is entirely independent from the connection K which gives rise to the exponential map $\exp \colon TM \supset \mathcal{D} \to M$ on which the differentiable structure of $\mathcal{P}(x_1, \dots, x_\nu)$ is based. Given any $\xi = ((z,p), v) \in \mathrm{pr}^* \mathcal{D} \subset \mathrm{pr}^* TM$, $v \in T_pM$, we consider the geodesic arc $t \mapsto \gamma(t) = \exp_p(tv)$ in M associated to K. The parallel translation along γ with respect to the Hermitian connection ∇^J defines an isomorphism

$$P(z,v) \colon \{z\} \times T_pM \xrightarrow{\cong} \{z\} \times T_{\gamma(t)}M, \quad t \in [0,1] \tag{2.1}$$

defined by the first order differential equation

$$\begin{gathered} \nabla^J_t X = \nabla^J_{\dot\gamma(t)} X = 0, \quad X(t) \in \mathrm{pr}^* TM_{(z,\gamma(t))}, \\ P(z,v)(X(0)) = X(t), \quad \text{i.e. } X \text{ vector field along } t \mapsto (z, \exp_p(tv)) . \end{gathered}$$

Thus, P is a smooth map $P \colon \mathrm{pr}^* \mathcal{D} \times \mathrm{pr}^* TM \to \mathrm{pr}^* TM$ satisfying

$$\tilde{\tau} \circ P(z, v_p)(w_p) = \exp_p v, \quad P(z, 0_p) = \mathrm{id}_{\{z\} \times T_pM} . \tag{2.2}$$

Since P is derived from a Hermitian connection, it holds

$$\begin{gathered} P(z,v) \circ J(z,p) \cdot w = J(z, \exp_p v) \circ P(z,v) \cdot w \\ \text{f.a. } w \in T_pM, \ (z,p) \in \Sigma \times M . \end{gathered}$$

This parallel translation now provides us with a smooth trivialization of the bundle $L^p_\Sigma(\mathcal{P}^*X^J)$.

Consider any $\alpha, \beta \in \mathcal{P}(x_1, \dots, x_\nu) \subset C^0_{x_1,\dots,x_\nu}(\overline{\Sigma}, M)$ such that $\beta = \exp_\alpha s$, $s \in C^0(\alpha^*\mathcal{D})$. Then, we define for $\phi_\alpha \in C^0(\alpha^* X^J)$

$$G_{\beta\alpha}(z) \cdot \phi_\alpha(z) = P(z, s(z)) \circ \phi_\alpha(z), \quad z \in \Sigma, \tag{2.3}$$

providing $G_{\beta\alpha} \cdot \phi_\alpha \in C^0(\beta^* X^J)$. In the appendix we prove that for $g = \exp_h s^h = \exp_k s^k \in \mathcal{P}(x_1, \dots, x_\nu)$, $h, k \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$, the composition $F_{kh} = G^{-1}_{gk} \circ G_{gh}$ provides a smooth local trivialization of $L^p_\Sigma(\mathcal{P}^*X^J)$ adapted to the atlas $\{\mathcal{U}_h, \psi_h\}_{h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)}$ of $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$.

2.3 The Construction of the Nonlinear Operator

Up to this point we have introduced the ambient space

$$\mathcal{P} = \mathcal{P}(x_1, \dots, x_\nu) = \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$$

for the solution set together with a set of bundles. We now proceed by defining the crucial map whose zero loci will present this solution space as a subspace of $\mathcal{P}$. This map is constructed as a smooth section in the bundle $L^p_\Sigma(\mathcal{P}^*X^J)$.

Since we are considering a fixed model surface $\Sigma = (\overline{\Sigma}, (\psi_i)_{i=1,\dots,\nu})$, we will usually identify in the following $z = \psi_i(s,t)$, $i = 1, \dots, \nu$ with the cylindrical coordinates (s,t) themselves as well as we use the basis $ds, dt \in \Omega^1(\Sigma_Z)$ for 1-forms on the cylindrical ends of Σ.

2.3.1 Proposition *Let $k \in C^\infty_\Sigma(L)$ be a smooth section in the homomorphism bundle over $\overline{\Sigma} \times M$ and $x_1, \dots, x_\nu \in \Omega^o(M)$ be loops, such that*

$$\lim_{\epsilon_i s \to \infty} k(\psi_i(s,t), x_i(t)) = -dt \otimes \frac{dx_i}{dt}(t) \quad \text{f.a. } t \in S^1, \ i = 1, \dots, \nu .$$

Then the mapping

$$C^\infty_{x_1,\dots,x_\nu}(\Sigma, M) \ni u \mapsto du + k(u) \in C^\infty_\Sigma(u^*L)$$

with $k(u)(z) = k(z, u(z))$ has a unique extension to a smooth section

$$d + k \colon \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) \to L^p_\Sigma(\mathcal{P}^*L) .$$

Here we use the notation du for the differential yielding a 1-form on Σ with values in the pull-back bundle u^*TM, i.e $du \in \Omega^1(u^*TM)$. The proof is carried out in the appendix. Recalling the smooth bundle map $\Lambda_J \colon L^p_\Sigma(\mathcal{P}^*L) \to L^p_\Sigma(\mathcal{P}^*X^J)$ associated to a $J \in \mathcal{J}$ we now are able to define the map which is central to the entire analysis.

2.3.2 Definition *Given any $J \in \mathcal{J}$ and $k, x_1, \dots, x_\nu$ as in Proposition 2.3.1, we define for $\mathcal{P} = \mathcal{P}(x_1, \dots, x_\nu)$*

$$\overline{\partial}_{J,k} = \Lambda_J \circ (d + k) \colon \mathcal{P} \to L^p_\Sigma(\mathcal{P}^*X^J)$$

as a smooth section in a Banach vector bundle over a Banach manifold, i.e.

$$u \mapsto \overline{\partial}_{J,k} u \in L^p_\Sigma(u^*X^J) .$$

At this stage we point out that not only the proof of Proposition 2.3.1 but also the Fredholm property of the linearization of $\overline{\partial}_{J,k}$ are sensitive to the imposed structures on the cylindrical ends of Σ. This concerns the conformal structure $j = i$ on $\Sigma_Z = \bigcup_{i=1,\dots,\nu} \psi_i(Z^{\epsilon_i})$ as well as the differentiable structure. In Proposition 2.3.1, the operator $d\colon u \mapsto du$ over the cylindrical ends is taken with respect to the coordinates $(s,t) \in \mathbb{R} \times S^1$, i.e.

$$dh(\psi_i(s,t)) = ds \otimes h_s(s,t) + dt \otimes h_t(s,t),$$
$$h_s = T(h \circ \psi_i) \cdot \tfrac{\partial}{\partial s}, \quad h_t = T(h \circ \psi_i) \cdot \tfrac{\partial}{\partial t},$$

for $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$, $(s,t) \in Z^{\epsilon_i}$.

After introducing the operator $\overline{\partial}_{J,k}$ in a general way on arbitrary surfaces with cylindrical ends, we now ask how this is related to the original operator $\overline{\partial}_{J,H} = \partial_s + J\partial_t + \nabla H$ from standard Floer theory on cylinders $\mathbb{R} \times S^1$. Let us consider local conformal coordinates $s + it \in \mathbb{C}$ on Σ. Since $X^J \xrightarrow{\sim} TM$, $\phi \mapsto \phi \cdot \frac{\partial}{\partial s}$ is an isomorphism, it is equivalent to the local representation

$$\begin{aligned}\overline{\partial}_{J,k}(u) \cdot \tfrac{\partial}{\partial s} &= \Lambda_J(du + k(u)) \cdot \tfrac{\partial}{\partial s} \\ &= \tfrac{\partial u}{\partial s} + J(z,u) \circ du \circ j(z) \cdot \tfrac{\partial}{\partial s} + \Lambda_J \circ k(u) \cdot \tfrac{\partial}{\partial s} \\ &= u_s + J(z,u)u_t + k(u) \cdot \tfrac{\partial}{\partial s} + J(z,u)k(u)\tfrac{\partial}{\partial t}\,.\end{aligned}$$

On the cylinder $\Sigma = \mathbb{R} \times S^1$ these coordinates $(s,t) \in \mathbb{C}/i\mathbb{Z}$ are global. Let $H \in C^\infty(S^1 \times M, \mathbb{R})$ be a t-dependent Hamiltonian with associated vector field X_{H_t}, i.e. $\omega(X_{H_t}, \cdot) = -dH_t$, $H_t = H(t,\cdot)$. Then setting

$$k = k(H) \in C^\infty_\Sigma(L), \quad k(z,p) = -dt \otimes X_{H_t}(p),$$

leads to the standard "gradient flow"-operator known from Floer theory,

$$\overline{\partial}_{J,k}(u) \cdot \tfrac{\partial}{\partial s} = u_s + J(u_t - X_{H_t}(u)) = u_s + Ju_t + \nabla H_t(u)\,. \tag{2.4}$$

We are trying to find a generalization of Floer's homology theory as an S^1-cobordism theory. The crucial point for the model surfaces with cylindrical ends is the following observation. We may construct $\overline{\partial}_{J,k}$ in such a way that we consider the gradient flow equation for the symplectic action as in (2.4) on each of the cylindrical ends. If we extend this $k = k(H)$ as above smoothly over the compact complement of Σ_Z we obtain an elliptic quasi-linear operator $\overline{\partial}_{J,k}$ which is equipped with all essential properties. For example, its linearization is a suitable Fredholm operator. The important point, however, is that we maintain the translation invariance of (2.4) asymptotically on the cylindrical ends. That is, for $T > 0$ large enough, $(\overline{\partial}_{J,k})_{|\psi_i(Z^{\epsilon_i}_T)}$ is invariant under the $\mathbb{R}$-shifting $(u * \tau)(s,t) = u(s+\tau, t)$. Here, we use the notation $Z^+_T = [T, \infty)$ and $Z^-_T = (-\infty, -T]$. This so-called asymptotic translation invariance will provide the commutativity relation with the (co-)chain δ-operator which is defined by means of the bounded energy trajectories of relative index 1. Thus, counting the zeros of $\overline{\partial}_{J,k}$ will result in a homological invariant for (M, ω).

Let us once again consider the cylinder $\mathbb{R} \times S^1$ and recall the following definitions from Floer homology theory.

2.3.3 Definition *Let $H \in C^\infty(S^1 \times M, \mathbb{R})$ and also $J \in \mathcal{J}$ be independent of the variable $s \in \mathbb{R}$, i.e. $J \in C^\infty(S^1 \times M)$. We recall the definition*

$$\mathcal{P}_1(H) = \{\, x \in \Omega^o(M) \mid \dot{x}(t) = X_{H_t}(x(t)) \quad \text{f.a. } t \in S^1 \,\}$$

of the 1-periodic orbits of the Hamilton equation. The Hamiltonian H is called ***regular*** *if all solutions $x \in \mathcal{P}_1(H)$ are non-degenerate, i.e. 1 is not a Floquet multiplier for any contractible 1-periodic solution. Moreover, the pair (J,H) is called* ***regular*** *if H is regular and the linearizations $\overline{\partial}'_{J,H}(u)$ are onto for all finite energy solutions of $\overline{\partial}_{J,H}(u) = 0$, i.e. the operator*

$$\overline{\partial}_{J,H} = \partial_s + J\partial_t + \nabla H \colon \mathcal{P}^{1,p}_{x,y}(\mathbb{R} \times S^1) \to L^p(\mathcal{P}^* X^J)$$

has only regular zeros for all $x, y \in \mathcal{P}_1(H)$.

In [48] it is shown that the space of regular pairs is dense with respect to the C^∞-topology. In the general concept for model surfaces Σ, we only consider operators $\overline{\partial}_{J,k}$ where (J,k) restricted to the cylindrical ends is given by regular pairs. However, this regularity of the cylindrical restrictions (J^i, H^i) is not sufficient for the so-called transversality result which states that all linearizations $\overline{\partial}'_{J,k}(u)$ are onto. This required property will be established in Section 4.2.

2.3.4 Definition *Let $J \in \mathcal{J}$ and $k \in C^\infty_\Sigma(L)$. We call the pair (J,k) T-* ***admissible*** *if the restrictions to $\Sigma_{Z_T} = \bigcup_{i=1}^\nu \psi_i(Z^{\epsilon_i}_T)$ for some $T > 0$ large enough are only depending on $(t,p) \in S^1 \times M$ and*

$$k_{|Z^{\epsilon_i}_T}(t,p) = -dt \otimes X_{H^i_t}(p) \quad \text{f.a. } (t,p) \in S^1 \times M,$$

for some $H^i \in C^\infty(S^1 \times M)$ such that (J^i, H^i), $i = 1, \dots, \nu$, are regular pairs with $J^i(t,p) = J(\psi_i(s,t),p)$ for all $\epsilon_i s \geqslant T$. We call (J,k) an admissible extension of $(J^i, H^i)_{i=1,\dots,\nu}$. Analogously, let us call the conformal structure j on Σ admissible if it extends the induced standard cylindrical structure $M(\psi_j)_* i$, $j = 1, \dots, \nu$.

An important point for the homotopy invariance results, concerning the entire concept, is that the topological space of all admissible extensions of a fixed family $(J^i, H^i)_{i=1,\dots,\nu}$ is contractible because the space of ω-compatible almost complex structures is contractible. Note that we can always construct an admissible extension of such a family of regular pairs. For example, let us choose cut-off functions $\beta^\pm \in C^\infty(\mathbb{R}, [0,1])$ with

$$\beta^+(s) = \begin{cases} 0, & s \leqslant 1 \\ 1, & s \geqslant 2 \end{cases}, \quad \beta'(s) > 0 \quad \text{for } 1 < s < 2, \quad \beta^-(s) = \beta^+(-s)\,.$$

We define the $(1,1)$-tensor $k^o(H) \in C^\infty_\Sigma(L)$ associated to $H = (H^i)_{i=1,\dots,\nu}$ with support in Σ_{Z_1} by

$$k^o(H)(\psi_i(s,t),p) = -\beta^{\epsilon_i}(s)dt \otimes X_{H^i_t}(p)\,.$$

2.3.5 Definition *We call this admissible extension $k^o(H)$ the **special** extension of $(H^i)_{i=1,\dots,\nu}$.*

Using that the space of ω-compatible almost complex structures on M is contractible we similarly may extend the J^i, for instance in a way such that $J = J(p)$ is z-independent on $\Sigma \setminus \Sigma_{Z_1}$ and $J_{|Z_2^{e_i}} = J^i$.

2.4 Linearization

The study of the operator $\overline{\partial}_{J,k}$ splits into two parts. On the one hand much of the homological information drawn from the solution spaces of $\overline{\partial}_{J,k}$ will be linked to the nonlinear properties. In this section, on the other hand, we begin discussing the linearized situation. The entire next chapter will be devoted to the crucial Fredholm property of the linearization of $\overline{\partial}_{J,k}$ at a solution $u \in \overline{\partial}_{J,k}^{-1}(0)$. Here we compute this linear operator.

By the linearization of a differentiable section of a vector bundle we mean a projection of the differential of the section at a zero. It is the projection of the tangent space at a point in the bundle onto its vertical subspace. This is well-defined if this point is contained in the zero section of the bundle, and it is canonically identified with the fibre of this bundle. However, in the situation of the infinite dimensional Banach space bundle $L^p_\Sigma(\mathcal{P}^*X^J)$ there are different approaches to construct a local trivialization and the respective linearizations have different looking forms. Here we choose the approach by a Hermitian connection such that we have to deal with torsion. In [48], the authors employ a Levi-Civita connection and have to take into consideration terms involving ∇J. We denote the linearization of $\overline{\partial}_{J,k}$ at the solution $\overline{\partial}_{J,k}(u) = 0$ by

$$D_u = D\overline{\partial}_{J,k}(u)\colon H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^J)\,.$$

We already introduced the local trivialization of

$$L^p_\Sigma(\mathcal{P}^*X^J) \to \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$$

in terms of the Hermitian parallel translation (2.1).

2.4.1 Proposition *Let $u = \exp_h s \in \mathcal{P}$ with $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ and $s \in H^{1,p}_\Sigma(h^*\mathcal{D})$ and let $G_{uh}^{-1}\colon L^p_\Sigma(u^*X^J) \xrightarrow{\cong} L^p_\Sigma(h^*X^J)$ be the associated trivialization map. Then the induced map*

$$F_h\colon H^{1,p}_\Sigma(h^*\mathcal{D}) \to L^p_\Sigma(h^*X^J), \qquad s \mapsto G_{uh}^{-1}(\overline{\partial}_{J,k}(\exp_h s)),$$

has the differential at $s = 0$

$$DF_h(0)\colon H^{1,p}_\Sigma(h^*TM) \to L^p_\Sigma(h^*TM), \quad DF_h(0)\cdot\xi = \Lambda_J(\nabla\xi + A)\,.$$

The zero order part $A = A(h) \in C^\infty_\Sigma(\mathrm{Hom}(h^*TM; h^*L))$ is given by

$$A(h)\cdot\xi = \mathrm{Tor}(dh, \xi) + \nabla_\xi k(\cdot, h)\,.$$

In order to give a proof and an explicit formula for A we first verify the following statement.

2.4.2 Lemma *Under the same assumptions as in Proposition 2.4.1 we compute the formal differential at 0 of the trivialization of d at h, i.e. $G_h = d_{triv}\colon C^\infty(h^*\mathcal{D}) \to C^\infty(h^*F)$, $s \mapsto G_{uh}^{-1}(d(\exp_h s))$, as*

$$DG_h(0)\cdot\xi = \nabla\xi + \mathrm{Tor}(dh, \xi)\,.$$

Proof. Given any $p \in M$, $\xi \in T_pM \cap \mathcal{D}$ we consider the smooth curve $\gamma\colon [0,1] \to M$, $\gamma(\lambda) = \exp_p(\lambda\xi)$. For $v \in C^\infty(\gamma^*TM)$ and $z \in \Sigma$ we obtain with the parallel translation from (2.2) for the curve $w(\lambda) = P(z,\xi)^{-1}\cdot v(\lambda)$,

$$\frac{d}{d\lambda}\Big|_{\lambda=\lambda_0} w(\lambda) = P(z,\xi)^{-1}\cdot\nabla_\lambda v(\lambda_0)\,. \tag{2.5}$$

Here ∇ denotes the covariant derivative associated to the Hermitian connection. The identity (2.5) follows from the computation with an arbitrary parallel frame $e_1, \dots, e_{2n}$ along γ, i.e. $\nabla_\lambda e_i = 0$, such that

$$P(z,\xi)^{-1}\left(\sum_{i=1}^{2n} a_i(\lambda)e_i(\lambda)\right) = \sum_{i=1}^{2n} a_i(\lambda)e_i(0) \qquad \text{and}$$
$$\nabla_\lambda\Big(\sum_{i=1}^{2n} a_i(\lambda)e_i(\lambda)\Big) = \sum_{i=1}^{2n}\frac{da_i}{d\lambda}(\lambda)e_i(\lambda)\,.$$

Next, we take any smooth curve $\sigma\colon [0,1] \to \Sigma$ and $\xi \in C^\infty(h^*\mathcal{D}) \cap H^{1,p}_\Sigma(h^*\mathcal{D})$ inducing the smooth map

$$u\colon [0,1]\times[0,1] \to M, \qquad (\tau,\lambda) \mapsto \exp_{h\circ\sigma(\tau)}\left(\lambda\xi(\sigma(\tau))\right).$$

Using the standard identity from Riemannian geometry applied to the connection $\nabla = \nabla^J$,

$$\mathrm{Tor}\left(\tfrac{\partial u}{\partial\lambda}, \tfrac{\partial u}{\partial\tau}\right) = \nabla_\tau\tfrac{\partial u}{\partial\lambda} - \nabla_\lambda\tfrac{\partial u}{\partial\tau},$$

we compute in view of (2.5)

$$\begin{aligned}
&\frac{d}{d\lambda}\Big|_{\lambda=0}\left(P(\sigma(\tau),\xi(\sigma(\tau)))^{-1}\cdot\tfrac{\partial u}{\partial\tau}\right) = (\nabla_\lambda\tfrac{\partial u}{\partial\tau})(\tau,0) = \\
&= \nabla_\tau\tfrac{\partial u}{\partial\lambda}(\tau,0) - \mathrm{Tor}\left(\tfrac{\partial u}{\partial\lambda}(\tau,0), \tfrac{\partial u}{\partial\tau}(\tau,0)\right) \\
&= \nabla_\tau\xi(\sigma(\tau)) + \mathrm{Tor}\left(dh(\sigma(\tau))\cdot\dot\sigma(\tau), \xi(\sigma(\tau))\right).
\end{aligned}$$

This is true for any σ. Hence we obtain $DG_h(0) = \nabla + \mathrm{Tor}(dh,\cdot)$. ■

Proof of Proposition 2.4.1. It suffices to combine Lemma 2.4.2 with the computation of the differential of

$$K_h\colon s \mapsto (G_{uh}^{-1}\circ k)(\exp_h s), \quad s \in H^{1,p}_\Sigma(h^*\mathcal{D})$$

at $s = 0$. Using again the fundamental identity (2.5), we obtain

$$DK_h(0)\cdot\xi = \nabla_\xi k(\cdot, h),$$

where $\nabla_v k(z,p) \in F(z,p)$ denotes the covariant derivative of k with respect to the second variable of k in direction of $v \in T_pM$. Altogether we derive

$$DF_h(0) \cdot \xi = \Lambda_J(\nabla\xi + \mathrm{Tor}(dh,\xi) + \nabla_\xi k) \in L^p_\Sigma(h^* X^J) \tag{2.6}$$

for $\xi \in H^{1,p}_\Sigma(h^*TM)$. In precise terms, $A = A(h) \in C^\infty_\Sigma(\mathrm{Hom}(h^*TM, h^*F))$ is defined as

$$(A(h)(z) \cdot \xi(z)) \cdot v = \mathrm{Tor}\,(dh(z) \cdot v, \xi(z)) + \nabla_{\xi(z)} k(z, h(z)) \cdot v$$

for $\xi \in H^{1,p}_\Sigma(h^*TM)$, $v \in T_z\Sigma$, $z \in \Sigma$. ■

Note that we have implicitly used the fact that F_h is a smooth map, due to Proposition 2.3.1.

We compute in local conformal coordinates $(s,t) \in \mathbb{C}$ on Σ

$$\begin{aligned}(DF_h(0) \cdot \xi)\tfrac{\partial}{\partial s} =& \nabla_s\xi + J(\cdot,h)\nabla_t\xi + \mathrm{Tor}\,(\tfrac{\partial h}{\partial s},\xi) + +J(\cdot,h)\,\mathrm{Tor}\,(\tfrac{\partial h}{\partial t},\xi)\\ &+ \nabla_\xi k(\cdot,h)\cdot\tfrac{\partial}{\partial s} + J(\cdot,h)\nabla_\xi k(\cdot,h)\cdot\tfrac{\partial}{\partial t}\,.\end{aligned} \tag{2.7}$$

We now should proceed by analyzing this linearization D_u at solutions $u \in \overline{\partial}^{-1}_{J,k}(0)$. For admissible extensions $(J, k(H))$ with regular Hamiltonian, it will turn out that D_u is in fact a Fredholm operator. But before, we summarize further properties of $\overline{\partial}_{J,k}$ and D_u concerning its properties as elliptic operators.

2.5 Ellipticity

Let us analyze the properties of local regularity which hold for the elliptic nonlinear operator $\overline{\partial}_{J,k}$ as well as for D_u. For that it is sufficient to work with local coordinates, that is without loss of generality $u\colon \Omega \to \mathbb{R}^{2n}$, where $\Omega \subset \mathbb{C}$ is an open domain with coordinates $s + it$. In this section we deal with the operators in local conformal coordinates

$$\overline{\partial}_{J,k}\colon u \mapsto \partial_s + J(s,t,u)\partial_t + k(s,t,u)\cdot\tfrac{\partial}{\partial s} \tag{2.8}$$

and the linear version

$$F\colon u \mapsto \partial_s + J(s,t)\partial_t + S(s,t)\cdot u, \quad S(s,t) \in \mathcal{L}_\mathbb{R}(\mathbb{R}^{2n}). \tag{2.9}$$

The operator $\overline{\partial}_{J,k}$ in (2.8) is, apart from $k(s,t,u)$ a quasi-linear operator with the ω-compatible $J \in C^\infty(\Omega \times \mathbb{R}^{2n})$ such that $J(s,t,u)$ is of the same differentiability class a u. For the fully linear operator F in (2.9) we assume that J, S both are smoothly depending on $(s,t) \in \Omega$. We recall that $u \in L^1_{\mathrm{loc}}(\Omega)$ is by definition a weak solution of $Fu = \eta$ for a given $\eta \in L^1_{\mathrm{loc}}(\Omega)$, if it holds

$$\int_\Omega \langle \partial_s\phi + J^t\partial_t\phi, u\rangle ds dt = -\int_\Omega \langle \phi, \eta - Su + (\partial_t J)u\rangle ds dt \tag{2.10}$$

for all $\phi \in C^\infty_o(\Omega, \mathbb{R}^{2n})$.

Let us first consider the linear case (2.9). For the standard Cauchy-Riemann operator

$$\overline{\partial}\colon C^\infty(\mathbb{C},\mathbb{C}^n) \to C^\infty(\mathbb{C},\mathbb{C}^n), \quad u \mapsto u_s + J_o u_t,$$

with $J_o = i \oplus \ldots \oplus i$ the standard structure on $\mathbb{R}^{2n} = \mathbb{C}^n$, there is the following classical estimate due to Douglis and Nirenberg, [12]. Let $\mathcal{D}_1 = C^\infty_o(B_1(0))$ denote the set of smooth functions with compact support in the open unit disk and

$$|u|^p_{k,p} = \sum_{|\alpha|\leqslant k} |D^\alpha|^p_{L^p(B_1(0))}, \quad k \in \mathbb{N},\ 1 \leqslant p < \infty\,.$$

2.5.1 Theorem (Douglis-Nirenberg) *For each $k \in \mathbb{N}$, $1 < p < \infty$ there is a constant $c = c(k,p)$ such that*

$$|u|_{k+1,p} \leqslant c\,|\overline{\partial}u|_{k,p} \qquad \text{f.a. } u \in \mathcal{D}_1\,.$$

Given a compact set $Q \Subset \Omega$ we can extend this estimate to $u \in W^{k,p}(Q)$. Moreover, based on the fact that every weak solution $u \in L^1_{\mathrm{loc}}$ of $\frac{\partial u}{\partial \bar z} = 0$ is holomorphic, we obtain the following result about local regularity. It is a quotation of Theorem B.3.4 in [38].

2.5.2 Theorem (Local regularity) *Let $1 < p < \infty$ and $k \geqslant 0$ be an integer. Every weak solution $u \in L^p_{\mathrm{loc}}$ of $\overline{\partial}u = f$ for $f \in W^{k,p}_{\mathrm{loc}}(\Omega)$ satisfies $u \in W^{k+1,p}_{\mathrm{loc}}$. For every compact set $Q \Subset \Omega$ there exists a constant $c = c(k,p,Q,\Omega) > 0$ such that*

$$\|u\|_{W^{k+1,p}(Q)} \leqslant c(\|\overline{\partial}u\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)})$$

for all $u \in C^\infty(\overline{\Omega})$.

PROOF. See Theorem B.3.4 in the appendix of [38]. ■

This result can be applied to the above operator F as follows.

2.5.3 Corollary *Let J, S from (2.9) be smooth on $\overline{\Omega}$, $1 < p < \infty$ and $k \geqslant 0$ an integer. If $u \in L^p_{\mathrm{loc}}(\Omega)$ is a weak solution of (2.10) with $\eta \in W^{k,p}_{\mathrm{loc}}$, then $u \in W^{k+1,p}_{\mathrm{loc}}(\Omega)$. Moreover, for every compact set $Q \Subset \Omega$ there exists a constant $c = c(k,p,Q,\Omega,J,A) > 0$ such that*

$$\boxed{\|u\|_{W^{k+1,p}(Q)} \leqslant c(\|Fu\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)})\,.} \tag{2.11}$$

PROOF. Applying the Gram-Schmidt orthogonalization procedure to n vector fields $X_1, \ldots, X_n \in C^\infty(\overline{\Omega}, \mathbb{C}^n)$ which are pointwise complex linear independent with respect to the complex structure $J(s,t)$, we find a smooth map $C \in C^\infty(\overline{\Omega}, \mathcal{GL}_\mathbb{R}(\mathbb{C}^n))$ such that

$$C(s,t)J(s,t) = J_o C(s,t) \qquad \text{f.a. } (s,t) \in \Omega.$$

Consequently $u \in L^p_{\mathrm{loc}}(\Omega)$ is a weak solution for $Fu = \eta$ if and only if Cu is a weak solution for $Gu = C\eta$ with

$$G = \partial_s + J_o\partial_t + B, \qquad B = C(\partial_s + J\partial_t + S)C^{-1}.$$

Since $f \in W^{k,p}_{\mathrm{loc}}(\Omega)$ if and only if $Cf \in W^{k,p}_{\mathrm{loc}}(\Omega)$ we can assume without loss of generality that $J(s,t) = J_o$ for all $(s,t) \in \Omega$. Thus, u is a weak solution of (2.10) with respect to η iff it is a weak solution of $\overline{\partial}u = f$ with $f = \eta - Su \in L^p_{\mathrm{loc}}(\Omega)$. Proceeding by induction on $k \in \mathbb{N}$ we deduce from Theorem 2.5.2 that $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$. Similarly, by induction on k,

$$\begin{aligned}\|u\|_{W^{k+1,p}(Q)} &\leqslant c_1\big(\|\overline{\partial}u + Su - Su\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)}\big)\\ &\leqslant c_1\big(\|Fu\|_{W^{k,p}(\Omega)} + \|Su\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)}\big)\end{aligned}$$

and employing that $S \in C^\infty(\overline{\Omega})$ we obtain the asserted estimate for a new constant c additionally depending on J and S. ■

We proceed now by using the well-known bootstrapping argument and deduce

2.5.4 Corollary *Given $J, S \in C^\infty(\overline{\Omega})$ as above, every weak solution $u \in L^p_{\mathrm{loc}}(\Omega)$ of $Fu = 0$ is a smooth classical solution.*

We note that the problem (2.10) could directly be related to the standard operator $\overline{\partial} = \partial_s + i\partial_t$, because we assumed J and S to be smooth.

Let us now turn to the quasi-linear operator $\overline{\partial}_J u = \partial_s u + J(s,t,u)\partial_t u$. Here we need a sharper regularity result, because $\tilde{J}(s,t) = J(s,t,u(s,t))$ can no longer be assumed to be smooth. We consider the partial differential equation (2.8) in the form

$$\partial_s u + \tilde{J}(s,t)\partial_t = \eta(s,t), \tag{2.12}$$
$$\eta(s,t) = -k(s,t,u(s,t)) \cdot \tfrac{\partial}{\partial s}, \quad \tilde{J}(s,t) = J(s,t,u(s,t)).$$

Thus, $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$ for $k \geqslant 1$, $1 < p < \infty$ implies that $\tilde{J}, \eta \in W^{k,p}_{\mathrm{loc}}(\Omega)$. We quote the following regularity theorem from [38].

2.5.5 Proposition *Assume $J \in W^{k,p}_{loc}$, $\eta \in W^{k,p}_{loc}$ with $p > 2$, $k \geqslant 1$. If $u \in L^p_{loc}$ is a weak solution of $\partial_s u + J\partial_t u = \eta$, then $u \in W^{k+1,p}_{loc}$. Moreover, for every compact set $Q \Subset \Omega$ there is a constant $c = c(p, Q, \Omega, \|J\|_{W^{k,p}})$ such that*

$$\|u\|_{W^{k+1,p}(Q)} \leqslant c\big(\|u\|_{W^{k,p}(\Omega)} + \|\eta\|_{W^{k,p}(\Omega)}\big).$$

However, in the case of $\tilde{J}$, we firstly have to assume that $u \in W^{1,p}_{\mathrm{loc}}$, and secondly, the constant c is no longer independent from u. We finally conclude by induction on $k \in \mathbb{N}$.

2.5.6 Corollary *Let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of $\overline{\partial}_{J,k}(u) = 0$. Then u is a smooth classical solution.*

This local regularity result consequently applies to the zeros of the section $\overline{\partial}_{J,k}\colon \mathcal{P} \to L^p_\Sigma(\mathcal{P}^*X^J)$. Thus, we have $\overline{\partial}_{J,k}^{-1} \subset C^\infty(\Sigma^\circ, M)$. However, the question whether we also obtain

$$\overline{\partial}_{J,k}^{-1}(0) \overset{?}{\subset} C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$$

is open so far. This will be proven by an analysis of the asymptotic behaviour such as suitably fast convergence along the cylindrical ends. We shall prove the following result.

2.5.7 Proposition *Let $(J, k(H))$ be an admissible extension of regular pairs $(J^i, H^i)_{i=1,\dots,\nu}$. Then, solutions $u \in \overline{\partial}_{J,k}^{-1}(0)$ converge exponentially fast towards the periodic solutions $x_i \in \mathcal{P}_1(H^i)$, $u(\psi_i(s,\cdot)) \to x_i$ for $\epsilon_i s \to \infty$, $i = 1,\dots,\nu$. In particular, it holds*

$$\overline{\partial}_{J,k}^{-1}(0) \subset C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)\,.$$

The proof will be carried out in Section 4.1, see 4.1.2. The crucial ingredients are the regularity of $(H^i)_{i=1,\dots,\nu}$ and that we geometrically deal with a gradient flow along the cylindrical ends.

The last result allows us to fix the coordinate charts for $\mathcal{P}(x_1,\dots,x_\nu)$ and the local trivialization of $L^p_\Sigma(\mathcal{P}^*X^J)$ at the solution $u \in \overline{\partial}_{J,k}^{-1}(0)$ itself. Thus, we profit from Proposition 2.4.1 which only yields the linearization at an $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$. Hence, for $h = u$, we obtain a formula for the linearization at a zero of the section $\overline{\partial}_{J,k}$.

CHAPTER 3

Fredholm Theory

This chapter is entirely devoted to the study of the linearized Cauchy-Riemann operator. At first, we prove the Fredholm property based on the local elliptic estimate and the asymptotic properties. This is essentially similar to the standard cylinder. The necessary L^p-estimate for the asymptotic translation invariant operator on the cylindrical ends is proven in detail also for $p > 2$. Compared to the original Floer theory, the new phenomenon is a more general index computation. Over a general model surface, the index cannot be computed anymore in terms of a spectral flow as it was for the cylinder. It is related to the Riemann-Roch formula which represents the case of closed model surfaces. The key method is the additivity of the Fredholm index with respect to gluing the model surfaces at their cylindrical ends. Using this gluing result we will deduce the general index formula by only computing explicitly the index for the cap. This is the model surface without genus with one end, i.e. of the type of a disk.

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3.1 The Fredholm Property

The proof of the Fredholm property is based on a combination of the local regularity result as given in Corollary 2.5.3, and the regularity of an asymptotically translation invariant operator A_i on each cylindrical end $i = 1, \ldots, \nu$. This operator A_i is related to the "gradient flow" in its geometrical sense, such that the regularity stems from the non-degeneracy condition for the Hessian of the action functional at its critical points. More precisely, we use the linear analogue of the regularity defined in Definition 2.3.3.

Let $(\mathbb{R}^{2n}, \omega_o)$ be the standard symplectic vector space with the standard complex structure $J_o = i \oplus \ldots \oplus i$, where $\mathbb{R}^{2n} \cong \mathbb{C} \oplus \ldots \oplus \mathbb{C}$.

3.1.1 Definition *A smooth loop of real matrices $S \in C^\infty(S^1, \mathcal{L}_\mathbb{R}(\mathbb{C}^n))$ is called regular, if the differential equation*

$$\dot{x}(t) = J_o S(t) x(t), \quad x(0) = x(1), \quad x\colon S^1 \to \mathbb{R}^{2n}, \tag{3.1}$$

*only admits the trivial solution $x = 0$. We call S **admissible** if it is regular and pointwise symmetric, i.e. $S(t) = S(t)^T$ for all $t \in S^1$.*

Equivalently, we could define S as admissible iff the first order operator

$$A = J_o \partial_t + S\colon C^\infty(S^1) \to C^\infty(S^1)$$

is injective and L^2-selfadjoint. Note that whenever we refer to the L^2-inner product or use Sobolev norms $\|\cdot\|_{k,p}$ we mean the standard Euclidean structure on $\mathbb{R}^{2n}$, $\langle\cdot,\cdot\rangle = \omega_o \circ (\mathrm{Id} \times J_o)$. However, we will also deal with different, nonconstant ω-compatible almost complex structures J. Those give rise to different L^2-scalar products based on $\langle\cdot,\cdot\rangle_J = \omega \circ (\mathrm{Id} \times J)$. We explicitly denote this structure by $\langle\cdot,\cdot\rangle_{J,L^2}$.

Based on the fact that the extended operator $A\colon H^{1,p}(S^1) \to L^p(S^1)$ for $1 \leqslant p < \infty$ is a continuous linear operator, which for $p = 2$ is selfadjoint with compact resolvent, we can deduce the following.

3.1.2 Lemma *Given an admissible $S \in C^\infty(S^1, \mathrm{End}(\mathbb{R}^{2n}))$, the associated operator $A_S = J_o\partial_t + S\colon H^{1,p}(S^1) \to L^p(S^1)$, $p \in (1, \infty)$, is an isomorphism with discrete spectrum $\sigma(A) \subset \mathbb{R} \setminus \{0\}$.*

Proof. The discrete spectrum is due to the compact resolvent. Since $H^{1,p}(S^1)$ embeds continuously into $C^0(S^1)$ for $p \geqslant 1$, the kernel of A consists of smooth solutions of (3.1) and thus is trivial due to the assumption. Next, we estimate

$$\begin{aligned}\|x\|_{1,p} &\leqslant (\|J\dot{x}\|_{0,p} + \|x\|_{0,p}) \leqslant (\|J\dot{x} + Sx\|_{0,p} + \|Sx\|_{0,p} + \|x\|_{0,p}) \\ &\leqslant c(\|Ax\|_{0,p} + \|x\|_{0,p}) \ .\end{aligned}$$

The compact embedding $H^{1,p}(S^1) \hookrightarrow L^p(S^1)$ together with Lemma 3.1.10 on 43 imply that A has a closed range for all $p \in (1, \infty)$. For $p = 2$,

$$\operatorname{coker} A = \ker A^* = \ker A = \{0\},$$

the L^2-selfadjointness of A yields the isomorphism. Thus, $L^2 \cap L^p$ lying dense within L^p implies dense ranges for all $p \geqslant 1$ and the lemma follows. ∎

Considering now an $S \in C^\infty(\mathbb{R} \times S^1, \mathcal{L}_\mathbb{R}(\mathbb{C}^n))$, such that the limits

$$S^\pm = \lim_{s \to \pm\infty} S(s, \cdot)$$

exist and are admissible, we obtain exactly the linear Floer type operator for the "gradient trajectories"

$$\frac{\partial}{\partial s} + A_S(s) = \frac{\partial}{\partial s} + J_o\frac{\partial}{\partial t} + S(s,t) \colon H^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n})\,.$$

This is a Fredholm operator with index given by the spectral flow of A, that is the number of eigenvalues crossing 0 counted with the sign of the crossing direction. For the exposition of this analysis in the case $p = 2$, the reader is referred to [48]. However, recalling the Sobolev embedding theorem, the final Floer analysis requires $p > 2$. The proof of the Fredholm property in this situation becomes more complicated, because one cannot use anymore Plancherel's theorem. Fourier transformation is the key feature for an elegant short proof. Nevertheless, for reasons of completeness we present a proof in our generalized situation, following the arguments of Floer, Lockhardt-McOwen and Maz'ya-Plamenewski in [17], [35] and [36].

3.1.1 The Generalized $\overline{\partial}$-Operator on Hermitian Bundles

The next aim is to find a generalization of the analytical setup for the Fredholm operator theory from the standard cylinder to the model surface Σ. We now want to deal with merely local conformal coordinates and we also need a concept for rank-$2n$ bundles over Σ generalizing $\mathbb{R}^{2n}$. We are aiming at the pull-back bundles u^*TM, $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$, carrying the Hermitian structure from TM.

Let $E \to B$ be a smooth rank-$2n$ bundle over a smooth manifold. By a Hermitian structure we understand a pair (ω, J) of a smooth bilinear form ω on E such that each fibre $(E_p, \omega(p))$, $p \in B$, is a symplectic vector space, together with an ω-compatible complex structure J on E, i.e. $J^2 = -\,\mathrm{Id}$, $\omega \circ (\mathrm{Id}\times J)$ is a Riemannian metric. This is equivalent to a Hermitian metric $(\cdot,\cdot)_{\omega,J}$ on the complex vector bundle (E, J). Both terms are related by

$$(v,w)_p = \omega(p)(v, J(p)w) - i\,\omega(p)(v,w),$$

for all $v, w \in E_p$, $p \in B$.

3.1.3 Definition *A* ***unitary trivialization*** *of such a Hermitian bundle $E \to B$ is a bundle isomorphism $\Phi \colon B \times \mathbb{R}^{2n} \xrightarrow{\sim} E$ with*

$$J(p)\Phi(p) = \Phi(p)J_o, \quad \omega(p)\circ(\Phi(p)\times\Phi(p)) = \omega_o$$

for all $p \in B$.

A simple fact is that every complex trivialization of E implies a unitary one,

3.1.4 Lemma *Assume the Hermitian bundle $E \to B$ equipped with pointwise complex linearly independent sections $\{Y_1,\dots,Y_n\}$. Then there exists a smooth unitary trivialization.*

Proof. Given a smooth complex frame $\{Y_1,\dots,Y_n\} \subset C^\infty(E)$ we obtain a unitary frame by applying the Gram-Schmidt procedure with respect to the Hermitian metric $(\cdot,\cdot)_{\omega,J}$. This constructive procedure provides a smooth unitary frame $\{Z_1,\dots,Z_n\} \subset C^\infty(E)$, so that we can define the trivialization $\Phi \colon B \times \mathbb{R}^{2n} \xrightarrow{\sim} E$ by

$$\Phi(p)e_i = Z_i(p), \quad \Phi(p)J_oe_i = J(p)Z_i(p),$$

$p \in B$, where $\{e_1, J_oe_1, \dots, e_n, J_oe_n\}$ denotes the canonical basis of $(\mathbb{R}^{2n}, \omega_o, J_o)$. ∎

As a simple consequence we conclude that over $[0,1]\times S^1$ every smooth ω_o-compatible almost complex structure J on $\mathbb{R}^{2n}$ is unitarily equivalent to J_o, because every orientable Hermitian rank-$2n$ bundle over $[0,1]\times S^1$ can be endowed with a complex trivialization. Thus, Definition 3.1.1 is equivalent to the general case of

$$A \colon C^\infty(S^1, \mathbb{R}^{2n}) \to C^\infty(S^1, \mathbb{R}^{2n}), \qquad A = J(t)\partial_t + S(t),$$

with $J, S \in C^\infty(S^1)$. Here A is regular and selfadjoint if it is injective and $S = \dot{J} + S^T$. Under a unitary transformation Φ, $\Phi J_o = J\Phi$, this is the case exactly if $J_o\Phi^{-1}\dot{\Phi} + \Phi^{-1}A\Phi$ is admissible in the sense of Definition 3.1.1.

Further, we observe that any Hermitian vector bundle $\xi \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$, in particular every u^*TM, $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ possesses a unitary trivialization over its cylindrical ends. Of course, if $\partial\Sigma \neq 0$, Σ has the homotopy type of a 1-dimensional cellular complex. Thus we could always find a unitary trivialization over the entire surface. However, we will require distinguished classes of trivializations over the cylindrical ends.

3.1.5 Definition *Let $\xi \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$ be a Hermitian bundle with fixed unitary trivializations $\Phi \colon \Sigma_Z \times \mathbb{R}^{2n} \to \xi_{|\Sigma_Z}$, $\Psi \colon \Sigma_Z \times \mathbb{R}^{2n} \to \xi_{|\Sigma_Z}$ over the cylindrical ends $\Sigma_Z = \bigcup_{i=1}^{\nu} \psi_i(Z^{\epsilon_i})$. We call Φ and Ψ equivalent, $\Phi \simeq \Psi$, if*

$$\Phi^{-1}\circ\Psi \colon \Sigma_Z \to \mathrm{U}(n) \subset \mathcal{GL}_{\mathbb{R}}(\mathbb{C}^n)$$

is homotopically trivial.

Since Σ_Z is homotopy equivalent to $\partial\Sigma = \bigsqcup_{i=1}^{\nu} S^1$ a disjoint union of circles and $\pi_1(\mathrm{U}(n), \mathbf{1}) \cong \mathbb{Z}$ by $\det \colon \mathrm{U}(n) \to S^1$, the set of equivalence classes $[\Phi]$ is isomorphic to $\mathbb{Z}^\nu$. The question of these equivalence classes will later play an important role for the index formula for the Fredholm operators.

The next effort of generalization is necessary for the appropriate definition of the class of elliptic partial differential operators. Whereas the linear Cauchy-Riemann type operator $\overline{\partial}_J + S = \partial_s + J\partial_t + S$ over the cylinder $\mathbb{R}\times S^1$ can be described in explicit conformal coordinates as an operator interrelating sections in a fixed bundle, we now face the lack of global conformal coordinates (s,t). That is, in general, the model surface Σ cannot be parallelized a way yielding the conformal vector fields $i\frac{\partial}{\partial s} = \frac{\partial}{\partial t}$ over the cylindrical ends Σ_Z. A possible way is to consider a generalized CR-operator D as an operator $D\colon C^\infty(E) \to \Omega^{0,1}(E)$ on the sections in the complex vector bundle $E \to \Sigma$, such that

$$D(fs) = fD(s) + \overline{\partial} f \otimes_J s \tag{3.2}$$

for all $f \in C^\infty(\Sigma)$, $s \in C^\infty(E)$, with $\overline{\partial} f \otimes_J s = df \otimes s + (df \circ j) \otimes Js \in \Omega^{0,1}(E)$, where $\Omega^{0,1}(E)$ is the module of differential-1-forms α on Σ with values in E such that $\alpha \circ j = -J \circ \alpha$, i.e. $\Omega^{0,1} = C^\infty(X^J(E))$. However, for the proof of the Fredholm property, we want to analyze D in such a way that we equally may consider the formally adjoint operator D^*. By the latter we understand the formally adjoint of D via Hermitian structures on E and

$$X^J(E) = T^{0,1}\Sigma \otimes_J E = \overline{\mathrm{Hom}_{\mathbb{C}}}(T\Sigma, E)\,.$$

In addition to the conformal structure j we choose any volume 2-form σ on Σ compatible with the standard structure on the cylindrical ends Σ_Z,

$$\sigma_{|\Sigma_Z} = \sum_{i=1}^{\nu} (\psi_i^{-1})^* ds \wedge dt\,.$$

Then we define the Hermitian structure $(\tilde{\omega}, \tilde{J})$ on $X^J(E)$ by

$$\tilde{J}\phi = J \circ \phi, \quad \tilde{\omega}_z(\phi, \phi') = \frac{\omega(\phi v, \phi' v)}{\sigma(v, jv)}$$

for all $z \in \Sigma$, $\phi, \phi' \in X^J(E)_z$, where $v \in T_z\Sigma$ can be chosen arbitrarily. Computing in local conformal coordinates one sees that the induced L^2_J-metric on $L^2_\Sigma(X^J(E))$ is independent of σ. It only depends on (ω, J) on E. The 2-form

$$\langle s, \bar{s}\rangle_J = \tilde{\omega}(s, \tilde{J}\bar{s})\sigma \tag{3.3}$$

on Σ associated to $s, \bar{s} \in L^2_{\mathrm{loc}}(X^J(E))$ is intrinsically well-defined, see also Section 4.3.2.

We want to describe $D\colon C^\infty(E) \to C^\infty(X^J(E))$ as belonging to a class of linear differential operators between Hermitian bundles such that the formally adjoint operator D^* belongs to the same class of operators. We introduce a different notion of a generalized Cauchy-Riemann operator based on the idea of its symbol. This means that such an operator in suitable local coordinates has the first order leading term $\partial_s + J_o\partial_t$. In view of the last section of this chapter, we notice that a Cauchy-Riemann operator on a bundle over a closed model surface is an elliptic operator whose index is uniquely determined by topological invariants of the surface and the first Chern class of the bundle E. Thus, the Hermitian structure is not relevant apart from the complex structure J. However, allowing cylindrical ends demands the refinement via a Hermitian structure and a description by means of unitary trivializations as studied above. The invariants which additionally determine the index of the Fredholm operator we are aiming at, are not invariant under arbitrary complex transformations but under symplectic ones, in particular under unitary transformations.

Let $\xi, \eta \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$ be Hermitian rank-$2n$ vector bundles over the model surface Σ and

$$F\colon C^\infty_\Sigma(\xi) \to C^\infty_\Sigma(\eta)$$

a first order linear differential operator.

3.1.6 Definition *Given unitary trivializations Φ and Ψ of the bundles $\xi_{|\Sigma_Z}$ and $\eta_{|\Sigma_Z}$ over the cylindrical ends we choose an extension to a family of local complex trivializations $\{\Phi_U, \Psi_U\}$ where $\{(U, \varphi)\}$ are conformal coordinate charts on Σ including $\psi_i\colon Z^{\epsilon_i} \to \Sigma$, $i = 1, \ldots, \nu$. We call F a* ***$\overline{\partial}$-operator*** *if the local complex trivializations can be chosen such that the local representation F_U on $\xi_{|U}$ with respect to $\Phi_U : U \times \mathbb{R}^{2n} \to \xi_{|U}$, $\Psi_U : U \times \mathbb{R}^{2n} \to \eta_{|U}$ and the conformal coordinates (s,t) on U is given by $F_U = \Psi_U^{-1} \circ F \circ \Phi_U$,*

$$F_U = \tfrac{\partial}{\partial s} + J_o \tfrac{\partial}{\partial t} + S_U$$

with $S_U \in C^\infty(U, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$. Moreover, we call such a $\overline{\partial}$-operator F ***admissible*** *if on the cylindrical ends $U_i = \psi_i(Z^{\epsilon_i})$ the zero order terms S_{U_i} yield admissible loops*

$$S_{i|\partial\Sigma} = \lim_{\epsilon_i s \to \infty} S_i(s) \in C^\infty(S^1, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$$

in the sense of Definition 3.1.1. For the fixed unitary trivializations Φ, Ψ over Σ_Z we call

$$\Omega = (\Phi_{|\partial\Sigma}, \Psi_{|\partial\Sigma}, (S_{i|\partial\Sigma})_{i=1,\ldots,\nu})$$

the asymptotic data of (F, Φ, Ψ) and we denote by $F_\Omega(\xi, \eta)$ the set of all $\overline{\partial}$-operators F with asymptotic data Ω.

Let $E \to \Sigma$ be a fixed Hermitian bundle over Σ and $\Phi_U : U \times \mathbb{C}^n \to E_{|U}$ be a unitary local trivialization. Then we immediately obtain a complex trivialization of $X^J(E)_{|U}$ by

$$\Psi_U : U \times \mathbb{C}^n \to X^J(E)_{|U}, \quad \Psi_U(w) = d\bar{z} \otimes_J \Phi(w),$$

where $d\bar{z} \otimes_J v = ds \otimes v - dt \otimes Jv$ for $v \in E_{|U}$ implicitly makes use of the conformal coordinates (s,t) on U. Obviously, it holds

$$\Psi_U \circ J_o = \tilde{J} \circ \Psi_U,$$

and $\sigma = ds \wedge dt$ on Σ_Z yields that Ψ_{U_i} is a unitary trivialization over the end $\Psi_i(Z^{\epsilon_i})$,

$$\tilde{\omega} \circ (\Psi_{U_i} \times \Psi_{U_i}) = \omega_o\,.$$

We now choose a Hermitian connection ∇ on $E \to \Sigma$, that is, by definition

$$\nabla J = 0,$$

and we consider the operator

$$\Gamma\colon C^\infty(E) \to \Omega^{0,1}(E) = X^J(E),$$
$$\Gamma s = \nabla s + J \circ \nabla s \circ j\,.$$

3.1.7 Proposition *The operators $\Gamma, D\colon C^\infty(E) \to \Omega^{0,1}(E)$ with D from (3.2) are $\overline{\partial}$-operators in the sense of Definition 3.1.6.*

Proof. Since $D - \Gamma = B \in C^\infty\big(\mathrm{Hom}(E, T^{0,1}\Sigma \otimes_J E)\big)$ is a smooth section in the homomorphism bundle, the local representation of B with respect to the complex trivializations Ψ_U, Φ_U yields

$$\Psi_U^{-1} \circ B \circ \Phi_U \in C^\infty\big(U, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n)\big)\,.$$

In other words, the difference $D - \Gamma$ is of lower order. Thus, D is a $\overline{\partial}$-operator if and only if Γ is. In order to prove this property of Γ we choose complex parallel frames $\{Z_1, \dots, Z_n\}$ for the Hermitian bundle $E_{|U}$ for each U, i.e.

$$\nabla Z_i = 0$$

for $i = 1, \dots, n$. In view of Lemma 3.1.4 we may assume the frame to be unitary, so that it yields the unitary trivialization

$$\Phi_U : U \times \mathbb{C}^n \to E_{|U}\,,$$
$$\Phi_U(v_1, \dots, v_{2n}) = \sum_{i=1}^{m} (v_{2i-1} Z_i + v_{2i} J Z_i)\,.$$

Using the associated complex trivialization $\Psi_U(w) = d\bar{z} \otimes_J \Phi_U(w)$ as defined above we obtain

$$\Psi_U^{-1} \circ \Gamma \circ \Phi_U = \partial_s + J_o \partial_t$$

due to $\nabla Z_i = 0$ and $\nabla J = 0$. Since we already remarked that Ψ_{U_i} over the cylindrical end $U_i = \psi_i(Z^{\epsilon_i})$ is unitary, it follows that Γ meets the conditions of Definition 3.1.6 and hence is a $\overline{\partial}$-operator. ∎

We finally have to verify that the class of $\overline{\partial}$-operators between Hermitian bundles is compatible with replacing an operator F by its formally adjoint F^*. More, precisely, it is invariant under this operation modulo replacing J by $-J$. Given the Hermitian bundles (ξ, ω_ξ, J_ξ) and $(\eta, \omega_\eta, J_\eta)$ over Σ and a linear differential operator $F\colon C^\infty(\xi) \to C^\infty(\eta)$ we obtain its formally adjoint operator $F^*\colon C^\infty(\eta) \to C^\infty(\xi)$ via the identity

$$\omega_\eta(v, J_\eta F w) = \omega_\xi(F^* v, J_\xi w)$$

for all $v \in C_o^\infty(\eta)$, $w \in C_o^\infty(\xi)$.

3.1.8 Lemma *If F is a $\overline{\partial}$-operator with respect to the standard structure J_o on $\mathbb{R}^{2n}$, its associated formally adjoint F^* is a $\overline{\partial}$-operator with respect to $-J_o$.*

Proof. Let $\langle\cdot,\cdot\rangle_\xi = \omega_\xi \circ (\mathrm{Id} \times J_\xi)$ and analogously $\langle\cdot,\cdot\rangle_\eta$ and $\langle\cdot,\cdot\rangle_o$ be the associated Riemannian metrics on ξ, η and $\mathbb{R}^{2n}$. Considering the complex local trivializations $\Phi_U\colon U \times \mathbb{R}^{2n} \to \xi_{|U}$ and $\Psi_U\colon U \times \mathbb{R}^{2n} \to \eta_{|U}$, the local representation $F_U = \Psi_U^{-1} \circ F \circ \Phi_U$ leads to

$$\begin{aligned}\langle F^* v, w\rangle_\xi &= \langle v, Fw\rangle_\eta = \langle v, \Psi_U F_U \Phi_U^{-1} w\rangle_\eta \\ &= \langle \Psi_U^T v, F_U \Phi_U^{-1} w\rangle_o \\ &= \langle (\Phi_U^{-1})^T F_U^* \Psi_U^T v, w\rangle_\xi\end{aligned}$$

for all $v \in C_o^\infty(\eta)$ and $w \in C_o^\infty(\xi)$ with $\operatorname{supp} v, \operatorname{supp} w \subset U$. Thus, the complex trivializations Φ_U and Ψ_U correspond to Ψ_U^T and $(\Phi_U^{-1})^T$ and the latter are unitary iff Φ_U, Ψ_U are. Hence, it remains to consider the formally adjoint operator F_U^* with respect to the standard structure $\langle\cdot,\cdot\rangle = \omega_o \circ (\mathrm{Id} \times J_o)$. Since it holds

$$(\partial_s + J_o \partial_t + S)^* = -\partial_s + J_o \partial_t + S^T,$$

the formally adjoint operator F^* satisfies the condition of a $\overline{\partial}$-operator if we replace J_o by $-J_o$ which is again an ω_o-compatible almost complex structure. ∎

We now proceed by stating the main result of this section. Since a $\overline{\partial}$-operator F is a first order linear differential operator, we can consider its closure with respect to L^p-spaces. In the last chapter we introduced the Banach manifolds $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ modeled on $H^{1,p}_\Sigma(\mathbb{R}^{2n})$. Therefore we extend F to $H^{1,p}_\Sigma(\xi)$ for $p > 2$. However, the crucial Fredholm property, provided F is admissible, can be proven most easily for $p = 2$ using the Hilbert space structure. Due to elliptic regularity and asymptotic nondegeneracy, we expect index and kernel of F to be independent of $p \geqslant 2$.

3.1.9 Theorem *Let $\xi, \eta \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$ be Hermitian vector bundles of rank $2m$ and F be an admissible $\overline{\partial}$-operator $F\colon C^\infty_\Sigma(\xi) \to C^\infty_\Sigma(\eta)$. Then, the induced operators*

$$F_p\colon H^{1,p}_\Sigma(\xi) \to L^p_\Sigma(\eta)$$

for $p \geqslant 2$ are Fredholm with

$$\ker F_p = \ker F_q, \quad \operatorname{ind} F_p = \operatorname{ind} F_q$$

for all $p, q \geqslant 2$. Moreover, operators $F, F' \in F_\Omega(\Phi, \Psi)$ with the same asymptotic data have the same index, $\operatorname{ind} F = \operatorname{ind} F'$.

Observe that for the special case of a closed model surface $\partial\Sigma = \emptyset$, the Fredholm property immediately follows from the ellipticity, a local property of F, as stated in the last chapter. Then the Fredholm index can be expressed in terms of rank and characteristic classes of the complex bundles. This is derived from the Theorem of Riemann-Roch. Thus, fixing a local trivialization fixes the index. However, in the case of cylindrical ends, we need further analysis of F which, on these ends, is of the form

$$\tfrac{\partial}{\partial s} + A(s)\colon H^{1,p}(\mathbb{R} \times S^1) \to L^p(\mathbb{R} \times S^1).$$

The above assumed regularity provides an asymptotic discrete, real spectrum

$$\sigma(A(\pm\infty)) \subset \mathbb{R} \setminus \{0\}$$

bounded away from 0. A result of Lockhardt and McOwen provides the Fredholm property of F for the model surface Σ with cylindrical ends. This argumentation was given by A. Floer in [17], p.779 and Section 4. The idea is to reduce the proof of the Fredholm property to the invertibility of the asymptotic translation invariant operators

$$F_{\pm\infty} = \tfrac{\partial}{\partial s} + A(\pm\infty).$$

However, the proof that $F_{\pm\infty}\colon H^{1,p}(\mathbb{R}\times S^1) \to L^p(\mathbb{R}\times S^1)$ is invertible for any $p \geqslant 1$ requires rather involved arguments which are given in [36]. Here, we intend to present a complete proof for $p \geqslant 2$. It is based on the local elliptic estimate in Corollary 2.5.3 in a refined version, on the result for $p = 2$, which can be proved in a straight forward manner, and on a lemma of Maz'ya and Plamenewski combining these two results.

The main part of the proof of Theorem 3.1.9 consists in establishing the Semi-Fredholm property. We recall that a bounded linear operator $F \in \mathcal{L}(X;Y)$ is called Semi-Fredholm if its kernel is finite-dimensional and its range is closed. We introduced the generalized notion of a $\overline{\partial}$-operator, so that we may apply Lemma 3.1.8 which implies that the formally adjoint operator F^* is also Semi-Fredholm. Thus, it only remains to relate the cokernel of F to the kernel of F^*, which is done by using the regularity results for weak solutions together with the analysis of the asymptotic decrease of weak solutions of F^*.

We have the following well-known criterion for the Semi-Fredholm property.

3.1.10 Lemma *Let X, Y and Z be Banach spaces and $F \in \mathcal{L}(X;Y)$, $K \in \mathcal{K}(X;Z)$ and $c > 0$ with*

$$\|x\|_X \leqslant c\,(\|Fx\|_Y + \|Kx\|_Z) \tag{3.4}$$

for all $x \in X$. Then F is a Semi-Fredholm operator.

A proof of this standard lemma can be found, for example, in [50], Section 2.2.1. This crucial estimate for the operators F_p, $p \geqslant 2$, is essentially based on the local regularity for $\overline{\partial}$-operators and the asymptotic nondegeneracy of $A_i = J_o\frac{\partial}{\partial t} + S_i$ on the cylindrical ends.

3.1.11 Theorem *For any $p \geqslant 2$ the admissible $\overline{\partial}$-operator $F_p\colon H^{1,p}_\Sigma(\xi) \to L^p_\Sigma(\eta)$ satisfies an estimate*

$$\|x\|_{H^{1,p}} \leqslant c(p)(\|F_p x\|_{L^p} + \|K_p\|_{Z(p)})$$

for all $x \in H^{1,p}_\Sigma(\xi)$, for some constant $c(p) > 0$ and a compact operator $K_p\colon H^{1,p}_\Sigma(\xi) \to Z(p)$ with some Banach space $Z(p)$.

3.1.2 The Asymptotic Operator

In the last subsection we introduced the generalized $\overline{\partial}$-operator for suitable bundles over the model surface. The idea is to describe the operator in question in local conformal coordinates so that it suffices to analyze operators $\partial_s + J_o\partial_t + S(s,t)$. In this subsection we concentrate on the necessary nondegeneracy condition to $A_i(s) = J_o\partial_t + S_i(s,t)$ on the cylindrical end $\psi_i(Z^{\epsilon_i})$ for $\epsilon_i s \to \infty$. More precisely, we consider the so-called asymptotic operators for $i = 1, \dots, \nu$,

$$\begin{aligned} G_i &= \tfrac{\partial}{\partial s} + A_i(\infty)\colon H^{1,p}(\mathbb{R}\times S^1) \to L^p(\mathbb{R}\times S^1),\\ A_i(\infty) &= J_o\tfrac{\partial}{\partial t} + S_i(\epsilon_i\infty), \end{aligned}$$

where the limit $S_i(\epsilon_i\infty) = \lim_{\epsilon_i s\to\infty} S_i(s) \in C^\infty(S^1, \mathcal{L}_\mathbb{R}(\mathbb{C}^n))$ exists, because we always understand smooth maps to be Σ-smooth in the sense of Definition 2.1.2.

The main concern in the proof of the Fredholm theorem shall be an injectivity estimate for all $p \geqslant 2$, that is the existence of a constant $c(p) > 0$ such that

$$\|u\|^{\mathbb{R}\times S^1}_{1,p} \leqslant c(p)\|G_i u\|^{\mathbb{R}\times S^1}_{0,p} \tag{3.5}$$

for all $u \in H^{1,p}(\mathbb{R}\times S^1)$. This is due to the translation invariance of G and the assumption that $S_i(\infty)$ is admissible. It is fairly easy to prove (3.5) for the Hilbert space situation $p = 2$. One possibility is to use Fourier transformation $\mathcal{F}_\mathbb{R}$ with respect to the variable $s \in \mathbb{R}$,

$$(\mathcal{F}_\mathbb{R} u)(\lambda, t) = \hat{u}(\lambda, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda s}\, u(s,t)ds$$

in order to prove that one obtains the inverse$\mathcal{O}$ of $G = \partial_s + A_S$

$$\mathcal{O}v(s,t) = \frac{1}{\sqrt{2\pi}}\int_\mathbb{R} \mathrm{e}^{i\lambda s} R(\lambda)(\hat{v}(\lambda,\cdot))d\lambda.$$

Here $R(\lambda)$ is the resolvent of the self-adjoint operator A_S for $\lambda \in \mathbb{R}$,

$$R(\lambda) = (i\lambda + A_S)^{-1}\colon L^2(S^1) \to H^{1,2}(S^1),$$

so that

$$\mathcal{F}_\mathbb{R} \circ G \circ \mathcal{F}_\mathbb{R}^{-1} = \mathcal{F}_\mathbb{R}(\partial_s + A_S)\mathcal{F}_\mathbb{R}^{-1} = (i\lambda + A_S)\,.$$

A different explicit proof follows below.

However, the proof of the estimate (3.5) is far more complicated for $p \neq 2$. In particular, it involves the nontrivial local estimate for the Cauchy-Riemann operator

$$\|\overline{\partial} u\|_{0,p} \geqslant c(p)\|u\|_{1,p}$$

for all smooth maps $u\colon \mathbb{C} \to \mathbb{C}^n$ with compact support within the unit disk $\{|z| < 1\}$, see Theorem 2.5.2 and Corollary 2.5.3 in Section 2.5. Then, by using arguments due to Maz'ya and Plamenewski, see [36], we can deduce the estimate (3.5) for $p > 2$ from that for $p = 2$.

3.1.12 Proposition *Given an admissible loop $S \in C^\infty(S^1, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$, the translation invariant operator*

$$G = \tfrac{\partial}{\partial s} + A_S \colon H^{1,2}(\mathbb{R} \times S^1, \mathbb{C}^n) \to L^2(\mathbb{R} \times S^1, \mathbb{C}^n)$$

is an isomorphism. In particular, there exists a constant $c(2) > 0$ such that

$$\|u\|_{1,2}^{\mathbb{R}\times S^1} \leqslant c\|Gu\|_{0,2}^{\mathbb{R}\times S^1}$$

for all $u \in H^{1,2}(\mathbb{R} \times S^1, \mathbb{C}^n)$.

Proof. Let $u \in C_o^\infty(\mathbb{R} \times S^1, \mathbb{C}^n)$ and define the function

$$a \colon C^\infty(S^1, \mathbb{C}^n) \to \mathbb{R},$$
$$a(x) = -\frac{1}{2}\langle Ax, x\rangle_{L^2(S^1)},$$

where $A \colon H^{1,2}(S^1) \to L^2(S^1)$ is the self-adjoint operator $A = J_o\frac{\partial}{\partial t} + S$ from Lemma 3.1.2. Viewing u as a map $\mathbb{R} \to C^\infty(S^1, \mathbb{C}^n)$ we compute

$$\begin{aligned}\frac{d}{ds}(a \circ u) &= -\frac{1}{2}\langle A\partial_s u, u\rangle_{L^2(S^1)} - \frac{1}{2}\langle Au, \partial_s u\rangle_{L^2(S^1)}\\ &= -\langle \partial_s u, Au\rangle_{L^2(S^1)} = \langle Au - Gu, Au\rangle_{L^2(S^1)},\end{aligned}$$

using the selfadjointness $A = A^*$. Integrating the last identity with respect to s over $\mathbb{R}$ thus yields for functions u with compact support

$$0 = \|Au\|^2_{L^2(\mathbb{R}\times S^1)} - \langle Gu, Au\rangle_{L^2(\mathbb{R}\times S^1)}$$

and therefore

$$\|Au\|_{L^2(\mathbb{R}\times S^1)} \leqslant \|Gu\|_{L^2(\mathbb{R}\times S^1)} \tag{3.6}$$

Now we use the fact, proven in Lemma 3.1.2, that A is an isomorphism defined on $H^{1,2}(S^1, \mathbb{C}^n)$, so that there exists a constant $c > 0$ such that

$$\|x\|_{H^{1,2}(S^1)} \leqslant c\|Ax\|_{L^2(S^1)}$$

for all $x \in H^{1,2}(S^1)$. Integrating over $\mathbb{R}$ amounts to

$$\|u\|^2_{L^2(\mathbb{R}\times S^1)} + \|\partial_t u\|^2_{L^2(\mathbb{R}\times S^1)} \leqslant c^2\|Au\|^2_{L^2(\mathbb{R}\times S^1)}\,. \tag{3.7}$$

Hence, combining

$$\|\partial_s u\|_{L^2(\mathbb{R}\times S^1)} \leqslant \|Gu\|_{L^2(\mathbb{R}\times S^1)} + \|Au\|_{L^2(\mathbb{R}\times S^1)}$$

with the estimate (3.6) leads to

$$\|\partial_s u\|_{L^2(\mathbb{R}\times S^1)} \leqslant 2\|Gu\|_{L^2(\mathbb{R}\times S^1)}$$

and in view of (3.7) to the asserted inequality

$$\|u\|^2_{H^{1,2}(\mathbb{R}\times S^1)} \leqslant (4 + c^2)\|Gu\|^2_{L^2(\mathbb{R}\times S^1)}\,. \tag{3.8}$$

It remains to prove the surjectivity of G. Since the range $\mathcal{R}(G)$ is closed due to (3.8), it suffices to prove the denseness. We consider an orthonormal basis $(e_k)_{k\in\mathbb{Z}}$ for $L^2(S^1, \mathbb{C}^n)$ of eigenvectors of the selfadjoint operator A. The space

$$X = \{\sum_{|k|\leqslant N} a_k e_k \mid a_k \in C_o^\infty(\mathbb{R}),\ N \in \mathbb{N}\}$$

lies dense in $L^2(\mathbb{R} \times S^1)$. Hence we have to prove that

$$Gu = v$$

is solvable with $u \in H^{1,2}(\mathbb{R} \times S^1, \mathbb{C}^n)$ for all $v \in X$. Setting $v = \sum_{|k|\leqslant N} a_k e_k$ we choose the Ansatz $u = \sum_{|k|\leqslant N} u_k e_k$ which leads to the ordinary differential equations

$$\dot{u}_k + \lambda_k u_k = a_k, \quad |k| \leqslant N,$$

where $\lambda_k \neq 0$ because S is admissible. Thus, picking solutions u_k of the initial value problems with $u_k(T) = 0$ for $\lambda_k > 0$ and $u_k(-T) = 0$ for $\lambda_k < 0$ where $(-T, T)$ contains the support of a_k, we see that the exponential decrease on the respective other side,

$$u_k(s) = u_k^o\, e^{\lambda_k s} \quad \text{for } \pm s \text{ large}$$

guarantees that $u = \sum_{|k|\leqslant N} u_k e_k$ belongs to $H^{1,2}(\mathbb{R} \times S^1, \mathbb{C}^n)$. ■

Our aim is to prove the isomorphism result for the translation invariant operator G also for $p > 2$. One can prove this result for all $p > 1$, for our purpose it is enough to have it for $p \geqslant 2$.

3.1.13 Theorem *Given an admissible loop $S \in C^\infty(S^1, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$, the operator*

$$G = \tfrac{\partial}{\partial s} + A_S \colon H^{1,p}(\mathbb{R} \times S^1, \mathbb{C}^n) \to L^p(\mathbb{R} \times S^1, \mathbb{C}^n)$$

is an isomorphism for all $p \geqslant 2$. If $u \in L^p(\mathbb{R} \times S^1, \mathbb{C}^n)$ is a weak solution of $Gu = v$ with $v \in L^p(\mathbb{R} \times S^1, \mathbb{C}^n)$ then $u \in H^{1,p}(\mathbb{R} \times S^1)$, and there exists a constant $c(p) > 0$ such that

$$\|u\|_{1,p}^{\mathbb{R}\times S^1} \leqslant c(p)\|Gu\|_{0,p}^{\mathbb{R}\times S^1}$$

for all $u \in H^{1,p}(\mathbb{R} \times S^1, \mathbb{C}^n)$.

Proof. It suffices to prove the estimate for $u \in H^{1,p}(\mathbb{R} \times S^1)$. First, from local regularity, see Corollary 2.5.3, it is clear that $u \in H^{1,p}_{\mathrm{loc}}$ if $u, Gu \in L^p$. Then we consider the sequence $\beta_n u \in H^{1,p}(\mathbb{R} \times S^1)$, where $\beta \in C_o^\infty((-1,1),[0,1])$ is a fixed cut-off function with $\beta_{|[-\frac{1}{2},\frac{1}{2}]} \equiv 1$ and $\beta_n(s) = \beta(\frac{s}{n})$. Then, due to

$$\begin{aligned}\big\|(\beta_m - \beta_n)u\big\|_{1,p} \leqslant c(p)\Big(&\big\|(\frac{1}{m}\partial_s\beta(\frac{\cdot}{m}) - \frac{1}{n}\partial_s\beta(\frac{\cdot}{n}))u\big\|_{0,p}\\ &+ \big\|(\beta_m - \beta_n)Gu\big\|_{0,p}\Big),\end{aligned}$$

$u, Gu \in L^p(\mathbb{R}\times S^1)$ together with Lebesgue's dominated convergence implies that $(\beta_n u)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^{1,p}(\mathbb{R}\times S^1)$ and

$$\beta_n u \to u \quad \text{in } H^{1,p}(\mathbb{R}\times S^1)\,.$$

The idea of the proof of the estimate relies on a combination of the global L^2-estimate from Proposition 3.1.12 and the local elliptic L^p-estimate for the Cauchy-Riemann operator. We shall restrict the inverse operator G^{-1} to the following dense subspace,

$$\mathcal{O}\colon L^2 \supset \{\, f\in L^p(\mathbb{R}\times S^1) \mid \operatorname{supp} f \text{ compact}\,\} \to L^p(\mathbb{R}\times S^1),$$

using the Sobolev embedding $H^{1,2}\hookrightarrow L^p$.

3.1.2.1 The Lemmata of Maz'ya and Plamenewski

Let us specify a partition of unity subordinate to the covering $\big([k-1,k+1]\big)_{k\in\mathbb{Z}}$ of $\mathbb{R}$. Choosing smooth, monotone cut-off functions $\beta^\pm\colon \mathbb{R}\to[0,1]$ with

$$\beta^+(s) = \begin{cases} 0, & s\geqslant 1 \\ 1, & s\leqslant \frac12 \end{cases}, \quad \beta^-(s)=\beta^+(-s) \quad \text{and } |\tfrac{d}{ds}\beta^\pm| \leqslant 3, \tag{3.9}$$

we set $\beta = \beta^+\cdot\beta^-$, so that $\operatorname{supp}\beta\subset[-1,1]$. We consider the translated functions $\beta_k(s)=\beta(s-k)$. Consequently, $\bigcup_{k\in\mathbb{Z}}\operatorname{supp}\beta_k$ covers $\mathbb{R}$ and

$$\psi_k = \frac{\beta_k}{\sum\limits_{i\in\mathbb{Z}}\beta_i}, \quad k\in\mathbb{Z},$$

represents a partition of unity as demanded, with

$$\tfrac12\beta_k\leqslant\psi_k\leqslant\beta_k \quad\text{and}\quad |\psi_k'|\leqslant 3,$$

because $\psi_{k|[k,k+\frac12]} = (1+\beta_{k+1})^{-1}$ and $\psi_{k|[k+\frac12,k+1]} = \beta_k(1+\beta_k)^{-1}$.

We use the shorthand notations for the Banach spaces $\mathcal{E}_0 = L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ with $\|u\|_0 = \|u\|_{0,p}^{\mathbb{R}\times S^1}$ and $\mathcal{E}_1 = H^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ with $\|u\|_1 = \|u\|_{1,p}^{\mathbb{R}\times S^1}$, where we have the fundamental estimates

$$\frac12\|u\|_0 \leqslant \Big(\sum_{k\in\mathbb{Z}}\|\psi_k u\|_0^p\Big)^{\frac1p} \leqslant 2^{\frac1p}\|u\|_0 \tag{3.10}$$

and

$$\|u\|_1 \leqslant c_o\Big(\sum_{k\in\mathbb{Z}}\|\psi_k u\|_1^p\Big)^{\frac1p} \leqslant c_o'\|u\|_1\,. \tag{3.11}$$

The first follows immediately from

$$\frac{1}{2^p}\int\limits_{[k-\frac12,k+\frac12]\times S^1}|u|^p dsdt \leqslant \|\psi_k u\|_0^p \leqslant \int\limits_{[k-1,k+1]\times S^1}|u|^p dsdt\,.$$

The second is deduced by computing

$$\begin{aligned}
&\frac{1}{2^p}\int\limits_{[k-\frac12,k+\frac12]\times S^1}\big(|u|^p + |\tfrac{\partial}{\partial t}u|^p + |\tfrac{\partial}{\partial s}u|^p\big)dsdt \\
&\leqslant \int\limits_{[k-1,k+1]\times S^1}\big(|\psi_k u|^p + |\psi_k\tfrac{\partial}{\partial t}u|^p + |\tfrac{\partial}{\partial s}(\psi_k u) - \tfrac{\partial\psi_k}{\partial s}u|^p\big)dsdt \\
&\leqslant c\Big(\|\psi_k u\|_1^p + \int\limits_{[k-\frac12,k+\frac12]\times S^1}|u|^p dsdt\Big),
\end{aligned}$$

so that

$$\begin{aligned}
\|u\|_1^p &\leqslant c\Big(\sum_{k\in\mathbb{Z}}\|\psi_k u\|_1^p + \|u\|_0^p\Big) \\
&\overset{(3.10)}{\leqslant} c_o^p\sum_{k\in\mathbb{Z}}\|\psi_k u\|_1^p\,.
\end{aligned}$$

The estimate $\sum_{k\in\mathbb{Z}}\|\psi_k u\|_1^p\leqslant \text{const}\,\|u\|_1^p$ follows with $|\psi_k'|\leqslant 3$ analogously to (3.10).

3.1.14 Definition *We call a linear operator*

$$\mathcal{O}\colon \{\, f\in\mathcal{E}_0 \mid \operatorname{supp} f \text{ compact}\,\}\to\mathcal{E}_0$$

admissible *if there exist $c_1,\epsilon>0$ such that*

$$\|\psi_k\mathcal{O}\psi_m v\|_0 \leqslant c_1\,\mathrm{e}^{-|m-k|\epsilon}\,\|\psi_m v\|_0$$

holds for all $v\in\mathcal{E}_0$ and $m,k\in\mathbb{Z}$.

This property leads to the following estimate.

3.1.15 Lemma *Given an admissible operator $\mathcal{O}$ there exists a constant $c_2>0$ such that*

$$\|\mathcal{O}v\|_0\leqslant c_2\|v\|_0$$

for all $v\in\mathcal{E}_0$ with compact support.

Proof. We compute for the admissible operator

$$\begin{aligned}
\|\mathcal{O}v\|_0 &= \Big\|\mathcal{O}\Big(\sum_{m\in\mathbb{Z}}\psi_m v\Big)\Big\|_0 \overset{(3.10)}{\leqslant} 2\Big(\sum_{k\in\mathbb{Z}}\|\psi_k\mathcal{O}\Big(\sum_{m\in\mathbb{Z}}\psi_m v\Big)\|_0^p\Big)^{\frac1p} \\
&\leqslant 2\Big(\sum_k\Big(\sum_m\|\psi_k\mathcal{O}\psi_m v\|_0\Big)^p\Big)^{\frac1p} \leqslant 2c_1\Big(\sum_k\Big(\sum_m \mathrm{e}^{-|m-k|\epsilon}\,\|\psi_m v\|_0\Big)^p\Big)^{\frac1p}.
\end{aligned}$$

Since we obtain a continuous mapping $l^p\to l^p$, $g\mapsto f*g$ from the convolution

$$(f*g)(k) = \sum_{m\in\mathbb{Z}} f(k-m)g(m), \qquad f(l)=\mathrm{e}^{-|l|\epsilon},$$

with $\|f * g\|_{l^p} \leqslant \|f\|_{l^1}\|g\|_{l^p}$, we can set $g(m) = \|\psi_m v\|_0$, that is $g \in l^p$ due to (3.10), and we conclude by

$$\|\mathcal{O}v\|_0 \leqslant c\|f * g\|_{l^p} \leqslant c(\epsilon)\|g\|_{l^p} = c\Big(\sum_{m\in\mathbb{Z}} \|\psi_i v\|_0^p\Big)^{\frac{1}{p}} \leqslant c\|v\|_0 .$$

Here we used c as a generic constant. ∎

3.1.16 Lemma *Let $\sigma_k = \psi_{k-1} + \psi_k + \psi_{k+1}$ for $k \in \mathbb{Z}$ and assume that there exists a constant $c > 0$ independent of $k \in \mathbb{Z}$ such that*

$$\|\psi_k \mathcal{O}v\|_1 \leqslant c(\|\sigma_k v\|_0 + \|\sigma_k \mathcal{O}v\|_0)$$

for all $v \in \mathcal{E}_0$ with compact support. If the operator $\mathcal{O}$ is admissible, its range lies in $\mathcal{E}_1$ and there exists a constant $c_3 > 0$ such that

$$\|\mathcal{O}v\|_1 \leqslant c_3\|v\|_0$$

for all $v \in \mathcal{E}_0$.

Proof. Since the functions $v \in \mathcal{E}_0 = L^p(\mathbb{R}\times S^1, \mathbb{C}^n)$ with compact support lie dense in $\mathcal{E}_0$, it suffices to prove the c_3-estimate for those functions. Combining all assumptions on $\mathcal{O}$ with the result of Lemma 3.1.15 we compute with c denoting again a generic constant, that

$$\begin{aligned}\|\mathcal{O}v\|_1 &\overset{(3.11)}{\leqslant} c\Big(\sum_k \|\psi_k \mathcal{O}v\|_1^p\Big)^{\frac{1}{p}} \leqslant c\Big(\sum_k (\|\sigma_k v\|_0 + \|\sigma_k \mathcal{O}v\|_0)^p\Big)^{\frac{1}{p}} \\ &\leqslant c\Big(\Big(\sum_k \|\psi_k v\|_0^p\Big)^{\frac{1}{p}} + \Big(\sum_k \|\psi_k \mathcal{O}v\|_0^p\Big)^{\frac{1}{p}}\Big) \\ &\overset{(3.10)}{\leqslant} c(\|v\|_0 + \|\mathcal{O}v\|_0) \leqslant c\|v\|_0,\end{aligned}$$

for all $v \in \mathcal{E}_0$ with compact support. The last estimate is due to Lemma 3.1.15. ∎

The last lemmata reduce the problem of finding the necessary L^p-estimate for the operator $\mathcal{O} = G^{-1}$ to the local elliptic estimate and to the question whether $\mathcal{O}$ is admissible in the sense of Definition 3.1.14.

First, we observe that in case of $p \geqslant 2$ there is a uniform constant $c_4 > 0$ such that the Sobolev embedding

$$\|u\|_{L^p(\mathbb{R}\times S^1)} \leqslant c_4\|u\|_{H^{1,2}(\mathbb{R}\times S^1)} \tag{3.12}$$

holds for all $u \in H^{1,2}(\mathbb{R}\times S^1)$. The standard Sobolev embedding result provides us with

$$\|\psi_k u\|_{0,p} \leqslant c_o\|\psi_k u\|_{1,2}$$

with $c_o > 0$ independent of $k \in \mathbb{Z}$ and $u \in H^{1,2}(\mathbb{R}\times S^1)$. Using the estimates (3.10) and (3.11) for the partition of unity $(\psi_k)_{k\in\mathbb{Z}}$ we find

$$\begin{aligned}\|u\|_{0,p} &\overset{(3.10)}{\leqslant} c\Big(\sum_k \|\psi_k u\|_{0,p}^p\Big)^{\frac{1}{p}} \\ &\leqslant c_o c\Big(\sum_k \|\psi_k u\|_{1,2}^p\Big)^{\frac{1}{p}} \\ &\leqslant \bar{c}\big\|(\|\psi_k u\|_{1,2})\big\|_{l^p} \\ &\leqslant \bar{c}\big\|(\|\psi_k u\|_{1,2})\big\|_{l^2}, \quad \text{because } p \geqslant 2 \\ &= \bar{c}\Big(\sum_k \|\psi_k u\|_{1,2}^2\Big)^{\frac{1}{2}} \\ &\overset{(3.11)}{\leqslant} c_4\|u\|_{1,2} .\end{aligned}$$

From (3.12) we deduce that

$$\mathcal{O}\colon \{\, v \in \mathcal{E}_0 \,|\, v \text{ compact support} \,\} \to \mathcal{E}_0$$

is well-defined. We have to provide the local estimate in the assumption of Lemma 3.1.16 and the admissibility of $\mathcal{O}$ in the sense of Definition 3.1.14. We reformulate the local elliptic estimate from Section 2.5. Let Ω be a bounded domain in $\mathbb{R}\times S^1$ and $S\colon \overline{\Omega} \to \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n)$ a continuous map.

3.1.17 Proposition *Given compact subsets Q, Q' of Ω with*

$$Q \subset \mathring{Q}' \subset Q' \subset \Omega$$

there exists a constant $c = c(Q, Q', S)$ such that

$$\|u\|_{1,p}^{Q} \leqslant c\,(\|\overline{\partial}u + Su\|_{0,p}^{Q'} + \|u\|_{0,p}^{Q'})$$

for all smooth maps $u\colon \Omega \to \mathbb{C}^n$.

Now, consider $v \in \mathcal{E}_0$ with compact support, so that $\mathcal{O}v \in \mathcal{E}_0$. Then, we choose a sequence $v_n \in C_o^\infty$ converging towards v in $L^p(\mathbb{R}\times S^1)$. Since the support may assumed to be uniformly bounded this sequence converges to v also in $L^2(\mathbb{R}\times S^1)$ and thus

$$\mathcal{O}v_n \xrightarrow{H^{1,2}} \mathcal{O}v .$$

In view of (3.12), we obtain that $\mathcal{O}v_n$ converges towards $\mathcal{O}v$ in $L^p(\mathbb{R}\times S^1)$. Therefore, by Proposition 3.1.17, there is a constant $c_5 > 0$ independent of $k \in \mathbb{Z}$, because G is translation invariant, so that

$$\|\psi_k \mathcal{O}v\|_{1,p} \leqslant c_5(\|\sigma_k v\|_{0,p} + \|\sigma_k \mathcal{O}v\|_{0,p})$$

for all $v \in \mathcal{E}_0 = L^p(\mathbb{R}\times S^1)$ with compact support. In particular, we have

$$\|u\|_{1,2} \leqslant c_5(\|Gu\|_{0,2} + \|u\|_{0,2}) \tag{3.13}$$

for all $u \in H^{1,2}(\mathbb{R} \times S^1)$ with $\operatorname{supp} u \subset [k-1, k+1] \times S^1$, $k \in \mathbb{Z}$.

It remains to verify that the operator

$$\mathcal{O}\colon L^p_{\mathrm{loc}}(\mathbb{R} \times S^1) \to H^{1,p}_{\mathrm{loc}}(\mathbb{R} \times S^1)$$

is admissible.

3.1.18 Lemma *There are uniform constants $\epsilon, c_1 > 0$ such that*

$$\|\psi_k \mathcal{O} \psi_m v\|_{0,p} \leqslant c\, \mathrm{e}^{-|m-k|\epsilon}\, \|\psi_m v\|_{0,p}$$

for all $m, k \in \mathbb{Z}$, $v \in L^p(\mathbb{R} \times S^1)$, that is, the operator $\mathcal{O} = G^{-1}$ is admissible.

PROOF. Combining the above estimates (3.12) and (3.13) we obtain

$$\begin{aligned}\|\psi_k \mathcal{O}\psi_m v\|_{0,p} &\leqslant c(\|G(\psi_k \mathcal{O}\psi_m v)\|_{0,2})\\ &\leqslant c(\|(\tfrac{\partial}{\partial s}\psi_k)\mathcal{O}\psi_m v\|_{0,2} + \|\psi_k\psi_m v\|_{0,2} + \|\psi_k \mathcal{O}\psi_m v\|_{0,2})\\ &\leqslant c(\|\sigma_k \mathcal{O}\psi_m v\|_{0,2} + \|\psi_k\psi_m v\|_{0,2})\end{aligned}$$

for a generic constant $c > 0$ independent from $k, m \in \mathbb{Z}$. Since for each $\epsilon > 0$ there is a $c(\epsilon) > 0$ such that

$$\sup_{[m-1,m+1]} |\psi_k| \leqslant c(\epsilon)\, \mathrm{e}^{-|m-k|\epsilon} \quad \text{for all } m, k \in \mathbb{Z},$$

it suffices to prove the estimate

$$\|\sigma_k \mathcal{O}\psi_m v\|_{0,2} \leqslant c(\epsilon)\, \mathrm{e}^{-|m-k|\epsilon}\, \|\psi_m v\|_{0,2} \tag{3.14}$$

for constants $c(\epsilon), \epsilon > 0$ independent from $m, k \in \mathbb{Z}$. Then we use that $p \geqslant 2$, so that

$$\|\psi_m v\|_{0,2} \leqslant c' \|\psi_m v\|_{0,p}$$

for all $m \in \mathbb{Z}$, $v \in L^p$, with a constant $c' = c'(|\operatorname{supp}\psi_m|, p)$, that is independent of $m \in \mathbb{Z}$. Let us define now the exponentially weighted norms for any $\delta > 0$

$$\|u\|_{k,2;\delta} = \|\,\mathrm{e}^{\delta s}\, u\|_{k,2}, \quad k = 1, 2\,.$$

with $H^{k,2}_\delta(\mathbb{R} \times S^1) = \{u \in H^{k,2}_{\mathrm{loc}} \,|\, \|u\|_{k,2;\delta} < \infty\}$. Then the isomorphism $G\colon H^{1,2} \xrightarrow{\cong} L^2$ from Proposition 3.1.12 applies to $u \in H^{1,2}_\delta$ and we obtain

$$\begin{aligned}\|u\|_{0,2;\delta} \leqslant \|u\|_{1,2;\delta} &= \|\,\mathrm{e}^{\delta s} u\|_{1,2}\\ &\leqslant c_2 \|G(\mathrm{e}^{\delta s} u)\|_{0,2} = c_2 \|\delta\, \mathrm{e}^{\delta s}\, u + \mathrm{e}^{\delta s}\, Gu\|_{0,2}\\ &\leqslant c_2 |\delta| \|u\|_{0,2;\delta} + c_2 \|Gu\|_{0,2;\delta}\,.\end{aligned}$$

Hence, for $|\delta| < \delta_o = \frac{1}{c_2}$,

$$\|u\|_{0,2;\delta} \leqslant c(\delta) \|Gu\|_{0,2;\delta} \tag{3.15}$$

for all $u \in H^{1,2}_{\mathrm{loc}}(\mathbb{R} \times S^1)$ with $\|u\|_{1,2;\delta} < \infty$ and $c(\delta) = \frac{c_2}{1 - c_2|\delta|}$.

Now, let $u = \mathcal{O}\psi_m v$, $v \in L^p(\mathbb{R} \times S^1)$, so that $\operatorname{supp} Gu \subset [m-1, m+1] \times S^1$ and thus $u \in H^{1,2}_\alpha(\mathbb{R} \times S^1)$ for all $|\alpha| < \delta_o$ in view of (3.15). We compute

$$\begin{aligned}\|\sigma_k \mathcal{O}\psi_m v\|^2_{0,2} &\leqslant \int_{k-2}^{k+2} \|u\|^2_{L^2(S^1)} ds = \int_{k-2}^{k+2} \mathrm{e}^{-2\alpha s}\, \|\,\mathrm{e}^{\alpha s}\, u\|^2_{L^2(S^1)} ds\\ &\leqslant \mathrm{e}^{-2\alpha(k \pm 2)}\, \|u\|^2_{0,2;\alpha} \quad (\pm \text{ depending on } \operatorname{sgn}\alpha)\\ &\overset{(3.15)}{\leqslant} \tilde{c}(\epsilon)\, \mathrm{e}^{-2\alpha k}\, \|Gu\|^2_{0,2;\alpha}, \quad \text{for} \quad |\alpha| \leqslant \epsilon < \delta_o,\\ &= \tilde{c}(\epsilon)\, \mathrm{e}^{-2\alpha k} \int_{m-1}^{m+1} \mathrm{e}^{2\alpha s}\, \|Gu\|^2_{L^2(S^1)} ds,\end{aligned}$$

because $\operatorname{supp}_{\mathbb{R}} Gu \subset [m-1, m+1]$,

$$\leqslant c(\epsilon)\, \mathrm{e}^{2\alpha(m-k)}\, \|Gu\|^2_{0,2} = c(\epsilon)\, \mathrm{e}^{2\alpha(m-k)}\, \|\psi_m v\|^2_{0,2}.$$

Thus, setting

$$\alpha = \alpha(m,k) = \begin{cases} \epsilon, & m < k,\\ -\epsilon, & m \geqslant k,\end{cases}$$

implies the proof of (3.14). ∎

Summing up, Lemma 3.1.18 provides the estimate for the operator $\mathcal{O} = G^{-1}$,

$$\begin{aligned}&\mathcal{O}\colon L^p(\mathbb{R} \times S^1) \to H^{1,p}(\mathbb{R} \times S^1),\\ &\|\mathcal{O}v\|_{1,p} \leqslant c\|v\|_{0,p}\,.\end{aligned}$$

Thus, Theorem 3.1.13 is proven. ∎

3.1.3 The Proof of the Fredholm Property

We have analyzed the asymptotic, translation-invariant situation on the cylindrical ends. Let us now return to the discussion of the admissible $\overline{\partial}$-operator F_p on the model surface Σ. We denote by $F_i = F_{|Z_i}$ the restriction to the cylindrical end $Z_i = \psi_i(Z^{\epsilon_i})$ with respect to the cylindrical coordinates,

$$\begin{aligned}F_i\colon H^{1,p}(Z^{\epsilon_i}, \mathbb{R}^{2n}) &\to L^p(Z^{\epsilon_i}, \mathbb{R}^{2n}),\\ v &\mapsto \tfrac{\partial}{\partial s} v + J_o \tfrac{\partial}{\partial t} v + S_i v,\end{aligned}$$

Let us consider the fixed cylindrical end Z_i, where we assume without loss of generality that $\epsilon_i = +1$. We denote $Z_T = [T, \infty) \times S^1$ for $T \geqslant 0$ and we obtain the following

3.1.19 Proposition *If S_i is admissible there is a $T \geqslant 0$ and a $c = c(T) > 0$ such that*

$$\|v\|_{H^{1,p}(Z_T)} \leqslant c(T) \|F_i v\|_{L^p(Z_T)}$$

for all $v \in H^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ with $v_{|(-\infty,T] \times S^1} = 0$.

PROOF. Given $\epsilon > 0$, we choose $T_\epsilon \geqslant 0$ large enough, such that

$$\left|S^+(t) - S(s,t)\right| \leqslant \epsilon \quad \text{f.a. } s \geqslant T_\epsilon,\ t \in S^1,$$

where $S_i(s,\cdot) \to S^+(t)$ uniformly in $C^\infty(S^1)$ for $s \to \infty$. Setting

$$G = \tfrac{\partial}{\partial s} + J_o \tfrac{\partial}{\partial t} + S^+ \colon H^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n})$$

we obtain

$$\begin{aligned}\|Gv\|_{L^p(\mathbb{R}\times S^1)} &\leqslant \|F_i v\|_{L^p} + \|(F_i - G)v\|_{L^p}\\ &= \|F_i v\|_{L^p(Z_T)} + \|(S_i - S^+)v\|_{L^p(Z_T)}\\ &\leqslant \|F_i v\|_{L^p(Z_T)} + \epsilon\|v\|_{L^p(Z_T)}\,.\end{aligned}$$

Theorem 3.1.13 now provides the estimate with $c = c(p, S^+)$

$$\begin{aligned}\|v\|_{H^{1,p}(Z_T)} = \|v\|_{H^{1,p}(Z)} &\leqslant c\|Gv\|_{L^p(Z)}\\ &\leqslant c\big(\|F_i v\|_{L^p(Z_T)} + \epsilon\|v\|_{H^{1,p}(Z_T)}\big).\end{aligned}$$

Hence, for $0 < \epsilon < \frac{1}{c}$ and $T \geqslant T(\epsilon) > 0$ large enough we deduce

$$\|v\|_{H^{1,p}(Z_T)} \leqslant \frac{c}{1 - c\epsilon}\|F_i v\|_{L^p(Z_T)}$$

for all $v \in H^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ with $v_{|Z\setminus Z_T} \equiv 0$. ■

We transfer this result to our model surface Σ and the $\overline{\partial}$-operator

$$F_p \colon H^{1,p}_\Sigma(\xi) \to L^p_\Sigma(\eta)\,.$$

We first define the compact subset Σ_T of Σ with cylindrical ends of finite length by

$$\Sigma_T = \overline{\left(\Sigma \setminus \textstyle\bigcup_{i=1}^{\nu} \psi_i(Z^{\epsilon_i}_{T_i})\right)},$$

for $T = (T_1, \dots, T_\nu) \in \mathbb{R}^\nu_+$.

3.1.20 Corollary *Let $F \in F_\Omega(\Phi,\Psi)$ be admissible and $p \geqslant 2$. Then there exists a $T \in \mathbb{R}^\nu_+$ and a constant $c = c(p, T, \Omega) > 0$ such that*

$$\|v\|_{H^{1,p}_\Sigma(\xi)} \leqslant c\,\|F_p v\|_{L^p_\Sigma(\eta)}.$$

for all $v \in H^{1,p}_\Sigma(\xi)$ with $v_{|\Sigma_T} = 0$.

PROOF. This is a straightforward consequence of Proposition 3.1.19 and we set $c(T) = \max(c(T_1), \dots, c(T_\nu))$. ■

We now are able to prove Theorem 3.1.9, that is the Semi-Fredholm property. This will be achieved by combining the asymptotic estimate from Corollary 3.1.20 with the elliptic estimate of local regularity (2.11).

PROOF OF THEOREM 3.1.9. Let $T = (T_1, \dots, T_\nu) \in \mathbb{R}^\nu_+$ as provided by Corollary 3.1.20. We choose monotone cut-off functions $\beta_i \in C^\infty(\overline{\Sigma}, [0,1])$, $i = 1, \dots, \nu$, satisfying

$$\beta_{j|Z^{\epsilon_i}_{T_i+1}} = \delta_{ij} \quad \text{and} \quad \beta_{i|\Sigma\setminus Z^{\epsilon_i}_{T_i}} = 0$$

for $i, j = 1, \dots, \nu$, where we identify $Z^{\epsilon_i}_T$ with $\psi_i(Z^{\epsilon_i}_T) \subset \Sigma$. Thus, applying Corollary 3.1.20 to $\beta_i v$ for $v \in H^{1,p}_\Sigma(\xi)$, we obtain

$$\begin{aligned}\|\beta_i v\|_{H^{1,p}_\Sigma(\xi)} &\leqslant c(T)\|F_p(\beta_i v)\|_{L^p_\Sigma(\eta)}\\ &\leqslant c(T)\Big(\|\beta_i F_p v\|^{Z^{\epsilon_i}_{T_i}}_{0,p} + c(\beta_i)\||d\beta_i||v|\|^{Z^{\epsilon_i}_{T_i}\setminus Z^{\epsilon_i}_{T_{i+1}}}_{0,p}\Big)\\ &\leqslant c_1\big(\|\beta_i F_p v\|^{Z^{\epsilon_i}_{T_i}}_{0,p} + \|v\|^{Z^{\epsilon_i}_{T_i}\setminus Z^{\epsilon_i}_{T_{i+1}}}_{0,p}\big)\end{aligned}$$

Considering $\hat\beta = 1 - \sum_{i+1}^{\nu}\beta_i$, i.e. $\hat\beta_{|\Sigma_T} \equiv 1$, $\hat\beta_{\Sigma\setminus\Sigma_{T+1}} \equiv 0$ for $T + 1 = (T_1 + 1, \dots, T_\nu + 1)$, we now combine all results from the cylindrical ends, yielding

$$\begin{aligned}\|v\|_{H^{1,p}_\Sigma(\xi)} &\leqslant \|\hat\beta v\|^{\Sigma}_{1,p} + \sum_{i=1}^{\nu}\|\beta_i v\|^{\Sigma}_{1,p}\\ &\leqslant \|\hat\beta v\|^{\Sigma_{T+1}}_{1,p} + c_1\sum_{i=1}^{\nu}\big(\|\beta_i F_p v\|^{Z^{\epsilon_i}_{T_i}}_{0,p} + \|v\|^{Z^{\epsilon_i}_{T_i}\setminus Z^{\epsilon_i}_{T_{i+1}}}_{0,p}\big)\\ &\leqslant \|\hat\beta v\|^{\Sigma_{T+1}}_{1,p} + c_1\big(\|(1-\hat\beta)F_p v\|^{\Sigma}_{0,p} + \|v\|^{\Sigma_{T+1}}_{0,p}\big)\,.\end{aligned}$$

Applying the local estimate (2.11) of interior regularity to the elliptic $\overline{\partial}$-operator F_p on the compact subset Σ_{T+1} we obtain

$$\begin{aligned}\|\hat\beta v\|^{\Sigma_{T+1}}_{1,p} &\leqslant c_2\big(\|F_p(\hat\beta v)\|^{\Sigma_{T+1}}_{0,p} + \|\hat\beta v\|^{\Sigma_{T+1}}_{0,p}\big)\\ &\leqslant c_3\big(\|\hat\beta F_p v\|^{\Sigma_{T+1}}_{0,p} + \|v\|^{\Sigma_{T+1}}_{0,p}\big).\end{aligned}$$

Altogether we compute

$$\|v\|_{H^{1,p}_\Sigma(\xi)} \leqslant c\big(\|F_p v\|_{L^p_\Sigma(\eta)} + \|v_{|\Sigma_{T+1}}\|_{L^p(\xi_{|\Sigma_{T+1}})}\big).$$

The second summand can be expressed in terms of the operator

$$K_p \colon H^{1,p}_\Sigma(\xi) \to L^p(\xi_{|\Sigma_{T+1}}), \quad v \mapsto v_{|\Sigma_{T+1}},$$

which is compact for $p \geqslant 1$ by Rellich's embedding. We recall that the Banach space topology of $L^p(\xi_{|\Sigma_{T+1}})$ is uniquely defined, although the norm $\|\cdot\|_{L^p(\xi_{|\Sigma_{T+1}})}$ depends on the chosen local trivialization. But it only changes by a constant after a change of the trivialization, because Σ_{T+1} is compact. Hence, after setting $Z(p) = L^p(\xi_{|\Sigma_{T+1}})$, the proof of Theorem 3.1.9 is complete ■

In order to complete the proof of the Fredholm property of F_p for $p \geqslant 2$ we first observe that we may simplify the operator F so that its asymptotic properties with respect to the cylindrical ends are easier to handle. Namely, we are allowed to deform the operator continuously as long as it remains Semi-Fredholm. For example, we deform F inside its class $F_\Omega(\xi,\eta)$.

3.1.21 Lemma *The equivalence class $F_\Omega(\xi,\eta) \subset \mathcal{L}(H^{1,p}_\Sigma(\xi); L^p_\Sigma(\eta))$ of admissible $\overline{\partial}$-operators with fixed asymptotic data Ω is path-connected within the space of bounded linear operators.*

PROOF. Given complex local trivializations Φ_U and Ψ_U, we may deform the operator $F \in F_\Omega(\xi,\eta)$ by continuous deformations of S_U where $\Psi_U^{-1} \circ F \circ \Phi_U = \partial_s + J_o\partial_t + S_U$ in the local representation. Fixing the asymptotic data Ω allows homotopy inside $F_\Omega(\xi,\eta)$. ∎

Thus we are able to deform $F_t \in F_\Omega(\xi,\eta)$, $t \in [0,1]$ so that, due to the fixed admissible asymptotic data, we remain in the class of Semi-Fredholm operators. This allows us to use Kato's Perturbation Theorem.

3.1.22 Proposition *The equivalence class F_Ω lies entirely within a path component of Fredholm operators $K \subset \mathcal{F}(H^{1,p}_\Sigma(\xi); L^p_\Sigma(\eta))$ if and only if it contains at least one element that is Fredholm. In this case, each of these Fredholm operators has the same index.*

PROOF. Kato's Perturbation Theorem states that any continuous 1-parameter family $(F_t)_{t\in[0,1]}$ of Semi-Fredholm operators lies entirely within the set of Fredholm operators, if F_{t_o} is Fredholm for some $t_o \in [0,1]$, cf. [33]. Hence, the path connectedness of F_Ω completes the proof. ∎

The last statement enables us to simplify operators $F \in F_\Omega(\xi,\eta)$ for fixed asymptotic data as follows. Considering the representation $F_i = \frac{\partial}{\partial s} + A_i(s)$ over the cylindrical ends $Z_i = \psi_i(Z^{\epsilon_i})$ we may assume from now on, without loss of generality, that $A_i(s) = J_o\frac{\partial}{\partial t} + S_i(s,t)$ with $\frac{\partial}{\partial s}S_i(s,t) = 0$ for $|s|$ large, i.e. we may assume that A_i is asymptotically constant. This assumption allows us to give a simple proof for

3.1.23 Lemma *Given admissible asymptotic data Ω, any smooth solution $u \in C^\infty_{loc}(Z^{\epsilon_i},\mathbb{R}^{2n})$ of $F_i u = 0$ satisfying*

$$\lim_{n\to\infty} \|u(\epsilon_i s_n,\cdot)\|_{L^2(S^1)} = 0$$

for some sequence $s_n \to \infty$, has asymptotic exponential L^2-decrease, that is, the function

$$\alpha_i\colon [0,\infty) \to \mathbb{R}, \quad \alpha_i(s) = \|u(\epsilon_i s,\cdot)\|_{L^2(S^1)},$$

decays like $\alpha_i(s) \leqslant c\,\mathrm{e}^{-\delta_i s}$ for some constants c, $\delta_i > 0$.

PROOF. We consider $u \in C^\infty([0,\infty)\times S^1,\mathbb{R}^{2n})$ to be a solution of

$$\tfrac{\partial u}{\partial s}(s,t) + J_o\tfrac{\partial u}{\partial t}(s,t) + S(s,t)u(s,t) = 0 \tag{3.16}$$

for all $(s,t) \in [0,\infty)\times S^1$, with $\frac{\partial S}{\partial s}(s,t) = 0$ for all $s \geqslant s_o$, $t \in S^1$ for some $s_o > 0$. Here $S(s,t) = S^+(t)$, $s \geqslant s_o$, $t \in S^1$ and S^+ is admissible, that is regular and pointwise symmetric as defined in Definition 3.1.1. Denoting

$$\alpha(s) = \frac{1}{2}\|u(s,\cdot)\|^2_{L^2(S^1)} = \frac{1}{2}\int_{S^1} |u(s,t)|^2 dt$$

we compute for $s \geqslant s_o$

$$\begin{aligned}
\alpha'(s) &= \langle \tfrac{\partial u}{\partial s}(s,\cdot), u(s,\cdot)\rangle_{L^2(S^1)} = -\langle (J_o\tfrac{\partial}{\partial t} + S^+(t))u(s), u(s)\rangle,\\
\alpha''(s) &= -\langle (J_o\tfrac{\partial}{\partial t} + S^+)\tfrac{\partial u}{\partial s}, u\rangle - \langle (J_o\tfrac{\partial}{\partial t} + S^+)u, \tfrac{\partial u}{\partial s}\rangle\\
&= 2\|A^+u(s)\|^2_{L^2(S^1)}
\end{aligned} \tag{3.17}$$

using $\frac{\partial u}{\partial s} = -A(s)u(s)$ from (3.16). From Lemma 3.1.2 we know that

$$A^+\colon H^{1,2}(S^1) \xrightarrow{\cong} L^2(S^1)$$

is an isomorphism. Hence, equation (3.17) yields the estimate

$$\alpha''(s) \geqslant 2\|A^+u(s)\|^2_{L^2(S^1)} \geqslant 4\delta^2\alpha(s)$$

for all $s \geqslant s_o$, where $\delta = \min\{\,|\lambda|\mid \lambda\in\sigma(A^+)\,\} > 0$ in view of Lemma 3.1.2. A simple maximum principle in combination with the assumption $\alpha(s_n) \to 0$ for $s_n \to \infty$ now provides the exponential decrease

$$\alpha(s) \leqslant c\,\mathrm{e}^{-2\delta s}$$

for all $s \geqslant 0$, for some constant $c > 0$. Namely, setting

$$\alpha_o(s) = \alpha(s_o)\,\mathrm{e}^{-2\delta(s-s_o)} \quad \text{and} \quad \triangle(s) = \alpha(s) - \alpha_o(s)$$

we obtain the estimate

$$\triangle''(s) \geqslant 4\delta^2\triangle(s)$$

for all $s \geqslant s_o$, and $\triangle(s_o) = 0$, $\lim_{s\to\infty}\triangle(s) = 0$. Since $\triangle$ cannot have a local maximum in (s_o,∞), the asserted exponential inequality follows. Thus, we obtain $\|u(s,\cdot)\|_{L^2(S^1)} \leqslant \tilde{c}\,\mathrm{e}^{-\delta s}$, concluding the proof. ∎

However, the exponential decay of the L^2-traces over S^1 does not immediately imply an asymptotic $H^{1,p}_\Sigma$-decay. We have to prove

3.1.24 Lemma *Under the assumptions of Lemma 3.1.23 it holds*

$$u \in H^{1,p}(Z^{\epsilon_i},\mathbb{R}^{2n})$$

for all $p \geqslant 2$.

PROOF. Without loss of generality we consider $\epsilon_i = +1$. From Lemma 3.1.23 we have the exponential decay $\|u(s,\cdot)\|_{L^2(S^1)} \leqslant c\,\mathrm{e}^{-\delta s}$ for $s \geqslant 0$, in particular $u \in L^2(Z^+,\mathbb{R}^{2n})$. Let $\beta^\pm \in C^\infty_o(Z^+,[0,1])$ be the cut-off functions as in (3.9) and define for $1 < \sigma < \tau < \infty$

$$\beta_{\sigma,\tau}(s) = \beta^+(s-\tau)\cdot\beta^-(s-\sigma),$$

that is, $\operatorname{supp}\beta_{\sigma,\tau} \subset (\sigma-1,\tau+1)$. Thus, $u \in C^\infty_{loc}(Z^+,\mathbb{R}^{2n})$ implies $\beta_{\sigma,\tau}u \in H^{1,p}(Z^+,\mathbb{R}^{2n})$. By applying Theorem 3.1.13 to

$$\tfrac{\partial}{\partial s} + A^+\colon H^{1,p}(\mathbb{R}\times S^1) \xrightarrow{\cong} L^p(\mathbb{R}\times S^1),$$

we obtain the estimate with some new constant $c > 0$

$$\begin{aligned}\|\beta_{\sigma,\tau} u\|_{1,p}^{Z+} &\leqslant c\|(\tfrac{\partial}{\partial s} + A^+)(\beta_{\sigma,\tau} u)\|_{0,p}^{Z+}\\ &\leqslant c\|(\tfrac{\partial}{\partial s} + A(s))(\beta_{\sigma,\tau} u) + (A^+ - A(s))\beta_{\sigma,\tau} u\|_{0,p}^{Z+}\\ &\leqslant c\left(\||\beta'_{\sigma,\tau}|u\|_{0,p}^{[\sigma-1,\tau+1]\times S^1} + \|\beta_{\sigma,\tau}(S^+(\cdot) - S(s,\cdot))u\|_{0,p}^{[\sigma-1,\tau+1]\times S^1}\right).\end{aligned}$$

This implies

$$\|\beta_{\sigma,\tau} u\|_{1,p}^{Z+} \leqslant \tilde{c}\,\|u\|_{0,p}^{[\sigma-1,\tau+1]\times S^1} \tag{3.18}$$

for some $\tilde{c}$ independent of σ, τ. Using the Rellich embedding $H^{1,2}(Q) \hookrightarrow L^q(Q)$ for any $1 \leqslant q < \infty$ with $\dim Q = 2$ such that

$$\|u\|_{0,p}^{[\sigma-1,\tau+1]\times S^1} \leqslant c_1\,\|\beta_{\sigma-2,\tau+2} u\|_{1,2}^{Z+}$$

and iterating (3.18) with $p = 2$, we conclude for any $2 \leqslant p < \infty$ that

$$\|\beta_{\sigma,\tau} u\|_{1,p}^{[\sigma,\infty)\times S^1} \leqslant c\,\|u\|_{0,2}^{[\sigma-3,\infty)\times S^1}$$

for all $3 < \sigma < \tau < \infty$, with c independent of τ. Thus, Lemma 3.1.23 gives rise to the exponential estimate

$$\|u\|_{1,p}^{Z_T^+} \leqslant c(p,u)\,\mathrm{e}^{-\delta T} \tag{3.19}$$

for $T > 1$. ■

3.1.25 Definition *An element $u \in H_\Sigma^{1,p}(\xi)$ is said to have exponential $H_\Sigma^{1,p}$-decay, if its representations over the cylindrical ends $\psi_i(Z^{\epsilon_i})$ with respect to the fixed trivialization satisfy an estimate of the type (3.19).*

We now combine the latter exponential decay result with the elliptic estimates of local regularity. Summing up we obtain

3.1.26 Proposition *(a) Every solution $u \in H_\Sigma^{1,q}(\xi)$ of $F_q u = 0$ for $q \geqslant 2$ is smooth and has exponential $H_\Sigma^{1,p}$-decay, in particular*

$$u \in H_\Sigma^{1,p}(\xi)$$

for all $p \geqslant 2$. (b) The same holds for every weak solution $u \in L_\Sigma^2(\xi)$, i.e.

$$\langle F_2^*\phi, u\rangle_{L_\Sigma^2} = 0 \quad \text{f.a.} \quad \phi \in C_o^\infty(\Sigma,\eta),$$

where F_2^* is the formally adjoint operator from Lemma 3.1.8.

Proof. In Section 2.5, we showed that strong solutions of $Fu = 0$ as well as weak solutions are smooth, that is $u \in C^\infty_{\mathrm{loc}}$. It remains to verify the exponential decay. First, $u \in H_\Sigma^{1,q}(\xi) \subset L_\Sigma^q(\xi)$ implies

$$\int_0^\infty \int_{S^1} |u_i|^q ds dt < \infty$$

for $u_i(s,t) = u_{|\psi_i(Z^{\epsilon_i})} \circ \psi_i(\epsilon_i s, t)$, $s \geqslant 0$, $t \in S^1$, $i = 1,\ldots,\nu$, so that there is a sequence $s_n \to \infty$ with $\|u_i(s_n,\cdot)\|_{L^q(S^1)} \to 0$. Since $q \geqslant 2$, u_i satisfies the condition in Lemma 3.1.23. Thus, Lemma 3.1.24 provides the exponential $H_\Sigma^{1,p}$-decay for all $p \geqslant 2$.

For (b), we already know that $u \in L_\Sigma^2(\xi) \cap C^\infty_{loc}(\xi)$. Thus it is a solution of $F_i u = 0$ with $\|u_i(s_n,\cdot)\|_{L^2(S^1)} \to 0$ for some $s_n \to \infty$ on each cylindrical end $i = 1,\ldots,\nu$. We conclude that $u \in H_\Sigma^{1,p}(\xi)$ for all $p \geqslant 2$ with exponential $H_\Sigma^{1,p}$-decay. ■

3.1.27 Corollary *The asymptotically constant operator*

$$F_2 \in \mathcal{L}(H_\Sigma^{1,2}(\xi); L_\Sigma^2(\eta))$$

is a Fredholm operator.

Proof. Since we know from Theorem 3.1.11 that F_2 is Semi-Fredholm, it remains to show that $\operatorname{coker} F_2$ has finite dimension. We are in the L^2-Hilbert space situation. Hence, we deduce from the closed range of the Semi-Fredholm operator F_2 that

$$\begin{aligned}\operatorname{coker} F_2 &\cong \mathrm{R}(F_2)^{\perp_{L^2}}\\ &= \{\, u \in L_\Sigma^2(\eta) \,|\, \langle F_2\phi, u\rangle_{L_\Sigma^2(\eta)} = 0 \quad \text{f.a. } \phi \in C_o^\infty(\xi_{|\Sigma}) \,\}.\end{aligned}$$

Thus, we have to show that the space of weak solutions of the formally adjoint operator F_2^* has finite dimension. After replacing J_o by $-J_o$ and changing the sign F_2^* is again an admissible $\overline{\partial}$-operator and therefore Semi-Fredholm. Hence, Proposition 3.1.26 (b) implies that

$$\operatorname{coker} F_2 \cong \ker\left(F_2^* \colon H_\Sigma^{1,2}(\eta) \to L_\Sigma^2(\xi)\right)$$

is finite dimensional. ■

3.1.28 Corollary *If $F \in F_\Omega(\Phi,\Psi)$ is asymptotically constant the induced operators $F_p \in \mathcal{L}(H_\Sigma^{1,p}(\xi); L_\Sigma^p(\eta))$ satisfy*

$$\ker F_p = \ker F_q$$

for all $p, q \geqslant 2$.

Proof. This is an immediate consequence of Proposition 3.1.26 (a). ■

The last corollary suggests the following

3.1.29 Definition *We denote the common kernel of the operators F_p, $p \geqslant 2$, and analogously for the formally adjoint operator F^* by*

$$\ker F = \bigcap_{p\geqslant 2} \ker F_p, \quad \textit{and} \quad \ker F^* = \bigcap_{p\geqslant 2} \ker F_p^*.$$

Using now the regularity and the asymptotic decrease property for $u \in \ker F^*$ we finally complete the proof of the Fredholm property by

3.1.30 Proposition *For any $p \geqslant 2$, the operator $F_p \in \mathcal{L}(H^{1,p}_\Sigma(\xi); L^p_\Sigma(\eta))$ has a finite dimensional cokernel, in particular, it is Fredholm.*

PROOF. We use the short hand notation $C = \ker F^*$, that is, $C \subset H^{1,p}_\Sigma(\eta)$ for all $p \geqslant 2$, and we define

$$\begin{aligned} T_p \colon H^{1,p}_\Sigma(\xi) \times C &\to L^p_\Sigma(\eta),\\ T_p(x,y) &= F_p x + y\,. \end{aligned}$$

Considering $(x,y) \in \ker T_p$, it follows that $F_p x = -y \in \mathcal{R}(F_2)^\perp$ and hence,

$$\|y\|^2_{L^2_\Sigma(\eta)} = -\langle y, F_p x\rangle = 0\,.$$

Thus we have $\ker T_p = \ker T_2 = \ker F \times \{0\}$ for all $p \geqslant 2$. Due to construction it follows that T_2 is onto. Let

$$T_2(x,y) = z \in C^\infty_o(\eta),$$

so that (x,y) satisfy asymptotically on the cylindrical ends $\psi_i(Z^{\epsilon_i}_T)$

$$F_2 x_{|\psi_i(Z^{\epsilon_i}_T)} = -y_{\psi_i(Z^{\epsilon_i}_T)}\,.$$

Hence, Theorem 3.1.13 applies to the asymptotically translation-invariant operator $(F_2)_{|\psi_i(Z^{\epsilon_i}_T)}$. Thus $y \in L^p_\Sigma(\eta)$ together with $x \in H^{1,2}_\Sigma(\xi) \subset L^p_\Sigma(\xi)$ for $p \geqslant 2$ imply that $x \in H^{1,p}_\Sigma(\xi)$. Therefore we have $C^\infty_o(\eta) \subset \mathcal{R}(T_p)$, that is, the range of T_p lies dense in $L^p_\Sigma(\eta)$. The Semi-Fredholm property of F_p yields the closedness of $\mathcal{R}(T_p)$. Hence, T_p is onto. Now, we finally deduce that for each $p \geqslant 2$

$$\phi_p \colon C \to \operatorname{coker} F_p, \quad y \mapsto [y]_{\mathcal{R}(F^o_p)},$$

is an isomorphism. $\phi_p(y) = 0$ implies $F_p(x) - y = 0$ for some $x \in H^{1,p}_\Sigma(\xi)$ and thus $y = 0$ due to $\ker T_p = \ker F \times \{0\}$. The surjectivity of ϕ_p follows from that of T_p. Hence, the proof is complete. ∎

3.1.4 Application to the Linearization of $\overline{\partial}_{J,k}$

We now apply the general Fredholm result to the situation where we consider the linearization of $\overline{\partial}_{J,k}$ as an operator

$$H^{1,p}_\Sigma(h^*TM) \to L^p_\Sigma(X^J(h^*TM))$$

for pull-back bundles. At first we recall that we already discussed the induced Hermitian bundle $\eta = X^J(\xi)$ for $\xi \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$. Let us now consider the smooth pull-back bundle h^*TM for $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$. This bundle h^*TM inherits its Hermitian structure (ω, J) from the symplectic structure (M,ω) together with a compatible almost complex structure J. Since we assume $x_1,\dots,x_\nu$ to be

contractible 1-periodic orbits of Hamiltonian equations, it follows that $h^*TM \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$. Moreover, we construct an asymptotic unitary trivialization $\Phi_{|\partial\Sigma}$ as follows. For each $i = 1,\dots,\nu$ we choose an extension $u_{x_i} \colon D^2 \to M$ with

$$u_{x_i|\partial D^2} = x_i, \quad i = 1,\dots,\nu.$$

Since D^2 is contractible, $u^*_{x_i}TM$ can be endowed with a complex frame, which may be extended over $h^*TM_{|\psi_i(Z^{\epsilon_i})}$. In view of Lemma 3.1.4 this gives rise to a unitary trivialization Φ_i of $h^*TM_{|\psi_i(Z^{\epsilon_i})}$ for all $i = 1,\dots,\nu$. From now on, in this section, we refer to such a trivialization Φ of $h^*TM_{|\Sigma_Z}$ constructed upon extensions u_{x_i}, $i = 1,\dots,\nu$. We apply the Fredholm theory of the last section to the linearization $DF_h(0)$ of the smooth section $\overline{\partial}_{J,k} \colon \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) \to L^p_\Sigma(\mathcal{P}^*X^J)$ for $k = k(H)$ as constructed in Proposition 2.4.1

3.1.31 Theorem *Let (J,k) be an admissible extension of (J^i, H^i) and $x_i \in \mathcal{P}_1(H^i)$ for $i = 1,\dots,\nu$. Given any $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ the linearization*

$$DF_h(0) \colon H^{1,p}_\Sigma(h^*TM) \to L^p_\Sigma(X^J(h^*TM))$$

is an admissible $\overline{\partial}$-operator if the $(H^i)_{i=1,\dots,\nu}$ are regular. Then, $DF_h(0)$ is a Fredholm operator.

PROOF. Since we already know that h^*TM and $X^J(h^*TM)$ carry suitable unitary local trivializations, it is sufficient to analyze the representation of $DF_h(0)$ over the cylindrical ends $\psi_i(Z^{\epsilon_i}_T)$, $i = 1,\dots,\nu$, for $T > 0$ large enough with respect to trivializations Φ_Z and Ψ_Z as in Definition 3.1.6. Let us consider the operator

$$D^i \colon H^{1,p}_\Sigma(h^*TM_{\psi_i(Z^{\epsilon_i}_T)}) \to L^p_\Sigma(h^*TM_{\psi_i(Z^{\epsilon_i}_T)})$$

obtained from $D^i = DF_h(0)_{\psi_i(Z^{\epsilon_i}_T)} \cdot \frac{\partial}{\partial s}$. We recall equation (2.7) with $k = k(H)$ such that $k_{\psi_i(Z^{\epsilon_i}_T)} \cdot \frac{\partial}{\partial s} = \nabla H^i$ for T large enough. We constantly use the shorthand notation $\frac{\partial}{\partial s} = T\psi_i(\frac{\partial}{\partial s})$ for the section in $T\Sigma_{\psi_i(Z^{\epsilon_i}_T)}$. For $(s,t) \in Z^{\epsilon_i}_T$ it holds

$$\begin{aligned} (D^i \cdot \xi)(s,t) = \nabla_s \xi + \bar{J}(s,t)\nabla_t \xi &+ \\ &+ \operatorname{Tor}(\tfrac{\partial h}{\partial s}, \xi) + \bar{J}(s,t)\operatorname{Tor}(\tfrac{\partial h}{\partial t}, \xi) + \nabla_\xi \nabla H^i(t,h), \end{aligned}$$

where $\bar{J}(s,t) = J^i(s,t,h(s,t))$. This is a representation of the linearization with respect to a Hermitian connection, i.e. $\nabla \bar{J} = 0$. Using the unitary trivialization $\Phi \colon Z^{\epsilon_i}_T \times \mathbb{R}^{2n} \xrightarrow{\sim} (h \circ \psi_i)^*TM$ with $\bar{J}\Phi = \Phi J_o$ we compute $D^i_{\mathrm{triv}} = \Phi^{-1} \circ D^i \circ \Phi$ as

$$D^i_{\mathrm{triv}} \cdot v = \partial_s v + J_o \partial_t v + \tilde{S}v + Sv$$

with

$$\begin{aligned} \tilde{S}v &= (\Phi^{-1}\nabla_s \Phi)v + \Phi^{-1}\operatorname{Tor}(\partial_s h, \Phi v),\\ Sv &= J_o(\Phi^{-1}\nabla_t \Phi)v + J_o\Phi^{-1}\operatorname{Tor}(\partial_t h, \Phi v) + \Phi^{-1}\nabla_{\Phi v}\nabla H^i(t,h)\,. \end{aligned}$$

Since $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$, Lemma 2.1.3 implies that $\tilde S(s,t) \to 0$ uniformly in $t \in S^1$ as $\epsilon_i s \to \infty$. Thus, it remains to prove that $\lim_{\epsilon_i s\to\infty} S(s,t)$ is admissible. First of all, h, Φ being C^∞_Σ-smooth yields that the uniform limit

$$\begin{aligned}\lim_{\epsilon_i s\to\infty} S(s,t) &= S^\infty(t) = \\ &= \Phi(t)^{-1}\left[\bar J \nabla_t \Phi(t) + \bar J(t)\,\mathrm{Tor}\,(\dot x(t), \Phi(t)\,\cdot\,) + \nabla_{\Phi(t)\cdot} \nabla H^i(t, x_i(t))\right] \quad (3.20)\end{aligned}$$

exists, where $\Phi\colon S^1 \times \mathbb{R}^{2n} \to x_i^* TM$ is the respective unitary trivialization with $\bar J(t) = J^i(t, x_i(t))$. We now prove the fact that S^∞ is admissible, based on the geometric idea, that $-J_o\frac{\partial}{\partial t} - S^\infty$ corresponds to the Hessian of the action functional $\mathcal{A}_H$ at the critical point x_i with $H = H^i$. We divide the proof into two parts.

3.1.32 Lemma *The operator $J_o\frac{\partial}{\partial t} + S^\infty$ is L^2-symmetric, that is*

$$\langle J_o \partial_t v + S^\infty v, w\rangle_{L^2(S^1)} = \langle v, J_o\partial_t w + S^\infty w\rangle_{L^2(S^1)}$$

for all $v, w \in C^\infty(S^1, \mathbb{R}^{2n})$.

Proof. We let $v, w \in C^\infty(S^1, \mathbb{R}^{2n})$ and consider the smooth vector fields along x_i, $\xi = \Phi v$, $\zeta = \Phi w$. Thus, it is equivalent to prove

$$\langle J\nabla_t \xi + J\,\mathrm{Tor}(\dot x_i, \xi) + \nabla_\xi \nabla H, \zeta\rangle_{L^2} = \langle \xi, J\nabla_t\zeta + J\,\mathrm{Tor}(\dot x_i, \zeta) + \nabla_\zeta \nabla H\rangle_{L^2}$$

where now $\langle\cdot,\cdot\rangle_{L^2}$ denotes the L^2-inner product on $C^\infty(x_i^*TM)$ with respect to the Hermitian structure $(\omega_{|x_i}, J = J^i(t, x_i))$. Now we define for small $\epsilon > 0$

$$\alpha\colon S^1 \times (-\epsilon,\epsilon)^2 \to M, \quad \alpha(t, \lambda_1, \lambda_2) = \exp_{x_i(t)}\left(\lambda_1 \xi(t) + \lambda_2 \zeta(t)\right),$$

and we recall the action functional $\mathcal{A}_H$ well-defined up to $\omega_{|\pi_2(M)}$,

$$\mathcal{A}_H(y) \equiv \int_{D^2} u_y^* \omega - \int_{S^1} H(t, y(t))dt \mod \min_{A\in\pi_2(M)} |\omega(A)|$$

for $y \in C^\infty(S^1, M)$, $u_y\colon D^2 \to M$, $u_{y|S^1} = y$. Notwithstanding the ambiguity we compute the well-defined directional derivative

$$\frac{\partial}{\partial\lambda_1}(\mathcal{A}_H \circ \alpha) = -\int_{S^1} \langle J\alpha_t + \nabla H(t,\alpha), \partial_{\lambda_1}\alpha\rangle dt$$

with $\alpha_t = \partial_t\alpha$. Furthermore,

$$\begin{aligned}\frac{\partial^2}{\partial\lambda_2\partial\lambda_1}(\mathcal{A}_H\circ\alpha)\big|_{\lambda_1=\lambda_2=0} &= \int_{S^1}\langle J\nabla_{\lambda_2}\alpha_t + \nabla_{\lambda_2}\nabla H(t,\alpha), \partial_{\lambda_1}\alpha\rangle dt\big|_{\lambda_1=\lambda_2=0} \\ &\quad + \int_{S^1}\langle J\alpha_t + \nabla H(t,\alpha), \nabla_{\lambda_2}\partial_{\lambda_1}\alpha\rangle dt\big|_{\lambda_1=\lambda_2=0}.\end{aligned}$$

The second term on the right hand side vanishes for $\lambda_1 = \lambda_2 = 0$ because $J\dot x_i + \nabla H(t, x_i) = 0$. Since $\mathcal{A}_H\circ\alpha\colon (-\epsilon,\epsilon)^2 \to \mathbb{R}$ is smooth we know that

$$\frac{\partial^2}{\partial\lambda_2\partial\lambda_1}(\mathcal{A}_H\circ\alpha) = \frac{\partial^2}{\partial\lambda_1\partial\lambda_2}(\mathcal{A}_H\circ\alpha).$$

Thus it holds

$$\int_{S^1}\langle J\nabla_{\lambda_2}\alpha_t + \nabla_\zeta\nabla H, \xi\rangle dt = \int_{S^1}\langle J\nabla_{\lambda_1}\alpha_t + \nabla_\xi \nabla H, \zeta\rangle dt$$

for $\lambda_1 = \lambda_2 = 0$. Now using the identity

$$\nabla_{\lambda_2}\alpha_t = \nabla_t\partial_{\lambda_2}\alpha + \mathrm{Tor}(\partial_t\alpha, \partial_{\lambda_2}\alpha) = \nabla_t\zeta + \mathrm{Tor}(\dot x, \zeta)$$

at $\lambda_1 = \lambda_2 = 0$ and analogously for λ_1 and ξ yields the assertion. ∎

3.1.33 Lemma *The operator $J\partial_t + S_i^\infty\colon C^\infty(S^1, \mathbb{R}^{2n}) \to C^\infty(S^1, \mathbb{R}^{2n})$ is regular, if $x_i \in \mathcal{P}_1(H^i)$ is non-degenerate.*

Proof. Let ψ be the flow generated by the Hamiltonian field X_{H^i},

$$\begin{aligned}\psi\colon \mathbb{R}\times M \to M, &\quad (t,p) \mapsto \psi_t(p), \\ \partial_t\psi(t,p) &= X_{H_t}(\psi_t(p)),\end{aligned}$$

so that $x_i(t) = \psi_t(x_i(0))$ and $x_i(0)$ is a fixed point of ψ_1. Considering the linear symplectic map

$$d\psi(t,p) = D_p\psi_t\colon T_pM \xrightarrow{\sim} T_{\psi_t(p)}M$$

we assert that it holds

$$(\nabla_t D\psi_t)(p)\cdot v = \nabla_{D\psi_t\cdot v} X_{H_t}(\psi_t(p)) + \mathrm{Tor}\left(D\psi_t(p)\cdot v, X_{H_t}(\psi_t(p))\right). \tag{3.21}$$

This can be seen as follows. Let $\gamma\colon(-\epsilon,\epsilon)\to M$ be a curve with $\gamma(0) = p$ and $\dot\gamma(0) = v \in T_pM$. This gives rise to

$$\alpha\colon [0,1]\times(-\epsilon,\epsilon) \to M, \quad (t,s)\mapsto \psi_t(\gamma(s))$$

so that it follows

$$\partial_t\alpha(t,s) = X_{H_t}(\psi_t(\gamma(s))) \quad\text{and}\quad \partial_s\alpha(t,0) = D\psi_t(p)\cdot v$$

for all $s \in (-\epsilon,\epsilon)$, $t\in[0,1]$. We consequently derive

$$(\nabla_t D\psi_t)(p)\cdot v = (\nabla_t\partial_s\alpha)(t,0) = (\nabla_s\partial_t\alpha)(t,0) + \mathrm{Tor}\left(\partial_s\alpha(t,0), \partial_t\alpha(t,0)\right)$$

thus proving (3.21). We now fix $x_0 = x_i(0)$ and define the symplectic path with respect to the unitary trivialization Φ from above,

$$\Psi(t) = \Phi(t)^{-1}\circ D\psi_t(x_0)\circ\Phi(0), \quad t\in[0,1]. \tag{3.22}$$

From this we compute with (3.21)

$$\begin{aligned}&\nabla_t\Phi(t)\Psi(t) + \Phi(t)\partial_t\Psi(t) \\ &= \nabla_{D\psi_t(x_0)\Phi(0)}X_{H_t}(\psi_t(x_0)) + \mathrm{Tor}\left(D\psi_t(x_0)\Phi(0), X_{H_t}(\psi_t(x_0))\right) \\ &= \nabla_{\Phi(t)\Psi(t)}X_{H_t}(x_i(t)) + \mathrm{Tor}\left(\Phi(t)\Psi(t), X_{H_t}(x_i(t))\right)\end{aligned}$$

yielding

$$\partial_t \Psi(t) = -\Phi^{-1}\nabla_t \Phi\Psi + \Phi^{-1} J \nabla_{\Phi\Psi} \nabla H_t(x_i) + \Phi^{-1}\,\mathrm{Tor}(\Phi\Psi, \dot{X}_i),$$

where we use that $\nabla J = 0$ due to the Hermitian connection. Thus we conclude with

$$J_o\partial_t\Psi + J_o\Phi^{-1}\nabla_t\Phi\Psi + \Phi^{-1}\nabla_{\Phi\Psi}\nabla H_t(x_i) + J_o\Phi^{-1}\,\mathrm{Tor}(\dot{x}_i, \Phi\Psi) = 0$$

that

$$(J_o\partial_t + S^\infty)(\Psi) = 0\,. \tag{3.23}$$

This identity signifies that $J_o\partial_t + S^\infty$ is injective exactly if there is no $v \in \mathbb{R}^{2n}$ such that $\Psi(1)v = v$, that is if and only if $\det(1 - \Psi(1)) \neq 0$. ■

Combining both lemmata we deduce that S_i^∞ is admissible for all $i = 1, \dots, \nu$. Hence $DF_h(0)$ is an admissible $\overline{\partial}$-operator and due to Theorem 3.1.9 a Fredholm operator. This completes the proof of Theorem 3.1.31. ■

3.2 The Linear Gluing Operation

The second half of the Fredholm analysis for our generalized $\overline{\partial}$-operators comprises the computation of the index. In the original Floer theory only cylindrical model surfaces are relevant. Therefore it is possible to express the index by the spectral flow of the operator $A(s)\colon H^{1,2}(S^1) \to L^2(S^1)$, where $F = \partial_s + A(s)\colon H^{1,2}(\mathbb{R}\times S^1) \to L^2(\mathbb{R}\times S^1)$, $s \in \mathbb{R}$. But now we deal with arbitrary topological classes of orientable surfaces, so that we need a more general concept of index computation. The main feature is the additivity of the index whenever pairs of model surfaces are glued along their cylindrical ends. Thus we already have to consider a gluing operation at this stage, although only in its linear version.

At first we have to construct a gluing operation for the underlying model surfaces. In the framework of standard Floer homology based on cylinders $\mathbb{R}\times S^1$ this point did not require any further mentioning. But now we have to give precise definitions in accordance with the construction of a model surface. Let us consider two model surfaces Σ^1 and Σ^2 with their labeled cylindrical ends $(\psi_i^1, \epsilon_i^1)_{i=1,\dots,\nu_1}$ and $(\psi^2, \epsilon_i^2)_{i=1,\dots,\nu_2}$. The prime condition for gluing cylindrical ends is to couple 'entrances' with 'exits', i.e. in order to guarantee compatible orientations we demand for ends (ψ_i^1, ϵ_i^1), (ψ_j^2, ϵ_j^2) to be glued, that $\epsilon_i^1 = -\epsilon_j^2$. Moreover, each gluing process for a specified end shall be parametrized by its own parameter ρ_i. We recall the notation

$$Z_{\rho_i}^{\epsilon_i} = \begin{cases} [\rho_i, \infty) \times S^1, & \text{for } \epsilon_i = +1, \\ (-\infty, -\rho_i] \times S^1, & \text{for } \epsilon_i = -1. \end{cases}$$

Specifying the r cylindrical ends to be glued by $1 \leqslant i_1, \dots, i_r \leqslant \nu_1$, $1 \leqslant j_1, \dots, j_r \leqslant \nu_2$ with $i_k \neq i_l$, $j_k \neq j_l$ for all $k \neq l \in \{1, \dots, r\}$ and choosing gluing parameters $\rho_k \geqslant 0$, $k \in \{1, \dots, r\}$ we call the r-tuples

$$\alpha = ((i_1, j_1), \dots, (i_r, j_r)), \quad R = (\rho_1, \dots, \rho_r)$$

the gluing data. Of course, we may also glue two opposite oriented ends from the same model surface. This gluing data is analogously specified as $(\alpha, R) = (i_1, j_1, \rho_1), \dots, (i_r, j_r, \rho_r))$ where now $1 \leqslant i_1, \dots, i_r, j_1, \dots, j_r \leqslant \nu$ are $2r$ different labels with $\epsilon_{i_k} = -\epsilon_{j_k}$ for $k = 1, \dots, r$.

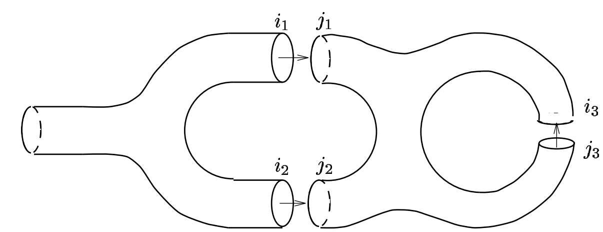


Figure 3.1: Gluing of cylindrical ends

3.2.1 Definition *Given two model surfaces Σ^1, Σ^2 and the gluing data (α, R) we define the glued surface $\Sigma^{1,2}_{(\alpha,R)} = \Sigma^1 \#_{(\alpha,R)} \Sigma^2$ as follows. Considering the truncated surfaces*

$$\Sigma^1_{(\alpha,R)} = \overline{\overset{o}{\Sigma}{}^1 \setminus \textstyle\bigcup_{i=1}^r \psi^1_{i_k}(Z^{\epsilon_{i_k}}_{\rho_k})},$$

analogously $\Sigma^2_{(\alpha,R)}$, and the gluing map $f = (f_1, \dots, f_r)$ being given by

$$f_k\colon \psi^1_{i_k}(\{\epsilon_{i_k}\rho_k\} \times S^1) \xrightarrow{\cong} \psi^2_{j_k}(\{\epsilon_{j_k}\rho_k\} \times S^1),$$
$$f(\psi^1_{i_k}(\epsilon_{i_k}\rho_k, t)) = \psi^2_{j_k}(\epsilon_{j_k}\rho_k, t) \quad \text{for } |s| = \rho_k,$$

$\Sigma^{1,2}_{(\alpha,R)}$ is defined as

$$\Sigma^1 \#_{(\alpha,R)} \Sigma^2 = \Sigma^1_{(\alpha,R)} \cup_f \Sigma^2_{(\alpha,R)}\,.$$

Compare also Figure 3.2 on page 68. In other terms,

$$\Sigma^{1,2}_{(\alpha,R)} = \Sigma^1_{(\alpha,R+\epsilon)} \sqcup \Sigma^2_{(\alpha,R+\epsilon)} \big/ \sim,$$

where $\sim$ denotes the equivalence relation

$$\psi^1_{i_k}(\epsilon_{i_k}(\rho_k + s), t) \sim \psi^2_{j_k}(\epsilon_{j_k}(\rho_k - s), t)$$

*for $|s| \leqslant \epsilon$, $k = 1, \dots, r$ and some fixed small $\epsilon > 0$. Analogously, referring to the same equivalence relation, we define a so-called **contraction operation** for a given model surface Σ with gluing data (α, R) by*

$$\mathrm{tr}_{(\alpha,R)}\,\Sigma = \Sigma_{(\alpha,R+\epsilon)} \big/ \sim$$

for some small $\epsilon > 0$. (See Figure 3.1 and Figure 3.2 on page 68.)

Obviously, the gluing $\Sigma^1\#_{(\alpha,R)}\Sigma^2$ with

$$(\alpha,R)=((i_1,j_1,\rho_1),\dots,(i_r,j_r,\rho_r))$$

yields the same as combining the contraction operation for

$$(\check\alpha,\check R)=((i_1,j_1,\rho_1),\dots,\widehat{(i_\mu,j_\mu,\rho_\mu)},\dots,(i_r,j_r,\rho_r))$$

with the gluing for $(\alpha_\mu,R_\mu)=((i_\mu,j_\mu,\rho_\mu))$, i.e.

$$\Sigma^1\#_{(\alpha,R)}\Sigma^2=\mathrm{tr}_{(\check\alpha,\check R)}\left(\Sigma^1\#_{(\alpha_\mu,R_\mu)}\Sigma^2\right).$$

Furthermore, gluing and contracting is associative and additive, respectively,

$$\begin{gathered}(\Sigma^1\#_{(\alpha,R)}\Sigma^2)\#_{(\beta,S)}\Sigma^3=\Sigma^1\#_{(\alpha,R)}(\Sigma^2\#_{(\beta,S)}\Sigma^3),\\ \mathrm{tr}_{(\alpha,R)}\left(\mathrm{tr}_{(\beta,S)}\Sigma\right)=\mathrm{tr}_{(\alpha\cup\beta,R\cup S)}\Sigma,\end{gathered}$$

with

$$(\alpha\cup\beta,R\cup S)=((i_1,j_1,\rho_1),\dots,(i_r,j_r,\rho_r),(i_1',j_1',\rho_1'),\dots,(i_s',j_s',\rho_s'))$$

for $(\alpha,R)=(i_k,j_k,\rho_k)_{k=1,\dots,r}$, $(\beta,U)=(i_k',j_k',\rho_k')_{k=1,\dots,s}$, provided that $i_k\neq i_l'$ for $k=1,\dots,r$ and $l=1,\dots,s$.

We now proceed in constructing the linear gluing operation by defining the gluing of asymptotically trivial vector bundles over our model surfaces.

3.2.2 Definition *Let $\xi_j\in\mathrm{Vec}^*_{C^\infty}(\Sigma^j)$, $j=1,2$, be Hermitian rank-$2n$-bundles over the model surfaces Σ^1, Σ^2 endowed with gluing data (α,R). Moreover let the asymptotic data Ω_1 and Ω_2 be given as in Definition 3.1.5, so that we have unitary trivializations of ξ_j over the cylindrical ends*

$$\Phi_j\colon\psi_i^j(Z^{\epsilon_i^j})\times\mathbb{R}^{2n}\xrightarrow{\cong}\xi_{j\,|\psi_i^j(Z^{\epsilon_i^j})},\quad i=1,\dots,\nu_j$$

compatible with Ω_j, $j=1,2$. We identify the bundles

$$\xi_{1\,|\psi^1_{i_k}(Z^{\epsilon^1_{i_k}}_{\rho_{i_k}})}\cong\xi_{2\,|\psi^2_{j_k}(Z^{\epsilon^2_{j_k}}_{\rho_{j_k}})}$$

via the bundle isomorphism $\Phi_2\circ\Phi_1^{-1}$ in accordance with the above gluing maps f_k, $k=1,\dots,r$. This provides the C^∞_Σ-smooth Hermitian bundle

$$\xi_1\#_{(\alpha,R)}\xi_2\to\Sigma^1\#_{(\alpha,R)}\Sigma^2$$

together with compatible unitary local trivializations over the cylindrical ends and the compact gluing region. We denote the associated asymptotic data by $\Omega_1\#_\alpha\Omega_2$. We proceed analogously with the contraction operation on Σ for given asymptotic data Ω for a Hermitian bundle $\xi\in\mathrm{Vec}^*_{C^\infty}(\Sigma)$. This yields $\mathrm{tr}_{(\alpha,R)}\,\xi$.

It is clear that the gluing of asymptotically trivial bundles depends on the given asymptotic trivializations Φ_1 and Φ_2.

Next we provide an appropriate setup for the gluing of admissible $\overline\partial$-operators. As we already observed in the previous section, we may restrict to asymptotically constant operators as far as we are concerned with questions of Fredholm class, Fredholm index, etc. Let us consider Hermitian bundles ξ_i,η_i over Σ^i together with asymptotic data $\Omega_i=(\Phi_i,\Psi_i,(S_k^i)_{k=1,\dots,r})$, $i=1,2$. Φ_i and Ψ_i are unitary trivializations of $\xi_{i|\Sigma^i_Z}$ and $\eta_{i|\Sigma^i_Z}$, i.e. over the cylindrical ends, and Ω_1 and Ω_2 fit together over the ends to be glued. This means that $S^1_{i_k}=S^2_{j_k}$ for $k\in\{1,\dots,r\}$.

3.2.3 Definition *Given asymptotically constant admissible $\overline\partial$-operators $K\in F_{\Omega_1}(\xi_1,\eta_1)$ and $L\in F_{\Omega_2}(\xi_2,\eta_2)$ we define*

$$K\#_{(\alpha,R)}L\in F_{\Omega_1\#_\alpha\Omega_2}(\xi_1\#_{(\alpha,R)}\xi_2,\eta_1\#_{(\alpha,R)}\eta_2)$$

as follows. By assumption, there is a $\rho_o>0$ such that for all $k=1,\dots,r$ we have

$$\begin{gathered}K_{i_k}=\Psi_1^{-1}\circ K\circ\Phi_1\colon C^\infty(Z^{\epsilon^1_{i_k}},\mathbb{R}^{2n})\to C^\infty(Z^{\epsilon^1_{i_k}},\mathbb{R}^{2n}),\\ K_{i_k}=\tfrac{\partial}{\partial s}+J_o\tfrac{\partial}{\partial t}+S^K_{i_k}(s,t),\quad S^K_{i_k}(s,t)=S^1_{i_k}(t)\end{gathered}$$

for all $\epsilon^1_{i_k}s\geqslant\rho_o$, $t\in S^1$, and analogously $L_{j_k}=\frac{\partial}{\partial s}+J_o\frac{\partial}{\partial t}+S^L_{j_k}$, $S^L_{j_k}(s,t)=S^2_{j_k}(t)$ for $\epsilon^2_{j_k}s\geqslant\rho_o$, $t\in S^1$. Assuming without loss of generality that $\epsilon^1_{i_k}=+1$, $\epsilon^2_{j_k}=-1$, we define for $\rho_k>\rho_o$ $K_{i_k}\#_{\rho_k}L_{j_k}$ over $(-\rho_k,\rho_k)\times S^1$ by

$$\begin{gathered}K_{i_k}\#_{\rho_k}L_{j_k}=\tfrac{\partial}{\partial s}+J_o\tfrac{\partial}{\partial t}+S^K_{i_k}\#_{\rho_k}S^L_{j_k},\\ S^K_{i_k}\#_{\rho_k}S^L_{j_k}(s,t)=\begin{cases}S^K_{i_k}(s+\rho_k,t), & s<0,\\ S^1_{i_k}(t)=S^2_{j_k}(t), & s=0,\\ S^L_{j_k}(s-\rho_k,t), & s>0.\end{cases}\end{gathered}$$

If $\rho_k>\rho_o$ for all $k=1,\dots,r$, we obtain an admissible $\overline\partial$-operator

$$K\#_{(\alpha,R)}L\colon C^\infty(\xi_1\#_{(\alpha,R)}\xi_2)\to C^\infty(\eta_1\#_{(\alpha,R)}\eta_2)$$

from $K_{i_k}\#_{\rho_k}L_{j_k}$. Analogously we define $\mathrm{tr}_{(\alpha,R)}K\in F_{\mathrm{tr}_\alpha\Omega}(\mathrm{tr}_{(\alpha,R)}\Phi,\mathrm{tr}_{(\alpha,R)}\Psi)$.

Referring to Lemma 3.1.4 we observe that the gluing of asymptotically constant operators is compatible with adjunction.

3.2.4 Lemma *Given two asymptotically constant operators K, L as in Definition 3.2.3, the associated formally adjoint operators K^*, L^* satisfy*

$$(K\#_{(\alpha,R)}L)^*=K^*\#_{(\alpha,R)}L^*.$$

Proof. This is a straightforward consequence of Lemma 3.1.8 and the definition of the gluing by means of the local cylindrical representation via the unitary trivializations. ∎

Summing up we notice that the set of admissible $\overline{\partial}$-operators is closed under the above defined gluing- and contraction-operations. Given the asymptotically constant Fredholm operators $K \in F_{\Omega_1}(\xi^1)$, $L \in F_{\Omega_2}(\xi^2)$, we obtain again a Fredholm operator $K\#_{(\alpha,R)}L \in F_{\Omega_1\#_\alpha\Omega_2}$, now over the model surface $\Sigma^1\#_{(\alpha,R)}\Sigma^2$.

The aim of this section is to find the relation between $\operatorname{ind} K$, $\operatorname{ind} L$ and $\operatorname{ind}(K\#L)$. As in the original purely cylindrical situation of Floer Homology, we expect additivity under gluing, i.e. symbolically

$$\operatorname{ind}\circ\# = + \circ \operatorname{ind}\,.$$

We prove this relation by establishing an isomorphism

$$\ker K \times \ker L \xrightarrow{\cong} \ker K\#L,$$

provided that all operators are onto. Hence, we need a further gluing operation for sections in the bundles ξ_1, ξ_2, $\xi_1\#\xi_2$, etc.

3.2.5 Definition *We define the so-called linear pre-gluing map*

$$\#^o_{(\alpha,R)}\colon H^{1,2}_\Sigma(\xi_1)\times H^{1,2}_\Sigma(\xi_2)\to H^{1,2}_\Sigma(\xi_1\#_{(\alpha,R)}\xi_2)$$

as follows. First, note that $H^{1,p}_\Sigma(\xi_1\#_{(\alpha,R)}\xi_2)$ and also $L^p_\Sigma(\dots)$ have well-defined norms or inner products for $p=2$, because ξ_1, ξ_2 were assumed with fixed unitary local trivializations and $\xi_1\#\xi_2$ thus obtains a well-defined unitary trivialization over the r different interior, finite cylinders $[-\rho_k,\rho_k]\times S^1$ where the gluing process takes place. We use cut-off functions $\beta^\pm\in C^\infty(\mathbb{R},[0,1])$ with

$$\beta^\epsilon(s)=\begin{cases}1, & \epsilon s\leqslant -1,\\ 0, & \epsilon s\geqslant 0\end{cases},\quad \text{strictly monotone for } \epsilon s\in(-1,0),\ \epsilon=\pm 1\,.$$

We set $v\#^o_{(\alpha,R)}w$ equal to v on $\Sigma^1_{(\alpha,0)}$ and and w on $\Sigma^2_{(\alpha,0)}$. Assuming $\epsilon_{i_k}=+1$, $\epsilon_{j_k}=-1$ without loss of generality and regarding the identification $\psi^1_{i_k}(\rho_k+s)=\psi^2_{j_k}(-(\rho_k-s))$ for $|s|\leqslant\epsilon$ we obtain

$$\psi_k\colon[-\rho_k,\rho_k]\times S^1 \xrightarrow{\approx} \psi^1_{i_k}(Z^+)\#_{(\alpha,R)}\psi^2_{j_k}(Z^-)\subset\Sigma^1\#_{(\alpha,R)}\Sigma^2,$$
$$(s,t)\mapsto\begin{cases}\psi^1_{i_k}(s+\rho_k,t), & s\leqslant 0,\\ \psi^2_{j_k}(s-\rho_k,t), & s\geqslant 0\,.\end{cases}$$

We finally define for $\rho_k\geqslant\rho_o>1$

$$(v\#^o_{(\alpha,R)}w)\circ\psi_k(s,t)=\beta^+(s)\cdot v(\psi^1_{i_k}(s+\rho_k,t))+\beta^-(s)\cdot w(\psi^2_{j_k}(s-\rho_k,t)),$$

see Figure 3.2.

From the point of view of the glued model surface we equivalently use the cut-off functions

$$\beta^1_R,\beta^2_R\in C^\infty(\Sigma^1\#_{(\alpha,R)}\Sigma^2,[0,1])$$

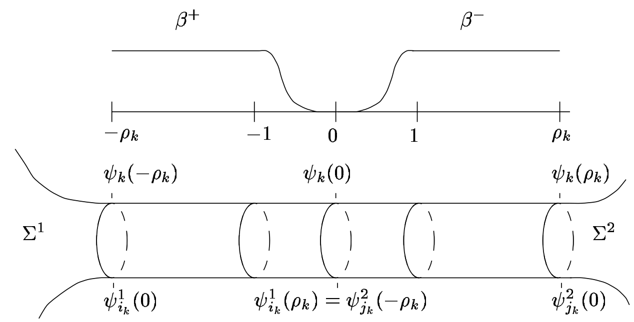


Figure 3.2: The cylindrical coordinates for the linear gluing

with $\beta^i_{R|\Sigma^j_{(\alpha,R-1)}}=\delta_{ij}$, $i,j=1,2$ for $\overline{\Sigma}^j_{(\alpha,R)}$ and

$$\beta^1_R(\psi_k(s,t))=\beta^{\epsilon_{i_k}}(s),\quad \beta^2_R(\psi_k(s,t))=\beta^{\epsilon_{j_k}}(s)$$

for $k=1,\dots,r$. For $\epsilon_{i_k}=+1$, $\epsilon_{j_k}=-1$ this means

$$\beta^1_R(\psi^1_{i_k}(s,t))=\beta^+(s-\rho_k),\quad \beta^2_R(\psi^2_{j_k}(s,t))=\beta^-(s+\rho_k)\,.$$

Hence, the mappings

$$L^2_\Sigma(\xi_1\#_{(\alpha,R)}\xi_2)\to L^2_\Sigma(\xi_i),\quad v\mapsto\beta^i_R\cdot v,$$
$$L^2_\Sigma(\xi_i)\to L^2_\Sigma(\xi_1\#_{(\alpha,R)}\xi_2),\quad v\mapsto\beta^i_R\cdot v$$

are well-defined. We briefly write

$$u\#^o_{(\alpha,R)}v=\beta^1_R u+\beta^2_R v\,.$$

The linear pre-gluing map allows us to prove the first gluing result. However, we first introduce a further extension of the Fredholm operators in question. We already mentioned that we can prove the index additivity for surjective operators. For general admissible $\overline{\partial}$-operators, i.e. Fredholm-operators we require a stabilization technique which eliminates the finite-dimensional cokernel.

3.2.6 Definition *Given an admissible $\overline{\partial}$-operator as above,*

$$K\in\mathcal{L}(C^\infty(\xi);C^\infty(\eta))\cap F_\Omega,$$

and $a=(a_1,\dots,a_k)$, $a_i\in L^2_\Sigma(\eta)$, $i=1,\dots,k$, $b=(b_1,\dots,b_l)$, $b_i\in L^2_\Sigma(\xi)$, $i=1,\dots,l$, we define

$$K_{a,b}\colon H^{1,2}_\Sigma(\xi)\times\mathbb{R}^k\to L^2_\Sigma(\eta)\times\mathbb{R}^l,$$
$$K_{a,b}(u,h)=(Ku+a\bullet h,u\bullet b),$$
$$a\bullet h=\sum_{i=1}^k h_i a_i,\quad u\bullet b=\sum_{i=1}^l\langle u,b_i\rangle_\xi e_i\,.$$

Here we denote by $\langle\cdot,\cdot\rangle_\eta$, $\langle\cdot,\cdot\rangle_\xi$ the L^2_Σ-scalar products on ξ, η well-defined in view of (3.3). $\{e_i\}$ denotes the standard unit vectors in $\mathbb{R}^l$.

3.2.7 Corollary *For any k, l, a and b as above, $K_{a,b}$ is again a Fredholm operator with the adjoint operator*

$$(K_{a,b})^* = K^*_{b,a}\colon H^{1,2}_\Sigma(\eta)\times\mathbb{R}^l \to L^2_\Sigma(\xi)\times\mathbb{R}^k\,.$$

For $l=0$ we find $k\geqslant 0$, $a_1,\dots,a_k \in \bigcap_{p\geqslant 2} H^{1,p}_\Sigma(\eta)$ such that

$$K_a = K_{a,0}\colon H^{1,2}_\Sigma(\xi)\times\mathbb{R}^k \to L^2_\Sigma(\eta)$$

is onto.

PROOF. This is an immediate consequence of the last section's Fredholm theorems together with the regularity results. ∎

3.2.8 Definition *In view of Definition 3.2.3 and 3.2.5 we construct the gluing operation for two operators $K_{a,b}\colon H^{1,2}_\Sigma(\xi_1)\times\mathbb{R}^k \to L^2_\Sigma(\eta_1)\times\mathbb{R}^l$ and $L_{a',b'}\colon H^{1,2}_\Sigma(\xi_2)\times\mathbb{R}^k \to L^2_\Sigma(\eta_2)\times\mathbb{R}^l$ as*

$$K_{a,b}\#_{(\alpha,R)}L_{a',b'}\colon H^{1,2}_\Sigma(\xi_1\#_{(\alpha,R)}\xi_2)\times\mathbb{R}^{2k} \to L^2_\Sigma(\eta_1\#_{(\alpha,R)}\eta_2)\times\mathbb{R}^{2l},$$
$$(u,(h,h'))\mapsto \Big((K\#_{(\alpha,R)}L)u + \sum_{i=1}^k (h_i a_i)\#^o_{(\alpha,R)}(h'_i a'_i), (\beta^1_R u\bullet b, \beta^2_R u\bullet b')\Big),$$

that is, $K_{a,b}\#_{(\alpha,R)}L_{a',b'} = (K\#L)_{a\#a',b\#b'}$ with

$$a\#a' = (\beta^1_R a_1,\dots,\beta^1_R a_k, \beta^2_R a'_1,\dots,\beta^2_R a'_k),$$

etc.

We now give the central result about the linear gluing operation which analyzes the multi-bump solutions of the glued Fredholm operators, provided that the gluing parameter is large enough.

3.2.9 Proposition *Given*

$$\Pi_{(\alpha,R)}\colon L^2_\Sigma(\xi_1\#_{(\alpha,R)}\xi_2)\times\mathbb{R}^{2k}\to \ker(K_{a,b}\#_{(\alpha,R)}L_{a',b'}),$$

the L^2_Σ-orthogonal projection with respect to $\langle\cdot,\cdot\rangle_{\xi_1\#_{(\alpha,R)}\xi_2}$ and the standard product on $\mathbb{R}^{2k}$, the linear mapping between finite dimensional vector spaces

$$\phi_{(\alpha,R)}\colon \ker K_{a,b}\times\ker L_{a',b'} \to \ker(K_{a,b}\#_{(\alpha,R)}L_{a',b'}),$$
$$((v,h),(w,k))\mapsto \Pi_{(\alpha,R)}(v\#^o_{(\alpha,R)}w,(h,k))$$

is surjective, if $R=(\rho_1,\dots,\rho_r)$ is large enough, i.e. $\min(\rho_1,\dots,\rho_r)\geqslant\rho_o$ for some $\rho_o>1$ large enough.

Note that we already chose ρ_o large enough, so that $K\#_{(\alpha,R)}L$ is well-defined with respect to the asymptotically constant operators K, L as in Definition 3.2.3. In the following we say $R\geqslant\rho_o$ if $\min_k\rho_k\geqslant\rho_o$.

3.2.10 Corollary *Given R large enough, it holds*

$$\dim\ker(K_{a,b}\#_{(\alpha,R)}L_{a',b'}) \leqslant \dim\ker K_{a,b} + \dim\ker L_{a',b'},$$

and if $K_a=K_{a,0}$, $L_{a'}=L_{a',0}$ are onto, the same is true for

$$K_a\#_{(\alpha,R)}L_{a'} = (K\#_{(\alpha,R)}L)_{a\#a'}\colon H^{1,2}_\Sigma(\xi_1\#_{(\alpha,R)}\xi_2)\times\mathbb{R}^{2k}\to L^2_\Sigma(\eta_1\#_{(\alpha,R)}\eta_2)\,.$$

PROOF. The first statement follows from Proposition 3.2.9 and the second by virtue of the identities $(K_a)^* = K^*_{0,a}$, $(L_{a'})^* = L^*_{0,a'}$. Thus, $K^*_{0,a}$ and $L^*_{0,a'}$ are injective and therefore also $K^*_{0,a}\#_{(\alpha,R)}L^*_{0,a'} = (K\#L)^*_{0,a\#a'} = \big((K\#_{(\alpha,T)}L)_{a\#a'}\big)^*$. ∎

PROOF OF PROPOSITION 3.2.9. We prove the existence of a $c>0$ and a $\rho_o>1$ such that for $\min(\rho_1,\dots,\rho_r)\geqslant\rho_o$ it holds

$$\begin{aligned}\big\|(K_{a,b}\#_{(\alpha,R)}L_{a',b'})(u,(h,k))\big\|_{L^2_\Sigma(\eta_1\#\eta_2)\times\mathbb{R}^{2l}}&\\ \geqslant c\big|(u,(h,k))\big|_{L^2_\Sigma(\xi_1\#\xi_2)\times\mathbb{R}^{2k}}&\end{aligned}\tag{3.24}$$

for all

$$\begin{gathered}(u,(h,k))\in\mathcal{R}(\#^o_{(\alpha,R)})^\perp = \\ \{\,(u,(h,k))\in L^2_\Sigma(\xi_1\#\xi_2)\times\mathbb{R}^{2k}\,|\,\langle u, v\#^o w\rangle_{\xi_1\#\xi_2} + \langle h,x\rangle + \langle k,y\rangle = 0\\ \text{for all } (v,x)\in\ker K_{a,b},\ (w,y)\in\ker L_{a',b'}\,\}\,.\end{gathered}$$

This property implying the surjectivity of $\phi_{(\alpha,R)}$ will now be proven by contradiction. Let us assume sequences

$$\begin{gathered}R_n = (\rho_{1,n},\dots,\rho_{r,n}),\quad \rho_{k,n}\overset{n\to\infty}{\longrightarrow}\infty,\quad k=1,\dots,r, \\ (u_n,(h_n,k_n))\in\big(H^{1,2}_\Sigma(\xi_1\#_{(\alpha,R)}\xi_2)\times\mathbb{R}^{2k}\big)\cap\mathcal{R}(\#^o_{(\alpha,R)})^\perp\quad\text{with}\\ \|u_n\|^2_{L^2_\Sigma}+|h_n|^2+|k_n|^2=1\quad\text{for all } n\in\mathbb{N}\quad\text{and}\\ \big\|(K_{a,b}\#_{(\alpha,R)}L_{a',b'})(u_n,(h_n,k_n))\big\|_{L^2_\Sigma}\to 0\quad\text{for } n\to\infty\,.\end{gathered}\tag{3.25}$$

The main part of the proof consists of finding a $(v,h)\in\ker K_{a,b}$ such that

$$\|\beta^1_{R_{n_k}}u_{n_k}-v\|_{L^2_\Sigma(\xi_1)}\to 0,\quad h_{n_k}\to h\tag{3.26}$$

for a suitable subsequence $n_k \to \infty$. We estimate

$$\begin{aligned}
&\big\|K_{a,b}(\beta^1_{R_n}u_n, h_n)\big\|_{L^2_\Sigma(\eta_1)\times\mathbb{R}^l}\\
\leqslant&\big\|(K(\beta^1_{R_n}u_n) + \beta^1_{R_n}a \bullet h_n, \beta^1_{R_n}u_n \bullet b)\big\|_{L^2_\Sigma(\eta_1)\times\mathbb{R}^l}\\
&\quad + \big\|(1-\beta^1_{R_n})a \bullet h_n\big\|_{L^2_\Sigma(\eta_1)}\\
\leqslant&\big\|((K\#_{(\alpha,R_n)}L)(\beta^1_{R_n}u_n + \beta^2_{R_n}u_n) + \beta^1_{R_n}a \bullet h_n + \beta^2_{R_n}a' \bullet k_n,\\
&\quad \beta^1_{R_n}u_n \bullet b, \beta^2_{R_n}u_n \bullet b')\big\|_{L^2_\Sigma(\eta_1\#\eta_2)\times\mathbb{R}^{2l}}\\
&\quad + \big\|(1-\beta^1_{R_n})a \bullet h_n\big\|_{L^2_\Sigma(\eta_1)} + \big\|(1-\beta^2_{R_n})a' \bullet k_n\big\|_{L^2_\Sigma(\eta_2)}\\
\leqslant&\big\|(K_{a,b}\#_{(\alpha,R_n)}L_{a',b'})(u_n,(h_n,k_n))\big\|_{L^2_\Sigma(\eta_1\#\eta_2)\times\mathbb{R}^{2l}}\\
&\quad + \big\|(|d\beta^1_{R_n}| + |d\beta^2_{R_n}|)u_n\big\|_{L^2_\Sigma(\eta_1\#\eta_2)} + \big\|(1-\beta^1_{R_n})a\big\|_{L^2_\Sigma(\eta_1)}\cdot|h_n|\\
&\quad + \big\|(1-\beta^2_{R_n})a'\big\|_{L^2_\Sigma(\eta_2)}\cdot|k_n|\,.
\end{aligned}$$

If we show that $\sum_{k=1}^r \|u_n \circ \psi_k\|_{L^2_\Sigma([-2,2]\times S^1;\xi_1\#\xi_2)} \to 0$ for $n \to \infty$, we obtain from the above estimate

$$\big\|K_{a,b}(\beta^1_{R_n}u_n, h_n)\big\| \to 0 \quad \text{for } n \to \infty, \tag{3.27}$$

because $|h_n|, |k_n| \leqslant 1$, $a, a' \in L^2_\Sigma$ and $\rho_{k,n} \to \infty$ for $k = 1, \dots, r$.

Without loss of generality we may consider u_n with respect to the unitary trivializations Φ and Ψ as in Definition 3.2.3, i.e.

$$u_{n,k} = u_n \circ \psi_k \colon [-\rho_{k,n}, \rho_{k,n}] \times S^1 \to \mathbb{R}^{2n}\,.$$

Let $\beta_\tau \colon \mathbb{R} \to [0,1]$, $\tau > 0$, be the cut-off function

$$\begin{aligned}
&\beta_\tau(s) = \beta^o(\tfrac{s}{\tau}), \quad \beta^o(s) = \beta^+(s-2)\cdot\beta^-(s+2),\\
&(\tfrac{d}{ds}\beta_\tau)(s) = \frac{1}{\tau}\frac{d\beta^o}{ds}(\frac{s}{\tau}), \quad \operatorname{supp}\beta_\tau \subset [-2\tau, 2\tau], \quad \beta_{\tau|[-\tau,\tau]} \equiv 1\,.
\end{aligned}$$

Given $n_o \in \mathbb{N}$ such that $\rho_{k,n} \geqslant \tau$ for all $n \geqslant n_o$, $k = 1, \dots, r$, we find

$$\big\|(K\#_{(\alpha,R_n)}L)_{triv}(\beta_\tau u_{n,k})\big\|_{L^2(\mathbb{R}\times S^1,\mathbb{R}^{2n})} = \big\|(\tfrac{\partial}{\partial s} + J_o\tfrac{\partial}{\partial t} + S^1_{i_k})(\beta_\tau u_{n,k})\big\|_{L^2},$$

because $S^K_{i_k}$, $S^L_{j_k}$ are constant with respect to s and identical over $\operatorname{supp}\beta_\tau$ for n large. Via the isomorphism

$$\tfrac{\partial}{\partial s} + J_o\tfrac{\partial}{\partial t} + S^1_{i_k} \colon H^{1,2}(\mathbb{R}\times S^1) \xrightarrow{\cong} L^2(\mathbb{R}\times S^1)$$

and the estimate with a constant $c = c(\beta^o)$

$$\begin{aligned}
&\big\|(K\#_{(\alpha,R_n)}L)_{\mathrm{triv}}(\beta_\tau u_{n,k})\big\|_{L^2}\\
\leqslant&\frac{c}{\tau}\|u_{n,k}\|_{L^2} + \big\|\beta_\tau\cdot(K\#_{(\alpha,R_n)}L)_{\mathrm{triv}}(u_{n,k})\big\|_{L^2}\\
\leqslant&\frac{c}{\tau}\|u_{n,k}\|_{L^2} + \big\|\beta_\tau(K_{a,b}\#_{(\alpha,R_n)}L_{a',b'})_{\mathrm{triv}}(u_n,h_n,k_n)\big\|_{L^2}\\
&\quad + \big\|\beta_\tau(\beta^1_{R_n}a \bullet h_n + \beta^2_{R_n}a' \bullet k_n)\big\|_{L^2},
\end{aligned}$$

where the second and third term tend to zero and $\|u_{n,k}\|_{L^2} \leqslant 1$ uniformly in n, we obtain $\overline{\lim}_{n\to\infty}\|\beta_\tau u_{n,k}\|_{H^{1,2}} = \mathrm{O}(\frac{1}{\tau})$ for all $\tau > 0$, such that

$$u_{n,k} \to 0 \quad \text{in} \quad H^{1,2}_{loc}(\mathbb{R}\times S^1) \tag{3.28}$$

for all $k = 1, \dots, r$. Consequently we have proven (3.27) and we derive from the Semi-Fredholm property of $K_{a,b}$ and the boundedness of (u_n, h_n) the existence of a subsequence $n_k \to \infty$, $k \to \infty$, such that (3.26) holds for some $(v,h) \in \ker K_{a,b}$. We proceed analogously for $L_{a',b'}$, i.e. there is a $(w,k) \in \ker L_{a',b'}$ such that up to the choice of a suitable subsequence, say $(u_k) = (u_{n_k})$,

$$\|\beta^2_{R_n}u_n - w\|_{L^2_\Sigma(\xi_2)} \to 0, \quad k_n \to k \tag{3.29}$$

for $n \to \infty$. We deduce now the contradiction by combining (3.26) and (3.29),

$$\begin{aligned}
1 &= \lim_{n\to\infty} \|u_n\|^2_{L^2_\Sigma} + |h_n|^2 + |k_n|^2\\
&= \lim_{n\to\infty} \langle(\beta^1_{R_n})^2u_n + (\beta^2_{R_n})^2u_n + (1-(\beta^1_{R_n})^2-(\beta^2_{R_n})^2)u_n, u_n\rangle_{L^2_\Sigma(\xi_1\#\xi_2)}\\
&\qquad + \langle h_n, h_n\rangle + \langle k_n, k_n\rangle\\
&= \lim_{n\to\infty} (\langle\beta^1_{R_n}v, u_n\rangle_{L^2_\Sigma(\xi_1)} + \langle h, h_n\rangle + \langle\beta^2_{R_n}w, u_n\rangle_{L^2_\Sigma(\xi_2)} + \langle k, k_n\rangle)\\
&\qquad + \lim_{n\to\infty}\sum_{k=1}^r \big\|(1-\beta^+-\beta^-)u_{n,k}\big\|^2_{L^2([-2,2]\times S^1)}\\
&= \lim_{n\to\infty} (\langle v\#^o_{(\alpha,R_n)}w, u_n\rangle_{L^2_\Sigma(\xi_1\#\xi_2)} + \langle h, h_n\rangle + \langle k, k_n\rangle)\\
&= 0, \quad \text{because} \quad (u_n,(h_n,k_n)) \in \mathrm{R}(\#^o_{(\alpha,R_n)})^\perp \quad \text{f.a. } n \in \mathbb{N}.
\end{aligned}$$

Hence, (3.24) is proven. ∎

We now proceed towards the index additivity. Since we are dealing with Fredholm operators K, L, we certainly always find $a, a' \in \times^k_{\nu=1}L^2_\Sigma(\eta_i)$ such that K_a, $L_{a'}$ are onto. In principle, $\operatorname{span}(a_1, \dots, a_k)$ and $\operatorname{span}(a'_1, \dots, a'_k)$ must contain the cokernels of K and L.

3.2.11 Corollary *Suppose*

$$K_a \colon H^{1,2}_\Sigma(\xi_1)\times\mathbb{R}^k \to L^2_\Sigma(\eta_1) \quad \textit{and} \quad L_{a'} \colon H^{1,2}_\Sigma(\xi_2)\times\mathbb{R}^k \to L^2_\Sigma(\eta_2)$$

are onto. Then,

$$\phi_{(\alpha,R)} \colon \ker K_a \times \ker L_{a'} \to \ker(K\#_{(\alpha,R)}L)_{a\#a'}$$

is an isomorphism, if $R \geqslant R_o$ is large enough.

Proof. Let $R_o > 0$ be large enough so that in view of Corollary 3.2.10 also $K_a\#_{(\alpha,R)}L_{a'}$ is surjective for all $R \geqslant R_o$. In other terms,

$$(K_a)^* = K^*_{0,a}, \quad (L_{a'})^* = L^*_{0,a'} \quad \text{and} \quad (K^*\#_{(\alpha,R)}L^*)_{0,a\#a'}$$

are all injective. Moreover, there is a $c > 0$ independent of R, such that for all $R \geqslant R_o$

$$\begin{aligned} &\|u\|_{L^2_\Sigma(\eta_1 \#_{(\alpha,R)}\eta_2)} \\ &\leqslant c\big\|(K^*\#_{(\alpha,R)}L^*)_{0,a\#a'}(u)\big\|_{L^2_\Sigma(\xi_1\#_{(\alpha,R)}\xi_2)\times\mathbb{R}^{2k}} \end{aligned} \tag{3.30}$$

for all $u \in H^{1,2}_\Sigma(\eta_1\#_{(\alpha,R)}\eta_2)$. Namely, suppose there exists sequences R_n and u_n with

$$\begin{aligned} R_n \to \infty, \quad &\|u_n\|_{L^2_\Sigma(\xi_1\#_{(\alpha,R_n)}\xi_2)} = 1 \\ \text{and} \quad &\big\|(K^*\#_{(\alpha,R_n)}L^*)_{0,a\#a'}(u_n)\big\|_{L^2_\Sigma\times\mathbb{R}^{2k}} \to 0\,. \end{aligned}$$

Then, proceeding as in the proof of Proposition 3.2.9, we deduce the existence of a subsequence $(n_k) \subset \mathbb{N}$ such that

$$\begin{aligned} \beta^1_{T_{n_k}} u_{n_k} \xrightarrow{L^2_\Sigma} x \in \ker K^*_{0,a}, \quad &\beta^2_{T_{n_k}} u_{n_k} \xrightarrow{L^2_\Sigma} y \in \ker L^*_{0,a'}, \\ (1-\beta^1_{T_{n_k}} - \beta^2_{T_{n_k}})u_{n_k} &\xrightarrow{H^{1,2}_{loc}} 0\,. \end{aligned}$$

By assumption, $\ker K^*_{0,a} = \ker L^*_{0,a'} = \{0\}$, hence it follows $u_{n_k} \xrightarrow{L^2_\Sigma} 0$ in contradiction to $\|u_n\|_{L^2_\Sigma} = 1$.

It remains to prove that for R large, $\phi_{(\alpha,R)}$ is injective. Let us assume the contrary, i.e. sequences $R_n \to \infty$, $((u_n,h_n),(v_n,k_n)) \subset \ker K_a \times \ker L_{a'}$ satisfying

$$\begin{aligned} \|(u_n,h_n)\|^2_{L^2_\Sigma(\xi_1)\times\mathbb{R}^k} + \|(v_n,k_n)\|^2_{L^2_\Sigma(\xi_2)\times\mathbb{R}^k} = 1 \quad \text{with} \\ (\beta^1_{R_n}u_n + \beta^2_{R_n}v_n,(h_n,k_n)) \in \ker(K_a\#_{(\alpha,R_n)}L_{a'})^\perp \quad \text{f.a. } n \geqslant 0\,. \end{aligned}$$

Since, for $A = \ker K_a\#_{(\alpha,R_n)}L_{a'}$, $(\ker A)^\perp = \mathcal{R}(A^*)$ and A^* is injective, we find unique $\phi_n \in H^{1,2}_\Sigma(\eta_1\#_{(\alpha,R_n)}\eta_2)$, $n \geqslant 0$, with

$$(\beta^1_{R_n}u_n + \beta^2_{R_n}v_n,(h_n,k_n)) = (K^*\#_{(\alpha,R_n)}L^*)_{0,a\#a'}(\phi_n) \tag{3.31}$$

for all $n \geqslant 0$. We finally are led to the contradiction

$$\begin{aligned} 1 &= \lim_{n\to\infty} \Big\langle (\beta^1_{R_n}u_n + \beta^2_{R_n}v_n,(h_n,k_n)),(K_a\#_{(\alpha,R_n)}L_{a'})^*(\phi_n)\Big\rangle_{L^2_\Sigma(\xi_1\#\xi_2)\times\mathbb{R}^{2k}} \\ &= \lim_{n\to\infty} \Big\langle (K_a\#_{(\alpha,R_n)}L_{a'})(\beta^1_{R_n}u_n + \beta^2_{R_n}v_n,(h_n,k_n)),\phi_n\Big\rangle_{L^2_\Sigma} \\ &\overset{(3.30)}{\leqslant} \lim_{n\to\infty}\Big(\big\|K(\beta^1_{R_n}u_n) + \beta^1_{R_n}a\bullet h_n\big\|_{L^2_\Sigma(\xi_1)} + \big\|L(\beta^2_{R_n}v_n) + \beta^2_{R_n}a'\bullet k_n\big\|_{L^2_\Sigma(\xi_2)}\Big) \\ &\qquad \cdot c\big\|(K_a\#_{(\alpha,R_n)}L_{a'})^*(\phi_n)\big\|_{L^2_\Sigma\times\mathbb{R}^{2k}} \\ &= 0, \end{aligned}$$

because $\big\|K(\beta^1_{R_n}u_n) + \beta^1_{R_n}a\bullet h_n\big\| \to \big\|K_a(u_n,h_n)\big\| = 0$ due to $\|u_n\|^2 + |h_n|^2 \leqslant 1$, analogously for $L_{a'}$ and $\big\|(K_a\#_{(\alpha,R_n)}L_{a'})^*(\phi_n)\big\| \leqslant 1$ due to (3.31). Thus, the proof is complete. ■

Let us draw the conclusion in form of the

3.2.12 Theorem *Given the classes of admissible $\overline{\partial}$-operators F_{Ω_1} and F_{Ω_2} as above together with the gluing data (α,R), the Fredholm indices for any operators $K \in F_{\Omega_1}$, $L \in F_{\Omega_2}$, $K\#L \in F_{\Omega_1\#_\alpha\Omega_2}$ satisfy*

$$\operatorname{ind}(K\#L) = \operatorname{ind}K + \operatorname{ind}L\,.$$

Proof. As we already noticed above, we may consider any asymptotically constant operators

$$K \in F_{\Omega_1} \subset \mathcal{L}(C^\infty(\xi_1);C^\infty(\eta_1)), \quad L \in F_{\Omega_2} \subset (C^\infty(\xi_2);C^\infty(\eta_2))\,.$$

Next, we choose $a \in \times^k_{i=1}L^2_\Sigma(\eta_1)$, $a' \in \times^k_{i=1}L^2_\Sigma(\eta_2)$ such that K_a and $L_{a'}$ are surjective. In view of Corollary 3.2.10 and 3.2.11, we obtain the surjective operator $K_a\#_{(\alpha,R)}L_{a'} = (K\#_{(\alpha,R)}L)_{a\#a'}$ together with the isomorphism

$$\phi_{(\alpha,R)}\colon \ker K_a \times \ker L_{a'} \xrightarrow{\cong} \ker(K\#_{(\alpha,R)}L)_{a\#a'} \tag{3.32}$$

for large $R > 0$. We now consider the exact sequence of finite dimensional vector spaces

$$0 \to \ker K \xrightarrow{i} \ker K_a \xrightarrow{\varphi} \mathbb{R}^k \xrightarrow{j} \operatorname{coker}K \to 0$$

with $i(u) = (u,0)$ the canonical inclusion, $\varphi(u,h) = h$, and $j(h) = K_a(0,h) + \mathcal{R}(K)$. We obviously have exactness $\ker\varphi = \operatorname{im}i$ and $\ker j = \operatorname{im}\varphi$ by construction, and j is onto, because K_a is onto. From the exactness we gain the identity

$$\dim\ker K + k = \dim\ker K_a + \dim\operatorname{coker}K,$$

and conclude

$$\operatorname{ind}K = \dim\ker K_a - k\,. \tag{3.33}$$

We have analogous identities for $L_{a'}$ and $(K\#_{(\alpha,R)}L)_{a\#a'}$, so that we finally obtain

$$\begin{aligned} \operatorname{ind}K\#_{(\alpha,R)}L &\overset{(3.33)}{=} \dim\ker(K\#_{(\alpha,R)}L)_{a\#a'} - 2k \\ &\overset{(3.32)}{=} \dim\ker K_a - k + \dim\ker L_{a'} - k \\ &\overset{(3.33)}{=} \operatorname{ind}K + \operatorname{ind}L\,. \end{aligned}$$

Thus, the proof is complete. ■

We certainly may ask for a similar relationship for the above defined contraction operation with gluing data (α,R). By absolutely analogous methods we are able to prove

3.2.13 Theorem *Let $K \in F_\Omega$ be a surjective, admissible $\overline{\partial}$-operator, which is asymptotically constant over the cylindrical ends and admissible for the $\mathrm{tr}_{(\alpha,R)}$-operation. Then, for R large,*

$$\phi_{(\alpha,R)}\colon \ker K \to \ker \mathrm{tr}_{(\alpha,R)} K, \quad u \mapsto \Pi^{L^2}_{\ker}(\beta_{(\alpha,R)} u),$$

is an isomorphism. In particular,

$$\mathrm{ind}\, K = \mathrm{ind}\, M$$

for all $K \in F_\Omega(\Sigma)$, $M \in F_{\mathrm{tr}_{(\alpha,R)}\Omega}(\mathrm{tr}_{(\alpha,R)}\Sigma)$.

Here, $\beta_{(\alpha,R)} \in C^\infty(\mathrm{tr}_{(\alpha,R)}\Sigma, [0,1])$ is a cut-off function which is identically 1 on the complement of the glued cylindrical ends, and on the glued cylindrical end indexed by i_k and j_k, with

$$\psi_k\colon [-\rho_k, \rho_k] \times S^1 \to \psi_{i_k}(Z^+)\#_{(\alpha,R)}\psi_{j_k}(Z^-)$$

as in Definition 3.2.5, it is determined by

$$\beta_{(\alpha,R)} \circ \psi_k(s,t) = \begin{cases} \beta^+(s), & s \leqslant 0, \\ \beta^-(s), & s \geqslant 0\,. \end{cases}$$

Again, we identify implicitly $u \circ \psi_{i_k}(s,t) = u \circ \psi_k(s - \epsilon_{i_k}\rho_k, t)$. $\Pi^{L^2}_{\ker}$ denotes the orthogonal projection onto $\ker \mathrm{tr}_{(\alpha,R)} K$ with respect to the inner product of $L^2_\Sigma(\mathrm{tr}_{(\alpha,R)}\xi)$.

3.3 The Computation of the Index

The last step in the analysis of the occuring Fredholm operators concerns the computation of the index. The idea is to reduce this problem for general model surfaces with arbitrary asymptotic data to a sufficiently small class of simpler operators. An important element herein will be the theorem of Riemann-Roch providing the Fredholm index for closed surfaces. This is due to the fact that any $\overline{\partial}$-operator over a closed surface is equivalent to a Cauchy-Riemann operator in the classical sense. That is, we can transform all data, j on Σ, J on the bundle $\xi \to \Sigma$ and the zero order term without changing the Fredholm index, such that we deal with a classical complex analytical problem. Thus the Riemann-Roch formula adapted to our terms reads

$$\mathrm{ind}\, F = 2n(1-g) + 2c_1(\xi)[\Sigma]\,.$$

Here, g is the genus of Σ, $c_1(\xi)$ the first Chern class of the complex bundle[1] ξ of rank n, and $[\Sigma]$ is the fundamental class of Σ. Note that we give the index over the field of real numbers. However, it is also possible to compute the index merely by means of the available Fredholm analysis. We nevertheless present the relation with the classical Riemann-Roch formula.

[1] for algebraic geometers the negative canonical class $-K$.

We are able to retrieve the index of an admissible $\overline{\partial}$-operator over a model surface Σ with ν boundary components by using the index additivity from the last section, provided that we already know the index of operators over model surfaces of the simplest type, that are needed to compactify Σ by gluing. These are the disks $D^\pm \approx \{|z| < 1\}$ with either orientation of the boundary. Since any rank-$2n$ bundle with a Hermitian structure over $D^\pm$ is unitarily isomorphic to $D^\pm \times \mathbb{C}^n$ with the standard structure (ω_o, J_o), the index computation over $D^\pm$ will be sufficient for admissible $\overline{\partial}$-operators

$$K\colon H^{1,2}_\Sigma(D^\pm \times \mathbb{C}^n) \to L^2_\Sigma(D^\pm \times \mathbb{C}^n)\,.$$

In the following, we call the model surface $D^\pm$ a cap.

3.3.1 The Computation for the Cap

First of all we have to construct the function spaces for the Fredholm analysis. Let us start with the analysis for the so-called 'exit' cap, that is the model surface D^+ with one positively oriented cylindrical end.

Since the Fredholm analysis asymptotically requires the cylindrical conformal structure and measure, i.e. of $Z = \mathbb{R} \times S^1 = \mathbb{C}/i\mathbb{Z}$ we have to find an extension to the interior compact complement. We will use the fixed canonical conformal structure i on the set $D^+ = \mathbb{C}$. But we have to equip D^+ with a different measure which combines the Lebesgue measure of $\mathbb{C}$ on the interior with the cylindrical measure on the 'end'. This analysis will be carried out by using the polar coordinates (r, φ).

3.3.1 Definition *We define the standard exit cap D^+ by gluing the positive half cylinder to the unit disk D^2,*

$$D^+ = D^2 \cup_f Z^+, \quad D^2 = \{|z| \leqslant 1\}, \quad Z^+ = [0,\infty) \times S^1 \subset \mathbb{C}/i\mathbb{Z}\,,$$
$$f\colon \mathbb{C}/i\mathbb{Z} \xrightarrow{\approx} \mathbb{C} \setminus \{0\}, \quad (s,t) \mapsto \mathrm{e}^{2\pi(s+it)} = r\,\mathrm{e}^{i\varphi}\,.$$

Using the polar coordinates (r,φ) on $D^+ \setminus \{0\}$ we define the volume form $\sigma = \gamma^+(r) dr \wedge d\varphi$ on D^+ by $\gamma^+ \in C^\infty((0,\infty),(0,\infty))$ given as

$$\gamma^+(r) = \begin{cases} r, & r \leqslant \frac{1}{2}, \\ \frac{1}{4\pi^2 r}, & r > 1, \end{cases}$$

with smooth, positive interpolation on $[\frac{1}{2}, 1]$. By virtue of this measure $d\mu = \gamma^+ dr d\varphi$ we obtain the function spaces

$$L^2_\Sigma(D^+ \times \mathbb{C}^n) = \{\, u \in L^2_{loc}(D^+ \times \mathbb{C}^n) \mid \textstyle\int_{D^+} |u|^2 d\mu < \infty \,\},$$
$$H^{1,2}_\Sigma(D^+ \times \mathbb{C}^n) = \{\, u \in H^{1,2}_{loc} \mid u, u' \in L^2_\Sigma(D^+ \times \mathbb{C}^n) \,\}\,.$$

In terms of the polar coordinates on $D^+ \setminus \{0\}$ we find

3.3.2 Auxiliary Proposition *We have $u \in H^{1,2}_\Sigma(D^+ \times \mathbb{C}^n)$ exactly if $u \in H^{1,2}_{loc}(D^+ \times \mathbb{C}^n)$ and*

$$\int_1^\infty \int_0^{2\pi} (|u|^2 + 4\pi^2 r^2 |\tfrac{\partial}{\partial r} u|^2 + 4\pi^2 |\tfrac{\partial}{\partial \varphi} u|^2) \tfrac{1}{4\pi^2 r} dr d\varphi < \infty .$$

PROOF. Transform the condition

$$\int_0^\infty \int_0^{2\pi} (|u|^2 + |\tfrac{\partial}{\partial s} u|^2 + |\tfrac{\partial}{\partial t} u|^2) ds dt < \infty$$

using $\frac{\partial}{\partial s} = 2\pi r \frac{\partial}{\partial r}$ and $\frac{\partial}{\partial t} = 2\pi \frac{\partial}{\partial \varphi}$. ∎

Note that the factor 2π appears because of the choices of the asymptotic cylindrical structure $\mathbb{C}/i\mathbb{Z}$. This is due to the convention to consider 1-periodic t-curves as generators of Floer homology. We have chosen a conformal coordinate change from (r, φ) to (s, t) so that we deal with the fixed complex structure i on $\mathbb{C}$. In polar coordinates, this is expressed as

$$i(\tfrac{\partial}{\partial r}) = \tfrac{1}{r} \tfrac{\partial}{\partial \varphi} .$$

Associated to the trivial rank-$2n$ bundle $D^+ \times \mathbb{C}^n$ with the standard Hermitian structure (ω_o, J_o), $J_o = i \oplus \ldots \oplus i$, we have

$$X^{J_o}(D^+ \times \mathbb{C}^n) = X^{J_o} = \{\, \phi \in T^*D^+ \otimes \mathbb{C}^n \mid \phi \circ i = -J_o \circ \phi \,\},$$

and we define the $\overline{\partial}$-operator in question as follows.

3.3.3 Definition *Given $\beta \in C^\infty(D^+, [0,1])$ with*

$$\beta(r) = \begin{cases} 0, & r \leqslant 2, \\ 1, & r \geqslant 3, \end{cases} \quad \text{and strictly monotone on } (2,3),$$

and given $\omega \in \mathbb{R}$, we define

$$K^+_\omega \colon C^\infty(D^+ \times \mathbb{C}^n) \to C^\infty(X^{J_o}),$$
$$K^+_\omega = \overline{\partial}_{J_o} + \beta ds \otimes (\omega \,\mathrm{Id}_{\mathbb{C}^n}) = \overline{\partial}_{J_o} + \tfrac{\beta}{2\pi r} dr \otimes (\omega \,\mathrm{Id}_{\mathbb{C}^n}) .$$

Considering the induced linear operator $K^+_\omega \colon H^{1,2}_\Sigma(D^+ \times \mathbb{C}^n) \to L^2_\Sigma(X^{J_o})$ we find a suitable trivialization as follows. Let $v^+ \in C^\infty(TD^+_{|D^+ \setminus \{0\}})$ be a smooth and L^∞-bounded extension of the vector field $\frac{\partial}{\partial s}$ over the cylindrical end, for example $v(r, \varphi) = \alpha(r) \frac{\partial}{\partial r}$ with $\alpha \in C^\infty((0,\infty), \mathbb{R})$ given by

$$\alpha(r) = \sqrt{\frac{r}{\gamma^+(r)}} = \begin{cases} 1, & r \leqslant \frac{1}{2}, \\ 2\pi r, & r \geqslant 2, \end{cases} .$$

In view of the canonical L^2-norm $\|\cdot\|_{J,L^2}$ on $L^2_\Sigma(X^J)$, see 3.3 on page 39,

$$\|\phi\|^2_{J,L^2} = 2 \int_{\mathbb{C}} |\phi \cdot \tfrac{\partial}{\partial r}|^2_J r dr d\varphi = 2 \textstyle\int_Z |\phi \cdot \tfrac{\partial}{\partial s}|^2_J ds dt$$

this provides us with the well-defined isomorphism

$$\lambda \colon L^2_\Sigma(X^{J_o}) \xrightarrow{\cong} L^2_\Sigma(D^+ \times \mathbb{C}^n),$$
$$\phi \mapsto \phi \cdot v,$$

thus yielding

$$\lambda \circ K^+_\omega \colon H^{1,2}_\Sigma(\mathbb{C}^n) \to L^2_\Sigma(\mathbb{C}^n),$$
$$\lambda \circ K^+_\omega = \alpha\big(\tfrac{\partial}{\partial r} + J_o \tfrac{1}{r} \tfrac{\partial}{\partial \varphi} + \tfrac{\beta}{2\pi r} \omega \,\mathrm{Id}\big) .$$

Let us now ask whether K^+_ω is admissibly defined.

3.3.4 Auxiliary Proposition *For every $\omega \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ the operator K^+_ω is an admissible $\overline{\partial}$-operator and $K^+_\omega \in F_{\Omega(\omega)}(D^+ \times \mathbb{C}^n)$ with the asymptotic data $\Omega(\omega) = \{\omega \,\mathrm{Id}_{\mathbb{C}^n}\}$.*

PROOF. Clearly, K^+_ω is a $\overline{\partial}$-operator. It remains to verify that the selfadjoint operator $S = i\frac{\partial}{\partial t} + \omega \colon H^{1,2}(S^1, \mathbb{C}^n) \to L^2(S^1, \mathbb{C}^n)$, $\omega \in \mathbb{R}$, is regular. This is equivalent to the demand that $\omega \notin 2\pi\mathbb{Z}$. ∎

Note that, due to λ, the Fredholm operators K^+_ω and $\lambda \circ K^+_\omega$ have identical kernels and $\operatorname{coker}(\lambda \circ K^+_\omega) \subset C^\infty(D^+ \setminus \{0\}) \cap L^\infty(D^+)$ is isomorphic to $\operatorname{coker} K^+_\omega$. Hence, the replacement $\lambda \circ K^+_\omega$ for K^+_ω will simplify the proof of

3.3.5 Proposition *Given $\omega \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ we have*

$$\dim_{\mathbb{R}} \ker K^+_\omega = 2n \cdot \#\{\, k \in \mathbb{Z} \mid 0 \leqslant k < \frac{\omega}{2\pi} \,\},$$
$$\dim_{\mathbb{R}} \operatorname{coker} K^+_\omega = 2n \cdot \#\{\, k \in \mathbb{Z} \mid \frac{\omega}{2\pi} < k \leqslant -1 \,\} .$$

In particular, we have

$$\operatorname{ind} K^+_\omega = \operatorname{ind} K^+_{(2l-1)\pi} = n \cdot 2l \quad \text{for} \quad l - 1 < \frac{\omega}{2\pi} < l,\ l \in \mathbb{Z} .$$

PROOF. In the following we write K^+_ω instead of $\lambda \circ K^+_\omega$. Let us recall the regularity result from Proposition 3.1.26 that $\ker K_\omega$, $\operatorname{coker} K_\omega \subset C^\infty(D^+ \setminus \{0\}, \mathbb{C}^n)$ with exponential $H^{1,2}(S^1)$-decay at infinity. Moreover, we may consider without loss of generality $n = 1$, because the zero order term $\beta\omega \,\mathrm{Id}_{\mathbb{C}^n}$ of the linear differential operator is in diagonal form in accordance with the splitting $J_o = i \oplus \ldots \oplus i$ on $\mathbb{C}^n$. Thus, any $u \in \ker K^+_\omega$ can be represented as a Fourier series

$$u = \sum_{k \in \mathbb{Z}} A_k(r)\, e^{ik\varphi}, \quad A_k \in C^\infty((0,\infty), \mathbb{C}^n), \tag{3.34}$$

where, in view of Auxiliary Proposition 3.3.2, $u \in H^{1,2}_\Sigma(D^+)$ provides the necessary conditions

$$\int_0^{\frac{1}{2}} \Big[(1+\frac{k^2}{r^2})|A_k|^2 + |A_k'|^2\Big] r\,dr < \infty$$

$$\text{and} \quad \int_1^\infty \Big[(1+4\pi^2k^2)|A_k|^2 + 4\pi^2 r^2 |A_k'|^2\Big]\frac{1}{4\pi^2 r}dr < \infty \tag{3.35}$$

for all $k \in \mathbb{Z}$. From the identity for $u \in \ker K^+_\omega$,

$$0 = K^+_\omega u = \begin{cases} (r + \frac{i}{r}\frac{\partial}{\partial\varphi}) \sum\limits_{k\in\mathbb{Z}} A_k\,\mathrm{e}^{ik\varphi}, & r \leqslant \frac{1}{2}, \\ (2\pi r\frac{\partial}{\partial r} + 2\pi i \frac{\partial}{\partial\varphi} + \beta\omega) \sum\limits_{k\in\mathbb{Z}} A_k\,\mathrm{e}^{ik\varphi}, & r \geqslant 3, \end{cases}$$
$$= \sum_{k\in\mathbb{Z}} \begin{cases} (A_k' - \frac{k}{r}A_k)\,\mathrm{e}^{ik\varphi}, & r \leqslant \frac{1}{2}, \\ (2\pi r A_k' - 2\pi k A_k + \omega A_k)\,\mathrm{e}^{ik\varphi}, & r \geqslant 3, \end{cases}$$

we obtain equivalently

$$A_k' = A_k \cdot \begin{cases} k r^{-1}, & 0 < r \leqslant \frac{1}{2}, \\ (k - \frac{\omega}{2\pi}) r^{-1}, & r \geqslant 3, \end{cases} \quad k \in \mathbb{Z}\,. \tag{3.36}$$

This leads to

$$A_k(r) = \begin{cases} \mathrm{const}\, r^k, & 0 < r \leqslant \frac{1}{2}, \\ \mathrm{const}\, r^{(k-\frac{\omega}{2\pi})}, & r \geqslant 3, \end{cases}, \quad k \in \mathbb{Z},$$

and the necessary condition (3.35) implies

$$\int_0^{\frac{1}{2}} (r^{2k+1} + 2k^2 r^{2k-1})\,dr < \infty \quad \text{and}$$
$$\int_3^\infty ((1+4\pi^2k^2) + (k - \frac{\omega}{2\pi})^2) r^{2k-\frac{\omega}{\pi}-1}\,dr < \infty\,.$$

Therefore, we deduce as a necessary condition for $u \in \ker K^+_\omega$

$$k \geqslant 0 \quad \text{and} \quad 2\pi k < \omega\,. \tag{3.37}$$

Thus, if $\omega < 0$ we obtain $\ker K^+_\omega = \{0\}$. Moreover, since $A_k \not\equiv 0$ for at most finitely many $k \in \mathbb{Z}$, and since any $k \geqslant 0$ gives rise to a solution of $\overline{\partial}_{j^+,J_o} u = 0$ at 0, (3.37) also serves as a sufficient condition for $u \in H^{1,2}_\Sigma(D^+)$. Consequently, we have proven the identity

$$\dim_{\mathbb{C}} \ker K_\omega = n \cdot \#\{\, k \in \mathbb{Z} \,|\, 0 \leqslant k < \frac{\omega}{2\pi} \,\}\,. \tag{3.38}$$

Instead of considering the adjoint operator of K^+_ω we directly compute the cokernel of K^+_ω. Given any $v \in \operatorname{coker} K^+_\omega \subset L^2_\Sigma(D^+, \mathbb{C}^n)$ we have

$$0 = \int_o^\infty \int_0^{2\pi} \langle K^+_\omega \phi, v\rangle d\mu = \int_0^\infty \int_0^{2\pi} \langle \alpha(\tfrac{\partial}{\partial r} + J_o \tfrac{1}{r}\tfrac{\partial}{\partial\varphi} + \tfrac{\beta}{2\pi r}\omega)\phi, v\rangle \gamma^+ dr\,d\varphi$$

for all $\phi \in H^{1,2}_\Sigma(D^+, \mathbb{C}^n)$. We obtain that $v \in \operatorname{coker} K^+_\omega$ exactly if

$$v \in C^\infty(D^+ \setminus \{0\}, \mathbb{C}^n) \cap L^\infty(D^+, \mathbb{C}^n), \qquad \int_0^\infty\int_0^{2\pi} |v|^2 \gamma^+ dr\,d\varphi < \infty \quad \text{and}$$
$$(\alpha(-\tfrac{\partial}{\partial r} + J_o \tfrac{1}{r}\tfrac{\partial}{\partial\varphi}) - \alpha\tfrac{(\gamma^+)'}{\gamma^+} + \beta\omega - \alpha')v = 0\,.$$

An analogous reduction to $n = 1$, $v = \sum_{k\in\mathbb{Z}} A_k\,\mathrm{e}^{ik\varphi}$, leads to the identities

$$0 = \begin{cases} (-\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{1}{r}) \sum\limits_{k\in\mathbb{Z}} A_k\,\mathrm{e}^{ik\varphi}, & 0 < r \leqslant \frac{1}{2}, \\ (-2\pi r\frac{\partial}{\partial r} + 2\pi i\frac{\partial}{\partial\varphi} + \omega) \sum\limits_{k\in\mathbb{Z}} A_k\,\mathrm{e}^{ik\varphi}, & r \geqslant 3, \end{cases}$$
$$= \sum_{k\in\mathbb{Z}} \begin{cases} (-A_k' - \frac{k+1}{r}A_k)\,\mathrm{e}^{ik\varphi}, & 0 < r \leqslant \frac{1}{2}, \\ (-2\pi r A_k' + (\omega - 2\pi k)A_k)\,\mathrm{e}^{ik\varphi}, & r \geqslant 3\,. \end{cases}$$

Thus we deduce

$$A_k(r) = \begin{cases} \mathrm{const}\cdot r^{-k-1}, & 0 < r \leqslant \frac{1}{2}, \\ \mathrm{const}\cdot r^{\frac{\omega}{2\pi}-k}, & r \geqslant 3\,. \end{cases}$$

The necessary condition

$$\int_0^\infty |A_k|^2\gamma^+ dr < \infty, \text{ i.e. } \int_0^{\frac{1}{2}} r^{-2k-1} dr < \infty \quad \text{and} \quad \int_3^\infty r^{\frac{\omega}{\pi}-2k-1} dr < \infty,$$

yields $k \leqslant -1$ and $\frac{\omega}{2\pi} < k$. For the converse, this is also sufficient for $A_k\,\mathrm{e}^{ik\varphi} \in \operatorname{coker} K^+_\omega$ and we obtain the formula

$$\dim_{\mathbb{C}} \operatorname{coker} K^+_\omega = n \cdot \#\{\, k \in \mathbb{Z} \,|\, \frac{\omega}{2\pi} < k \leqslant -1 \,\}\,. \tag{3.39}$$

The index formula follows immediately from (3.38) and (3.39). ∎

Next, we study the reverse situation, that is the 'entry' cap D^-,

$$D^- = Z^- \cup_{\bar f} D^2, \quad Z^- = (-\infty, 0] \times S^1 \subset \mathbb{C}/i\mathbb{Z}\,,$$
$$\bar f\colon \mathbb{C}/i\mathbb{Z} \xrightarrow{\approx} \mathbb{C} \setminus \{0\}, \quad (s,t) \mapsto \mathrm{e}^{2\pi(-s+it} = r\,\mathrm{e}^{\varphi}\,.$$

Of course, we could derive the index immediately from D^+, S^2 and the index additivity. Let us, however, carry out the direct computation as a test. We impose the same measure $d\mu = \gamma^- dr\,d\varphi$, $\gamma^-(r) = \gamma^+(r)$, but we use the conformal structure $j^- = -i$ obtained by reversing the orientation,

$$j^- \tfrac{\partial}{\partial r} = -\tfrac{1}{r}\tfrac{\partial}{\partial\varphi}, \quad j^- \tfrac{\partial}{\partial\varphi} = r\tfrac{\partial}{\partial r},$$
$$\text{with} \quad \tfrac{\partial}{\partial s} = -2\pi r\tfrac{\partial}{\partial r}, \quad \tfrac{\partial}{\partial t} = 2\pi\tfrac{\partial}{\partial\varphi}\,.$$

This yields the Hermitian bundle $X^{J_o}(D^- \times \mathbb{C}^n) = X^{J_o}_{j^-}$ and

$$K^-_\omega : C^\infty(D^- \times \mathbb{C}^n) \to C^\infty(X^{J_o}_{j^-}), \quad K^-_\omega = \overline{\partial}_{j^-,J_o} + \beta ds \otimes (\omega\, \mathrm{Id}_{\mathbb{C}^n}),$$

in straight analogy to above. Again, we obtain K^-_ω as an admissible $\overline{\partial}$-operator exactly if $\omega \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. The precisely analogous analysis leads to

3.3.6 Proposition *Given $\omega \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, the admissible $\overline{\partial}$-operator K^-_ω satisfies*

$$\operatorname{ind} K^-_\omega = 2n \cdot \left(\#\{\, k \in \mathbb{Z} \,|\, \frac{\omega}{2\pi} < k \leqslant 0\,\} - \#\{\, k \in \mathbb{Z} \,|\, 1 \leqslant k < \frac{\omega}{2\pi}\,\}\right),$$

that is

$$\operatorname{ind} K^-_\omega = \operatorname{ind} K^-_{(2l-1)\pi} = 2n(1-l)$$

for $l - 1 < \frac{\omega}{2\pi} < l$, $l \in \mathbb{Z}$.

PROOF. Instead of the extension v^+ of $\frac{\partial}{\partial s}$, we now use

$$v^- \in C^\infty(TD^-_{|D^- \setminus \{0\}})$$

given by $v^-(r,\varphi) = -\alpha(r)\frac{\partial}{\partial r}$. Thus, we obtain the trivialized operator

$$\lambda \circ K^-_\omega = \alpha(-\tfrac{\partial}{\partial r} + J_o \tfrac{1}{r}\tfrac{\partial}{\partial \varphi}) + \beta\omega\,.$$

Following the proof of Proposition 3.3.5 we have to replace (3.36) by

$$A'_k = A_k \cdot \begin{cases} -kr^{-1}, & 0 < r \leqslant \frac{1}{2}, \\ (\frac{\omega}{2\pi} - k)r^{-1}, & r \geqslant 3, \end{cases}$$

$k \in \mathbb{Z}$, where we consider again the series expansion $\sum A_k \, e^{ik\varphi} \in \ker(\lambda \circ K^-_\omega)$. We henceforth compute

$$\dim_{\mathbb{C}} \ker K^-_\omega = n \cdot \{\, k \in \mathbb{Z} \,|\, \frac{\omega}{2\pi} < k \leqslant 0\,\}\,. \tag{3.40}$$

Likewise $\sum A_k \, e^{ik\varphi} \in \operatorname{coker}(\lambda \circ K^-_\omega)$ yields

$$A_k = \begin{cases} \mathrm{const} \cdot r^{k-1}, & 0 < r \leqslant \frac{1}{2}, \\ \mathrm{const} \cdot r^{k - \frac{\omega}{2\pi}}, & r \geqslant 3, \end{cases}$$

implying the formula

$$\dim_{\mathbb{C}} \operatorname{coker} K^-_\omega = n \cdot \{\, k \in \mathbb{Z} \,|\, 1 \leqslant k < \frac{\omega}{2\pi}\,\}\,. \tag{3.41}$$

Hence, the proposition follows. ■

As a test for our index theory we deduce the index for the glued operator $K^+_\omega \#_T K^-_\omega \in F((D^+ \#_T D^-) \times \mathbb{C}^n)$ over the closed model surface $D^+ \#_T D^- \approx$

S^2. By virtue of the central Theorem 3.2.12 about the index additivity we obtain from Proposition 3.3.5 and 3.3.6 for $l - 1 < \frac{\omega}{2\pi} < l$, $l \in \mathbb{Z}$,

$$\operatorname{ind} K^+_\omega \#_T K^-_\omega = \operatorname{ind} K^+_{(2l-1)\pi} \#_T K^-_{(2l-1)\pi} = 2nl + 2n(1-l) = 2n, \tag{3.42}$$

i.e. independent of $\omega \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.

Since $D^+ \#_T D^-$ is a model surface without ends and with genus 0, we find the formula $\operatorname{ind} K = 2n$ for all admissible $\overline{\partial}$-operators K on Hermitian rank-$2n$ bundles ξ over S^2 whose first Chern class $c_1(\xi)$ vanishes. This follows from (3.42) because $c_1(\xi) = 0$ exactly if the restrictions $\xi_{D^\pm}$ to the hemispheres $D^+ \cup D^- = S^2$ admit trivializations coinciding over $S^1 = D^+ \cap D^-$. Thus, our first simple index formula proves consistent with the classical Riemann-Roch formula which yields the real index

$$\boxed{\operatorname{ind} K = 2n(1-g) + 2c_1(\xi)[\Sigma]} \tag{3.43}$$

for a $\overline{\partial}$-operator K on a complex rank-n bundle ξ over a closed Riemannian surface Σ of genus g. This formula can be deduced from the classical canonical $\overline{\partial}$-operator on a holomorphic vector bundle. First, we recall from Section 3.1.1 the definition of a generalized $\overline{\partial}$-operator[2]. This is an operator $D\colon C^\infty(E) \to C^\infty(X^J(E))$ on a complex vector bundle $E \to \Sigma$ such that

$$D(fs) = fD(s) + \overline{\partial} f \otimes_J s$$

for all $f \in C^\infty(\Sigma)$, $s \in C^\infty(E)$. Proposition 3.1.7 proved that every two such $\overline{\partial}$-operators merely differ by a lower order operator. For a closed surface Σ, every $\overline{\partial}$-operator is elliptic and therefore Fredholm. Moreover, the lower order difference term yields a compact operator. Therefore, the index of a $\overline{\partial}$-operator is uniquely determined by the complex bundle E. In order to find a formula for the index we use the fact that every complex vector bundle over a Riemann surface is isomorphic to a holomorphic bundle. Thus, without loss of generality, we may assume that $E \to \Sigma$ is holomorphic and every $\overline{\partial}$-operator D on E has the same index as the canonical Cauchy-Riemann operator $\overline{\partial}_E\colon C^\infty(E) \to \Omega^{0,1}(E)$ uniquely associated to E. For this canonical $\overline{\partial}$-operator, we can use the classical results, see for example Gunning or Griffiths-Harris, [27],[25]. We obtain the complex index

$$\operatorname{ind}_{\mathbb{C}} \overline{\partial} = \frac{\operatorname{rank}_{\mathbb{C}} E \, \chi(\Sigma)}{2} + c_1(E),$$

where $\chi(\Sigma)$ is the genus of Σ. This proves the formula (3.43) for the real Fredholm index.

Note that it is possible to derive this index formula directly from the Fredholm analysis developed up to this stage, exactly like we deduced the special case for $g = 0$ in (3.42). The idea is that every Riemann surface can be decomposed into elementary building blocks given by caps, cylinders and pairs of pants. This will play a crucial role in the last chapter when we will have

[2]See Definition 3.1.6 or equivalently 3.2.

developed the full S^1-cobordism theory for Floer homology, see Section 5.5. The index formula for cylinders is given by the relative index from standard Floer homology and the index for the pair of pants, i.e. a model surface of type $(2,1,0)$, can be obtained by gluing with a cap in view of the index additivity. The eventually nontrivial Chern number $c_1(\xi)[\Sigma]$ has to be computed from the changes of the unitary trivializations over the cylindrical ends where the pieces are glued together. Let us content ourselves, however, with deriving in the following section the general index formula from the classical Riemann-Roch formula.

We now deduce a general index formula for any admissible $\overline{\partial}$-operator $K_S^\pm\colon C^\infty(D^\pm\times\mathbb{C}^n)\to C^\infty(X_{j^\pm}^{J_o})$ of the form

$$K_S^\pm = \overline{\partial}_{j^\pm,J_o} + \beta S\otimes ds, \tag{3.44}$$

where $S\in C^\infty(\bar{Z}^\pm,\mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$ such that $S^\pm = S(\pm\infty,\cdot)$ is admissible in the sense of Definition 3.1.1. The idea is to relate the index of $K_S^\pm$ to the Conley-Zehnder index μ_{CZ} in analogy with the original Floer theory. There, for the model surface $\mathbb{R}\times S^1$ and $K=\partial_s + J_o\partial_t + S(s,t)$ the Fredholm index can be computed by the spectral flow of $A(s)=J_o\partial_t+S(s,\cdot)$ and identified as the relative Conley-Zehnder index. We emphasize that we use the Conley-Zehnder index μ_{CZ} as considered in [21] and [22].

Let us recall the following definitions. Sp^* is the subset of $\mathrm{Sp}(n)$ consisting of all linear symplectic maps on $(\mathbb{C}^n,\omega_o)$ which do not have 1 in their spectrum. We consider the space of homotopy classes of continuous symplectic arcs

$$\mathcal{A} = \{\,[\Phi]\mid \Phi\colon[0,1]\to\mathrm{Sp},\ \Phi(0)=\mathrm{Id},\ \Phi(1)\in\mathrm{Sp}^*\,\}$$

Denoting by α, β the arcs

$$\alpha(t) = \mathrm{e}^{\pi i t}\,\mathrm{Id}$$

and

$$\beta(t) = \begin{pmatrix} \mathrm{e}^{-t}\,\Re + i\,\mathrm{e}^{t}\,\Im & & 0 & \\ & \mathrm{e}^{\pi i t} & & \\ 0 & & \ddots & \\ & & & \mathrm{e}^{\pi i t}\end{pmatrix}$$

as in [21] and by

$$\sigma(t) = \begin{pmatrix} \mathrm{e}^{2\pi i t} & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1\end{pmatrix}$$

the generator of $\pi_1(\mathrm{Sp}(n),\{\mathrm{Id}\})\cong\mathbb{Z}$, we quote the theorem from [21] and [22],

3.3.7 Theorem *There exists a unique map $\mu_{CZ}\colon\mathcal{A}\to\mathbb{Z}$ satisfying*

$$\mu_{CZ}([\sigma][\Phi]) = 2+\mu_{CZ}([\Phi]),$$
$$\mu_{CZ}([\alpha]) = n \quad\textit{and}\quad \mu_{CZ}([\beta]) = n-1\,.$$

This refers to the group action

$$\pi_1\times\mathcal{A}\to\mathcal{A}\colon [\sigma][\Phi]=[\sigma\Phi],$$
$$(\sigma\Phi)(t) = \sigma(t)\Phi(t)\,.$$

Note that this definition of the Conley-Zehnder index relies on the symplectic structure $(\mathbb{C}^n,\omega_o)$ with $J_o = i\oplus\ldots\oplus i$, whereas in [48] the index $\mu_1\colon\mathcal{A}\to\mathbb{Z}$ is defined by means of

$$J = \begin{pmatrix} 0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{pmatrix},$$

such that it holds

$$\mu_1 = -\mu_{CZ}\,.$$

Moreover, let us recall the relation with the Morse index. Given an admissible $S\in C^\infty(S^1,\mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$ we obtain an arc $\Phi_S(t)$, $t\in[0,1]$, from the initial value problem

$$\Phi_S(0)=\mathrm{Id},\quad \dot{\Phi}_S(t) = J_oS(t)\Phi_S(t)\,. \tag{3.45}$$

$\Phi_S(1)\in\mathrm{Sp}^*$ is equivalent to the regularity of S. If S is a constant symmetric nonsingular matrix with $|S|<2\pi$ we have

$$\mu_{CZ}([\Phi_S]) = -\mu_1([\Phi_S]) = n-\mu^-(S) \tag{3.46}$$

where $\mu^-(S)$ denotes the Morse index of S, see [48] Theorem 3.3.

We now are able to express the index formula for admissible $\overline{\partial}$-operators over the cap $D^\pm$.

3.3.8 Proposition *The Fredholm indices of the admissible $\overline{\partial}$-operators $K_S^\pm$ satisfy*

$$\operatorname{ind}K_S^- = n-\mu_{CZ}(\Phi_{S^-}),$$
$$\operatorname{ind}K_S^+ = n+\mu_{CZ}(\Phi_{S^+})\,.$$

The identity (3.46) immediately provides the

3.3.9 Corollary *If $S^o=S(\pm\infty,\cdot)$ is a constant nonsingular symmetric matrix with $|S^o|<2\pi$, the indices of the cap-operators are given as*

$$\operatorname{ind}K_S^- = \mu^-(S^o)\quad\textit{and}\quad \operatorname{ind}K_S^+ = 2n-\mu^-(S^o)\,.$$

Proof of Proposition 3.3.8. These index formulae are a direct consequence of the index additivity proven as Theorem 3.2.12 in the last section. For the computation of K_S^- we choose an $\tilde{S}\in C^\infty(\mathbb{R}\times S^1,\mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$ with

$$\tilde{S}(-\infty,t) = \omega\,\mathrm{Id}_{\mathbb{C}^n},\quad \omega\in\mathbb{R}\setminus2\pi\mathbb{Z},$$

and $\tilde{S}(+\infty,\cdot)=S(-\infty,\cdot)$. Thus, the admissible $\overline{\partial}$-operator $F_{\tilde{S}}=\partial_s+J_o\partial_t+\tilde{S}$ leads to the identity

$$\operatorname{ind}F_{\tilde{S}}+\operatorname{ind}K_S^- = \operatorname{ind}K_\omega^-\,. \tag{3.47}$$

The standard Floer theory, see Theorem 4.1 in [48], yields

$$\operatorname{ind} F_{\tilde{S}} = \mu_1(\Phi_{\omega\,\mathrm{Id}_{\mathbb{C}^n}}) - \mu_1(\Phi_S) = \mu_{CZ}(\Phi_S) - \mu_{CZ}(\Phi_{\omega\,\mathrm{Id}_{\mathbb{C}^n}})\,.$$

In Proposition 3.3.5 and Proposition 3.3.6 we directly computed $\operatorname{ind} K_\omega^\pm$, so that we deduce from Theorem 3.3.7 for $l-1 < \frac{\omega}{2\pi} < l$, $l \in \mathbb{Z}$ that

$$\begin{aligned} \operatorname{ind} K_\omega^+ &= \quad 2ln \quad &&= n + \mu_{CZ}(\Phi_{\omega\,\mathrm{Id}}),\\ \operatorname{ind} K_\omega^- &= 2n - 2ln &&= n - \mu_{CZ}(\Phi_{\omega\,\mathrm{Id}})\,. \end{aligned}$$

Thus the proposition follows from the index additivity (3.47). ■

3.3.2 The General Index Formula for Pull-Back Bundles

Up to now we merely considered Hermitian bundles ξ over $D^\pm$ which necessarily are trivial and each two trivializations are isomorphic. The computation of the Fredholm index does not depend on the specific choice of the trivialization. For general model surfaces this is more complicated. We encounter the problem that, although a Hermitian bundle over a model surface with boundary is always trivial, the computation of the index depends on the trivialization class of the restriction to the cylindrical ends. It turns out that on the one hand we require these asymptotic trivializations in order to employ the above Conley-Zehnder index. On the other hand also the topological isomorphism class of the complex bundle $\xi \to \Sigma$ plays a role as it is known for closed surfaces from Riemann-Roch, see (3.43).

Here we already consider pull-back bundles

$$u^*TM \quad \text{for } u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M).$$

This allows us to make the notion of asymptotic data more concrete. In order to present the index formula in full generality for the Floer theory we consider the covering $\widetilde{\Omega}_c^o \to \Omega^o(M)$ of the space of smooth contractible loops in M with the group of deck transformations

$$\Gamma = \pi_2(M)/\ker\phi_c, \quad \phi_c\colon \pi_2(M) \to \mathbb{Z},\ \{w\} \mapsto \int_{S^2} w^*\sigma,$$

where σ is any closed 2-form on M representing the integral cohomology-2-class $c_1 \in H^2(M,\mathbb{Z})$. ϕ_c is the fundamental homomorphism associated to the first Chern class $c_1 = c_1(TM,J)$ which describes the index ambiguity in the Floer theory. We define

$$\widetilde{\Omega}_c^o(M) = \{\, [x,u_x] \,|\, x \in \Omega^o(M),\ u_x \in C^\infty(D^2,M),\ u_{x|S^1} = x \,\},$$

with the equivalence relation

$$(x,u_x) \sim_c (y,u_y) \quad \Leftrightarrow \quad x = y, \quad \int_{D^2} u_x^*\sigma = \int_{D^2} u_y^*\sigma\,.$$

Let ξ be the Hermitian bundle $\xi = u^*TM$, $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)$, for $x_i \in \Omega^o(M)$, $i = 1,\dots,\nu$. For the bundles ξ and $\eta = X^J(\xi)$ we consider asymptotic data

$\Omega = (\Phi_i, S_i)_{i=1,\dots,\nu}$ such that the asymptotic unitary trivializations Φ_i can be extended over representatives u_{x_i} for $[x_i,u_{x_i}] \in \widetilde{\Omega}_c^o(M)$. Note that the equivalence class of Φ_i is uniquely determined by the homotopy class of u_{x_i} relative to x_i.

In general, let $\Omega = (\Phi, S)$ be associated to (x, u_x),

$$\Phi\colon S^1 \times \mathbb{C}^n \to x^*TM, \quad S \in C^\infty(S^1, \mathcal{L}_\mathbb{R}(\mathbb{C}^n))\,.$$

Then the Conley-Zehnder index $\mu_{CZ}(S) = \mu_{CZ}(\Psi_S)$ which was defined above by the initial value problem

$$\Psi_S(0) = \mathrm{Id}_{\mathbb{C}^n}, \quad \frac{d}{dt}\Psi_S(t) = J_o S(t)\Psi_S,$$

is already uniquely determined by S and the class $[x,u_x]$. This can be seen as follows. First, changing the trivialization from Φ to $\phi\Phi$ by multiplying with $\phi \in C^\infty(S^1, \mathrm{U}(n))$, $\phi(0) = \mathrm{Id}$, transforms Ψ_S to $\Psi_{\phi(S)} = \phi\cdot\Psi$. Consequently, it follows from Theorem 3.3.7 that

$$\mu_{CZ}(\Psi_{\phi(S)}) = 2\mu(\phi) + \mu_{CZ}(\Psi_S), \tag{3.48}$$

where $\mu\colon \pi_1(\mathrm{U}(n)) \xrightarrow{\cong} \mathbb{Z}$ is the isomorphism induced by $\det\colon \mathrm{U}(n) \to S^1$. Suppose now that Φ is associated to (x,u_x) and $\phi\Phi$ to $(x,\tilde{u}_x)$. Then it holds

$$\mu(\phi) = c_1(\tilde{u}_x \# (-u_x))$$

where $\tilde{u}_x\#(-u_x)\colon S^2 \to M$ is a continuous map obtained by gluing $\tilde{u}_x$ with u_x orientation reversed along x. Hence, if both Φ and $\phi\Phi$ are associated to the same class $[x,u_x]$, $\mu(\phi)$ vanishes.

Let $[x_1,u_{x_1}],\dots,[x_\nu,u_{x_\nu}] \in \widetilde{\Omega}_c^o(M)$ and $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)$ be given. We define the closed capped surface

$$\hat{\Sigma} = \Sigma \#_{i=1}^\nu D^{-\epsilon_i}$$

and the map $\hat{u} \in C^0(\hat{\Sigma}, M)$, $\hat{u} = u\#_{i=1}^\nu u_{x_i}$, where the representatives u_{x_i} are chosen with the appropriate orientation ϵ_i. It is sufficient to determine the topological class of $\hat{\Sigma}$ and the homotopy class of $\hat{u}$ such that $\alpha = \hat{u}_*(\{\hat{\Sigma}\}) \in H_2(M,\mathbb{Z})$ is well-defined for given representatives. Consequently, the integer

$$c_1(\alpha) = \int_{\hat{\Sigma}} \hat{u}^*\sigma \in \mathbb{Z}$$

with $\{\sigma\} = c_1$ is uniquely determined by $[u]_{\mathrm{rel}\,\partial\Sigma}$ and $\big([x_i,u_{x_i}]\big)_{i=1,\dots,\nu}$.

3.3.10 Proposition *Let $S_i \in C^\infty(S^1, \mathcal{L}_\mathbb{R}(\mathbb{C}^n))$ and $[x_i,u_{x_i}]$, $i = 1,\dots,\nu$ be given as asymptotic data Ω. Then every admissible $\overline{\partial}$-operator $K \in F_\Omega(u^*TM)$ satisfies*

$$\operatorname{ind} K + \sum_{i=1}^\nu (n - \epsilon_i\mu_{CZ}(S_i)) = 2n(1-g) + 2c_1(\alpha),$$

where g is the genus of $\hat{\Sigma}$.

Proof. This is now a direct application of the index additivity proven in the last section. Given the asymptotic data $\Omega_i = ([x_i, u_{x_i}], S_i)$ for each end $i = 1, \dots, \nu$, we consider the operator $K_{S_i}^{-\epsilon_i} \in F_{\Omega_i}(D^{-\epsilon_i})$ on the bundle $D^{-\epsilon_i} \times \mathbb{C}^n$ and a unitary trivialization Φ_i of $x_i^* TM$ extendible over a representative u_{x_i}. Proposition 3.3.8 yields

$$\operatorname{ind} K_{S_i}^{-\epsilon_i} = n - \epsilon_i \mu_{CZ}(S_i)\,.$$

Thus we can apply the index additivity from Theorem 3.2.12 so that the index formula from Riemann-Roch, that is (3.43) above, yields the asserted identity. ■

Finally we derive the index formula for the linearization

$$DF_u(0)\colon H_\Sigma^{1,p}(u^*TM) \to L_\Sigma^p(X^J(u^*TM))$$

of the nonlinear $\overline{\partial}$-operator $\overline{\partial}_{J,k(H)}$ at a $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$. In section 3.1.4 we proved that $DF_u(0)$ is indeed an admissible $\overline{\partial}$-operator and therefore Fredholm. We refer to the representation (3.20) of the asymptotic data $(S_i^\infty)_{i=1,\dots,\nu}$. Let us recall the definition of the Conley-Zehnder index associated to a non-degenerate periodic solution $x \in \mathcal{P}_1(H) \subset \Omega^o(M)$, see [48]. Fixing an extension $u_x\colon D^2 \to M$ we find a unitary trivialization Φ of x^*TM such that we obtain the arc of symplectic maps $\Psi(t) \in \mathrm{Sp}$

$$\Psi(t) = \Phi(t)^{-1} \circ D\Psi_t(x(0)) \circ \Phi(0), \quad t \in [0,1],$$

from the linearized flow $\psi_t\colon M \to M$ of the Hamiltonian vector field, see (3.22). The index $\mu_{CZ}([x, u_x])$ is defined as

$$\mu_{CZ}([x, u_x]) = \mu_{CZ}(\Psi)\,.$$

This definition is independent of the choice of Φ compatible with the class $[x, u_x]$. This can be seen as follows. From (3.23) in section 3.1.4 we know that Ψ satisfies

$$\dot{\Psi}(t) = J_o S(t) \Psi(t)$$

with S defined in (3.20) as

$$S(t) = J_o \Phi^{-1} \nabla_t \Phi + J_o \Phi^{-1} \operatorname{Tor}(\dot{x}, \Phi) + \nabla_\Phi \nabla H$$

for a Hermitian connection ∇. Hence we obtain the identity

$$\mu_{CZ}([x, u_x]) = \mu_{CZ}(\Psi_S)$$

where the right hand side is uniquely determined by $[x, u_x]$ as we have shown above. Combining Proposition 3.3.10 with the computations from Section 3.1.4 we can sum up

3.3.11 Theorem *Let $v \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ and $x_i \in \mathcal{P}_1(H^i)$, $i = 1, \dots, \nu$, be non-degenerate 1-periodic solutions. Then, given the classes*

$$[x_1, u_{x_1}], \dots, [x_\nu, u_{x_\nu}] \in \widetilde{\Omega}_c^o(M)$$

the Fredholm operator

$$DF_v(0)\colon H_\Sigma^{1,p}(v^*TM) \to L_\Sigma^p(X^J(v^*TM))$$

has the index

$$\boxed{\operatorname{ind} DF_v(0) = \sum_{i=1}^{\nu} \epsilon_i \mu_{CZ}([x_i, u_{x_i}]) + n(2 - 2g - \nu) + 2c_1(\hat{v}_*\{\hat{\Sigma}\})}\,.$$

CHAPTER 4

The Solution Spaces

This chapter is the core of the work. Here the topological properties of the solutions spaces defining the cohomology operations are discussed. The analytical scheme is analogous to that of classical Floer theory. First, we have to prove that we obtain the solutions of the well-posed Cauchy-Riemann problem as the zeros of a smooth section in a Banach space bundle. It allows us to apply the Implicit Function Theorem provided that this section is transverse to the zero section. The transversality result requires the same Unique Continuation method as for the gradient-type trajectories of bounded energy in the standard Floer theory. After having obtained the manifold property for the solution spaces, we study compactness properties. These are based on an a priori energy estimate generalized for the model surfaces with relative gradient flow on their cylindrical ends. It is the compactness result out of which the finiteness of the solution space in dimension 0 and the splitting-off of unparametrized "gradient flow" trajectories in dimension 1 arise. Here at most one cylindrical solution of relative index 1 splits off. This is related with the reverse problem of gluing. By a compactness-gluing-cobordism in dimension 1 which is analogous to the relative gradient flow in standard Floer homology we prove that broken solutions of local dimension 1 always occur in even number. Moreover we also consider a general gluing operation for two model surfaces of general type. This will serve for the composition rule of the cohomology operations defined in the next chapter.

Contents

4.1 The Definition of the Solution Spaces

Let $\Sigma = (\overline{\Sigma}, (\psi_i))$ be a model surface of type (a, b, g) with admissible conformal structure j and $(J^1, H^1), \dots, (J^\nu, H^\nu)$, $\nu = a + b$, regular pairs of ω-compatible almost complex structures $J^i \in C^\infty(S^1 \times M, \mathrm{End}(TM))$ and Hamiltonians $H^i \in C^\infty(S^1 \times M, \mathbb{R})$ as considered in Definition 2.3.3. We recall that $\epsilon_i = -1$ for $i = 1, \dots, a$ and $\epsilon_i = +1$ for $i = a+1, \dots, \nu$. Moreover, let $\beta^\pm \in C^\infty(\mathbb{R}, [0,1])$ be the cut-off functions from Section 2.3 and let us choose an extension $J \in \mathcal{J}$ of $(J^i)_{i=1,\dots,\nu}$. In the following, given $u \in C^\infty(\Sigma, M)$, we denote

$$u_i = u \circ \psi_i \colon Z^{\epsilon_i} \to M, \quad i = 1, \dots, \nu,$$

and $\Sigma_0 = \overline{\Sigma \setminus \Sigma_Z}$ with $\Sigma_Z = \bigcup_{i=1}^{\nu} \psi_i(Z^{\epsilon_i})$.

4.1.1 Definition *Given non-degenerate periodic solutions $x_i \in \mathcal{P}_1(H^i)$, $i = 1, \dots, a$, and $y_i \in \mathcal{P}_1(H^{a+i})$ for $i = 1, \dots, b$, we define the solution set for the special extension $k^o(H)$ (cf. Definition 2.3.5)*

$$\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J, k^o(H)) = \{\, u \in C^\infty(\Sigma, M) \mid u \text{ satisfies } (*) \,\}$$

with respect to the problem $(*)$

$$\overline{\partial}_J(u)(z) = 0 \quad \text{for } z \in \Sigma_0, \tag{$*_1$}$$

$$(\partial_s u_i + J^i(t, u_i)\partial_t u_i + \beta(\epsilon_i s)\nabla H^i(t, u_i))(s,t) = 0 \quad \text{for } (s,t) \in Z^{\epsilon_i}, \quad i = 1, \dots, \nu, \tag{$*_2$}$$

$$u_i(-s) \overset{C^1(S^1)}{\longrightarrow} x_i \quad \text{for } i = 1, \dots, a \quad \text{and} \quad u_i(s) \overset{C^1(S^1)}{\longrightarrow} y_i \quad \text{for } i = 1, \dots, b \quad \text{as } s \to \infty, \tag{$*_3$}$$

where $\overline{\partial}_J(u) = Tu + J \circ Tu \circ j$. For an arbitrary T-admissible extension $k = k(H)$ $(*_1)$ and $(*_2)$ have to be replaced by

$$(\overline{\partial}_J(u) + k(u) + J(u) \circ k(u) \circ j)(z) = 0 \quad \text{for all } z \in \Sigma_T \tag{$*_1'$}$$

and

$$(\partial_s u_i + J^i(t, u_i)\partial_t u_i + \nabla H^i(t, u_i))(s,t) = 0 \tag{$*_2'$}$$

for all $(s,t) \in Z^{\epsilon_i}_T$, $i = 1, \dots, \nu$.

This nonlinear partial differential equation is a well-posed problem of Cauchy-Riemann type. Based on the propositions of Chapter 1 we now are able to reformulate it in a way such that the solution space $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$ can be viewed as a subset of the Banach manifold $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ where $p > 2$ is fixed and $x_i = y_{i-a}$ for $i = a+1, \dots, \nu$.

Given an admissible extension (J, k) of $(J^i, H^i)_{i=1,\dots,\nu}$, analyzing the smooth section $\overline{\partial}_{J,k(H)} \colon \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) \to L^p_\Sigma(\mathcal{P}^* X^J)$ leads to the following equivalence.

4.1.2 Theorem *Given $(J, k(H))$ and $x_1, \ldots, x_\nu$ as above,*

$$\mathcal{M}^{x_1,\ldots,x_a}_{y_1,\ldots,y_b}(J, k(H)) = \{\, u \in \mathcal{P}(x_1, \ldots, x_\nu) \,|\, \overline{\partial}_{J,k(H)}(u) = 0 \,\}\,.$$

PROOF. This result is a restatement of Proposition 2.5.7 on page 34. The direction "$\supset$" is largely due to Corollary 2.5.6. It remains to verify condition $(*_3)$. At the first glance $u \in \mathcal{P}^{1,p}_{x_1,\ldots,x_\nu}(\Sigma, M)$ only provides asymptotic $C^0(S^1, M)$-convergence, whereas convergence in $C^1(S^1, M)$ is demanded by $(*_3)$. As well as Proposition 4.1.3 below, the proof of this step is postponed to Section 4.3 where the required techniques are developed in relation with the (non-)bubbling-off analysis. We refer to Corollary 4.3.6 on page 117. The remaining part of the proof analyses further the asymptotic convergence on the cylindrical ends. This is crucially based on the assumed nondegeneracy of the periodic solution $x_i \in \mathcal{P}_1(H^i)$ which delimits Floer homology theory from the Liusternik-Shnirelman variational approach.

It is enough to consider a fixed cylindrical end $\psi_i(Z^{\epsilon_i})$, without loss of generality $\epsilon_i = +1$, and we shortly denote $H = H^i$, $J^o = J^i$. Let $u \in C^\infty([T, \infty) \times S^1, M)$ be a fixed solution of

$$\partial_s u + J^o(t, u)\partial_t u + \nabla H(t, u) = 0 \tag{4.1}$$

for some $T > 0$ large enough. The following result will be a prime step for the compactness discussion. Thus we shall present the proof in Section 4.3. We just recall the following relations. Denoting by

$$\Phi(u) = \int_T^\infty \int_{S^1} |\partial_s u|_J^2 \, ds dt$$

the "flow" energy of u we have the fundamental "gradient flow" identity

$$\Phi(u_{|[T_1,T_2]\times S^1}) = \mathcal{A}_H(u(T_2)) - \mathcal{A}_H(u(T_1))$$

because (4.1) represents geometrically the relative positive gradient flow for the symplectic action $\mathcal{A}_H$. Note that the difference of $\mathcal{A}_H$ is well-defined also for $\phi_\omega \neq 0$ as long as we indicate the connecting path $s \mapsto u(s) \in \Omega^o(M)$.

4.1.3 Proposition *Given a solution u of (4.1) with regular H, $\Phi(u) < \infty$ exactly if there exists an $x \in \mathcal{P}_1(H)$ such that $u(s) \to x$ uniformly in $C^1(S^1, M)$ as $s \to \infty$. Moreover, then also $u(s) \to x$ and $\partial_s u(s) \to 0$ uniformly in $C^\infty(S^1)$.*

This result will be proven as Proposition 4.3.9. Using again the nondegeneracy assumption for the 1-periodic solutions of the Hamilton equation associated to H, we can improve it as follows.

4.1.4 Proposition *Given $u \in C^\infty([T, \infty) \times S^1, M)$ as a solution of (4.1) with $u(s) \to x \in \mathcal{P}_1(H)$ in $C^\infty(S^1, M)$ for $s \to \infty$, the convergence is exponentially fast of order $\mathrm{e}^{-\delta s}$ for some $\delta > 0$, if the 1-periodic solution x is non-degenerate.*

PROOF. Since $u(s) \to x$ in $C^0(S^1)$ we may represent $u(s)$ for $s \geqslant T$ large enough with respect to exponential coordinates at x. Moreover, we choose a unitary trivialization Φ of x^*TM such that we have

$$u(s,t) = \exp_{x(t)}\left(\Phi(t) \cdot v(s,t)\right) \quad \text{f.a. } s \geqslant T,\, t \in S^1,$$

with $v \in C^\infty([T, \infty) \times S^1, \mathbb{R}^{2n})$, $\|v\|_{C^k(S^1)} \to 0$, $\|\partial_s v\|_{C^k(S^1)} \to 0$ for all $k \in \mathbb{N}$ as $s \to \infty$. We compute that the partial differential equation (4.1) amounts to

$$\partial_s v(s,t) + J(s,t)\partial_t v(s,t) + F(t, v(s,t)) = 0$$

where

$$\begin{aligned}
J(s,t) &= \Phi(t)^{-1}\nabla_2\mathrm{exp}\left(\Phi(t)v(s,t)\right)^{-1} J^o(t, u(s,t))\nabla_2\mathrm{exp}\left(\Phi(t)v(s,t)\right)\Phi(t),\\
F(t,p) &= \Phi(t)^{-1}\nabla_2\mathrm{exp}\left(\Phi(t)\,p\right)^{-1} J^o(t, \Phi(t)\,p)\nabla_1\mathrm{exp}\left(\Phi(t)\,p\right)\dot{x}(t)\\
&\quad + \left(\Phi^{-1}\nabla_2\mathrm{exp}^{-1} J^o\nabla_2\mathrm{exp}\,\nabla_t\Phi\right)p + \Phi^{-1}\nabla_2\mathrm{exp}^{-1}\,\nabla H(t, \exp_x\Phi p)
\end{aligned}$$

for $(t,p) \in S^1 \times \mathbb{R}^{2n}$, $|p| \leqslant \epsilon$ small enough. We use the splitting of the differential map $D\exp(w)\colon T_wTM \to T_{\exp w}M$ into vertical and horizontal direction

$$T_wTM = T^h_wTM \oplus T^v_wTM, \quad T^v_wTM = \ker D\tau,$$

for $\tau\colon TM \to M$ and $T^h_wTM = \ker K_{LC}$. K_{LC} is the unique Levi-Civita connection associated to a fixed Riemannian metric which also defines the exponential map exp. We notify that we do not use a Hermitian connection now, so that the following computations of a linear $\overline{\partial}$-operator are not compatible with the linearization $DF_h(0)$ as before. We have the identities

$$\begin{aligned}
\nabla_1\mathrm{exp}(w) \cdot \zeta &= D\exp(w) \circ D\tau(w)^{-1} \cdot \zeta,\\
\nabla_2\mathrm{exp}(w) \cdot \xi &= D\exp(w) \circ K_{LC}^{-1} \cdot \xi,
\end{aligned}$$

see also (A.6) and (A.7) in the appendix. Since $\nabla_1\mathrm{exp}(0) = \nabla_2\mathrm{exp}(0) = \mathrm{id}_{T_mM}$ for $0 \in T_mM$ we obtain

$$F(t,0) = \Phi(t)^{-1}\left(J^o(t, x(t))\partial_t x + \nabla H_t(x(t))\right) = 0$$

due to $x \in \mathcal{P}_1(H)$. Thus we can write

$$\begin{aligned}
F(t, v(s,t)) &= S(s,t) \cdot v(s,t)\\
&= \left(\int_0^1 D_2F(t, \lambda v(s,t))d\lambda\right) \cdot v(s,t)
\end{aligned}$$

because $v(s,\cdot) \to 0$ in $C^0(S^1)$ for $s \to \infty$. Note that for s_o large enough, $S \in C^\infty([s_o, \infty) \times S^1, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$. Moreover, since $v(s) \to 0$ in $C^\infty(S^1)$, we have

$$\left.\begin{aligned}
S(s) &\overset{C^\infty(S^1)}{\longrightarrow} S(\infty) = D_2F(\cdot, 0)\\
J(s) &\overset{C^\infty(S^1)}{\longrightarrow} J_o
\end{aligned}\right\} \quad \text{as} \quad s \to \infty\,. \tag{4.2}$$

The PDE now has the form

$$\partial_s v + J(s,t)\partial_t v + S(s,t)\cdot v = 0$$

and we compute with (4.2)

$$S(\infty) = \Phi^{-1}\nabla_\Phi J \cdot \dot{x} + \Phi^{-1} J \nabla_t \Phi + \Phi^{-1}\nabla_\Phi \nabla H_t(x)\,. \tag{4.3}$$

This follows from the identities for the Levi-Civita connection $\nabla_i\nabla_j \exp(0) = 0$ and

$$\frac{d}{d\lambda}\Big|_{\lambda=0}(X(\exp_m(\lambda v))) = \nabla_v X(m)$$

for any vector-field at the point $m \in M$ and a vector $v \in T_m M$. In [48], within the proof of Theorem 5.3, it is explicitly verified that the loop of linear operators $S(\infty) \in C^\infty(S^1, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$ is regular in the sense of our Definition 3.1.1 if $x \in \mathcal{P}_1(H)$ is a non-degenerate periodic orbit. The computations are absolutely analogous to that of Lemma 2.4.2. However, here we use the Levi-Civita connection instead of a Hermitian connection, because we want to deal with the linearization of the exponential map. It follows that the operator $\partial_s + J(s,t)\partial_t + S(s,t)$ is an admissible $\overline{\partial}$-operator in the sense of our Fredholm analysis apart from the fact that $J(s,t)$ is variable. But it converges to J_o in $C^\infty(S^1)$ as $s \to \infty$.

Let us now provide a reinforcement of Lemma 3.1.23 for our present purpose.

4.1.5 Lemma *Let J, S, v be as above with $v(s) \to 0$, $\partial_s v \to 0$ in $C^\infty(S^1)$ as $s \to \infty$. Then there exists a $\delta > 0$ independent of v and $s_o(\delta, v)$, $c(\delta, v) > 0$ such that*

$$\|v\|_{H^{1,p}([\sigma,\infty)\times S^1)} \leqslant c\,\mathrm{e}^{-\delta\sigma}$$

for all $\sigma \geqslant s_o$. In particular, $v \in H^{1,p}(Z^+, \mathbb{R}^{2n})$.

Proof. We define $\alpha(s) = \frac{1}{2}\|v(s)\|^2_{L^2(S^1)}$, where $\|\cdot\|$ is taken with respect to the standard metric $\langle\cdot,\cdot\rangle_{J_o}$, and compute the derivatives

$$\alpha'(s) = \langle \partial_s v, v\rangle_{L^2(S^1)} = -\langle J\partial_t v + Sv, v\rangle_{L^2(S^1)}$$

and

$$\begin{aligned}\alpha''(s) =& \|J\partial_t v + Sv\|^2_{L^2(S^1)} \\ &- \langle (J\partial_t + S)\partial_s v, v\rangle - \langle \partial_s J \partial_t v + \partial_s S v, v\rangle\,.\end{aligned}$$

Let us denote by $A^\infty = J_o\frac{\partial}{\partial t} + S(\infty)$ the operator $A^\infty\colon H^{1,2}(S^1) \to L^2(S^1)$ which is a selfadjoint isomorphism due to the nondegeneracy of $x \in \mathcal{P}_1(H)$, see also Lemma 3.1.2. Hence, we find

$$\delta_o = \min\{\,|\lambda| \mid \lambda \in \sigma(A^\infty)\,\} > 0\,.$$

Moreover, the assumption that $\|v\|_{C^k(S^1)}$, $\|\partial_s v\|_{C^k(S^1)} \to 0$ as $s \to \infty$ for $k \in \mathbb{N}$ implies that for $s \to \infty$ we have

$$\begin{gathered}\|\partial_s J\|_{C^0(S^1)},\ \|\partial_s S\|_{C^0(S^1)},\ \|J - J_o\|_{C^0(S^1)}, \\ \|S - S(\infty)\|_{C^0(S^1)},\ \|S^T - S(\infty)\|_{C^0(S^1)}, \quad \text{and} \quad \|\partial_t J\|_{C^0(S^1)} \longrightarrow 0\,.\end{gathered}$$

This is due to the definition of J and A where the s-dependence is implicitly given by v. Hence, we find a $c = c(s) > 0$ with $c(s) \to 0$ as $s \to \infty$ such that

$$\alpha''(s) \geqslant 2\|A^\infty v\|^2_{L^2(S^1)} - c\,\|v\|^2_{H^{1,2}(S^1)}\,.$$

Using the nondegeneracy assumption, so that

$$\|A^\infty v\|_{L^2(S^1)} \geqslant \delta_o \|v\|_{H^{1,2}(S^1)}$$

we obtain

$$\alpha''(s) \geqslant (4\delta_o^2 - c(s))\alpha(s)\,.$$

Thus we again are able to apply the maximum principle because $\alpha(s) \to 0$ for $s \to \infty$. For any small $\epsilon > 0$ there exists an $s_o(\epsilon) > T$ such that

$$\|v(s)\|_{L^2(S^1)} \leqslant \|v(s_o)\|\,\mathrm{e}^{-(\delta_o-\epsilon)(s-s_o)} \quad \text{f.a. } s \geqslant s_o\,. \tag{4.4}$$

We consequently obtain $v \in L^2([0,\infty)\times S^1, \mathbb{R}^{2n})$ with

$$\|v\|_{L^2([\sigma,\infty)\times S^1)} \leqslant \|v(s_o)\|\frac{\mathrm{e}^{(\delta_o-\epsilon)s_o}}{\sqrt{2(\delta_o-\epsilon)}}\,\mathrm{e}^{-(\delta_o-\epsilon)\sigma}$$

for $\sigma \geqslant s_o$.

Moreover, interpolating the L^2-estimate (4.4) with the decrease $\|v(s)\|_{C^k(S^1)} \to 0$ given by assumption we are also provided with an L^p-estimate for $p \geqslant 2$ and $\delta > 0$ arbitrarily close to $\delta_o - \epsilon$,

$$\|v(s)\|_{L^p(S^1)} \leqslant c(s_o, p, \delta)\,\mathrm{e}^{-\delta s} \quad \text{f.a. } s \geqslant s_o \tag{4.5}$$

and likewise $v \in L^p([0,\infty)\times S^1)$ with

$$\|v\|_{L^p([\sigma,\infty)\times S^1)} \leqslant \tilde{c}(s_o, p, \delta)\,\mathrm{e}^{-\delta\sigma} \quad \text{f.a. } \sigma \geqslant s_o\,. \tag{4.6}$$

Finally, we advance from the L^p-estimate to the asserted $H^{1,p}$-estimate using the injectivity estimate from Theorem 3.1.13. Thus, for all $w \in H^{1,p}(Z^+, \mathbb{R}^{2n})$

$$\|w\|_{1,p} \leqslant c_o(S(\infty))\,\|(\partial_s + A^\infty)w\|_{0,p}$$

where $c_o(S(\infty))$ is a uniform constant merely depending on the asymptotic data fixed by x, H and Φ. Let us fix the cut-off functions β_r, $\beta_l \in C^\infty(\mathbb{R}, [0,1])$

$$\beta_l(s) = \begin{cases} 0, & s \leqslant 0, \\ 1, & s \geqslant 1 \end{cases}, \quad \beta_r(s) = \begin{cases} 1, & s \leqslant 0, \\ 0, & s \geqslant 1 \end{cases},$$

and $\beta_{\sigma,\tau}(s) = \beta_l(s-\sigma)\cdot\beta_r(s-\tau)$ for $0 < \sigma < \tau$. Then we compute

$$\begin{aligned}\|\beta_{\sigma,\tau} v\|_{1,p} &\leqslant c_0\|(\partial_s + A^\infty)(\beta_{\sigma,\tau} v)\|_{0,p} \\ &= c_0\,\|(\partial_s\beta_{\sigma,\tau})v + (A^\infty - A(s,t))\beta_{\sigma,\tau} v + \beta_{\sigma,\tau}(\partial_s v + A(s)v)\|_{0,p}\end{aligned}$$

with $A(s) = J(s,t)\partial_t + S(s,t)$

$$\begin{aligned}
&\leqslant c_1(S^\infty, \beta_l, \beta_r)\,\|v\|_{L^p([\sigma,\infty)\times S^1)} \\
&\quad + \big\|\beta_{\sigma,\tau}\big[(J_o - J(s,t))\partial_t v + (S(\infty) - S(s,t))\big]v\big\|_{0,p} \\
&\leqslant c_1\|v\|_{L^p([\sigma,\infty))} + \bar\epsilon(\sigma)\,\|\beta_{\sigma,\tau}v\|_{1,p}
\end{aligned}$$

where $\bar\epsilon(\sigma) \to 0$ as $\sigma \to \infty$. This leads to

$$\|\beta_{\sigma,\tau}v\|_{1,p} \leqslant \frac{c_1}{1-\bar\epsilon(\sigma)}\|v\|_{L^p([\sigma,\infty))} \leqslant c_2\,\mathrm{e}^{-\delta\sigma}\ .$$

Hence we obtain $v \in H^{1,p}([0,\infty)\times S^1)$ and the desired estimate

$$\|v\|_{H^{1,p}([\sigma,\infty)\times S^1)} \leqslant \mathrm{const}\,\mathrm{e}^{-\delta\sigma}\ . \quad \blacksquare$$

Using iteratively the elliptic estimate (2.11) from Corollary 2.5.3 and using trace theorems we consequently obtain the estimate

4.1.6 Corollary *If v is as above, there is a constant $c_k(v) > 0$ for each $k \in \mathbb{N}$ and a constant $\tilde\delta > 0$ independent of v such that*

$$\|v(s)\|_{C^k(S^1)} \leqslant c_k\,\mathrm{e}^{-\tilde\delta s} \quad \text{f.a. } s \geqslant 0\,.$$

In particular, we finally finished the proof of Proposition 4.1.4. $\blacksquare$

4.1.7 Corollary *The solution u of (4.1) given in Proposition 4.1.4 is C^∞_Σ-smooth.*

Proof. In view of Definition 2.1.2 we have to prove that

$$(s,t) \mapsto u\Big(\frac{s}{\sqrt{1-s^2}}, t\Big), \quad (s,t) \in [s_o, 1) \times S^1,$$

is smoothly extendible to $[s_o, 1] \times S^1$. Thus, we have to show that

$$(s,t) \mapsto D^\alpha u\Big(\frac{s}{\sqrt{1-s^2}}, t\Big)\cdot(1-s^2)^{-\frac{b}{2}}$$

is continuous at $s = 1$ for all multi-indices α and $b \in \mathbb{N}$. But Corollary 4.1.6 yields that

$$\sup_{t\in S^1}\Big|\big(\tfrac{\partial}{\partial t}\big)^a u(s,t)\Big| \to 0$$

converges exponentially fast for every $a \in \mathbb{N}$ and due to

$$\partial_s u = -J(\partial_t u - X_H(u))$$

also $\sup_{t\in S^1}|D^\alpha u|$ for every multi-index with $|\alpha| \geqslant 2$. $\blacksquare$

Altogether this implies

$$\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b} \subset C^\infty_{x_1,\dots,x_\nu}(\Sigma, M) \subset \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)\,. \tag{4.7}$$

In particular, the proof of Theorem 4.1.2 is complete. $\blacksquare$

We can extract a further corollary from the proof of Proposition 4.1.4 above. Let us recall the differential of the representation of $\overline\partial_{J,k}$ in local coordinates,

$$DF_h(0)\colon H^{1,p}_\Sigma(h^*TM) \to L^p_\Sigma(h^*X^J),$$

computed in the first chapter in Proposition 2.4.1 for $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$. In Theorem 3.1.31 we saw that these operators, which represent the linearization D_u of $\overline\partial_{J,k}$ at a solution $u \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$, are admissible $\overline\partial$-operators. Considering the last computations we now can easily prove the exponential decay property for the elements in the kernel of the Fredholm operators $DF_h(0)$ for $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ without assuming that h is a solution of $\overline\partial_{J,k}u = 0$. In the section about the Fredholm property itself we only proved the exponential decay property for asymptotically constant operators.

4.1.8 Corollary *Given $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$, every solution v of*

$$DF_h(0)\cdot v = 0, \quad v \in H^{1,p}_\Sigma(h^*TM),$$

has exponential $H^{1,p}_\Sigma$-decay in the sense of Definition 3.1.25. In particular, it holds

$$\ker DF_h(0) \subset L^q_\Sigma(h^*TM)$$

for all $q \geqslant 1$.

Proof. We consider $v \in \ker DF_h(0)$ restricted to the cylindrical end $\psi_i(Z^{\epsilon_i}_T)$. Due to Proposition 2.4.1, v solves the equation

$$\nabla_s v + J(\cdot,h)\nabla_t v + \mathrm{Tor}(\partial_s h, v) + J(\cdot,h)\,\mathrm{Tor}(\partial_t h, v) + \nabla_v\nabla H(\cdot,h) = 0$$

which, after choosing a suitable unitary trivialization of v_i^*TM looks like

$$\partial_s v + J(s,t)\partial_t v + S(s,t)\cdot v = 0$$

as in the proof of Proposition 4.1.4. Here we have again

$$J(s) \overset{C^\infty(S^1)}{\longrightarrow} J_o, \quad S(s) \overset{C^\infty(S^1)}{\longrightarrow} S(\infty)$$

as well as

$$\partial_s J,\, \partial_s S \overset{C^0(S^1)}{\longrightarrow} 0 \quad \text{for } s \to \infty\,.$$

Since $DF_h(0)$ is an admissible $\overline\partial$-operator we repeat the argumentation in the proof of Proposition 4.1.4, based on the fact, that $v \in H^{1,p}_\Sigma(h^*TM)$ implies

$$\|v(s)\|_{L^\infty(S^1)} \to 0 \quad \text{for } s \to \infty\,. \tag{4.8}$$

Again, $S(\infty)$ is admissible and $\lim_{s\to\infty}\alpha(s) = 0$ for $\alpha(s) = \frac12\|v(s)\|^2_{L^2(S^1)}$ as in the proof of Lemma 4.1.5. The same arguments lead to the exponential decay for some constant $c, \epsilon > 0$

$$\|v(s)\|_{L^q(S^1)} \leqslant c(q,v)\,\mathrm{e}^{-\epsilon s}, \quad s \geqslant T,$$

for all $1 \leqslant q < \infty$, hence $v \in L^q(Z_T^+)$. Proceeding analogously to Lemma 4.1.5 we obtain for $p \geqslant 2$

$$\left\|v(s)\right\|_{H^{1,p}([\sigma,\infty)\times S^1)} \leqslant \text{const}\, \mathrm{e}^{-\epsilon\sigma}$$

for all $\sigma \geqslant T$. ■

4.2 Transversality

In this section we discuss sufficient conditions which provide us with the solution spaces $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$ as component-wise finite-dimensional manifolds. We consider 1-periodic orbits $x_1,\dots,x_a,y_1,\dots,y_b$ with

$$\begin{aligned} x_i &\in \mathcal{P}_1(H^i), \quad i=1,\dots,a,\\ x_{i+a} = y_i &\in \mathcal{P}_1(H^{i+a}), \quad i=1,\dots,b, \end{aligned}$$

where $(J^i,H^i)_{i=1,\dots,a+b}$ are given regular pairs. Moreover, we assume a fixed admissible extension k of $(H^i)_{i=1,\dots,\nu}$, where $k^o(H)$ always stands for the special extension by the cut-off functions $\beta^\pm$ as specified in Definition 4.1.1. Summarizing the last section, an admissible extension J of $(J^i)_{i=1,\dots,\nu}$ gives rise to the solution space

$$\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J,k) \subset \left\{\, u \in C^\infty(\Sigma,M) \,\middle|\, \overline{\partial}_{J,k}(u) = 0 \,\right\}$$

as a closed subspace of $\mathcal{P} = \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M)$. Let us recall the linearization

$$D_u = D^J_u \colon H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^J)$$

which is well-defined for each solution $\overline{\partial}_{J,k}(u) = 0$ and which is a Fredholm operator due to Theorem 3.1.31.

4.2.1 Definition *Let $\mathsf{J}_c(TM,\omega) \to M$ be the subspace of $\mathrm{End}(TM) \to M$ of ω-compatible almost complex structures on T_pM for each $p \in M$, and let $\pi^*\mathsf{J}_c \to \Sigma \times M$ be the pull-back bundle with respect to the projection $\pi\colon \Sigma\times M \to M$. Then $\mathcal{J}$ was defined as the space of Σ-smooth sections in $\pi^*\mathsf{J}_c$, $\mathcal{J} = C^\infty_\Sigma(\pi^*\mathsf{J}_c)$. We call an admissible extension $J \in \mathcal{J}$ of $(J^i)_{i=1,\dots,\nu}$* ***regular*** *for (Σ,j,k), if the linearizations D^J_u of $\overline{\partial}_{J,k}$ at solutions u are onto for all $u \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$ and $x_i \in \mathcal{P}_1(H^i)$, $i=1,\dots,\nu$.*

This notion of regularity then entails the manifold property for all solution spaces at once.

4.2.2 Theorem *Let $J \in \mathcal{J}$ be a regular admissible extension. Then for every ν-tuple $x_1,\dots,x_\nu$ the solution space $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b} = \overline{\partial}^{-1}_{J,k}(0)$ is a Banach submanifold of $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M)$. Moreover, each connected component is a finite-dimensional smooth manifold and the dimension of the component of u is given by*

$$\dim_u \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b} = \operatorname{ind} D^J_u\,.$$

Proof. This transversality theorem is a standard consequence from the Implicit Function Theorem combined with the Fredholm property of D^J_u. The regularity condition to J states that $\overline{\partial}_{J,k}$ is a smooth section of $L^p_\Sigma(\mathcal{P}^*X^J) \to \mathcal{P}$ intersecting the zero section transversely. Thus, the Implicit Function Theorem provides for each $u \in \overline{\partial}^{-1}_{J,k}(0)$ a diffeomorphism from an open neighbourhood $V(0) \subset \ker D^J_u$ onto an open neighbourhood $U(u) \subset \overline{\partial}^{-1}_{J,k}(0)$. The tangent space is identified with $\ker D^J_u$,

$$T_u\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J,k) = \ker\left(D^J_u\colon H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^J)\right). \quad ■$$

In Section 4.4 about gluing techniques we will study more refined variants of the Implicit Function Theorem. We will see that the solution manifold $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$ can even be described by local diffeomorphisms defined on $\ker D^J_{\tilde u}$ for linearizations at maps $\tilde u \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M)$ which are only sufficiently close to solutions. Here, we refer to the non-canonical linearization $D_{\tilde u} = DF_{\tilde u}(0)$ as derived in Section 2.4.

Our present problem is to study existence of regular admissible extensions J. The idea is to prove by means of the infinite dimensional version of Sard's theorem, i.e. the Sard-Smale Theorem, that there is a generic set of regular extensions. However, there is a delicate dependence on how we define the notion of genericity, that is, in which Baire space consisting of almost complex structures we want to search for a residual subset of regular extensions J.

Moreover, we give a refined version of perturbation so that we can confine the set $U \subset \Sigma$ over which we allow variation of J. Note that, in general, it is necessary to allow explicit variation of J in the variable $t \in S^1$, when the standard cylindrical solutions for the chain boundary operator are considered and H shall be prescribed. Now we allow dependence of J on the variable $z \in \Sigma$.

4.2.3 Definition *Let U be an open subset of our model surface Σ such that $U \subset \Sigma_T$ for some $T > 0$. Given $J_o \in \mathcal{J} = C^\infty_\Sigma(\pi^*\mathsf{J}_c)$ we define*

$$\mathcal{J}(U,J_o) = \left\{\, J \in \mathcal{J} \,\middle|\, (J-J_o)_{|\Sigma\setminus U} \equiv 0 \,\right\}.$$

This will be the set of variations of a given admissible extension J_o in which we look for regular extensions.

Let us consider the general framework for the application of the infinite dimensional version of Sard's theorem.

4.2.4 Proposition *Let $\mathcal{B} \subset C^\infty(\pi^*\mathsf{J}_c)$ be a smooth Banach manifold of almost complex structures and*

$$E\colon \mathcal{B}\times\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M) \to L^p_\Sigma(\mathcal{P}^*\,\mathrm{End}(TM)), \quad (J,u) \mapsto J(u)$$

be a smooth section. Then

$$\mathcal{E} = \bigcup_{(J,u)\in\mathcal{B}\times\mathcal{P}} \{(J,u)\} \times L^p_\Sigma(u^*X^J)$$

is a smooth subbundle of $\mathcal{F} = \mathcal{B} \times L^p_\Sigma(\mathcal{P}^*L)$ *and*

$$G\colon \mathcal{B} \times \mathcal{P} \to \mathcal{E}, \quad (J,u) \mapsto \overline{\partial}_{J,k}(u)$$

is a smooth section. Moreover, denoting by $\Pi\colon T_{(J,u,0)}\mathcal{E} \to \mathcal{E}_{(J,u)}$ *the projection associated to the canonical splitting along the zero section*

$$T_{(J,u,0)}\mathcal{E} = (T_J\mathcal{B} \times T_u\mathcal{P}) \oplus \mathcal{E}_{(J,u)},$$

we obtain the linearization

$$D_{(J,u)} = \Pi \circ DG(J,u)\colon T_J\mathcal{B} \times T_u\mathcal{P} \to \mathcal{E}_{(J,u)}$$

of G *at a solution* (J,u) *of* $\overline{\partial}_{J,k}(u) = 0$ *in the form*

$$D_{(J,u)}\colon T_J\mathcal{B} \times H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^J),$$
$$D_{(J,u)} \cdot (Y,\xi) = Y(u) \circ (du + k(u)) \circ j + D^J_u \cdot \xi\,.$$

Proof. We extend the Banach bundles which were considered in Theorem 2.2.5. By exactly the same methods as in the proof of Theorem 2.2.5 in the appendix we obtain the smooth bundle $L^p_\Sigma(\mathcal{P}^*\operatorname{End}(TM))$. The assumption of smoothness for the section E then provides us with smooth bundle homomorphisms $P^\pm\colon \mathcal{F} \to \mathcal{F}$ for the bundle $\mathcal{F} = \mathcal{B} \times L^p_\Sigma(\mathcal{P}^*L)$, where $L = \operatorname{Hom}(T\Sigma, TM) \to \Sigma \times M$,

$$P^\pm_{(J,u)}\colon L^p_\Sigma(u^*L) \to L^p_\Sigma(u^*X^{\pm J}) \subset L^p_\Sigma(u^*L),$$
$$\phi \mapsto \frac{1}{2}(\phi \pm J(u) \circ \phi \circ j),$$

that is $P^\pm(J,u,\phi) = \frac{1}{2}(\phi \pm E(J,u) \circ \phi \circ j)$. These maps $P^\pm$ thus are smooth projections onto the subsets $\mathcal{E}^\pm$ of $\mathcal{F}$,

$$\mathcal{E}^\pm = \bigcup_{(J,u)\in\mathcal{B}\times\mathcal{P}} \{(J,u)\} \times L^p_\Sigma(u^*X^{\pm J}) \to \mathcal{B} \times \mathcal{P}\,.$$

Fixing a $J_o \in \mathcal{B}$ we define the smooth section in the homomorphism bundle

$$\Theta\colon \mathcal{B} \times \mathcal{P} \to \mathcal{L}(\mathcal{F};\mathcal{F}),$$
$$\Theta(J,u) = P^+(J,u) \circ P^+(J_o,u) + P^-(J,u) \circ P^-(J_o,u)\,.$$

Since $\Theta(J_o,u) = \mathrm{id}_{\mathcal{F}_{|\{J_o\}\times\mathcal{P}}}$ we find an open neighbourhood $U(J_o) \subset \mathcal{B}$ such that

$$\Theta_{|U(J_o)}\colon U(J_o) \times L^p_\Sigma(\mathcal{P}^*X^{\pm J_o}) \xrightarrow{\cong} \mathcal{E}^\pm_{|U(J_o)}$$

are fibre-wise isomorphisms. Hence, the smooth maps $P^\pm$ provide us with smooth local trivializations of the Banach space bundles $\mathcal{E}^\pm \to \mathcal{B} \times \mathcal{P}$. We obtain the topological bundle splitting

$$\mathcal{F} = \mathcal{E}^+ \oplus \mathcal{E}^- \to \mathcal{B} \times \mathcal{P}$$

Next, the assumption that E is a smooth section immediately yields the smoothness of

$$G\colon \mathcal{B} \times \mathcal{P} \to \mathcal{E}^+ = \mathcal{E},$$
$$(J,u) \mapsto \overline{\partial}_{J,k}(u) = 2P^+(J,u)(du + k(u)),$$

because we already have the smoothness of $d + k\colon \mathcal{P} \to L^p_\Sigma(\mathcal{P}^*L)$ from Proposition 2.3.1. It remains to compute the linearization of G at a solution of $\overline{\partial}_{J,k}(u) = 0$. Considering G as the restriction of the respective map $\mathcal{B} \times \mathcal{P} \to \mathcal{F}$ to the subbundle $\mathcal{E}^+$ we obtain

$$DG(J,u)\colon T_J\mathcal{B} \times T_u\mathcal{P} \to T_{G(J,u)}\mathcal{E}^+ \subset T_{G(J,u)}\mathcal{F},$$
$$DG(J,u) \cdot (Y,\xi) = D_1G(J,u) \cdot Y + D_2G(J,u) \cdot \xi, \tag{4.9}$$

where

$$D_2G(J,u) = D(G(J,\cdot))(u) = D(\overline{\partial}_{J,k})(u) \tag{4.10}$$

and

$$D_1G(J,u) \cdot Y = D(G(\cdot,u))(J) \cdot Y = Y(u) \circ (du + k(u)) \circ j\,.$$

Note that, in general, we only have $D_1G(J,u) \cdot Y \in T_{G(J,u)}\mathcal{E}^+$. But, if (J,u) is a zero of G we can apply the identification

$$T_{(J,u,0)}\mathcal{E}^+ = (T_J\mathcal{B} \times T_u\mathcal{P}) \oplus L^p_\Sigma(u^*X^J)\,. \tag{4.11}$$

Since $Y \in T_J\mathcal{B}$ and $\overline{\partial}_{J,k}(u) = 0$ imply the identities

$$Y \circ J + J \circ Y = 0 \quad \text{and} \quad (du + k(u)) \circ j = J(u) \circ (du + k(u)),$$

it follows that the principal part of $D_1G(J,u) \cdot Y$ satisfies

$$D_1G(J,u) \cdot Y \in L^p_\Sigma(u^*X^J) \quad \text{for } G(J,u) = 0$$

in accordance with (4.11). On the other hand, by applying the canonical projection

$$\Pi\colon T_{(J,u,0)}\mathcal{E}^+ \to \mathcal{E}^+_{(J,u)}$$

to the second summand of $DG(J,u)$ in (4.9), the identity (4.10) gives rise to

$$\Pi \circ D_2G(J,u) = D^J_u\colon H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^J)\,.$$

Hence, the linearization of G at a solution of $\overline{\partial}_{J,k}(u) = 0$ can be expressed as asserted. ■

We now proceed by using the infinite dimensional version of Sard's theorem.

4.2.5 Proposition *Given a smooth separable Banach manifold* $\mathcal{B}$ *as in Proposition 4.2.4 such that* $G\colon \mathcal{B} \times \mathcal{P}(x_1,\dots,x_\nu) \to \mathcal{E}$ *is a smooth section intersecting the zero section transversely* $0_{\mathcal{B}\times\mathcal{P}} \subset \mathcal{E}$ *for all 1-periodic orbits* $x_1,\dots,x_\nu$*, i.e. all linearizations* $D_{(J,u)}$ *for* $\overline{\partial}_{J,k}(u) = 0$ *are onto, then there is a residual subset* $\mathcal{R} \subset \mathcal{B}$ *consisting of regular almost complex structures.*

Proof. Due to the assumption and the Fredholm property of the linear operators $D_u^J\colon T_u\mathcal{P} \to \mathcal{E}^+_{(J,u)}$, the solution set

$$\mathcal{M} = G^{-1}(0) \subset \mathcal{B} \times \mathcal{P}$$

is a smooth separable Banach submanifold and the projection

$$p\colon \mathcal{M} \to \mathcal{B}$$

is a smooth map between Banach manifolds. Moreover, by a standard argument, see for example [50], pp. 43-44, or [38], p. 36, the Fredholm property of D_u^J is passed on to the differential of p and $Dp(J,u)$ has the same Fredholm index as D_u^J. Hence, we can apply the Sard-Smale theorem, see [52], which states that, if p is a C^l-differentiable Fredholm map with $\operatorname{ind} Dp \leqslant l+1$, the set of regular values is residual in $\mathcal{B}$. In particular, it is a dense subset by Baire's theorem. This set of regular values of p coincides with the set of regular almost complex structures in the sense of Definition 4.2.1 but for fixed loops $x_1, \dots, x_\nu$. Since the Banach manifold $\mathcal{B}$ was given independently from these 1-periodic orbits, and the set of these orbits is finite (countability is sufficient !), we obtain the final set of regular J's as a countable intersection of residual subsets, which is again residual, in particular dense in $\mathcal{B}$. ∎

In order to find now a sufficiently rich set of regular almost complex structures, we have to find a separable Banach manifold $\mathcal{B}$ of variations of J such that the assumptions in Proposition 4.2.5 are satisfied. This set of variations has to be large enough in order to provide the transversality $G \pitchfork 0_{\mathcal{B}\times\mathcal{P}}$. In this exposition we choose Floer's original approach by constructing a Banach manifold lying sufficiently dense in $\mathcal{J}(U, J_o)$. A different, appealing approach is the analysis for $\mathcal{J}^k = C^k_\Sigma(\pi^*\mathsf{J}_c)$. This way has been chosen in [38] and [23]. However, it requires several arguments involving compactness and gluing results for the solution spaces. Moreover, considering at first the Banach manifold $\mathcal{B}^k \subset C^k(\pi^*\mathsf{J}_c)$ only leads to a C^{k-1}-submanifold $G^{-1}(0) \subset \mathcal{B}^k \times \mathcal{P}$. Hence, one has to assure that all solutions of $\overline{\partial}_{J,k}(u) = 0$ occuring have local index $\operatorname{ind} D_u \leqslant k-2$. This requires restriction of $\mathcal{P}(x_1, \dots, x_\nu)$ to components with solutions of fixed homological type. Then, after finding residual subsets for each homological type, one finally forms a countable intersection and obtains a set of regular almost complex structures which is still residual in the Baire space $\mathcal{J}(U, J_o) \subset C^\infty_\Sigma$. Here, we avoid this argumentation involving regularity theory for the class C^k. For our purposes it suffices to construct a Banach manifold $\mathcal{B} \subset \mathcal{J}$ which lies dense with respect to a suitably strong topology. Actually, it will be dense in all Sobolev spaces $H^{m,p}_o$ and therefore also in C^∞_o.

4.2.1 Floer's C^∞_ϵ-norm

4.2.6 Definition *Let $\epsilon = (\epsilon_n)_{n\in\mathbb{N}}$ be a given sequence of positive real numbers ϵ_n and $\Omega \subset \mathbb{R}^n$ be a fixed bounded domain. We define*

$$C^\infty_\epsilon(\Omega) = \{\, f \in C^\infty(\Omega, \mathbb{R}^n) \mid \|f\|_\epsilon < \infty \,\}$$

for

$$\|f\|_\epsilon = \sum_{k\geqslant 0} \epsilon_k \sup_{x\in\Omega} |D^k f(x)|\,.$$

Obviously, the embeddings

$$C^\infty_\epsilon(\Omega) \hookrightarrow C^k(\Omega), \quad k \in \mathbb{N}, \quad \text{and} \quad C^\infty_\epsilon(\Omega) \hookrightarrow C^\infty(\Omega)$$

are continuous.

4.2.7 Lemma *The normed vector space $(C^\infty_\epsilon(\Omega), \|\cdot\|_\epsilon)$ is complete.*

4.2.8 Lemma *There exists a sequence $\epsilon = (\epsilon_n)_{n\in\mathbb{N}}$ such that $C^\infty_\epsilon(\Omega) \cap H^{m,p}_o(\Omega)$ lies dense in every $H^{m,p}_o(\Omega)$, $m \in \mathbb{N}$, $1 \leqslant p < \infty$.*

Proof of Lemma 4.2.7. Given a Cauchy sequence $(f_n) \subset C^\infty_\epsilon$, it is a Cauchy sequence in every $C^k(\Omega)$, $k \in \mathbb{N}$. Thus, there is a unique $f \in C^\infty(\Omega)$ such that

$$f_n \to f \quad \text{in } C^k(\Omega) \quad \text{for all } k \in \mathbb{N}\,.$$

Considering the truncated norm on $C^m(\Omega)$, $m \in \mathbb{N}$,

$$\|g\|_{m,\epsilon} = \sum_{k\leqslant m} \epsilon_k \|D^k g\|_\infty,$$

we find an $n_o(\delta, m)$ for each $\delta > 0$, $m \in \mathbb{N}$, such that

$$\|f - f_n\|_{m,\epsilon} < \frac{\delta}{2} \quad \text{f.a. } n \geqslant n_o\,.$$

In particular, there is an increasing sequence $n(m) \to \infty$ for $m \to \infty$ such that

$$\|f - f_{n(m)}\|_{m,\epsilon} < \frac{\delta}{2} \quad \text{f.a. } m \in \mathbb{N}\,.$$

Since (f_n) is a Cauchy sequence, i.e. there is an $n_o(\delta)$ with $\|f_k - f_l\|_\epsilon < \frac{\delta}{2}$ for all $k, l > n_o(\delta)$, we deduce that

$$\|f - f_l\|_{m,\epsilon} < \delta \quad \text{for all } l \geqslant n_o(\delta),\ m \in \mathbb{N}\,.$$

Hence, we have $f \in C^\infty_\epsilon$ and $\|f - f_n\|_\epsilon \to 0$ for $n \to \infty$. ∎

Proof of Lemma 4.2.8. Let $\rho \in C^\infty_o(\mathbb{R}^n, \mathbb{R})$ be a fixed cut-off function with compact support in the unit ball $B_1(0)$ and with

$$\int_{\mathbb{R}^n} \rho\, d^n x = 1\,.$$

We denote by ρ_δ for $\delta > 0$ the rescaled function

$$\rho_\delta(x) = \rho(\tfrac{x}{\delta})\,.$$

Given any smooth function $u \in C^\infty_o(\Omega, \mathbb{R})$ with compact support in the interior of Ω we compute for $\delta > 0$ small enough, i.e. smaller than the distance of the support of u to the boundary of Ω,

$$\begin{aligned} \left|D^k(\rho_\delta * u)(x)\right| &= \Big|\int_{\mathbb{R}^n} (D^k\rho_\delta)(x-y)u(y)d^n y\Big| \\ &\leqslant \|u\|_{L^\infty} \|D^k\rho_\delta\|_{L^\infty} \operatorname{const}(u), \end{aligned}$$

and

$$\|D^k\rho_\delta\|_{L^\infty} \leqslant \frac{1}{\delta^k}\|D^k\rho\|_{L^\infty}\,.$$

Setting $a_k = \|D^k\rho\|_{L^\infty}$ we obtain for all $x \in \Omega$, $k \in \mathbb{N}$ and $\delta < \operatorname{dist}(\partial\Omega, \operatorname{supp} u)$

$$\left|D^k(\rho_\delta * u)(x)\right| \leqslant c\frac{a_k}{\delta^k}$$

with some constant $c > 0$. Hence, defining $\epsilon_k = (a_k k^k)^{-1}$, we conclude that

$$\epsilon_k \left\|D^k(\rho_\delta * u)\right\|_{L^\infty(\Omega)} \leqslant \frac{c}{(k\delta)^k} \leqslant c(\tfrac{1}{2})^k$$

for $k \geqslant 2\delta^{-1}$ and therefore

$$\rho_\delta * u \in C^\infty_\epsilon(\Omega, \mathbb{R})\,.$$

Thus the assertion follows from the fact that

$$\rho_{\delta_n} * u \to u \quad \text{in } H^{m,p}_o(\Omega) \quad \text{as } \delta_n \to 0,$$

for all $m \in \mathbb{N}$ and $1 \leqslant p < \infty$. ■

4.2.9 Proposition *Given a bounded domain $\Omega \subset \mathbb{R}^n$, the Banach space $C^\infty_\epsilon(\Omega, \mathbb{R})$ is separable.*

Proof. Due to the Theorem of Stone-Weierstraß, the set of continuous maps $C^0(\overline{\Omega}, V)$ from the compact space $\overline{\Omega}$ into a finite dimensional vector space V is separable. Let V_k be the vector space of k-linear maps from $\mathbb{R}^n$ to $\mathbb{R}$

$$V_k = \mathcal{L}(\otimes^k\mathbb{R}^n, \mathbb{R})$$

and X_k be a countable dense subspace of $C^0(\overline{\Omega}, V_k)$ for every $k \geqslant 0$. Then we consider the space

$$\begin{aligned} \Sigma &= \{\, (f_i)_{i\geqslant 0} \mid \sum_i \epsilon_i \|f_i\|_{C^0} < \infty \,\} \\ &\subset C^0(\overline{\Omega}, V_0) \times C^0(\overline{\Omega}, V_1) \times \ldots \end{aligned}$$

and the subset $X \subset \Sigma$,

$$X = \bigcup_{k\geqslant 0} X_0 \times \ldots \times X_k \times \{0\} \times \{0\} \times \ldots$$

which is countable because it is a countable union of finite products of countable sets. We observe that X is dense in Σ with respect to the norm $\|f\|_\epsilon = \sum_i \epsilon_i \|f_i\|_{C^0}$. Namely, for given $f = (f_i) \in \Sigma$, we find an approximating sequence $(u(n))_{n\in\mathbb{N}} \subset X$, $u(n) = (u(n)_i)_{i\geqslant 0}$, with

$$\begin{aligned} &\left\|u(n)_0 - f_0\right\|_{C^0(\overline{\Omega}, V_0)} < \frac{1}{n}, \\ &\left\|u(n)_i - f_i\right\|_{C^0(\overline{\Omega}, V_i)} < \frac{n^{-i}}{\epsilon_i}, \quad 1 \leqslant i \leqslant n, \\ &u(n)_i = 0, \quad i > n\,. \end{aligned}$$

Since

$$\left\|f - u(n)\right\|_\epsilon < \frac{1}{n} + \sum_{i=1}^n (\frac{1}{n})^i + \sum_{i\geqslant n+1} \epsilon_i \|f_i\|_{C^0(\overline{\Omega}, V_i)} \to 0$$

for $n \to \infty$, it follows that Σ is separable with respect to $\|\cdot\|_\epsilon$. Using the isometric embedding

$$C^\infty_\epsilon(\Omega, \mathbb{R}) \to \Sigma, \quad f \mapsto (D^k f)_{k\geqslant 0}$$

we deduce that $C^\infty_\epsilon(\Omega, \mathbb{R})$ is separable. ■

4.2.2 The Variation of the Almost Complex Structure

We now use this ϵ-norm for C^∞-maps in relation with the variation of the almost complex structure J. However, in order to apply this Banach space topology to the space $\mathcal{J}(U, J_o)$, we need a vector space structure. Therefore we resort to a technique of identifying the set of ω-compatible almost complex structures $\mathrm{J}_c(V, \omega)$ on a symplectic vector space by means of a Cayley transformation. This mapping is described in [5] in order to present a simple proof for the fact that $\mathrm{J}_c(TM, \omega)$ is contractible.

4.2.10 Definition *Given a fixed $J_o \in \mathcal{J}$ we define the subbundle $S_{J_o} \subset \operatorname{End}(TM)$ by*

$$S_{J_o} = \{\, A \in \operatorname{End}(TM) \mid J_o A + A J_o = 0,\ \omega \circ (A \times J_o) = -\omega \circ (J_o \times A) \,\}$$

together with the open neighbourhood of the zero section

$$S^1_{J_o} = \{\, A \in S_{J_o} \mid \|A\|_{J_o} < 1 \,\},$$

where $\|A\|_{J_o}$ at $m \in M$ denotes the operator norm of $A(m) \in \operatorname{End}(T_m M)$ with respect to the norm $\|\cdot\|_{J_o}$ on $T_m M$. We then define the map

$$\begin{aligned} \Phi_{J_o} &: \mathrm{J}_c(TM, \omega) \to S^1_{J_o}, \\ &J \mapsto (J + J_o)^{-1}(J - J_o)\,. \end{aligned}$$

It is easy to compute that for any $v \in T_m M$ we have

$$\left\|\Phi_{J_o}(J)v\right\|^2_{J_o} = \|v\|^2_{J_o} - 4\omega(y, J_o y) \tag{4.12}$$

where $y = (J + J_o)^{-1} J_o v$. Note that $J + J_o$ is invertible due to the condition that $\omega(x, Jx) > 0$ for all $x \in TM \setminus \{0\}$, $J \in \mathsf{J}_c(TM,\omega)$. Thus, it follows from (4.12) that

$$\left\|\Phi_{J_o}(J)\right\|_{J_o} < 1 \quad \text{for all } J \in \mathsf{J}_c(TM,\omega),$$

and the inverse map is given by

$$\Phi^{-1}_{J_o}(A) = J_o \circ (\mathrm{Id} + A) \circ (\mathrm{Id} - A)^{-1} \tag{4.13}$$

which is well-defined if $A \in S^1_{J_o}$. This diffeomorphism between $\mathsf{J}_c(TM,\omega)$ and $S^1_{J_o}$ allows us to equip $C^\infty_\Sigma(\pi^* \mathsf{J}_c)$ with the Banach manifold structure of $C^\infty_\epsilon(\pi^* S^1_{J_o})$.

4.2.11 Definition *We define $\mathcal{B} = \mathcal{J}_\epsilon(U, J_o)$ by*

$$\mathcal{J}_\epsilon(U, J_o) = \{\, J \in \mathcal{J} \mid \Phi_{J_o}(J) \in C^\infty_\epsilon(\pi^* S^1_{J_o}),\ \operatorname{supp} \Phi_{J_o}(J) \subset U \times M \,\},$$

where the norm $\|\cdot\|_\epsilon$, $\epsilon = (\epsilon_k)_{k\in\mathbb{N}}$,

$$\|A\|_\epsilon = \sum_{k\in\mathbb{N}} \epsilon_k \cdot \sup_{(z,m)\in\Sigma\times M} \left|\nabla^k A(z,m)\right|_{J_o}$$

is defined as above, for a fixed connection ∇ on $\pi^* TM_{|\Sigma_T \times M}$. The sequence ϵ is chosen in such a way that $C^\infty_\epsilon(\pi^* S_{J_o})$ lies dense in

$$\{\, A \in C^\infty(\pi^* S^1_{J_o}) \mid \operatorname{supp} A \subset U \times M \,\}$$

in the sense of Lemma 4.2.8.

We have now defined a Banach manifold $\mathcal{B} = \mathcal{J}_\epsilon(U, J_o)$ such that the tangent space at $J \in \mathcal{B}$ is given by

$$T_J\mathcal{B} = \{\, Y \in C^\infty(\pi^* S_J) \mid D\Phi_{J_o}(J)\cdot Y \in C^\infty_\epsilon(\pi^* S_{J_o}),\ Y_{|\Sigma \setminus U} \equiv 0 \,\}. \tag{4.14}$$

Due to the diffeomorphism property of Φ_{J_o}, $T_J\mathcal{B}$ lies dense within

$$\{\, Y \in L^2(\pi^* S_J) \mid \operatorname{supp} Y \subset U \times M \,\},$$

and the section

$$E\colon \mathcal{J}_\epsilon(U, J_o) \times \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) \to L^p_\Sigma(\mathcal{P}^* \operatorname{End}(TM))$$

is smooth because of the smoothness of

$$\Phi^{-1}_{J_o}\colon S^1_{J_o} \to \mathsf{J}_c(TM,\omega) \subset \operatorname{End}(TM)$$

and $\mathcal{B} \subset C^\infty_\Sigma(\pi^* \operatorname{End}(TM))$. Thus, for a given T-admissible extension k of $(H^i)_{i=1,\dots,\nu}$ and specified set $U \subset \Sigma_T$ it remains to verify the surjectivity of the linearization

$$D_{(J,u)}\colon T_J\mathcal{J}_\epsilon(U, J_o) \times T_u\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) \to L^p_\Sigma(u^* X^J)$$

for each solution $(J, u) \in \mathcal{J}_\epsilon(U, J_o) \times \mathcal{P}(x_1, \dots, x_\nu)$ of $\overline{\partial}_{J,k}(u) = 0$ and for all $x_i \in \mathcal{P}_1(H^i)$, $i = 1, \dots, \nu$.

Before we begin with proving this regularity statement we need an appropriate local characterization of the occuring solutions $u \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ of the Cauchy-Riemann type PDE $\overline{\partial}_{J,k}(u) = 0$. The fundamental observation is the following quality which generalizes a well-known fact from holomorphic functions. It is often called the "Unique Continuation". It can be based on a similarity principle which we quote here from [31] and [23]. It replaces the use of a theorem of Aronszajn in the original work by Floer.

Given $p > 2$, let $B_\epsilon = \{\, z \in \mathbb{C} \mid |z| < \epsilon \,\}$, $C \in L^p(B_\epsilon, \mathcal{L}_\mathbb{R}(\mathbb{C}^n))$ and let $z \mapsto J(z)$ be in the Sobolev space $W^{1,p}(B_\epsilon, \mathcal{L}_\mathbb{R}(\mathbb{C}^n))$ such that $J(z)$ is a complex structure, $J(z)^2 = -\mathrm{Id}$, for every z. Then we consider for $z = s + it$ the perturbed Cauchy-Riemann type PDE for maps $u\colon B_\epsilon \to \mathbb{C}^n$,

$$\partial_s u(z) + J(z)\partial_t u(z) + C(z)u(z) = 0\,. \tag{4.15}$$

The Similarity Principle states the following.

4.2.12 Theorem (Similarity Principle) *Let $u \in W^{1,p}(B_\epsilon, \mathbb{C}^n)$ be a solution of (4.15) with $u(0) = 0$. Then there exists a constant $0 < \delta < \epsilon$, a map $\Phi \in W^{1,p}(B_\delta, \mathcal{GL}_\mathbb{R}(\mathbb{C}^n))$ and a holomorphic map $\sigma\colon B_\delta \to \mathbb{C}^n$ such that*

$$u(z) = \Phi(z)\sigma(z), \quad \sigma(0) = 0, \quad J(z)\Phi(z) = \Phi(z)i$$

for $z \in B_\delta$.

Proof. See Appendix A.6 in [31], where $J \in C^\infty$ and $C \in L^\infty(B_\epsilon)$. The same proof for $J \in W^{1,p}$, $C \in L^p$ is given in [23]. ■

For our present purpose it is sufficient to use the following conclusion from the Similarity Principle.

4.2.13 Corollary (Unique Continuation) *Let $u \in W^{1,p}(B_\epsilon, \mathbb{C}^n)$ be a solution of (4.15) with $u(0) = 0$. Then either $u \equiv 0$ or there exists a $0 < \delta < \epsilon$ such that $u(z) \neq 0$ for all $0 < |z| < \delta$. Moreover, if $C = 0$ and $u \not\equiv 0$ there exists a constant $0 < \delta < \epsilon$ such that $du(z) \neq 0$ for $0 < |z| < \delta$.*

In order to prove the existence of generic regular extensions J of $(J^i)_{i=1,\dots,\nu}$ further conditions must be satisfied. This can be easily seen from the following example.

Let us consider the closed model surface $\Sigma = T^2 = S^1 \times S^1$, that is, $\Sigma_Z = \emptyset$. If we choose the admissible extension $k \equiv 0$ in the sense of Definition 2.3.4 which is the canonical special k^o, we obtain the solution set

$$\mathcal{M} = \{\, u \in C^\infty(\Sigma, M) \,|\, \overline{\partial}_J u = 0 \,\}\,.$$

The formula of Riemann-Roch provides the virtual dimension

$$\operatorname{ind} D_u = 2c_1(u^*TM)([T^2])$$

for $u \in \mathcal{M}$. In particular, we have the canonical embedding

$$M \hookrightarrow \mathcal{M}, \quad p \mapsto (u(z) = p),$$

with $\operatorname{ind} D_p = 0$ for all $p \in M$, and $\dim M = 2n$. Hence, there is no regular J in this framework.

We can, however, avoid this dilemma by exploiting the freedom of choosing a different admissible extension k.

4.2.14 Definition *Given a T-admissible*[1] *extension $(k(H), J_o)$ of (H^i, J^i) we call $k = k(H)$* ***regular*** *if either there exists an open subset $U \subset \Sigma_T$ with the following property:*

For all $J \in \mathcal{J}(U, J_o)$, $x_i \in \mathcal{P}_1(H^i)$, $i = 1, \ldots, \nu$ and $u \in \mathcal{M}^{x_1,\ldots,x_a}_{x_{a+1},\ldots,x_\nu}(J,k)$ there exists a $z_o \in U$ with

$$(du + k(u))(z_o) \neq 0,$$

or if for all admissible extensions J of (J^i) all linearizations

$$D_u^J \colon H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^J)$$

at solutions u of $du + k(u) = 0$ are surjective.

By means of the above Similarity Principle we now can prove that the special admissible extension $k^o(H)$ of $(H^i)_{i=1,\ldots,\nu}$ from Section 2.3 always admits a slight perturbation as a regular extension $k(H)$.

4.2.15 Corollary *Let $u \in C^\infty(Z^+, M)$ be a solution of $\overline{\partial}_{J,k}(u) = 0$ for a T-admissible extension $k(H)$ of H^i. If $\partial_s u \equiv 0$ on $[0, T+\epsilon] \times S^1$, $\partial_s u$ vanishes on the entire Z^+. In particular, $u(s,t) = x(t)$ is a 1-periodic solution for $X_{H^i_t}$.*

Proof. Due to the condition on $\partial_s u$ it suffices to consider $u_{|Z_T}$ solving

$$\partial_s u + J(t,u)(\partial_t u - X_{H_t}(u)) = 0\,.$$

Let ψ be the flow on M generated by the vector field X_H on $S^1 \times M$,

$$\partial_t \psi(t,p) = X_{H_t}(\psi(t,p)), \quad \psi(0,p) = p \quad \text{f.a. } p \in M\,.$$

[1] see Definition 2.3.4

Denoting

$$T\psi(t,p) = \tfrac{\partial}{\partial p}\psi(t,p) \colon T_pM \xrightarrow{\cong} T_{\psi(t,p)}M$$

we obtain for the smooth map $\xi \colon [T,\infty) \times \mathbb{R} \to M$ defined by

$$u(s,t) = \psi(t, \xi(s,t))$$

the identity

$$\begin{aligned} 0 &= T\psi(t,\xi)\partial_s\xi + J(t,\psi(t,\xi))\,[\partial_t\psi + T\psi \cdot \partial_t\xi - X_{H_t}(u)] \\ &= T\psi\partial_s\xi + J(t,\psi(t,\xi))T\psi \cdot \partial_t\xi\,. \end{aligned}$$

Thus, $\xi \in C^\infty([T,\infty) \times \mathbb{R}, M)$ solves

$$(\partial_s + \tilde{J}\partial_t)\xi = 0$$

for $\tilde{J}(t,p) = T\psi(t,p)^{-1}J(t,\psi(t,p))T\psi(t,p)$, and by assumption

$$d\xi_{|(T,T+\epsilon)\times S^1} \equiv 0\,.$$

Hence, the Unique Continuation result, Corollary 4.2.13, implies that ξ is constant and thus, u is independent of s. ■

4.2.16 Lemma *Given an open connected domain $\Omega \subset \mathbb{C}$ and a vector field $X \colon \Omega \times M \to TM$ with compact support, then any solution $u \in C^\infty$ of*

$$du + k(u) = 0 \quad \textit{with} \quad k = dt \otimes X$$

is constant, $u \equiv p$, with $X_{|\Omega \times \{p\}} = 0$.

Proof. Since du vanishes on a connected neighbourhood of $\partial\Omega$ in Ω and $\partial_s u \equiv 0$ on Ω the assertion follows immediately. ■

4.2.17 Proposition *Let $k = k^o(H)$ be the special extension of $(H^i)_{i=1,\ldots,\nu}$. Then, there is an arbitrary small perturbation $\tilde{k}$ of k on Σ_0 which is regular and admissible. The same is true for $k^o(H) = 0$ when Σ is a closed model surface.*

Proof. By Definition 2.3.4 $k^o(H)$ is T-admissible for $T \geqslant 2$. Then, for any extension J of (J^i) and solution $u \in \mathcal{M}^{\cdots}_{\cdots}(J, k^o(H))$ satisfying

$$(du + k^o(H))_{|\Sigma_{2+\epsilon}} \equiv 0,$$

we deduce from Corollary 4.2.15 that u must be constant,

$$u(\Sigma) = p \in C = \bigcap \{\, \operatorname{Crit} H^i_t \,|\, i = 1, \ldots, \nu,\ t \in S^1 \,\},$$

with $C = M$ if Σ is closed. This already leads to a contradiction if C is empty. In case of $C \neq \emptyset$ we find finitely many vector fields $X_1, \ldots, X_r$ such that $C \cap \bigcap_{i=1}^r \operatorname{Crit} X_i = \emptyset$. Then we can construct a regular perturbation of $k^o(H)$ by means of suitable cut-off functions $\rho_i \in C^\infty_o(\Sigma_0)$ and Lemma 4.2.16. ■

We have thus proven the existence of admissible regular extensions $k = k(H)$ of $(H^i)_{i=1,\dots,\nu}$. This is essential in order to guarantee that it is possible to find a generic variation of the extension J of $(J^i)_{i=1,\dots,\nu}$ such that all linearizations D_u are surjective for solutions of $\overline{\partial}_{J,k}(u) = 0$.

4.2.18 Proposition *Let $k(H)$ be a regular T-admissible extension of (H^i) and $U \subset \Sigma_T$ be an open subset as in Definition 4.2.14. Given an admissible extension J_o of $(J^i)_{i=1,\dots,\nu}$, then for every solution pair*

$$(J,u) \in \mathcal{J}_\epsilon(U,J_o) \times \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M), \quad \overline{\partial}_{J,k(H)}(u) = 0,$$

the linearization

$$D_{(J,u)} : T_J\mathcal{J}_\epsilon(U,J_o) \times H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^J)$$

is surjective.

Proof. From Proposition 4.2.4 we know that we have to prove the surjectivity of the operator

$$D_{(J,u)} : T_J\mathcal{B} \times H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^J),$$
$$D_{(J,u)} \cdot (Y,\xi) = Y(u) \circ (du + k(H)(u)) \circ j + D^J_u \cdot \xi$$

where without loss of generality $(du + k(H)(u))_{|U} \not\equiv 0$ in view of Definition 4.2.14. Here Y ranges through an L^2-dense subset of

$$\mathcal{X} = \{\, C^\infty(\pi^*S_J) \,|\, Y_{|\Sigma\setminus U} \equiv 0 \,\}\,.$$

First of all, we know that D^J_u is a Fredholm operator, so that also the range of $D_{(J,u)}$ is closed in $L^p_\Sigma(u^*X^J)$ with finite dimensional cokernel which is contained in coker D^J_u. We have to show that the range is also dense. If this is false then there exists a functional $\varphi \in (L^p_\Sigma(u^*X^J))$, $\varphi \neq 0$, vanishing on the range of $D_{(J,u)}$. We now use the well-defined L^2-product on u^*X^J, see for example Section 3.1.1 or Definition 4.4.7. We obtain an $\eta \in L^q_\Sigma(u^*X^J)$, $\frac{1}{q} + \frac{1}{p} = 1$ satisfying

$$\big\langle D_{(J,u)} \cdot (Y,\xi), \eta \big\rangle_{J,L^2_\Sigma} = 0 \quad \text{f.a. } (Y,\xi) \in T_J\mathcal{B} \times H^{1,p}_\Sigma(u^*TM)\,.$$

In particular, we deduce for $Y = 0$ that $\eta \in L^p_\Sigma(u^*X^J)$ is a weak solution for the formally adjoint F^* of the $\overline{\partial}$-operator $F = D^J_u$. In local conformal coordinates (s,t), $\tilde{\eta} = \eta \cdot \frac{\partial}{\partial s}$ is a weak solution of

$$\nabla_s\tilde{\eta} + \bar{J}(s,t)\nabla_t\tilde{\eta} + C(s,t)\tilde{\eta} = 0 \tag{4.16}$$

in the sense of (2.10) in Section 2.5. Here $\bar{J} = -J$ and C are smooth. Hence, we obtain from the elliptic regularity result in Corollary 2.5.3 that $\tilde{\eta}$ is a smooth solution of (4.16), i.e.

$$\eta \in L^q_\Sigma(u^*X^J) \cap C^\infty\,.$$

Moreover, locally, $\eta \in H^{1,p}_{\mathrm{loc}}$ allows the application of the Unique Continuation result, Corollary 4.2.13, to $\tilde{\eta} = \eta \cdot \frac{\partial}{\partial s}$. Thus, η must be identically to the zero section iff it vanishes on a non-discrete set of points in Σ. By assumption of regularity of $k(H)$, there is a $z_o \in U$ with

$$(du + k(H)(u))(z_o) \neq 0\,.$$

Choosing conformal coordinates (s,t) on a small neighbourhood Ω of z_o we can replace η by $\eta \cdot \frac{\partial}{\partial s}$ or $\eta \cdot \frac{\partial}{\partial t}$ so that with $\phi = (du + k(H)(u)) \cdot \frac{\partial}{\partial s}$ or $\cdot \frac{\partial}{\partial t}$ we obtain

$$\langle Y \cdot \phi, \eta \rangle_{J,L^2} = 0 \tag{4.17}$$

for all $Y \in \mathcal{X}$. The idea is now to prove by means of (4.17) and $\phi(z_o) \neq 0$ that η must vanish on an open neighbourhood of z_o. This requires the following algebraic lemma which we take from [48].

4.2.19 Lemma *Let (V,ω) be a symplectic vector space with an ω-compatible almost complex structure J, and let*

$$S_J(V) = \{\, Y \in \mathrm{End}(V) \,|\, JY + YJ = 0,\ \omega \circ (Y \times \mathrm{Id}) = -\omega \circ (\mathrm{Id} \times Y) \,\}\,.$$

Then, for every pair $0 \neq v, w \in V$ there exists a $Y \in S_J(V)$ such that

$$\langle v, Yw \rangle_J = \omega(v, JYw) \neq 0\,.$$

Proof. We repeat the proof from [48], pp. 1346–1347. If $\langle v, w \rangle_J \neq 0$, we choose $Y = \Lambda_J(P_w)$, where

$$\Lambda_J\colon\ \mathrm{End}(V) \to \mathrm{End}(V), \quad A \mapsto A + JAJ,$$

maps the subspace of symmetric operators onto S_J, and P_w is the orthogonal projection

$$P_w\colon V \to \mathbb{R}w, \quad u \mapsto w\frac{\langle u, w \rangle_J}{\langle w, w \rangle_J}\,.$$

If $\langle v, Jw \rangle_J \neq 0$, then let $Y = \Lambda_J(P_{Jw})$, and if v is orthogonal to the plane span$\{w, Jw\}$, we then choose $Y = \Lambda_J(P_{v+w})$. ∎

Let us choose Ω small enough such that $\phi(z) \neq 0$ for all $z \in \Omega$. Then η must vanish on Ω due to (4.17). Namely, suppose that $\eta(z) \neq 0$ for some $z \in \Omega$. Then, by Lemma 4.2.19, there is a $Y_o \in S_{J(z,u(z))}(T_{u(z)}M)$ such that

$$\Big\langle Y_o\phi(z), \eta(z) \Big\rangle_J > 0\,.$$

After extending Y_o to some $Y \in C^\infty(\pi^*S_J)$ we can find a cut-off function $\beta \in C^\infty_o(\Omega)$ with support small enough so that

$$\Big\langle \beta Y\phi, \eta \Big\rangle_{J,L^2} > 0$$

in contradiction to (4.17). ∎

It is necessary to remark that the simplifying element of this proof is the admitted explicit dependence of the variation Y on the variable $z \in U$. This allowed us to use the cut-off function β above in the variable $z \in \Omega$. In order to sum up the last results, we state the following

4.2.20 Theorem *Given a regular T-admissible extension $k(H)$ of $(H^i)_{i=1,\dots,\nu}$ and an admissible extension J_o of $(J^i)_{i=1,\dots,\nu}$, there is an open subset $U \subset \Sigma_T$ and a Banach manifold $\mathcal{B}$ of smooth admissible extensions J lying C^∞-dense in the metric space $\mathcal{J}(U, J_o) = \{ J \in \mathcal{J} \mid J_{|\Sigma\setminus U} \equiv J_{o|\Sigma\setminus U} \}$, such that $\mathcal{B}$ contains a residual subset $\mathcal{R}$ of admissible extensions J of $(J^i)_{i=1,\dots,\nu}$ which are regular for $(\Sigma, j, k(H))$. In particular, there is a regular $J \in \mathcal{J}(U, J_o)$ arbitrarily close to J_o in the C^∞-topology.*

We remark at this stage, that a countable intersection of residual sets is again residual, in particular dense by Baire's theorem. Hence, Theorem 4.2.20 also guarantees the existence of admissible extensions J which are simultaneously regular for a given countable family $(\Sigma, j_\lambda, k_\lambda)$, $\lambda \in \mathbb{N}$.

4.2.3 Homotopy

Up to now we discussed the variation of the almost complex structure J when all other parameters for Σ and $\overline{\partial}_{J,k}$ are fixed. In the final chapter we will require a generalization allowing dependence on a further parameter $\lambda \in [0,1]$ not only for J but also for the conformal structure j on Σ and for the admissible extension $k = k(H)$.

Let $(j_\lambda, J_\lambda, k_\lambda)$, $\lambda \in [0,1]$, be a smooth 1-parameter family of admissible extensions. In view of the above remark we may ask whether we can find $J_\lambda = J$ independent of λ and uniformly regular for $(\Sigma, j_\lambda, k_\lambda)$ for all $\lambda \in [0,1]$. This would be an ideal situation enormously simplifying the proof of homotopy invariance for the cohomology operations in Section 5.2. However, this cannot be expected, but we can easily deduce the existence of regular 1-parameter families $(J_\lambda)_{\lambda\in[0,1]}$ from the above analysis.

At first we have to add the interval $[0,1]$ to the analytic framework. In the following we use the more precise notation

$$u^*X^{j,J} = \{ \phi \in T^*\Sigma \otimes u^*TM \mid \phi(z)j(z) = -J(z,u(z))\phi(z) \}$$

and for $(j_\lambda, J_\lambda, k_\lambda)$ the short forms $u^*X^{j_\lambda,J_\lambda} = u^*X^\lambda$ and $\overline{\partial}_{j_\lambda,J_\lambda,k_\lambda} = \overline{\partial}_\lambda$.

4.2.21 Proposition *Given $(j_\lambda, J_\lambda, k_\lambda)$ as above, the set*

$$\mathcal{E} = \bigcup_{(\lambda,u)\in[0,1]\times\mathcal{P}} \{(\lambda,u)\} \times L^p_\Sigma(u^*X^\lambda)$$

is a smooth subbundle of $[0,1] \times \mathcal{F} \to [0,1] \times \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}$ and

$$F\colon [0,1] \times \mathcal{P} \to \mathcal{E}, \quad (\lambda,u) \mapsto \overline{\partial}_\lambda(u)$$

is a smooth section. Referring to the canonical splitting along the zero section

$$T_{(\lambda,u,0)}\mathcal{E} = (\mathbb{R} \times T_u\mathcal{P}) \oplus \mathcal{E}_{(\lambda,u)}$$

we obtain the linearization of F at a solution (λ,u) of $\overline{\partial}_\lambda(u) = 0$ in the form

$$D_{(\lambda,u)}\colon \mathbb{R} \times H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^\lambda),$$
$$D_{(\lambda,u)} \cdot (\tau,\xi) = A(\lambda,u)\cdot\tau + D^\lambda_u \cdot \xi$$

for some $A(\lambda,u) \in L^p_\Sigma(u^*X^\lambda)$ and $D^\lambda_u = D(\overline{\partial}_\lambda)(u)$.

Since this proposition follows in straight analogy with Proposition 4.2.4 we can omit the proof.

4.2.22 Definition *We call a 1-parameter family of admissible extensions*

$$\lambda \mapsto (j_\lambda, J_\lambda, k_\lambda), \quad \lambda \in [0,1],$$

regular *if (J_0,k_0) and (J_1,k_1) are regular with respect to j_0 and j_1, and if the linearizations $D_{(\lambda,u)}$ are surjective for all solutions (λ,u) of $\overline{\partial}_\lambda(u) = 0$.*

Let us state the transversality result which will be needed for the later homotopy invariance discussion.

4.2.23 Theorem *Given any admissible regular extensions (j_0,J_0,k_0) and (j_1, J_1,k_1) of $(\psi_l^* i)_{l=1,\dots,\nu}$ and $(J^l,H^l)_{l=1,\dots,\nu}$ there is an admissible and regular extension family $(j_\lambda,J_\lambda,k_\lambda)_{\lambda\in[0,1]}$.*

Proof. As we noticed in the foundational Section 2.2, the space of complex structures j on Σ extending $(\psi_k^* i)_{k=1,\dots,\nu}$ is contractible. Thus we may choose any smooth arc of extensions $(j_\lambda)_{\lambda\in[0,1]}$ connecting j_0 and j_1. Next, we argue analogously to Proposition 4.2.17 that we find a smooth 1-parameter family k_λ of T-admissible extensions connecting k_0 and k_1 such that every k_λ is regular in the sense of Definition 4.2.14. It remains to find an argument analogous to Theorem 4.2.20 providing us with a 1-parameter family (J_λ) connecting J_0 and J_1 such that the whole triple $(j_\lambda,J_\lambda,k_\lambda)$ is regular.

First, the existence of any connecting arc $(J^o_\lambda)_{\lambda\in[0,1]}$ follows from the contractibility of the space of ω-compatible almost complex structures. Next, we have to find a regular perturbation of (J^o_λ) by means of the above transversality arguments based on Sard's theorem as used in Proposition 4.2.4. We carry out the same program of extending the section F from Proposition 4.2.21 to the space $[0,1]\times\mathcal{B}\times\mathcal{P}$ where now $\mathcal{B}$ is a Banach manifold of perturbations of (J^o_λ) which are smooth maps $J\colon [0,1]\times\Sigma\times M \to \mathrm{End}(TM)$. This follows along the lines of Propositions 4.2.4 and 4.2.21. Generalizing Proposition 4.2.18, we have to verify that the linearization $D_{(\lambda,J,u)}$ of the section

$$(\lambda,(J_\lambda),u) \mapsto \overline{\partial}_\lambda(u)$$

is surjective at solutions $(\lambda, (J_\lambda), u)$ of $\overline{\partial}_\lambda(u) = 0$. But this follows directly from Proposition 4.2.18 because

$$D_{\left(\lambda,(J_\lambda),u\right)}(\tau, Y, \xi) = A(\lambda, (J_\lambda), u) \cdot \tau + D_{(J_\lambda,u)} \cdot (Y, \xi)\,.$$

Since k_λ is regular for all $\lambda \in [0,1]$ the surjectivity of $D_{(\lambda,(J_\lambda),u)}$ at a solution $(\lambda, (J_\lambda), u)$ follows from that of $D_{(J_\lambda,u)}$. Thus we find a regular admissible arc $(J_\lambda)_{\lambda\in[0,1]}$ connecting J_0 and J_1 and arbitrarily close to (J^o_λ). ■

4.3 Compactness

In this section we continue studying topological properties of the solution spaces. This is absolutely analogous to the standard Floer theory, where the moduli spaces $\widehat{\mathcal{M}}_{x,y}(J, H)$ are analyzed. On the one hand one has to prove compactness in dimension 0, in order to define the boundary operator δ. On the other hand, it is necessary to find a description of the compactification of the 1-dimensional components by simply broken trajectories, so that the relation $\delta \circ \delta = 0$ follows by means of a cobordism argument.

Here we proceed in the same way. We want to prove compactness properties for components of $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J, k(h))$ which are 0- respectively 1-dimensional. The foundation of the proofs is given by elliptic methods for the quasi-linear Cauchy-Riemann type operator $\overline{\partial}_J$ and by the bubbling-off analysis. For these methods, which now have been worked out extensively, we refer to [17, 19, 37, 29, 31, 38] especially to [31]. It is pointed out that here we rather deal with a non-bubbling-off analysis in the sense that we prohibit by requiring that $\phi\omega \equiv 0$ the existence of J-holomorphic spheres eventually interfering with the solution spaces in question.

Let $\Omega \subset \mathbb{C}$ be an open domain, $J \in C^\infty(\Omega \times M)$ be an ω-compatible almost complex structure and $Y \in C^\infty(\Omega \times M)$ be a vector field on M, both explicitly depending on Ω such that J, Y are C^∞-bounded. Then we consider the Cauchy-Riemann type PDE for $u \in C^\infty(\Omega, M)$,

$$\partial_s u + J(s, t, u)\partial_t u + Y(s, t, u) = 0 \quad \text{for } z = s + it \in \Omega\,. \tag{4.18}$$

In the following we use the induced Riemannian metric on the bundle $\pi^*TM \to \Omega \times M$ induced by the projection $\pi\colon \Omega \times M \to M$,

$$\begin{gathered}\langle\cdot,\cdot\rangle_J = g_J = \omega \circ (\mathrm{Id} \times J), \quad |v|^2_J = \omega(p)(v, J(z,p)v)\\ \text{for} \quad z \in \Omega,\ p \in M,\ v \in T_pM\,.\end{gathered}$$

4.3.1 Theorem (bounded gradient compactness) *Given Ω, J and Y as above and a family $X \subset C^\infty(\Omega, M)$ of solutions of (4.18) such that the gradient is uniformly bounded, i.e. there exists a constant $c > 0$ with*

$$\sup_{z\in\Omega} \big|\nabla u(z)\big|_J \leqslant c \quad \text{for all } u \in X,$$

then $X \subset C^\infty_{loc}(\Omega, M)$ is bounded. This means that for every compact $K \Subset \Omega$ and $n \in \mathbb{N}$ there is a constant $C_{K,n} > 0$ such that

$$\sup_{z\in K,\, |\alpha|\leqslant n} \big|D^\alpha u(z)\big| \leqslant C_{K,n} \quad \text{for all } u \in X\,.$$

For the norm $|D^\alpha u|$ we view M without loss of generality as a properly embedded submanifold of $\mathbb{R}^N$. Since $C^\infty(\Omega, M)$ has the Heine-Borel property, we consequently obtain that every sequence $(u_n) \subset C^\infty(\Omega, M)$ of solutions of (4.18) which has a uniform gradient bound possesses a subsequence (u_{n_k}) such that

$$u_{n_k} \to u \quad \text{for } k \to \infty,$$

uniformly in $C^l(K, \Omega)$ for all $l \in \mathbb{N}$, $K \Subset \Omega$. Clearly, u is again a solution of (4.18).

Proof. The proof of Theorem 4.3.1 can be found in [31]. It is part of the compactness analysis in Chapter 6 from page 236 to 243. There the proof is given within Lemma 6 for the gradient field $Y = \nabla H$ but it also works for a general bounded vector field. One proceeds by establishing uniform bounds in $W^{l,p}(K)$ for every compact subset $K \Subset \Omega$ by induction on l. The assumption of uniform gradient bounds yields C^0-compactness due to Arzela-Ascoli and therefore localization. Then the elliptic estimates for the $\overline{\partial}_J$-operator can be applied yielding a standard bootstrapping argument. ■

Referring to the compactness proof in [31] quoted above we may generalize Theorem 4.3.1. This will be needed for the discussion of homotopy invariance in Section 5.2.

4.3.2 Theorem *Let Ω be as above and $J_n, Y_n \in C^\infty(\Omega \times M)$ be converging sequences*

$$J_n \xrightarrow{C^\infty} J, \quad Y_n \xrightarrow{C^\infty} Y\,.$$

Given a family $(u_n)_{n\in\mathbb{N}} \subset C^\infty(\Omega, M)$ of solutions of

$$\partial_s u_n + J_n \partial_t u_n + Y_n = 0, \quad n \in \mathbb{N},$$

and a constant $c > 0$ such that

$$\sup_{z\in\Omega} \big|\nabla u_n(z)\big| \leqslant c$$

uniformly for all $n \in \mathbb{N}$, then (u_n) is bounded in $C^\infty_{loc}(\Omega, M)$.

Proof. Again, the uniform gradient bound allows the localization of the solutions u_n. Then, introducing the almost complex structures $\tilde{J}_n \in C^\infty(\Omega \times M)$ on the symplectic manifold $\Omega \times M$,

$$\tilde{J}_n(z, m)(a, b, h) = \big(-b, a, J_n(z, m)h + bY_n(z, m) + aJ_n(z, m)Y_n(z, m)\big)$$

we may equivalently formulate the originally given PDE as

$$\partial_s w_n + \tilde{J}_n(w_n)\partial_t w_n = 0 \tag{4.19}$$

where $w_n \in C^\infty(\Omega, \Omega \times M)$ is defined as $w_n(z) = (z, u_n(z))$. Hence, the proof can directly be reduced to the proof of Lemma 6 in [31]. ■

The second important result which we only quote here is related to the so-called bubbling-off analysis. It deduces the above uniform gradient bound from a given energy estimate. We define the energy of a map $u \in C^\infty(\Omega, M)$ as follows. Let J be as above, then the energy is given by

$$E(u) = \tfrac{1}{2}\int_\Omega (|\partial_s u|_J^2 + |\partial_t u|_J^2)\,dsdt\,.$$

We recall that $E(u)$ is invariant under conformal reparametrizations of u and for a J-holomorphic map, $\overline{\partial}_J(u) = 0$, we have $E(u) = \int_\Omega u^*\omega$.

The question whether a similar a priori energy estimate adapted to solutions of (4.18) implies a uniform gradient bound recovers the crucial role of J-holomorphic spheres. In this work we want to avoid considering the influence of J-holomorphic spheres on the Floer theory. Therefore, at a later point, we will assume that the homomorphism

$$\phi_\omega : \pi_2(M) \to \mathbb{R}, \quad \{u\} \mapsto \int_{S^2} u^*\omega,$$

describing the action of $\pi_2(M)$ on the energy, vanishes. However, in the moment we keep our considerations general. We define for a map $u \in C^\infty(\Omega, M)$

$$\Phi_Y(u) = \tfrac{1}{2}\int_\Omega (|\partial_s u|_J^2 + |\partial_t u - J(u)Y(u)|_J^2)\,dsdt,$$

also related as the "generalized flow energy".

4.3.3 Theorem (no bubbling allowed) *Let Ω, J, Y be as above and $(u_n) \subset C^\infty(\Omega, M)$ be a sequence of solutions of (4.18). Then, there is a C^∞_{loc}-convergent subsequence (u_{n_k}) if one of the following conditions holds:*

(a) $\lim_{n\to\infty} \Phi_Y(u_n) = 0$ *or*

(b) $(\Phi_Y(u_n))_{n\in\mathbb{N}}$ *is bounded and* $\phi_\omega \equiv 0$.

Proof. We again refer to [31]. Part (b) can directly be deduced from Theorem 8 in Chapter 6 of [31]. For (a) we use the assumption that J and Y are bounded on $\Omega \times M$. Thus we can apply Lemma 6 from [31]. If the gradient of u_n restricted to a fixed compact subset $K \Subset \Omega$ is not uniformly bounded we find

$$\begin{aligned}
&v_k \in C^\infty(D_k, M), \quad D_k = \{\, z \in \mathbb{C} \,|\, |z| \leqslant R_k \,\},\\
&|\nabla v_k(z)| \leqslant 2 \quad \text{for } z \in D_k,\\
&|\nabla v_k(0)| = 1,\\
&R_k \to \infty \quad \text{as } k \to \infty,
\end{aligned}$$

by conformal reparametrization as discussed on the pages 237–238 in [31]. Moreover,

$$\partial_s v_k + J(z, v_k)\partial_t v_k = \mathrm{O}(\tfrac{1}{R_k})\,.$$

Since the energy E is conformally invariant we obtain

$$E(v_k) \to 0 \quad \text{as } k \to \infty$$

from the assumption (a). Thus the J-holomorphic map $v \in C^\infty(\mathbb{C}, M)$ which arises by Lemma 6 from a suitable subsequence $v_{n_k} \to v$ in C^∞_{loc} must be constant in contradiction to $|\nabla v_k(0)| = 1$ for all $k \in \mathbb{N}$. ■

4.3.1 The Cylindrical Ends

Let us first apply these compactness results to solutions of cylindrical type. Given a regular pair (J, H), i.e. $J, H \in C^\infty(S^1 \times M)$, we denote for $u \in C^\infty([T, \infty) \times S^1, M)$

$$\overline{\partial}_{J,H}(u) = \partial_s u + J(t, u)\partial_t u + \nabla H_t(u)$$

as in Definition 2.3.3 and we consider the flow energy

$$\Phi(u) = \int_T^\infty \int_{S^1} |u_s|_J^2\,dsdt = \tfrac{1}{2}\int_T^\infty \int_{S^1} (|\partial_s u|_J^2 + |\partial_t u - X_{H_t}(u)|_J^2)\,dsdt\,.$$

4.3.4 Proposition *Let $u \in C^\infty([T, \infty) \times S^1, M)$ be a trajectory with finite flow energy*

$$\overline{\partial}_{J,H}(u) = 0, \quad \Phi(u) < \infty\,.$$

Then the translation family $(v_\tau)_{\tau \geqslant 0}$

$$v_\tau \in C^\infty([T - \tau, \infty) \times S^1, M), \quad v_\tau(s, t) = u(s + \tau, t)$$

is C^∞_{loc}-bounded, i.e. for every $\rho > 0$ and $n \in \mathbb{N}$ there is a constant $C_{n,\rho} > 0$ such that

$$\sup\{\, |D^\alpha v_\tau(s, t)| \,|\, |s| \leqslant \rho,\ t \in S^1,\ |\alpha| \leqslant n \,\} \leqslant C_{n,\rho}$$

for all $\tau \geqslant T + \rho$.

Proof. This follows immediately from Theorem 4.3.1 and Theorem 4.3.3. Let $\rho > 0$ be fixed, then $v_{\tau|[T-\tau)\times S^1}$ satisfies

$$\int_{-\rho}^\infty \int_{S^1} |\partial_s v_\tau|_J^2\,dsdt = \int_{\tau-\rho}^\infty \int_{S^1} |\partial_s u|_J^2\,dsdt \to 0$$

for $\tau \to \infty$. Hence $v_{\tau|[-\rho,\infty)\times S^1}$ meets condition (a) in Theorem 4.3.3. ■

From the Heine-Borel property of $C^\infty(\mathbb{R} \times S^1, M)$ due to Arzela-Ascoli we immediately obtain

4.3.5 Corollary *Let $u \in C^\infty([T,\infty)\times S^1, M)$ be as above with $u(T) \in \Omega^\circ(M)$, i.e. contractible, and $s_n \to \infty$. Then there is a subsequence $s_{n_k} \to \infty$ and a 1-periodic solution $x \in \mathcal{P}_1(H)$ such that for all $\rho > 0$*

$$u_{|[s_{n_k}-\rho, s_{n_k}+\rho]\times S^1} \to x$$

uniformly in $C^\infty([s_{n_k}-\rho, s_{n_k}+\rho]\times S^1, M)$ where $x(s,t) = x(t)$ for all $(s,t) \in \mathbb{R}\times S^1$. In particular, $u(s) \to x$ in $C^\infty(S^1, M)$ and $\partial_s u(s) \to 0$ in $C^\infty(S^1, TM)$ as $s = s_{n_k} \to \infty$.

Proof. Proposition 4.3.4 yields the C^∞_{loc}-convergence of a subsequence of

$$v_n \in C^\infty([-\rho,\rho]\times S^1, M), \quad v_n(s,t) = u(s+s_n, t),$$

i.e. $v_{n_k} \to v$ in C^∞. Since Φ is translation invariant we have $\Phi(v_n) \to 0$ and consequently $\Phi(v) = 0$. Therefore $\partial_s v \equiv 0$ and $\partial_t v = X_{H_t} \circ v$. ■

This argumentation can similarly be applied to the remaining problem of Theorem 4.1.2.

4.3.6 Corollary *Given $x \in \mathcal{P}_1(H)$, let the solution $u \in C^\infty([T,\infty)\times S^1, M)$ of $\overline{\partial}_{J,H}(u) = 0$ satisfy the condition*

$$u \in \mathcal{P}^{1,p}_x([T,\infty)\times S^1, M).$$

Then, $u(s) \to x$ in $C^\infty(S^1, M)$ and $\partial_s u(s) \to 0$ in $C^\infty(S^1, M)$ as $s \to \infty$.

Proof. By assumption, u satisfies

$$\int_T^\infty \int_{S^1} (|\partial_s u|_J^p + |\partial_t u|_J^p) ds dt < \infty .$$

Thus, there is a constant $c(\rho) > 0$ for every $\rho > 0$ such that

$$\int_{-\rho}^{\rho}\int_{S^1} |\partial_s u_\tau|_J^2 ds dt \leqslant c(\rho) \quad \text{for all } \tau \geqslant T + \rho .$$

This estimate follows from the inequality $\|f\|_{L^2(\Omega)} \leqslant |\Omega|^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p(\Omega)}$ for $p > 2$ and $\Omega \subset \mathbb{C}$ a bounded domain. We hence can repeat the arguments from Proposition 4.3.4 and Corollary 4.3.5 for the family

$$u_\tau : [-\rho,\rho]\times S^1 \to M, \quad u_\tau(s,t) = u(s+\tau, t).$$

Now, the limit $u(s) \to x$ in $C^\infty(S^1, M)$ as $s \to \infty$ is unique because already $u \in \mathcal{P}^{1,p}_x([T,\infty)\times S^1, M)$ implies $u(s) \to x$ in $C^0(S^1, M)$. ■

Let $H^{1,2}(S^1, M)$ be the Hilbert manifold

$$H^{1,2}(S^1, M) = \{\, u \in C^0(S^1, M) \mid \Psi \circ u \in H^{1,2}(S^1, \mathbb{R}^N) \,\},$$

where $\Psi \colon M \hookrightarrow \mathbb{R}^N$ is any fixed smooth proper embedding, and let us consider the continuous function

$$f \colon H^{1,2}(S^1, M) \to \mathbb{R}, \quad x \mapsto \|\dot{x} - X_{H_t}(x)\|_{L^2(S^1)},$$

where we use the L^2-norm $\|\xi\|_{L^2(S^1)} = \int_{S^1} |\xi|^2_{J_t} dt$ for $\xi \in L^2(x^*TM)$. Note that this L^2-norm is equivalent to the induced norm from the proper embedding. We consider the embedding of the contractible 1-periodic solutions

$$\mathcal{P}_1(H) \hookrightarrow \operatorname{Fix}\psi_1, \quad x \mapsto x(0),$$

and observe that $\mathcal{P}_1(H) = f^{-1}(0) \cap \Omega^\circ(M) \subset H^{1,2}(S^1, M)$. Here ψ_1 is the time-1-map of the Hamiltonian flow $\psi_t \colon M \to M$ associated to the Hamiltonian vector field X_{H_t}. It holds $f^{-1}(0) \subset C^\infty(S^1, M)$ because every weak $H^{1,2}$-solution of

$$\dot{x} = X_{H_t}(x)$$

is automatically smooth.

The crucial point is that for a regular Hamiltonian H, i.e. all contractible 1-periodic solutions are non-degenerate, $\mathcal{P}_1(H)$ is necessarily a finite set provided that M is compact. Let $m = \#\mathcal{P}_1(H)$. We find isolating open neighbourhoods $U_1, \ldots, U_m \subset H^{1,2}(S^1, M)$ such that

$$U_i \cap \mathcal{P}_1(H) = \{x_i\}, \quad i = 1, \ldots, m .$$

4.3.7 Proposition *Given a sequence $(z_n)_{n\in\mathbb{N}} \subset H^{1,2}(S^1, M)$ of contractible loops with $f(z_n) \to 0$ for $n \to \infty$ it finally lies within $U_1 \cup \ldots \cup U_m$. Moreover, if $(z_n) \subset U_i$ it converges to $x_i \in \mathcal{P}_1(H)$,*

$$z_n \xrightarrow{H^{1,2}} x_i,$$

for $n \to \infty$.

Proof. For sake of simplicity we consider again the smooth embedding $\Psi \colon M \hookrightarrow \mathbb{R}^N$ defining

$$\|y\|_{1,2} = \|\Psi \circ y\|_{H^{1,2}(S^1, \mathbb{R}^N)} \quad \text{for } y \in H^{1,2}(S^1, M).$$

Since M is compact and $(f(z_n))$ is bounded $(\|z_n\|_{1,2})$ is bounded. Hence there exists an $x \in H^{1,2}(S^1, M)$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ with

$$z_{n_k} \rightharpoonup x \quad \text{weakly in } H^{1,2}(S^1, \mathbb{R}^N),$$
$$z_{n_k} \to x \quad \text{strongly in } C^0(S^1, M).$$

Moreover, $\|\dot{z}_n - X_{H_t}(z_n)\|_{L^2} \to 0$ implies that $(\dot{z}_{n_k})$ is an L^2-Cauchy sequence, so that

$$z_{n_k} \to x \quad \text{strongly in } H^{1,2}(S^1, M).$$

Hence, it holds $f(x) = 0$, i.e. $x = x_i$. This argumentation works for every subsequence of (z_n). Therefore $(z_n) \subset U_i$ implies

$$z_n \xrightarrow{H^{1,2}} x_i, \quad n \to \infty . \quad ■$$

4.3.8 Definition *For every $a>0$ we define the open subsets*

$$U^a = \{\, z \in H^{1,2}(S^1,M) \,|\, f(z) < a \,\}$$

and $U_i^a = U^a \cap U_i$, $i = 1,\dots,m$.

Proposition 4.3.7 implies that given any isolating neighbourhoods U_i there is an $a_o > 0$ such that

$$U^a \subset \bigcup_{i=1}^m U_i \quad \text{f.a. } 0 < a \leqslant a_o\,.$$

Since the U_i separate the finitely many 1-periodic solutions $x_i \in \mathcal{P}_1(H)$ due to the assumption of nondegeneracy, we find an $s \in (s_1,s_2)$ with $f(\gamma(s)) \geqslant a_o$ for every continuous path $\gamma\colon [s_1,s_2] \to H^{1,2}(S^1,M)$ with

$$\gamma(s_1) \in U_i, \quad \gamma(s_2) \in U_j, \quad i \neq j\,.$$

We now are prepared for proving Proposition 4.1.3 which relates finite flow energy with asymptotic convergence towards a unique 1-periodic solution.

4.3.9 Proposition *Let $u \in C^\infty([T,\infty)\times S^1,M)$ be a solution of $\overline{\partial}_{J,H}(u) = 0$ where (J,H) is a regular pair so that all $x \in \mathcal{P}_1(H)$ are non-degenerate. Then the following conditions are equivalent.*

(a) $\Phi(u) < \infty$ and $u(T) \in C^\infty(S^1,M)$ is contractible.

(b) There exists a unique $x \in \mathcal{P}_1(H)$ such that $u(s) \to x$ in $C^1(S^1,M)$ as $s \to \infty$.

Moreover, then it holds

$$u(s) \overset{C^\infty(S^1)}{\longrightarrow} x \quad \text{and} \quad \partial_s u(s) \overset{C^\infty(S^1)}{\longrightarrow} 0 \quad \text{for } s \to \infty\,.$$

Proof. The implication (b) $\Rightarrow$ (a) is obvious, because

$$\int_{s_1}^{s_2}\int_{S^1} |\partial_s u|_J^2\, ds dt \;=\; \mathcal{A}_H(u(s_2)) - \mathcal{A}_H(u(s_1))\,.$$

The relative action along the path $u \in C^0([T,\infty),\Omega^o(M))$ is also well-defined if $\phi_\omega \neq 0$. Thus, the convergence

$$u(s) \to x \quad \text{for } s \to \infty$$

in $C^1(S^1,M)$ implies

$$\Phi(u) = \mathcal{A}_H(x) - \mathcal{A}_H(u(T)) < \infty,$$

because $\mathcal{A}_H$ is continuous along paths in $C^1(S^1)$.

Now let $\Phi(u) < \infty$ and $u(T) \in \Omega^o(M)$. Given any sequence $s_n \to \infty$. Corollary 4.3.5 yields a subsequence $s_{n_k} \to \infty$ and an $x_i \in \mathcal{P}_1(H)$ with

$$u(s_{n_k}) \overset{C^\infty(S^1)}{\longrightarrow} x_i, \quad k \to \infty\,. \tag{4.20}$$

Moreover, it holds

$$f(u(s)) \to 0 \quad \text{for } s \to \infty\,.$$

Otherwise, there is a sequence $\bar{s}_n \to \infty$ with $f(u(\bar{s}_n)) \geqslant c$ for all $n \in \mathbb{N}$ for some $c > 0$ contradicting Corollary 4.3.5. In view of (4.20) it suffices to prove that we have a unique $C^0(S^1)$-limit

$$u(s) \overset{C^0(S^1)}{\longrightarrow} x \in \mathcal{P}_1(H) \quad \text{for } s \to \infty\,.$$

Let us assume that this is false. Then, for any $x_i \in \mathcal{P}_1(H)$, there exists an $a > 0$ and a sequence $\bar{s}_n \to \infty$ with $u(\bar{s}_n) \notin U_i^a$ for all $n \in \mathbb{N}$. But again (4.20) implies that we have for a suitable subsequence $\bar{s}_{n_k} \to \infty$ and some $x_j \in \mathcal{P}_1(H)$

$$u(\bar{s}_{n_k}) \overset{C^\infty(S^1)}{\longrightarrow} x_j, \quad k \to \infty$$

with necessarily $j \neq i$. Thus, without loss of generality, there are sequences (s_n) and $(\bar{s}_n)$ with

$$s_n < \bar{s}_n, \quad s_n \to \infty \quad \text{for } n \to \infty,$$

$$\begin{matrix} u(s_n) \to x_i \\ u(\bar{s}_n) \to x_j \end{matrix} \quad \text{in } C^0(S^1,M), \quad i \neq j\,.$$

Consequently, due to the above considerations after Definition 4.3.8, there exists a sequence (σ_n) with $s_n < \sigma_n < \bar{s}_n$ such that

$$f(u(\sigma_n)) \geqslant a_o \quad \text{for } n \text{ large enough}\,.$$

This contradicts $f(u(s)) \to 0$ for $s \to \infty$. ■

Before we analyze the question of an a priori energy estimate for the solution spaces $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$, we relate the notion of C^∞_{loc}-convergence to the topology given by the surrounding Banach manifold $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M)$.

4.3.10 Proposition *Let $(u_n) \subset C^\infty([T,\infty)\times S^1,M)$ be a sequence of solutions of $\overline{\partial}_{J,H}(u_n) = 0$ with $u_n(s) \to x$ in $C^1(S^1)$ for $s \to \infty$ for all $n \in \mathbb{N}$. Assume that (u_n) converges uniformly in $C^0([T,\infty)\times S^1,M)$ towards a solution $u \in C^\infty([T,\infty)\times S^1,M)$, $\overline{\partial}_{J,H}(u) = 0$, $u(s) \to x$ in $C^1(S^1)$ for $s \to \infty$, that is,*

$$\sup_{(s,t)\in[T,\infty)\times S^1} d(u_n(s,t),u(s,t)) \to 0,$$

where d is the distance associated to the Riemannian metric on M. If $x \in \mathcal{P}_1(H)$ is non-degenerate then $u_n \to u$ converges in $\mathcal{P}^{1,p}_x([T+1,\infty)\times S^1,M)$, i.e.

$$u_n = \exp_u \xi_n, \quad \xi_n \to 0 \quad \text{in } H^{1,p}_\Sigma(u^*TM_{[T+1,\infty)\times S^1})\,.$$

Note that we can consider u^*TM endowed with a fixed C^∞_Σ-smooth unitary trivialization

$$\Phi\colon [T,\infty]\times S^1\times\mathbb{C}^n \to u^*TM$$

so that $\Phi(\infty)\colon S^1\times\mathbb{C}^n \to x^*TM$ can be extended over a given extension $u_x\colon D^2\to M$. This is possible because $u\in C^\infty_x([T,\infty)\times S^1,M)$ due to the exponentially fast convergence towards x. We prove the proposition in analogy to Proposition 4.1.4. Here we can consider $\Sigma=[T,\infty)\times S^1$ as a model surface with the canonical cylindrical coordinates, as we are only interested in the asymptotic behaviour. We write $\overline{\Sigma}=[T,\infty]\times S^1$.

PROOF. Since $u_n\to u$ in $C^0([T,\infty)\times S^1,M)$ we can represent u_n for $n\geqslant n_o$ large by

$$u_n(s,t)=\exp_{u(s,t)}\xi_n(s,t),\quad \xi_n\in C^\infty(u^*TM)$$

and with respect to the fixed trivialization Φ of u^*TM as above

$$u_n=\exp_u(\Phi\cdot v_n),\quad v_n\in C^\infty_\Sigma([T,\infty)\times S^1,\mathbb{C}^n)\,. \tag{4.21}$$

It holds

$$v_n(s),\partial_s v_n(s)\overset{C^\infty(S^1)}{\longrightarrow}0\quad\text{for } s\to\infty\quad\text{and} \tag{4.22}$$

$$\sup_{(s,t)\in[T,\infty)\times S^1}|v_n(s,t)|\to 0\quad\text{for } n\to\infty\,. \tag{4.22'}$$

Using the Levi-Civita connection ∇ for the linearization of the exponential map exp we transform the identity $\overline{\partial}_{J,H}(u_n)=0$ into the PDE

$$\partial_s v_n+J_n(s,t)\partial_t v_n+F(s,t,v_n)=0 \tag{4.23}$$

analogously to Proposition 4.1.4. Here we have

$$J_n(s,t)=\Phi^{-1}(s,t)\nabla_2\exp\left(\xi_n(s,t)\right)^{-1}J(t,u_n(s,t))\nabla_2\exp\left(\xi_n(s,t)\right)\Phi(s,t),$$

and

$$\begin{aligned}F(s,t,p)=\Phi^{-1}\nabla_2\exp(\Phi p)^{-1}\Big[&\nabla_1\exp(\Phi p)\cdot\partial_s u\\&+J(t,\exp_u(\Phi p))\nabla_1\exp(\Phi p)\cdot\partial_t u+J(\dots)\nabla_2\exp(\Phi p)\nabla_t\Phi\cdot p\\&+\nabla H_t(\exp_u\Phi p)\Big]\;+\Phi^{-1}\nabla_s\Phi\cdot p\end{aligned}$$

for $(s,t,p)\in[T,\infty)\times S^1\times\mathbb{C}^n$. Since $\overline{\partial}_{J,H}(u)=0$ it holds

$$F(s,t,0)=\Phi^{-1}(\overline{\partial}_{J,H}(u))=0\,.$$

Thus we can define $S_n\in C^\infty([T,\infty)\times S^1,\mathcal{L}_\mathbb{R}(\mathbb{C}^n))$ by

$$S_n(s,t)=\int_0^1 D_3F(s,t,\lambda v_n(s,t))d\lambda,$$

so that $F(s,t,v_n(s,t))=S_n(s,t)\cdot v_n(s,t)$ for all $(s,t)\in[T,\infty)\times S^1$. After computing

$$D_3F(s,t,0)=\Phi^{-1}\left[\partial_s u+\nabla_\Phi J(t,u)\partial_t u+J(t,u)\nabla_t\Phi+\nabla_\Phi H(t,u)\right] \tag{4.24}$$

we deduce that, due to (4.22), $S_n(s)\overset{C^\infty(S^1)}{\longrightarrow}S(\infty)$ for $s\to\infty$, where

$$S(\infty)=\Phi^{-1}(\nabla_\Phi J\cdot\dot{x}+J\nabla_t\Phi+\nabla_\Phi\nabla H)$$

is the same regular loop $S(\infty)\in C^\infty(S^1,\mathcal{L}_\mathbb{R}(\mathbb{C}^n))$ as in (4.3) in the proof of Proposition 4.1.4. Summing up we observe that

$$J_n(s)\to J_o,\,S_n(s)\to S(\infty)\quad\text{in } C^\infty(S^1)\quad\text{for } s\to\infty, \tag{4.25}$$

$$\sup_{(s,t)\in[T,\infty)\times S^1}\Big(|J_n(s,t)-J_o|,|S_n(s,t)-S(s,t)|\Big)\to 0\quad\text{for } n\to\infty\,. \tag{4.25'}$$

The convergence (4.25′) follows from (4.22′) with $S(s,t)=D_3F(s,t,0)$ as computed in (4.24). Let us now consider a cut-off function $\beta\in C^\infty(\mathbb{R},[0,1])$,

$$\beta(s)=\begin{cases}0, & s\leqslant T,\\ 1, & s\geqslant T+1,\end{cases}\quad 0\leqslant\beta'(s)\leqslant 2\quad\text{for all } s\in\mathbb{R}\,.$$

Since $S(\infty)$ is admissible in the sense of Definition 3.1.1 we can apply the fundamental estimate for $G=\partial_s+J_o\partial_t+S(\infty)$ from Theorem 3.1.13. That is, there is a constant $c>0$ such that

$$\|\beta v_n\|_{1,p}\leqslant c\big\|(\partial_s+J_o\partial_t+S(\infty))\beta v_n\big\|_{0,p}$$

for all $n\in\mathbb{N}$. Let $0<\epsilon<\frac{1}{2c}$ and $n_o=n_o(\epsilon)$ be such that

$$\|J_o-J_n\|_\infty^{Z_T^+},\|S-S_n\|_\infty^{Z_T^+}<\epsilon\quad\text{for all } n\geqslant n_o(\epsilon)\,.$$

Using (4.23) we then can estimate

$$\begin{aligned}\|\beta v_n\|_{1,p}^{Z_T^+}\leqslant c\Big(&\|J_o-J_n\|_\infty^{Z_T^+}\cdot\|\beta\partial_t v_n\|_{0,p}^{Z_T^+}+\|S-S_n\|_\infty^{Z_T^+}\cdot\|\beta v_n\|_{0,p}^{Z_T^+}\\&+\|\beta' v_n\|_{0,p}^{[T,T+1]\times S^1}\Big).\end{aligned}$$

Thus, for $n\geqslant n_o(\epsilon)$ we have

$$\|v_n\|_{1,p}^{Z_{T+1}^+}\leqslant\frac{2c}{1-2c\epsilon}\sup_{(s,t)\in[T,T+1]\times S^1}|v_n(s,t)|\;\to 0$$

for $n\to\infty$. ∎

We can now relate the weak convergence on cylindrical ends, i.e. in C^∞_{loc} with the strong convergence in the Banach manifold $\mathcal{P}_x^{1,p}$, provided that a topological a priori condition is satisfied. Namely, either we assume $\phi_\omega=0$ so that the action functional is well-defined, or we guarantee explicitly that during the weak convergence the homological class of u_n and u relatively to the boundary loops at T and ∞ is the same up to $\ker\phi_\omega$.

4.3.11 Proposition *Let $(u_n) \subset C^\infty([T,\infty) \times S^1, M) \subset \mathcal{P}^{1,p}_x$ be a sequence of solutions of $\overline{\partial}_{J,H}(u_n) = 0$ converging in C^∞_{loc} towards a solution $u \in C^\infty([T,\infty) \times S^1, M)$. If $\Phi(u_n) \to \Phi(u)$ and $u(\infty) = x$ they also converge strongly,*

$$u_n \to u \quad \text{in} \quad \mathcal{P}^{1,p}_x([T,\infty) \times S^1, M)$$

for $n \to \infty$.

Proof. Due to Proposition 4.3.10 it is sufficient to prove the uniform convergence in $C^0([T,\infty) \times S^1, M)$. Suppose that this is false. Then we find a sequence $(s_n, t_n) \subset [T,\infty) \times S^1$ and a constant $c > 0$ such that

$$d(u_n(s_n,t_n), u(s_n,t_n)) > c \quad \text{for all } n \in \mathbb{N}, \tag{4.26}$$

where necessarily $s_n \to \infty$. The weak convergence together with the energy convergence imply that $\Phi(u_{n|[s,\infty)}) \to \Phi(u_{|[s,\infty)})$ for all $s \geqslant T$. Hence,

$$\lim \Phi(u_{n|[s_n,\infty)}) \leqslant \lim \Phi(u_{n|[s,\infty)}) = \Phi(u_{|[s,\infty)})$$

for $s > T$, so that

$$\Phi(u_{n[s_n,\infty)}) \to 0 \quad \text{for } n \to \infty\,. \tag{4.27}$$

Similarly,

$$\Phi(u_{n[s_n-\rho,s_n+\rho]}) \to 0 \quad \text{for } n \to \infty$$

for all $\rho > 0$. Thus, by Corollary 4.3.5, there exists a subsequence (s_{n_k}) such that

$$u_{n_k}(s_{n_k}) \stackrel{C^\infty(S^1)}{\longrightarrow} y \in \mathcal{P}_1(H), \quad k \to \infty\,.$$

But arguing analogously to the proof of Proposition 4.3.9 the convergence (4.27) implies that $y = x$ in contradiction to (4.26). ■

We now are prepared to recall the classical theorem from Floer theory. Let us denote the partial ordering among the 1-periodic orbits $x, y \in \mathcal{P}_1(H)$ by

$$x \leqslant y \;\Leftrightarrow\; \mathcal{M}_{x,y}(J,H) \neq \emptyset\,.$$

At the present stage this relation is only known to be reflexive and antisymmetric, that is $x \leqslant y$ and $x \geqslant y$ imply $x = y$. The transitivity follows from the gluing operation in the next section, see 4.4.1.

Here we reach the point, where we assume $\phi_\omega = 0$ in order to avoid the discussion of J-holomorphic spheres. If the following theorem is to be generalized, for example for monotone symplectic manifolds, one needs a more refined energy and index balance, see for instance Theorem 3.3 in [29] and Proposition 3c in [19].

4.3.12 Theorem *Let (J,H) be a regular pair and $(u_n) \subset \mathcal{P}^{1,p}_x([T,\infty) \times S^1, M)$ be a sequence of solutions, $\overline{\partial}_{J,H}(u_n) = 0$. If*

$$\phi_\omega \equiv 0 \quad \text{and} \quad \mathcal{A}_H(u_n(T)) \geqslant c \quad \text{for all } n \in \mathbb{N}$$

for some real constant $c \in \mathbb{R}$, the following alternative is true. Either there is a subsequence (u_{n_k}) converging strongly

$$u_{n_k} \to u \quad \text{in } \mathcal{P}^{1,p}_x([T+1,\infty) \times S^1)$$

to a solution $u \in \mathcal{P}^{1,p}_x([T+1,\infty) \times S^1, M)$, or there are finitely many 1-periodic orbits $x = x_o \geqslant x_1 \geqslant \ldots \geqslant x_r \in \mathcal{P}_1(H)$ and sequences $(\tau_{k,i})_{k\in\mathbb{N}} \subset \mathbb{R}$ with $\lim_{k\to\infty} \tau_{k,i} = \infty$, $i = 1, \ldots, r$, such that for a subsequence (u_{n_k}) we have convergence towards a solution v of $\overline{\partial}_{J,H}(v) = 0$,

$$u_{n_k} \to v \in \mathcal{P}^{1,p}_{x_r}([T+1,\infty) \times S^1, M) \quad \text{for } k \to \infty$$

and

$$u_{n_k} * \tau_{k,i} \stackrel{C^\infty_{\mathrm{loc}}}{\longrightarrow} v_i \in \mathcal{M}_{x_i, x_{i-1}}(J,H), \quad i = 1, \ldots, r,$$

where $(u_n * \tau)(s,t) = u_n(s+\tau, t)$.

Proof. Using Theorem 4.3.3 we deduce C^∞_{loc}-convergence for suitable subsequences of $(u_n * \tau_n)_{n\in\mathbb{N}}$ for any $\tau_n \geqslant 0$ from the energy bound

$$\Phi(u_n) \leqslant \mathcal{A}_H(x) - c \quad \text{for all } n \in \mathbb{N}\,.$$

Then the proof works by an iterative argumentation. After taking a subsequence we can assume that

$$u_n \stackrel{C^\infty_{\mathrm{loc}}}{\longrightarrow} u \in C^\infty([T+1,\infty) \times S^1, M)\,.$$

Since $c \leqslant \mathcal{A}_H(u_n(s)) \leqslant \mathcal{A}_H(x)$ for all $n \in \mathbb{N}$, $s \in [T+1,\infty)$, it holds $\Phi(u) \leqslant \mathcal{A}_H(x) - c$ and therefore $u \in \mathcal{P}^{1,p}_y([T+1,\infty) \times S^1)$ for some $y \in \mathcal{P}_1(H)$ due to Proposition 4.3.9. Moreover, $\mathcal{A}_H(y) \leqslant \mathcal{A}_H(x)$. If $y = x$ the proof is completed by Proposition 4.3.11. Let us show that also $\mathcal{A}_H(y) = \mathcal{A}_H(x)$ implies $y = x$ and thus the end of the proof. Assume that $x \neq y$ and choose an $a > 0$ such that we obtain the isolating neighbourhoods $U^a(x)$ and $U^a(y)$ from Definition 4.3.8. For every $k \in \mathbb{N}$ we choose an $s_1(k)$ such that $u(s_1(k)) \in U^a(y)$ and $0 < \mathcal{A}_H(y) - \mathcal{A}_H(u(s_1)) < \frac{1}{k}$. Since $u_n \to u$ in C^∞_{loc} we thus find a subsequence $n_k \in \mathbb{N}$ with

$$u_{n_k}(s_1(k)) \in U^a(y) \quad \text{and} \quad 0 < \mathcal{A}_H(y) - \mathcal{A}_H(u_{n_k}(s_1(k))) < \frac{1}{k}\,.$$

On the other side we can choose $s_2(k) > s_1(k)$ such that for this subsequence

$$u_{n_k}(s_2(k)) \in U^a(x) \quad \text{and} \quad 0 < \mathcal{A}_H(x) - \mathcal{A}_H(u_{n_k}(s_2(k))) < \frac{1}{k}\,.$$

Altogether we obtain

$$\int_{s_1(k)}^{s_2(k)} \int_{S^1} |\partial_s u_{n_k}|_J^2 \, dt\, ds < \frac{2}{k} \to 0\,. \tag{4.28}$$

But due to the choice of the isolating neighbourhoods there must be a sequence $\sigma_k \in (s_1(k), s_2(k))$ with $u_{n_k}(\sigma_k) \notin U^a(x) \cup U^a(y)$, that is

$$\|\partial_s u_{n_k}(\sigma_k)\|_{L^2(S^1)} \geqslant a\,.$$

However, in view of Theorem 4.3.3 and Corollary 4.3.5, the energy estimate 4.28 implies that a suitable subsequence of the contractible loops $u_{n_k}(\sigma_k)$ must converge to a 1-periodic orbit $\tilde{x} \in \mathcal{P}_1(H)$ contradicting the choice of σ_k.

Thus, either the proof is complete, or we have $\mathcal{A}_H(y) < \mathcal{A}_H(x)$. In the latter case we continue by choosing a reparametrization sequence such that

$$\mathcal{A}_H\big((u_n * \tau_k)(0)\big) = \frac{\mathcal{A}_H(x) + \mathcal{A}_H(y)}{2}\,.$$

Again there is a subsequence converging in C^∞_{loc} towards a $v \in \mathcal{M}_{z,\bar{z}}(J,H)$ with

$$\mathcal{A}_H(y) \leqslant \mathcal{A}_H(z) \leqslant \mathcal{A}_H(\bar{z}) \leqslant \mathcal{A}_H(x)\,.$$

Like before we have to argue that $\mathcal{A}_H(y) = \mathcal{A}_H(z)$ implies $y = z$. Here we choose a subsequence n_k and sequence $s_1(k), s_2(k)$ satisfying

$$\begin{aligned} u_{n_k}(s_1(k)) &\in U^a(y), & 0 &< \mathcal{A}_H(y) - \mathcal{A}_H(u_{n_k}(s_1(k))) < \tfrac{1}{k} \\ u_{n_k}(s_2(k) + \tau_{n_k}) &\in U^a(z), & 0 &< \mathcal{A}_H(u_{n_k}(s_2(k) + \tau_{n_k})) - \mathcal{A}_H(z) < \tfrac{1}{k} \end{aligned}$$

Then the same argumentation as above proves $y = z$.

Repeating the reparametrization argument finitely many times due to the finiteness of $\mathcal{P}_1(H)$, the second statement of the asserted alternative follows after suitable renumbering. ■

We have sufficiently analyzed the necessary compactness properties of cylindrical "gradient flow" solutions on $[T,\infty) \times S^1$ for regular pairs (J,H). Let us now return to the general model surface Σ with cylindrical ends and admissible extensions (J,k) of given pairs $(J^i,H^i)_{i=1,\dots,\nu}$. We compute the fundamental a priori energy estimate from which we can deduce the compactness theorems.

4.3.2 The General Energy Estimate

Analogously to the general index formula independent of $\phi_c\colon \pi_2(M) \to \mathbb{Z}$, one can find an energy estimate without assumptions on $\phi_\omega\colon \pi_2(M) \to \mathbb{R}$.

Let $\Sigma = (\overline{\Sigma}, (\psi_i)_{i=1,\dots,\nu})$ be a model surface with a fixed conformal structure j such that

$$T\psi_k \circ i = j(\psi_k) \circ T\psi_k, \quad k = 1,\dots,\nu,$$

and let $E \to \Sigma$ be a Riemannian vector bundle with inner product $\langle\cdot,\cdot\rangle$. Given $\xi \in \Omega^1(E)$, a differential-1-form with values in E, and a conformal coordinate chart $f\colon \Sigma \supset U_f \to \mathbb{C}$ such that

$$\xi_{|U_f} = f^*(v_f ds + w_f dt),$$

we define the 2-form

$$|\xi|^2\sigma_j = f^*\big[(|v_f|^2 + |w_f|^2) ds \wedge dt\big]\,.$$

One easily verifies that $|\xi|^2\sigma_j$ is well-defined independently from the chart f.

4.3.13 Definition *Let $E \to \Sigma$ be a Hermitian bundle, then we define for $\xi \in \Omega^1(E)$*

$$\|\xi\|^2_{J,L^2} = \int_\Sigma |\xi|^2_J \sigma_j\,.$$

Let $k \in C^\infty_\Sigma(F)$ be a smooth section in the homomorphism bundle over $\Sigma \times M$ as in Proposition 2.3.1. Then we define the generalized "flow energy" for $u \in C^\infty(\Sigma, M)$ as

$$\Phi_k(u) = \frac{1}{2}\big\|(d+k)(u)\big\|^2_{J,L^2}$$

provided that it is finite.

Note that this generalized flow energy depends on the given structures j, J and k although we only indicate k. In the case of $\Sigma = \mathbb{R} \times S^1$ and $k = k(H) = -dt \otimes X_{H_t}$ we have for $u \in C^\infty(\mathbb{R} \times S^1, M)$

$$\Phi_k(u) = \Phi(u) = \frac{1}{2}\int_{\mathbb{R}\times S^1} \big(|\partial_s u|^2_J + |\partial_t u - X_{H_t}(u)|^2_J\big) ds dt,$$

so that

$$\Phi_k(u) = \int_{\mathbb{R}\times S^1} |\partial_s u|^2_J ds dt < \infty$$

for trajectories $u \in \mathcal{M}(J,H) = \bigcup_{x,y \in \mathcal{P}_1(H)} \mathcal{M}_{x,y}(J,H)$.

Let us again consider a covering of the loop space component $\Omega^o(M)$ which is now adapted to ϕ_ω. Let $\widetilde{\Omega}^o(M)$ denote the universal covering with transformation group $\pi_2(M)$, i.e.

$$\widetilde{\Omega}^o(M) = \big\{\, [x, u_x] \,\big|\, x \in \Omega^o(M),\ u_x\colon D^2 \to M,\ u_{x|S^1} = x \,\big\},$$

where

$$(x, u_x) \sim (x, \bar{u}_x) \ \Leftrightarrow\ 0 = \{u_x \#(-\bar{u}_x)\} \in \pi_2(M)\,.$$

We construct the quotient

$$\widetilde{\Omega}^o_\omega(M) = \widetilde{\Omega}^o(M)\big/_{\ker\phi_\omega},$$

a covering of $\Omega^o(M)$ with covering group $\Gamma_\omega = \pi_2(M)/\ker\phi_\omega$. That is, $[x, u_x] \in \widetilde{\Omega}^o_\omega(M)$ consists of representatives

$$(x, u_x) \sim_\omega (x, \bar{u}_x) \ \Leftrightarrow\ \int_{D^2} u_x^*\omega = \int_{D^2} \bar{u}_x^*\omega\,.$$

On this covering of $\Omega^o(M)$, the action

$$\mathcal{A}_H\colon \widetilde{\Omega}^o_\omega(M) \to \mathbb{R}$$

is well-defined and real-valued.

Given $\tilde{x}_1,\dots,\tilde{x}_\nu \in \widetilde{\Omega}^o(M)$, $\tilde{x}_i = [x_i, u_{x_i}]$, and a map $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$, we define the homotopy class $\{\bar{u}\} \in [\Sigma \#_{i=1}^\nu D^{-\epsilon_i}, M]$ by

$$\{\bar{u}\} = \{u \#_{i=1}^\nu (\epsilon_i u_{x_i})\}\,.$$

Here, $\pm u_x$ denotes the appropriate orientation of the homotopy class of u_x relative to x. The glued map needs only to be well-defined as a continuous map. We view $\{\bar{u}\}$ as a homology class in $H_2(M,\mathbb{Z})$. Thus, given $\tilde{x}_1,\dots,\tilde{x}_\nu \in \widetilde{\Omega}^o_\omega(M)$ and $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)$, the real number

$$\omega(\bar{u}) = \{\omega\}(\{\bar{u}\}) \in \mathbb{R}$$

is well-defined.

We now compute an energy estimate for solutions $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)$ of $\overline{\partial}_{J,k}(u) = 0$, where (J,k) is an admissible extension of the regular pairs $(J^i,H^i)_{i=1,\dots,\nu}$. Let us recall

$$\overline{\partial}_{J,k} = \Lambda_J \circ (d+k), \quad \overline{\partial}_{J,k}(u) \in \Omega^{0,1}_J(u^*TM),$$

and we compute in conformal coordinates $f\colon \Sigma \supset U_f \to \mathbb{C}$, $f(z) = (s,t)$,

$$\left|\overline{\partial}_{J,k}(u)\right|^2_J \sigma_j = 2\Big(\left|(d+k)(u)\right|^2_J \sigma_j - 2\omega(u_s+k_s, u_t+k_t)ds\wedge dt\Big), \tag{4.29}$$

where $k_s = k(u)\circ(Tf)^{-1}\cdot\frac{\partial}{\partial s}$ and $k_t = k(u)\circ(Tf)^{-1}\cdot\frac{\partial}{\partial t}$. In particular, we consider the special extension $k^o(H) = -\alpha\otimes X_{H_t}$ as defined in Definition 2.3.5 on page 29 by means of the cut-off functions $\beta^\pm$,

$$\beta(s) = \beta^+(s) = \begin{cases} 0, & s \leqslant 1, \\ 1, & s \geqslant 2, \end{cases},$$

$$0 < \beta'(s) < 2 \quad \text{for } 1 < s < 2, \quad \beta^-(s) = \beta^+(-s),$$

so that

$$\alpha\otimes X_{H_t} = \sum_{i=1}^{\nu} (\psi_i^{-1})^*(\beta^{\epsilon_i} dt)\otimes X_{H^i_t}.$$

We obtain the estimate

4.3.14 Lemma *Given $\tilde{x}_1,\dots,\tilde{x}_a,\tilde{y}_1,\dots,\tilde{y}_b \in \widetilde{\Omega}^o_\omega(M)$ with $x_i \in \mathcal{P}_1(H^i)$, $y_i \in \mathcal{P}_1(H^{i+a})$ for regular Hamiltonians $(H^i)_{i=1,\dots,a+b}$, there exists a constant*

$$0 < K \leqslant 2\max_i \|H^i\|_{C^0(S^1\times M)}$$

such that the following estimate holds. Let $(J,k^o(H))$ be the above special admissible extension of $((J^i,H^i))$, then every solution

$$u \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J,k(H))$$

has the finite energy

$$\boxed{\Phi_{k^o(H)}(u) = \omega(\bar{u}) + \sum_{i=1}^{b}\mathcal{A}_{H^{i+a}}(\tilde{y}_i) - \sum_{i=1}^{a}\mathcal{A}_{H^i}(\tilde{x}_i) + c(u)}$$

with $|c(u)| \leqslant K$. In the general case of an arbitrary T-admissible extension $(J,k(H))$ we find an estimate

$$\Phi_{k(H)}(u) \leqslant c(\omega(\bar{u}), x_1,\dots,y_b, k(H))$$

with a bound $c > 0$ only depending on the value of ω on the homology class $\{\bar{u}\} \in H_2(M)$, the relative action $\sum_{i=1}^b \mathcal{A}_{H^{i+a}}(\tilde{y}_i) - \sum_{i=1}^a \mathcal{A}_{H^i}(\tilde{x}_i)$ and the extension $k(H)$.

Proof. From (4.29) we compute for $u \in C^\infty(\Sigma,M)$ with $u_i = u\circ\psi_i$, $k_s = 0$ and $k_t = -\beta^{\epsilon_i}X_{H^i_t}$ on Z^{ϵ_i} and $k_t = 0$ on $\mathring{\Sigma}$ that

$$\Phi(u) = \frac{1}{4}\left\|\overline{\partial}_{J,k(H)}(u)\right\|^2_{J,L^2} + \int_\Sigma u^*\omega - \sum_{i=1}^{\nu}\int_{Z^{\epsilon_i}} \beta^{\epsilon_i}\frac{d}{ds}(H^i_t\circ u_i)dsdt.$$

Hence, using the given classes $\tilde{x}_1,\dots,\tilde{y}_b$ and the definition of $\omega(\bar{u})$, we obtain for a solution $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)$ with $x_{i+a} = y_i$ for $i = 1,\dots,b$,

$$\Phi(u) = \omega(\{\bar{u}\}) + \sum_{i=1}^{\nu}\epsilon_i\mathcal{A}_{H^i}(\tilde{x}_i) + \sum_{i=1}^{\nu}\epsilon_i\int_1^2\beta'(s)\int_{S^1}H^i_t(u_i(\epsilon_i s,t))dsdt.$$

In the case of an arbitrary T-admissible extension $k(H)$ we decompose it as $k = k^o(H) + \kappa$ with $\kappa \in C^\infty_o(\Sigma\times M)$ having support in $\Sigma_T\times M$. Thus it suffices to estimate in a rough way

$$\begin{aligned}\frac{1}{2}\left\|du + k(u)\right\|^2_{J,L^2} &= \frac{1}{2}\left\|du + k^o(H)(u) + \kappa(u)\right\|^2_{J,L^2} \\ &\leqslant \left\|du + k^o(H)\right\|^2_{J,L^2} + \left\|\kappa(u)\right\|^2_{J,L^2} \\ &\leqslant 2\Phi_{k^o(H)}(u) + c(\kappa)\end{aligned}$$

for some constant $c(\kappa) < \infty$ depending on the C^0-norm of κ and $T > 0$. ■

4.3.15 Definition *Let $(J,k(H))$ be an admissible extension of the ν-tuple $(J^i,H^i)_{i=1,\dots,\nu}$. Then we define*

$$\begin{aligned}\mathcal{M}(J,k(H)) \;=\; \{\, & u \in C^\infty(\Sigma,M) \mid \Phi(u) < \infty, \\ & u_{|\partial\Sigma} \text{ component-wise contractible for some } T > 0\,\}.\end{aligned}$$

Combining the estimate from Lemma 4.3.14 with Proposition 4.3.9 we obtain

4.3.16 Proposition

$$\mathcal{M}(J,k(H)) \;=\; \bigcup_{\substack{x_i\in\mathcal{P}_1(H^i),\,i=1,\dots,a \\ y_i\in\mathcal{P}_1(H^{i+a}),\,i=1,\dots,b}} \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J,k(H)).$$

4.3.17 Definition *We denote by $\mathcal{S} \subset H_2(M,\mathbb{Z})$ the image of $\pi_2(M)$ under the Hurewicz homomorphism and call a map $u \in C^\infty_\Sigma(\overline{\Sigma}, M)$ with contractible restriction $u_{|\partial\Sigma}$* ***spherical*** *if the associated homology classes $\{\bar{u}\} \in H_2(M,\mathbb{Z})$ lie in the spherical subgroup $\mathcal{S}$. Given a class from the quotient group $A \in H_2(M,\mathbb{Z})/\mathcal{S}$ we define the subset of* **A-solutions**

$$\mathcal{M}_A(J,k(H)) = \Big\{ u \in \mathcal{M}(J,k(H)) \,\big|\, \{\bar{u}\} + \mathcal{S} = A \Big\},$$

in particular, for $A = [0]_\mathcal{S}$ we denote by $\mathcal{M}_\mathcal{S}(J,k(h)) = \mathcal{M}_{[0]_\mathcal{S}}$ the set of spherical solutions.

We notify that $\{\bar{u}\}$ is only well-defined for given extension classes $[x_i, u_{x_i}] \in \widetilde{\Omega}^o(M)$ for each component $x_i = u_{|\partial\Sigma}$, but all these classes $\{\bar{u}\}$ are spherical if one is. More precisely, $\{\bar{u}\} + \mathcal{S} \in H_2(M,\mathbb{Z})/\mathcal{S}$ is well-defined. In the last chapter in Theorem 5.4.3, we will see that the specific cohomology operations on Floer homology associated to a model surface Σ are based only on the spherical solutions. Here we define the notion of an A-solution in order to be able to distinguish precisely spherical solutions from non-spherical ones. Note that each $\mathcal{M}_A$ forms an isolated component of the entire solution set. That is, each connected component of $\mathcal{M}(J,k(H))$ intersecting $\mathcal{M}_A$ lies entirely within $\mathcal{M}_A$. Theorem 5.4.3 shows that the components of

$$\mathcal{M} \setminus \mathcal{M}_\mathcal{S} = \bigcup_{A \neq [0]_\mathcal{S}} \mathcal{M}_A$$

are cohomologically irrelevant for this theory. This approach, however, is only useful when we assume that $\phi_\omega, \phi_c = 0$. Under this condition, $\phi_\omega(A)$ and $\phi_c(A)$ are well-defined. In general, a more refined analysis is necessary also for the different spherical classes.

Let us now draw the conclusions from the energy estimate in Lemma 4.3.14.

4.3.18 Theorem *Let $\phi_\omega \equiv 0$ and $A \in H_2(M,\mathbb{Z})/\mathcal{S}$. Then the set of A-solutions*

$$\mathcal{M}_A(J,k(H)) \Subset C^\infty(\Sigma^o, M)$$

is compact with respect to uniform convergence in all C^k on compact subsets of Σ^o.

Fixing the class A yields a constant value $\omega(A) = \omega(\{\bar{u}\})$ for all $u \in \mathcal{M}_A$. Thus Theorem 4.3.18 is a consequence of Theorem 4.3.3, when bubbling-off is not allowed. By combining the latter result about local C^∞-convergence with the result of Theorem 4.3.12 about the cylindrical ends, we are able to prove the crucial compactness theorems for the 0- and 1-dimensional components of the A-solution spaces $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J,k(H)) \cap \mathcal{M}_A$. Given a solution $u \in \mathcal{M}(J,k(H))$ we call the index of the linearization D_u of $\overline{\partial}_{J,k(H)}$ at u its **local dimension**,

$$\dim_{\mathrm{loc}} u = \operatorname{ind} D_u\,.$$

4.3.19 Theorem *Assuming $\phi_\omega \equiv 0$, for given 1-periodic orbits $x_1,\dots,x_a$, $y_1,\dots,y_b$ and a fixed class A as above, the set of A-solutions $u \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b} \cap \mathcal{M}_A$ of local dimension 0 is finite.*

Proof. Since solutions of local dimension 0 are isolated in the Banach manifold $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M)$ we have to prove compactness for the specified set. Theorem 4.3.18 yields the C^∞_{loc}-compactness. Hence, we have to consider the alternative from Theorem 4.3.12 in order to finish the proof. If there is a sequence (u_n) of A-solutions in $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$ with $\dim_{\mathrm{loc}} u_n = 0$ but converging in C^∞_{loc} to a $v \in \mathcal{M}^{\bar{x}_1,\dots,\bar{x}_a}_{\bar{y}_1,\dots,\bar{y}_b}$ with $\bar{x}_i \neq x_i$ or $\bar{y}_i \neq y_i$ at least once, there must be a chain of connecting orbits $\bar{y}_i < z_1 < \ldots < z_r < y_i$ or $\bar{x}_i > \ldots > x_i$ as stated by Theorem 4.3.12. Without loss of generality, we obtain

$$(v_1,\dots,v_r) \in \mathcal{M}_{\bar{y}_i,z_1}(J^i,H^i) \times \ldots \times \mathcal{M}_{z_r,y_i}(J^i,H^i)$$

with $\operatorname{ind} D_{v_j} > 0$ for $j = 1,\dots,r$. But then index additivity implies that

$$\operatorname{ind} D_{u_n} \geqslant \operatorname{ind} D_v + \sum_{j=1}^r \operatorname{ind} D_{v_j} > 0 \quad \text{for large } n\,.$$

This is based on a gluing theorem which will be proven in the next section, see Corollary 4.4.2. Roughly speaking, the sequence (u_n) converges towards a so-called broken solution in a topology for which $\dim_{\mathrm{loc}} u_n$ is locally constant. For this broken trajectory containing the solutions v and (v_j) the index additivity holds as proven in Theorem 3.2.12 contradicting the assumption that $\operatorname{ind} D_{u_n} = 0$. This holds without restriction to $\phi_c\colon \pi_2(M) \to \mathbb{Z}$ because we only work with the uniquely defined relative indices which equal the local dimensions. If we assumed that $\phi_c \equiv 0$ so that the Conley-Zehnder index is well-defined and integer-valued, the proof works directly via the index additivity without using the argument about the convergence to a broken solution. Finally, it follows that $\{\bar{v}\}$ belongs to the same class $A = \{\bar{u}_n\} + \mathcal{S}$. ■

Before we state the analogous compactness result for 1-dimensional components $K \subset \mathcal{M}_A(J,k(h))$, we introduce an appropriate notion of convergence related to the weak topology of uniform convergence on compact subsets with all derivatives, $C^\infty_{\mathrm{loc}}(\Sigma^o, M)$.

4.3.20 Definition *Let $(u_n) \subset \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J,k(H))$ be a C^∞_{loc}-convergent sequence of solutions with fixed ends $x_1,\dots,x_\nu$. We say that (u_n)* ***converges geometrically towards a broken solution of degree*** *1, if there is a $u \in \mathcal{M}^{\bar{x}_1,\dots,\bar{x}_a}_{\bar{y}_1,\dots,\bar{y}_b}$ with $\bar{x}_i \neq x_i$ or $\bar{y}_i \neq y_i$ exactly once and $\dim_{loc} u = 0$, a nonconstant connecting orbit $v \in \mathcal{M}_{\bar{y}_i,y_i}(J^{i+a},H^{i+a})$ or respectively $v \in \mathcal{M}_{x_i,\bar{x}_i}(J^i,H^i)$ with $\dim_{loc} v = 1$ and a shifting sequence $(\tau_n) \subset \mathbb{R}$ with $\epsilon_i \cdot \tau_n \to \infty$ such that*

$$u_n \xrightarrow{C^\infty_{loc}} u \quad \text{and} \quad (u_n \circ \psi_i) * \tau_n \xrightarrow{C^\infty_{loc}} v$$

for $n \to \infty$. We denote this convergence by

$$u_n \rightharpoonup (u,\hat{v}) \text{ resp. } (\hat{v},u), \quad n \to \infty,$$

where $\hat{v}$ stands for the unparametrized trajectory $[v]$ in the moduli space $\widehat{\mathcal{M}}_{\bar{y}_i,y_i}$ $= \mathcal{M}_{\bar{y}_i,y_i}/\mathbb{R}$ or $\widehat{\mathcal{M}}_{x_i,\bar{x}_i}$.

Note that the curve u within such a broken solution $(u, \hat{v})$ is uniquely determined by the geometrically converging sequence (u_n). The same holds for the unparametrized trajectory $\hat{v}$. Namely, assume that there is another reparametrization sequence (σ_n) for $u_n^i = u_n \circ \psi_i$ such that

$$u_n^i * \sigma_n \xrightarrow{C^\infty_{\text{loc}}} w \in \mathcal{M}_{z,\bar{z}}(J^{i+a}, H^{i+a}).$$

Then it either holds that $(|\sigma_n - \tau_n|)$ is bounded and therefore necessarily $[w] = \hat{v}$ with $z = \bar{y}_i$, $\bar{z} = y_i$. Or, by the same arguments as in the proof of Theorem 4.3.3, we have $z = w = \bar{z} = \bar{y}_i$ with $\sigma_n - \tau_n \to -\infty$, or $z = w = \bar{z} = y_i$ with $\sigma_n - \tau_n \to \infty$.

We point out that this definition of geometrical convergence merely describes the most simple case of $\dim_{\text{loc}} u = 0$ and $\dim_{\text{loc}} v = 1$. This special case of a broken solution of degree 1 is sufficient for our purpose to describe the cobordism relations for 1-dimensional components $K \subset \mathcal{M}_A(J, k(H))$. In general, we would have to distinguish between the different possibilities of multiply broken connecting orbits, for example $\bar{y}_i < z_1 < \ldots < z_r < y_i$ as in the above proof of Theorem 4.3.19. These multiply broken trajectories can occur at several ends $\{x_{i_1}, \ldots, x_{i_l}\} \subset \{x_1, \ldots, x_\nu\}$ in dependence of the local dimension. It is clear that the space of A-solutions $\mathcal{M}_A$ is closed with respect to geometrical convergence, that is, the solutions u in the pair $(u, \hat{v})$ is again an A-solution.

By the same argumentation we obtain

4.3.21 Theorem *Let $K \subset \mathcal{M}_A(J, k(H))$ be a 1-dimensional path-component of A-solutions for a fixed class A, $K \subset \mathcal{M}^{x_1,\ldots,x_a}_{y_1,\ldots,y_b}(J, k(H))$. Then, every sequence $(u_n) \subset K$ which is not relatively compact possesses a geometrically convergent subsequence of degree 1,*

$$u_{n_k} \rightharpoonup (u, \hat{v}), \quad k \to \infty,$$
$$(u, \hat{v}) \in \mathcal{M}^{x_1,\ldots,x_a}_{y_1,\ldots,\bar{y}_i,\ldots,y_b} \times \widehat{\mathcal{M}}_{\bar{y}_i,y_i}(J^{i+a}, H^{i+a})$$

or respectively $\hat{v} \in \mathcal{M}_{x_i,\bar{x}_i}(J^i, H^i)$, etc. In short, K is geometrically precompact.

Let us point out at this stage, that the non-compact components K concerned in this theorem necessarily are diffeomorphic to the interval $\mathbb{R}$,

$$\phi\colon K \xrightarrow{\sim} (-\infty, \infty).$$

By the combination with the gluing result of Theorem 4.4.1, it will turn out that the geometrically convergent sequences above correspond exactly to sequences $(\phi(s_n))$ with $s_n \to \pm\infty$. Moreover, the geometric limits $\lim_{s\to\pm\infty} \phi(s)$ are uniquely determined and necessarily different. This will be the heart of the proof of Theorem 5.1.3 where we show that the operations to be finally defined act on the level of Floer (co-)homology.

4.3.3 The Compactness Result for Homotopies

We already observed in the last section's energy estimate 4.3.14 that the upper bound can be found independently of the chosen structures j and J on Σ and M. In view of the discussion of homotopy invariance in the final chapter, we have to provide a compactness result for solutions $(u_n)_{n\in\mathbb{N}}$ of $\overline{\partial}_{j_n,J_n,k_n}(u_n) = 0$ with varying structures (j_n, J_n, k_n). We already treated the variation of (J_n, k_n) in Theorem 4.3.2. Now we also allow deformation of j. If (j_n) converges we do not come upon a new phenomenon. It suffices to find suitable converging j_n-holomorphic coordinates on Σ.

4.3.22 Theorem *Let (J_n, k_n) and (J, k) be T-admissible extensions of the ν-tuple $(J^i, H^i)_{i=1,\ldots,\nu}$ such that*

$$(J_n, k_n) \to (J, k)$$

converges in $C^\infty_{\text{loc}}(\Sigma \times M)$ as $n \to \infty$. Moreover, let j and $(j_n)_{n\in\mathbb{N}}$ be conformal structures on Σ extending $(\psi_k^* i)_{k=1,\ldots,\nu}$ over Σ_0 such that $j_n \to j$ converges in $C^\infty_{\text{loc}}(\Sigma)$. Then every sequence $(u_n) \subset C^\infty(\Sigma, M)$ of A-solutions with fixed ends $x_i \in \mathcal{P}_1(H^i)$, $y_i \in \mathcal{P}_1(H^{i+a})$,

$$u_n \in \mathcal{M}^{x_1,\ldots,x_a}_{y_1,\ldots,y_b}(j_n, J_n, k_n) \cap \mathcal{M}_A, \quad n \in \mathbb{N},$$

is relatively compact in $C^\infty_{\text{loc}}(\Sigma, M)$ if $\phi_\omega \equiv 0$.

Proof. On the cylindrical ends $\Sigma \setminus \Sigma_T$ we can immediately use the compactness results above because

$$(j_n, J_n, k_n)_{|\Sigma\setminus\Sigma_T} = (j, J, k)$$

is given by the standard structures i and $(J^k, H^k)_{k=1,\ldots,\nu}$. Thus, it suffices to consider $u_{n|\Sigma_T} \in C^\infty(\Sigma_T, M)$. Let us first discuss the variation of the complex structures j_n. Here we use an integrability result which can be deduced from a more general version, 4.5c in [41]. Let us present a short proof based on Gromov's compactness theorem for J-holomorphic curves, cf. [26, 32].

4.3.23 Proposition *Let $\Omega \subset \mathbb{C}$ be an open domain and $j_n \in C^\infty(\Omega)$ be complex structures converging towards a complex structure j in the C^∞-topology. Then for each $z \in \Omega$ there exists an open neighbourhood $U(z) \subset \Omega$ and complex coordinates*

$$\zeta_n\colon U \to \zeta_n(U) \subset \mathbb{C}$$

which are j_n-holomorphic and converge towards j-holomorphic complex coordinates $\zeta\colon U \to \zeta(U) \subset \mathbb{C}$ in the C^∞-topology. Moreover, $\zeta_n(z) = \zeta(z)$ for all $n \in \mathbb{N}$.

Proof. Let $B_R(0)$ be a large open disk in $\mathbb{C}$ containing Ω. We may extend the conformal structures j_n and j over $\mathbb{C}$ such that they are identical to the standard structure i on $\mathbb{C} \setminus B_R(0)$ and j_n converges to j on the whole plane $\mathbb{C}$

in the C^∞_{loc}-topology. Thus we consider $j_n \to j$ as conformal structures on the Riemannian sphere $S^2 = \mathbb{C} \cup \{\infty\}$. By Weierstraß's Uniformization Theorem there are smooth diffeomorphisms $u_n \colon S^2 \to S^2$ such that

$$Tu_n \circ j = j_n \circ Tu_n .$$

Let us fix a point $p \in \partial B_R(0)$ and we may assume that

$$u_n(0) = 0, \quad u_n(p) = p \quad \text{and} \quad u_n(\infty) = \infty \tag{4.30}$$

for all $n \in \mathbb{N}$. In view of the assertion we have to show that u_n converges towards the identity in $C^\infty(S^2, S^2)$. We choose a volume form ω_o on S^2 and set

$$\omega_n = \frac{1}{2}\big(\omega_o + \omega_o \circ (j_n \times j_n)\big) .$$

These are volume forms on S^2 converging in $C^\infty(S^2)$ towards $\omega = \frac{1}{2}(\omega_o + \omega_o \circ (j \times j))$. The j_n are ω_n-compatible and yield the Riemannian metrics $g_n = \omega_n \circ (\mathrm{id} \times j_n)$. It follows that the energies

$$\Phi_{j,j_n}(u_n) = \int_{S^2} u_n^* \omega_n$$

of the conformal maps $u_n \colon (S^2, j) \to (S^2, j_n)$ are uniformly bounded because

$$\Big| \Phi_{j,j_n}(u_n) - \int_{S^2} u_n^* \omega \Big| \to 0$$

and $\int_{S^2} u_n^* \omega = \text{const} > 0$ is independent of n. In general, (u_n) converges to a cusp-curve, due to Gromov's theory. Here, since we fixed the mapping degree 1, u_n converges either to a point map or again to a diffeomorphism. Due to (4.30), u_n converges towards the identity. ■

It follows from this proposition that there exists a compact neighbourhood

$$V \subset \zeta(U) \cap \bigcap_{n \in \mathbb{N}} \zeta_n(U)$$

such that $\zeta_n^{-1}{}_{|V}$ converges in $C^0(V, U)$ towards $\zeta^{-1}{}_{|V}$. This implies

$$\zeta_n^{-1}{}_{|V} \xrightarrow{C^\infty} \zeta^{-1}{}_{|V} , \tag{4.31}$$

and there is a neighbourhood

$$W(z) \subset \zeta^{-1}(V) \cap \bigcap_{n \in \mathbb{N}} \zeta_n^{-1}(V) \subset U(z) . \tag{4.32}$$

We now carry out the reduction to the original compactness results.

4.3.24 Proposition *Assume that the gradient of $u_{n|\Sigma_T}$ is uniformly bounded,*

$$\sup_{z \in \Sigma_T} \big|\nabla u_n(z)\big| < c$$

for all $n \in \mathbb{N}$, for some $c > 0$. Then there is a $C^\infty(\Sigma_{T-1})$-convergent subsequence

$$u_{n_k} \to u \quad \text{in } C^\infty(\Sigma_{T-1}, M)$$

towards a solution u of $\overline{\partial}_{j,J,k}(u) = 0$.

Proof. In view of Proposition 4.3.23, we find finitely many $z_1, \dots, z_N \in \Sigma_T$ such that the compact surface

$$\Sigma_T \subset \bigcup_{i=1}^N W(z_i)$$

is covered by the complex coordinates, without loss of generality $W(z_i) \subset U(z_i) \subset \Sigma_{T+1}$,

$$\zeta^i, \zeta_n^i \colon U(z_i) \to \mathbb{C} \supset V_i, \quad i = 1, \dots, N .$$

The maps

$$v_n^i \colon V_i \to M, \quad v_n^i = u_{n|U(z_i)} \circ (\zeta_n^i)^{-1}_{|V_i}$$

have uniformly bounded gradients,

$$\sup_{z \in V_i} \big|\nabla v_n^i(z)\big| < c_i \quad \text{for all } n \in \mathbb{N},$$

and they satisfy the equations

$$\partial_s v_n^i(z) + \tilde{J}_n^i(z, v_n^i)\partial_t v_n^i + Y_n^i(z, v_n^i) = 0$$

with

$$\begin{aligned} Y_n^i(z,m) &= k_n\big((\zeta_n^i)^{-1}(z), m\big) \circ D(\zeta_n^i)^{-1}(z) \cdot \tfrac{\partial}{\partial s} \quad \text{and} \\ \tilde{J}_n^i(z,m) &= J_n\big((\zeta_n^i)^{-1}(z), m\big) . \end{aligned}$$

Due to (4.31), we have the C^∞-convergence

$$\begin{aligned} \tilde{J}_n^i &\to \tilde{J}^i = J \circ (\zeta^i)^{-1} \quad \text{and} \\ Y_n^i &\to Y^i = \big(k \circ (\zeta^i)^{-1}\big) \cdot (D\zeta^i)^{-1} \cdot \tfrac{\partial}{\partial s} . \end{aligned}$$

Hence we are able to apply Theorem 4.3.2 yielding a subsequence $v_{n_k}^i$ converging towards a v^i in $C^\infty(\bar{V}_i)$ so that v^i solves

$$\partial_s v^i + \tilde{J}^i \partial_t v^i + Y^i(v^i) = 0 .$$

After choosing again a suitable subsequence (n_k) we obtain C^∞-convergence

$$u_{n_k|W(z_i)} = v_{n_k}^i \circ \zeta^i_{n_k|W(z_i)} \to v^i \circ \zeta^i{}_{|W(z_i)} = u_{|W(z_i)}$$

for all $i = 1, \dots, N$ for some $u \in C^\infty(\Sigma_T)$ as $k \to \infty$. ■

The proof of Theorem 4.3.22 is completed by the following result based on a uniform energy estimate.

4.3.25 Proposition *The gradient of $u_{n|\Sigma_T}$ is uniformly bounded provided that $\phi_\omega \equiv 0$.*

PROOF. We observed in 4.3.2 that we have an a priori estimate for the 'flow energy' $\Phi_{j,J,k}$ associated to (j,J,k) if we consider solutions with fixed boundary conditions. In view of Definition 4.3.13 we have

$$\Phi_{j,J,k}(u) = \frac{1}{2}\int_\Sigma \left|(d+k)(u)\right|_J^2 \sigma_j \tag{4.33}$$

which is bounded by a constant $C = C(x_1, \dots, y_b, J, k, A)$ independently of $u \in \mathcal{M}_A$ if $\phi_\omega \equiv 0$ and $A \in H_2(M,\mathbb{Z})/\mathcal{S}$ is fixed.

Let us argue indirectly. Assuming that there is no uniform gradient bound for $u_{n|\Sigma_T}$ we now proceed in analogy with the proof of Theorem 8 in [31], Chapter 6. In this case there is a subsequence (n_k) and a sequence $(z_k) \subset \Sigma_T$ with

$$z_k \to z_o \in \Sigma_T, \quad \left|\nabla u_{n_k}(z_k)\right|_J \to \infty$$

as $k \to \infty$. By Proposition 4.3.23 we again find complex coordinates

$$\zeta_n, \zeta \colon \Sigma \supset U(z_o) \to \mathbb{C}$$

satisfying (4.31). Then, setting $v_k = u_{n_k} \circ \zeta_{n_k}^{-1} \in C^\infty(\bar{V}, M)$ we find $\epsilon_k > 0$ with

$$\epsilon_k \to 0 \quad \text{and} \quad \epsilon_k \left|\nabla v_k(\zeta_{n_k}(z_k))\right|_J \to \infty, \tag{4.34}$$

whereas (v_k) satisfy

$$\partial_s v_k + \tilde{J}_k \partial_t v_k + Y_k(v_k) = 0 \tag{4.35}$$

with

$$\tilde{J}_l \xrightarrow{C^\infty} J \circ \zeta^{-1}, \quad Y_l \to (k \circ \zeta^{-1}) \cdot (D\zeta)^{-1} \cdot \tfrac{\partial}{\partial s}$$

for $l \to \infty$ as above. Since $J_n \to J$, $j_n \to j$ and $k_n \to k$ we obtain the energy estimate

$$\begin{aligned}
\int_{B_{\epsilon_n}(\zeta_n(z_n))} \left|\nabla v_n\right|^2_{J\circ\zeta^{-1}} &\leqslant \text{const} \int_{U(z_o)} \left|du_n\right|^2_{J_n} \sigma_{j_n} \\
&\leqslant \text{const}\left(4\Phi_{j_n,J_n,k_n}(u_n) + 2\int_{U(z_o)} \left|k_n(u_n)\right|^2_{J_n} \sigma_{j_n}\right) \\
&\leqslant c
\end{aligned}$$

for some constant $c > 0$ independent of $n \in \mathbb{N}$ in view of (4.33). Here, we use the balls $B_\epsilon(z) = \{\, w \in \mathbb{C} \mid |w - z| < \epsilon \,\}$. Proceeding exactly as in [31] by means of a conformal reparametrization, we finally obtain in the limit a J-holomorphic finite energy map $v \colon \mathbb{C} \to M$ with

$$\begin{gathered}
\left|\nabla v(0)\right| = 1, \\
\left|\nabla v(z)\right| \leqslant \text{const}, \quad z \in \mathbb{C}, \\
\partial_s v + J(v)\partial_t v = 0, \\
\int_\mathbb{C} v^*\omega = \frac{1}{2}\int_\mathbb{C} \left|\nabla v\right|_J^2 < \infty,
\end{gathered}$$

contradicting the assumption $\phi_\omega \equiv 0$. ■

The combination of Propositions 4.3.24 and 4.3.25 yields the proof of Theorem 4.3.22. ■

4.4 Gluing

In this section we complete the analysis of the solution spaces $\mathcal{M}(J, k(H))$ as far as the 0- and 1-dimensional components are concerned. We have already proven the strong compactness of the 0-dimensional components of solutions of a specified class $A \in H_2(M,\mathbb{Z})/\mathcal{S}$ and we stated the weaker result of geometrical compactness for the 1-dimensional path-components. However, in order to deduce from this topological analysis the necessary combinatorial information proving the commutativity with the δ-operator, we have to give a precise description of the difference between strong and geometric compactness. This is accomplished in straight analogy to Floer's proof of the fundamental relation $\delta \circ \delta = 0$. We have to verify that each non-compact 1-dimensional path-component is geometrically bounded by two different broken solutions $(u, \hat{v})$ of degree 1, and that every such broken solution appears in the geometrical compactification of a 1-dimensional component.

We follow the standard scheme of proving this cobordism relation for simply broken solutions by constructing a gluing operation which reverses geometrical convergence. First, we set up a pregluing operation providing us with approximate solutions $u\#_\rho^o\hat{v}$. Then, by using a Newton type argument we find a unique correction term supplying a correct solution, $(u,\hat{v}) \mapsto u\#_\rho\hat{v}$, whenever the gluing parameter $\rho > 0$ is large enough. This leads to a smooth map from compact sets of broken solutions $(u,\hat{v})$ of degree 1 into a component of $\mathcal{M}_A(J, k(H))$, so that we obtain a smooth compactification of an end of a 1-dimensional component of $\mathcal{M}_A(J, k(H))$, where $\dim_{\text{loc}} u = 0$ and $\dim_{\text{loc}} v = 1$. The gluing operation will be constructed in general, for any given local dimension.

Before we start with constructing the pregluing operation we recall the identification for the moduli spaces $\widehat{\mathcal{M}}_{x,y}(J,H)$ of unparameterized connecting orbits. Given a regular pair (J,H), let $x \neq y \in \mathcal{P}_1(H)$, so that the proper shifting operation

$$\begin{gathered}
\mathcal{M}_{x,y}(J,H) \times \mathbb{R} \to \mathcal{M}_{x,y}(J,H), \\
(u,\tau) \mapsto u_\tau = u * \tau, \quad u_\tau(s,t) = u(s+\tau,t),
\end{gathered}$$

is free and

$$a \colon \mathcal{M}_{x,y}(J,H) \to \mathbb{R}, \quad u \mapsto \mathcal{A}_H(u(0)) - \lim_{s\to-\infty} \mathcal{A}_H(u(s))$$

is a well-defined smooth function with $da \neq 0$ on $\mathcal{M}_{x,y}(J,H)$. Thus we identify the smooth finite dimensional manifolds

$$\widehat{\mathcal{M}}_{x,y}(J,H) = \Big\{\, u \in \mathcal{M}_{x,y}(J,H) \;\Big|\; \mathcal{A}_H(u(0)) = \frac{1}{2}\lim_{s\to\infty}\Big[\mathcal{A}_H(u(s)) + \mathcal{A}_H(u(-s))\Big] \Big\}$$

component-wise. This is well-defined also if $\phi_\omega \neq 0$, because the normalization condition is equivalent to $a(u) = \frac{1}{2}\Phi_{J,H}(u)$, where $\Phi_{J,H}(u)$ is the flow energy of u.

The main difficulty in the construction of the gluing operation is that it is not canonical, there is no natural choice for the pregluing map such that we could obtain a universal construction uniformly for all solution spaces. It can only be defined for compact sets of broken solutions. For the purpose of considering only broken solutions of degree 1, i.e. local dimensions 0 respectively 1, we restrict the gluing map to finite sets of isolated pairs $(u, \hat{v})$. However, we give a more general gluing theorem for compact sets of arbitrary dimension. Without loss of generality, we always consider broken solutions

$$(u, \hat{v}) \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J, k(H)) \times \widehat{\mathcal{M}}_{y_i,z}(J^{i+a}, H^{i+a}),$$

that is, we only glue at an 'exit' end y_i.

4.4.1 Theorem *Given compact subsets of solutions $K \subset \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J, k(H))$ and $\bar{K} \subset \widehat{\mathcal{M}}_{y_i,z}(J^{i+a}, H^{i+a})$, there exists a constant $\rho_o(K, \bar{K}) > 0$ and a smooth embedding*

$$\begin{aligned} \#\colon K \times \bar{K} \times [\rho_o, \infty) &\hookrightarrow \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_{i-1},z,y_{i+1},\dots,y_b}(J, k(H)) \\ (u, \hat{v}, \rho) &\mapsto u\#_\rho \hat{v}, \end{aligned}$$

such that

$$u_n \#_{\rho_n} \hat{v}_n \rightharpoonup (u, \hat{v})$$

whenever $u_n \to u$, $\hat{v}_n \to \hat{v}$ and $\rho_n \to \infty$. In particular, $\#$ is a local diffeomorphism and every geometrically convergent sequence $(w_n) \subset \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b}$, $w_n \rightharpoonup (u, \hat{v}) \in K^\circ \times \bar{K}^\circ$, finally lies in the range of $\#$, i.e. for n_o large enough

$$(w_n)_{n \geqslant n_o} \subset \{\, \tilde{u}\#_\rho \tilde{v} \mid \rho \geqslant \rho_o,\ \tilde{u} \in K^\circ,\ \tilde{v} \in \bar{K}^\circ \,\}.$$

The local diffeomorphism property can be expressed by means of the local dimensions, so that we deduce

4.4.2 Corollary *If $(w_n) \subset \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$ converges geometrically,*

$$w_n \rightharpoonup (u, \hat{v}), \quad n \to \infty,$$

it holds for n large enough

$$\dim_{loc} w_n = \dim_{loc} u + \dim_{loc} v$$

for any representative $v \in [v] = \hat{v} \in \widehat{\mathcal{M}}_{y_i, \bar{y}_i}$.

4.4.3 Remark For the proof of Theorem 4.4.1 it suffices to assume the compact sets K and $\bar{K}$ to be path-connected. Namely, let K_1, K_2 be two different path components of K and

$$\#^i\colon K_i \times \bar{K} \times [\rho_i, \infty) \to \mathcal{M}$$

be the associated gluing maps for $i = 1, 2$. Then there exists a common parameter bound $\rho_o \geqslant \max(\rho_1, \rho_2)$ such that

$$\#^1(K_1 \times \bar{K} \times [\rho_o, \infty)) \cap \#^2(K_2 \times \bar{K} \times [\rho_o, \infty)) = \emptyset,$$

that is, $\#$ can be defined on $K_1 \cup K_2$ component-wise such that it satisfies all statements of Theorem 4.4.1. If such a ρ_o did not exist, it would follow that there are

$$(u^i_n, v^i_n, \rho^i_n) \subset K_i \times \bar{K} \times [\rho_i, \infty)$$

for $i = 1, 2$ with $\rho^i_n \to \infty$ satisfying

$$u^1_n \#_{\rho^1_n} v^1_n = u^2_n \#_{\rho^2_n} v^2_n \tag{4.36}$$

for all $n \in \mathbb{N}$. Without loss of generality we may assume

$$u^i_n \to u^i, \quad v^i_n \to v^i, \quad i = 1, 2,$$

in view of the compactness of K^i and $\bar{K}$. Due to the continuity of $\#^i$, it follows that

$$u^i_n \#_{\rho^i_n} v^i_n \rightharpoonup (u^i, v^i)$$

for $n \to \infty$ and $i = 1, 2$. Hence,

$$u^1 = u^2 \quad \text{and} \quad v^1 = v^2,$$

so that the path components K^1 and K^2 are identical in contradiction to the assumption.

4.4.1 Pregluing

Let the path-connected compact subsets

$$K \subset \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J, k(H)) \quad \text{and} \quad \bar{K} \subset \widehat{\mathcal{M}}_{y_i,z}(J^{i+a}, H^{i+a})$$

be fixed. Without loss of generality we assume that these compact sets are closures of open submanifolds of the respective solution space. We first construct the approximate pregluing operation

$$K \times \bar{K} \ni (u, v) \mapsto u\#^o_\rho v \in \mathcal{P}^{1,p}_{x_1,\dots,z,\dots,x_\nu}(\Sigma, M), \quad \rho \geqslant \rho_o,$$

which already exhibits the essential properties, i.e.

$$u\#^o_{\rho_n} v \rightharpoonup (u, v) \text{ as } \rho_n \to \infty \quad \text{and} \quad \operatorname{ind} D_{u\#^o_{\rho_n} v} = \operatorname{ind} D_u + \operatorname{ind} D_v\,.$$

Throughout this section we use the smooth cut-off functions $\beta^\pm\colon \mathbb{R} \to [0, 1]$,

$$\beta^-(s) = \begin{cases} 1, & s \leqslant -1, \\ 0, & s \geqslant 0, \end{cases} \qquad -\tfrac{3}{2} \leqslant \tfrac{d}{ds}\beta^- \leqslant 0, \quad \beta^+(s) = \beta^-(-s)\,.$$

Moreover, we denote for every mapping $f\colon \mathbb{R} \to X$ the shifting operation by

$$f_\rho(s) = f(s + \rho)\,.$$

For any $w \in C^0(\Sigma^o, M)$ we again use the notation

$$w_i \in C^0(Z^{\epsilon_i}, M), \quad w_i = w_{|\psi_i(Z^{\epsilon_i})} \circ \psi_i\,.$$

4.4.4 Definition *Given the compact sets K and $\bar K$ as above there exists a constant $\rho_o(K,\bar K) > 0$ such that the ranges of the maps $u_{i|Z^+_{\rho-1}}$ and $v_{|Z^-_{-2\rho+1}}$ lie in the image of the injectivity neighbourhood of*

$$\exp_{y_i}\colon y_i^*\mathcal{D} \to M$$

for all $\rho \geqslant \rho_o + 1$ and $(u,v) \in K \times \bar K$. We define $u\#^o_\rho v \in \mathcal{P}^{1,p}_{x_1,\ldots,z,\ldots,x_\nu}(\Sigma, M)$ by $u\#^o_\rho v_{|\Sigma\setminus Z^{\epsilon_i}} = u_{|\Sigma\setminus Z^{\epsilon_i}}$ and

$$(u\#^o_\rho v)_i(s,t) = \\ = \begin{cases} u_i(s,t), & s \leqslant \rho - 1, \\ \exp_{y_i}\left[\beta^-_{-\rho}(s)\exp^{-1}_{y_i}(u_i(s,t)) + \beta^+_{-\rho}\exp^{-1}_{y_i}(v_{-2\rho})\right], & |s-\rho| \leqslant 1, \\ v_{-2\rho}(s,t), & s \geqslant \rho + 1\,. \end{cases}$$

We use the shorthand notation

$$\#^o\colon K \times \bar K \times [\rho_o, \infty) \to \mathcal{P}^{1,p}_{x_1,\ldots,z,\ldots,x_\nu}(\Sigma, M), \\ (u,v,\rho) = \chi \mapsto w_\chi = u\#^o_\rho v,$$

for this pregluing map, and we denote by

$$\tilde u^l_\rho = u\#^o_\rho y_i, \qquad \tilde v^r_\rho = y_i\#^o_\rho v \tag{4.37}$$

the asymptotically constant mappings. (See Figure 4.1 where a rough sketch of $\exp^{-1}_{y_i}[(u\#^o_\rho v)_i]$ is given.)

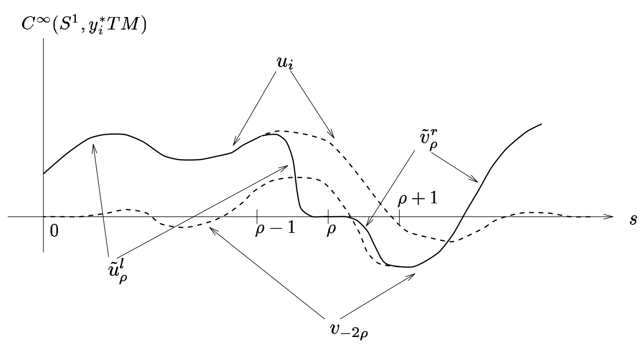


Figure 4.1: Sketch of pregluing.

First of all, we easily verify that this pregluing operation is a smooth map. This can be seen by using the local charts for $\mathcal{M}^{x_1,\ldots,x_a}_{y_1,\ldots,y_b}$, $\mathcal{M}_{y_i,z}$ and $\mathcal{P}^{1,p}_{x_1,\ldots,z,\ldots,x_\nu}$ containing u, v and $u\#^o_\rho v$, given by $\exp_{\tilde u}$, $\exp_{\tilde v}$ and $\tilde u\#^o_\rho\tilde v$ for suitable asymptotically constant mappings $\tilde u$ and $\tilde v$. See, for example, Definition 2.47 in [50].

Moreover, by construction, $u\#^o_{\rho_n} v \rightharpoonup (u,v)$ converges geometrically as $\rho_n \to \infty$. Regarding the asymptotically constant mappings $\tilde u^l_\rho \in \mathcal{P}^{1,p}_{x_1,\ldots,x_\nu}$ and $\tilde v^r_\rho \in \mathcal{P}^{1,p}_{y_i,z}$ we immediately verify the strong convergence

$$\tilde u^l_\rho \to u \quad\text{and}\quad (\tilde v^r_\rho)_{2\rho} \to v \quad\text{as } \rho \to \infty,$$

that is

$$\tilde u^l_\rho = \exp_u \xi_\rho, \quad \xi_\rho \to 0 \quad\text{in } H^{1,p}_\Sigma(u^*TM)$$

and analogously for $\tilde v^r_\rho$. Using local charts and local trivializations for the vector bundles $H^{1,p}_\Sigma(\mathcal{P}^*TM)$ and $L^p_\Sigma(\mathcal{P}^*X^J)$ at u and v we deduce that we obtain continuous 1-parameter families of Fredholm operators

$$\rho \mapsto D_{\tilde u^l_\rho}\colon H^{1,p}_\Sigma(\tilde u^{l*}_\rho TM) \to L^p_\Sigma(X^J(\tilde u^{l*}_\rho TM))$$

and likewise $\rho \mapsto D_{\tilde v^r_\rho}$ with

$$\operatorname{ind} D_{\tilde u^l_\rho} = \operatorname{ind} D_u \quad\text{and}\quad \operatorname{ind} D_{\tilde v^r_\rho} = \operatorname{ind} D_v\,.$$

We point out that the linearization $D_{\tilde u^l_\rho}$ is defined in Section 2.4 as $DF_{\tilde u^l_\rho}(0)$. In explicit terms,

$$D_w = \Lambda(w)(\nabla + \operatorname{Tor}(dw,\cdot) + \nabla k)\,.$$

Proposition A.0.9 in the appendix states that $w \mapsto D_w$ is a smooth section in a homomorphism bundle over $\mathcal{P}^{1,p}_{x_1,\ldots,x_\nu}(\Sigma, M)$.

Moreover, the operators $D_{\tilde u^l_\rho}$ and $D_{\tilde v^r_\rho}$ are admissible asymptotically constant $\overline\partial$-operators, so that we can directly apply the linear gluing operation. This was considered in Chapter 2 in order to derive the index additivity. In view of Theorem 3.1.31 $D_{u\#^o_\rho v} = DF_{w_\chi}(0)$ is a Fredholm operator, and Theorem 3.2.12 immediately leads to the following result.

4.4.5 Proposition *The pregluing operation $\chi = (u,v,\rho) \mapsto w_\chi = u\#^o_\rho v$ is compatible with the index additivity*

$$\operatorname{ind} D_{w_\chi} = \operatorname{ind} D_u + \operatorname{ind} D_v\,.$$

The pregluing operation $\#^o$ induces a smooth Banach-space bundle H over the compact base space $K \times \bar K \times [\rho_o, \infty)$,

$$H = \bigcup_{\chi=(u,v,\rho)} \{\chi\} \times H^{1,p}_\Sigma(w^*_\chi TM)\,.$$

Our aim in view of the proof of Theorem 4.4.1 is to construct a smooth section γ of the bundle H such that

$$\overline\partial_{J,k(H)}\big(\exp_{w_\chi}\gamma(\chi)\big) = 0 \quad\text{for all } \chi \in K \times \bar K \times [\rho_o,\infty)\,.$$

This γ is the correction term between the pregluing $\#^o$ and the final gluing map $\#$ yielding solutions of $\overline\partial_{J,k(H)}w = 0$. The existence of γ depends on the compactness of K and $\bar K$ and the lower parameter bound ρ_o. We will prove that for ρ_o large enough depending on $K \times \bar{K}$, the existence of γ is guaranteed. However, we also need the uniqueness of γ because of the required embedding property of the gluing map. We finally are aiming at a suitable compactification of non-strongly compact solution spaces $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$. Since the dimension of the target component within the manifold $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b}$ will be

$$\dim\ker D_w = \dim\ker D_u + \dim\ker D_v,$$

that is, in general positive, the solution $w = \exp_{w_\chi}\gamma(\chi)$ of $\overline{\partial}_{J,k(H)}w = 0$ cannot be expected to be unique without imposing further conditions on $\gamma(\chi)$. Namely, we construct a smooth closed sub-bundle $H^\perp \subset H$ with fibre-wise

$$\operatorname{codim} H^\perp_{w_\chi} = \dim\ker D_u + \dim\ker D_v$$

such that the section γ lying in $H^\perp$ will be unique.

In order to construct this sub-bundle $H^\perp$ we define the following **linear pregluing operation**.

4.4.6 Definition *Given $\chi = (u, v, \rho) \in K \times \bar{K} \times [\rho_o, \infty)$ we define*

$$\hat{\#}_\chi\colon H^{1,p}_\Sigma(u^*TM) \times H^{1,p}(v^*TM) \to H^{1,p}_\Sigma(w^*_\chi TM),$$
$$(h,k) \mapsto h\hat{\#}_\chi k = \beta^-_{-\rho+1}h + \beta^+_{-\rho-1}k_{-2\rho},$$

that is, $(h\hat{\#}_\chi k)_i \in H^{1,p}((w_\chi)^*_i TM)$ is given by

$$(h\hat{\#}_\chi k)_i(s,t) = \begin{cases} h_i(s,t), & s \leqslant \rho - 2,\\ \beta^-(s-\rho+1)h_i(s,t), & \rho-2 < s < \rho-1,\\ 0, & |s-\rho| \leqslant 1,\\ \beta^+(s-\rho-1)k(s-2\rho,t), & \rho+1 < s < \rho+2,\\ k(s,t), & s \geqslant \rho+2\,. \end{cases}$$

Next, we have to provide further technical structures on the spaces in question. The problem is that the restriction to a sub-bundle $H^\perp$ is not canonically supplied. It seems that there is no natural, canonically given construction for a gluing operation. This is also related to the fact we cannot expect the gluing map to satisfy a canonical associativity rule like

$$(u\#v)\#w \stackrel{?}{=} u\#(v\#w)\,.$$

In fact, our construction does not respect this relation. In order to find a concrete sub-bundle $H^\perp$ we introduce the notion of orthogonality by means of an L^2-scalar product.

Moreover, we will also need explicit norms $\|\cdot\|_{1,p}$ and $\|\cdot\|_{0,p}$ on the vector spaces $H^{1,p}_\Sigma(w^*TM)$ and $L^p_\Sigma(w^*X^J)$ which were only provided with a unique Banach space topology up to now, whereas there is no intrinsic notion of a norm. Now, these norms are necessary in order to handle explicit estimates for the

Fredholm operators D_u, D_v and D_{w_χ}. In the following we denote by $\sigma \in \Omega^2(\Sigma)$ a fixed 2-form on Σ which is compatible with the conformal structure j so that $\sigma \circ (\mathrm{id} \times j)$ is a Riemannian metric on Σ. Before we define the $\|\cdot\|_{1,p}$-norm we choose a fixed connection ∇ on $TM \to \Sigma \times M$, say a J-Hermitian one, together with a finite partition of unity $(\alpha_i)_{i\in I}$ on Σ which is subordinate to a finite covering of Σ by conformal coordinates extending $(\psi_i)_{i=1,\dots,\nu}$. The given family $(\alpha_i)_{i\in I}$ allows us to construct a Riemannian metric on the homomorphism bundle $T^*\Sigma \otimes TM \to \Sigma \times M$, again denoted by $\langle\cdot,\cdot\rangle_J$. This metric is an extension of the intrinsic structure on $X^J(TM) = T^{0,1}\Sigma \otimes_J TM$ which was already considered in Section 3.1.1, i.e. for $\phi, \phi' \in T^{0,1}_z\Sigma \otimes_J T_mM$,

$$\langle\phi,\phi'\rangle_{\bar{J}} = \frac{\langle\phi v, \phi' v\rangle_J}{\sigma(v, jv)} \quad \text{for any } v \in T_z\Sigma\,.$$

We conclude this preliminaries with

4.4.7 Definition *Let $w \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$, $\mathcal{P}^{1,p}_{x_1,\dots,z,\dots,x_\nu}(\Sigma, M)$ or $\mathcal{P}^{1,p}_{y_i,z}(Z, M)$. Given $\xi \in L^p_\Sigma(w^*TM)$ and $\zeta \in L^q_\Sigma(w^*TM)$ with $p^{-1} + q^{-1} = 1$, we define the L^2_J-scalar product*

$$\langle\xi,\zeta\rangle_{L^2,J} = \int_\Sigma \langle\xi,\zeta\rangle_J\sigma,$$

and the fibre-wise norms $\|\cdot\|_{0,p}$ on $L^p_\Sigma(X^J(w^*TM))$ by

$$\|\phi\|^p_{0,p} = \int_\Sigma |\phi|^p_{\bar{J}}\sigma\,.$$

If $w \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) \cap C^\infty_\Sigma$ we define $\|\cdot\|_{1,p}$ on $H^{1,p}_\Sigma(w^*TM)$ by

$$\|\xi\|^p_{1,p} = \int_\Sigma \left(|\xi|^p_J + |\nabla\xi|^p_J\right)\sigma$$

where $|\nabla\xi|_J$ is defined according to the above choices of ∇ and $(\alpha_i)_{i\in I}$.

Due to the choice of σ and $(\alpha_i)_{i\in I}$ it is clear that $\|\cdot\|_{0,p}$ and $\|\cdot\|_{1,p}$ are compatible with the Banach space topologies on $H^{1,p}_\Sigma(w^*TM)$ and $L^p_\Sigma(w^*X^J)$. We will prove the following properties in the appendix.

4.4.8 Proposition *Given a fixed Banach manifold $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ it holds that*

$$\langle\cdot,\cdot\rangle_{L^2,J}\colon L^p_\Sigma(\mathcal{P}^*TM) \otimes L^q_\Sigma(\mathcal{P}^*TM) \to \mathbb{R}$$

is smooth and there is a unique extension of $\|\cdot\|_{1,p}$ over $\mathcal{P}$ such that

$$\|\cdot\|_{1,p}\colon H^{1,p}_\Sigma(\mathcal{P}^*TM) \to \mathbb{R}, \quad \|\cdot\|_{0,p}\colon L^p_\Sigma(\mathcal{P}^*X^J) \to \mathbb{R}$$

are continuous. Moreover, $\|\cdot\|_{1,p}$ and $\|\cdot\|_{0,p}$ are Finsler structures. That is, given a fixed $u_o \in \mathcal{P}$ and a local trivialization

$$P\colon U(u_o) \times \mathcal{E}_{u_o} \xrightarrow{\cong} \mathcal{E}_U$$

of the bundles $\mathcal{E} = H^{1,p}_\Sigma(\mathcal{P}^*TM)$ *and* $L^p_\Sigma(\mathcal{P}^*X^J)$ *over a neighbourhood* $U(u_o)$, *the norms* $\|\cdot\| = \|\cdot\|_{1,p}$, $\|\cdot\|_{0,p}$ *satisfy for every* $\epsilon > 0$

$$\sup_{v\in\mathcal{E}_{u_o}\setminus\{0\}} \left\{ \frac{\|v\|_u}{\|v\|_{u_o}}, \frac{\|v\|_{u_o}}{\|v\|_u} \right\} < 1+\epsilon$$

if u *is sufficiently close to* u_o. *Here,* $\|\cdot\|_u = \|P\cdot\|$ *denotes the induced norm on the standard fibre* $\mathcal{E}_{u_o}$.

Before we construct the subbundle $H^\perp$ of H over $K\times\bar K\times[\rho_o,\infty)$ in which we will localize the unique correction section γ, we prepare this step by providing the necessary uniform estimates of the Fredholm operators $D_{\tilde u^l_\rho}$ and $D_{\tilde v^r_\rho}$ over the unglued parts of the mappings from (4.37).

4.4.9 Proposition *Let* $K^l_\rho = \{\, \tilde u^l_\rho \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M) \,|\, u\in K\,\}$ *for* $\rho \geqslant \rho_o$. *Then, for every* $w \in K^l_{[\rho_o,\infty]} = \bigcup_{\rho\geqslant\rho_o} K^l_\rho \cup K$ *the vector space*

$$\mathring{H}^\perp_w = \{\, \xi\in H^{1,p}_\Sigma(w^*TM) \,|\, \langle\xi,\zeta\rangle_{L^2,J} = 0 \text{ f.a. } \zeta\in\ker D_w \,\}$$

is a well-defined closed subspace of $H^{1,p}_\Sigma(w^*TM)$, *there exist* $\rho_o(K)>0$ *and a constant* $c(K)>0$ *such that*

$$\|\xi\|_{1,p} \leqslant c(K)\,\|D_w\xi\|_{0,p}$$

holds for all $\xi\in\mathring{H}^\perp_w$, $w\in K_{[\rho_o,\infty]}$. *The same result holds for the cylindrical maps* $\bar K^r_\rho = \{\, (\tilde v^r_\rho)_{2\rho} \in \mathcal{P}^{1,p}_{y_i,z}(Z,M) \,\big|\, v\in\bar K \,\}$.

Proof. Let us consider the smooth map

$$[\rho_o,\infty)\times K \to \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M),$$
$$(\rho,u)\mapsto \tilde u^i_\rho = \exp_{y_i}\left(\beta^-_{-\rho}(\exp^{-1}_{y_i} u)\right).$$

We use the short hand notations $H = H^{1,p}_\Sigma(\mathcal{P}^*TM)$ and $L = L^p_\Sigma(\mathcal{P}^*X^J)$ for the bundles over $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M)$. The above mapping $(u,\rho)\to\tilde u^i_\rho$ gives rise to the continuous map

$$\Gamma\colon [0,\tfrac{1}{\rho_o}]\times K \to \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M),$$
$$(\tau,u)\mapsto \tilde u_{\frac{1}{\tau}},$$

and the continuous pull-back bundles $\Gamma^*H\to[0,\frac{1}{\rho_o}]$, analogously Γ^*L. Moreover,

$$D\colon \Gamma^*H\to\Gamma^*L, \quad D(\tau,u)\cdot v = D_{\tilde u_{\frac{1}{\tau}}}\cdot v$$

is a continuous bundle homomorphism, because $D\colon H\to L$ is a smooth bundle homomorphism according to Proposition A.0.9. The Finsler structures $\|\cdot\|_{1,p}$ and $\|\cdot\|_{0,p}$ on H and L imply the continuity of the associated operator norm

$(\tau,u)\mapsto\|D(\tau,u)\|_{\mathcal{L}(H,L)}$. For $\rho_o(K)$ large enough all $D(\tau,u)$ are onto and we obtain the closed subbundle

$$\Gamma^*\ker D\to[0,\rho_o^{-1}]\times K$$

which has constant and finite fibre dimension, namely $\operatorname{rank}\Gamma^*\ker D = \operatorname{ind} D_{|K}$, because K was assumed to be connected. Due to Corollary 4.1.8 we know that $\ker D_w\subset L^q_\Sigma(w^*TM)$ for all $w\in K_{[\rho_o,\infty]}$, with $q^{-1} = 1-p^{-1}$. Hence the scalar product $\langle\cdot,\cdot\rangle_{L^2,J}$ is well-defined and continuous on $\Gamma^*H\otimes\Gamma^*\ker D$. This implies the continuous bundle splitting

$$\Gamma^*H = \Gamma^*\mathring{H}^\perp\oplus\Gamma^*\ker D,$$

that is, we obtain the closed subbundle

$$\Gamma^*\mathring{H}^\perp \cong (\mathring{H}^\perp\to K_{[\rho_o,\infty]}).$$

Moreover, the continuous family of Fredholm operators $D(\tau,u)$ gives rise to the bundle isomorphism

$$D_{|\mathring{H}^\perp}\colon \Gamma^*\mathring{H}^\perp \xrightarrow{\cong} \Gamma^*L.$$

Setting $G^K = (D_{|\mathring{H}^\perp})^{-1}$ we obtain the continuous family of operator norms

$$c(\tau,u) = \|G^K(\tau,u)\|_{\mathcal{L}(L;\mathring{H}^\perp)}, \quad (\tau,u)\in[0,\rho_o^{-1}]\times K.$$

Hence, we have the maximum

$$c(K) = \max_{w\in K_{[\rho_o,\infty]}} \|G^K\|.$$

■

We now are prepared to construct the appropriate 'normal' bundle $H^\perp$ over the set of approximative preglued solutions $K\times\bar K\times[\rho_o,\infty)$. Based on the last result we prove the crucial estimate for the pregluing operation. In view of the linear pregluing operation $\widehat{\#}_\chi$ from Definition 4.4.6 we obtain the following result.

4.4.10 Theorem *Given* $\chi\in K\times\bar K\times[\rho_o,\infty)$ *the vector space*

$$H^\perp_\chi = \{\, \xi\in H^{1,p}_\Sigma(w^*_\chi TM) \,|\, \langle\xi,\zeta\rangle_{L^2,J} = 0 \quad \text{f.a. } \zeta\in\ker D_u\widehat{\#}_\rho\ker D_v \,\}$$

is a well-defined closed subspace of $H_\chi = H^{1,p}_\Sigma(w^*_\chi TM)$, *so that*

$$H^\perp\to K\times\bar K\times[\rho_o,\infty)$$

is a closed smooth subbundle of H. *There exists a gluing parameter* $\rho_{K,\bar K}\geqslant\rho_o$ *and a constant* $c_{K,\bar K}>0$ *such that*

$$\|\xi\|_{1,p}\leqslant c_{K,\bar K}\|D_{w_\chi}\xi\|_{0,p}$$

for all $\xi\in H^\perp_\chi$, $\chi\in K\times\bar K\times[\rho_{K,\bar K},\infty)$.

This theorem is the core of the gluing construction. It provides us with a uniform right inverse of the linear operators D_{w_χ} over the preglued broken solutions. The proof uses the uniform estimate we have proven in the last proposition for each half of the preglued pair (u,v). It essentially remains to verify that these estimates for $\tilde{u}^l_\rho$ and $\tilde{v}^r_\rho$ can be combined. The idea of this combination has already been used in the linear gluing section within the Fredholm chapter.

PROOF. First, we point out that, analogously to Proposition 4.4.8, the L^2_J-scalar product is well-defined for pairs

$$(\xi,\zeta) \in H^{1,p}_\Sigma(w_\chi^* TM) \otimes (\ker D_u \widehat{\#}_\rho \ker D_v)$$

over $w_\chi = (u,v,\rho)$ because, asymptotically, ζ consists of elements in the kernels of D_u and D_v which have exponential decay due to Corollary 4.1.8. We use the shorthand notation for the smooth bundle R over $K \times \bar{K} \times [\rho_o,\infty)$,

$$R_\chi = \ker D_u \widehat{\#}_\rho \ker D_v \subset H^{1,p}_\Sigma(w_\chi^* TM)\,.$$

Since $\langle\cdot,\cdot\rangle_{L^2,J}$ is a smooth structure, we obtain the splitting

$$H = H^\perp \oplus R \to K \times \bar{K} \times [\rho_o,\infty)$$

into closed, smooth vector bundles where the rank of R is given by

$$\operatorname{rank} R = \operatorname{ind} D_{|K} + \operatorname{ind} D_{|\bar{K}},$$

because the linear pregluing is injective. This holds because, in view of Unique Continuation, cf. 4.2.13, a section $\xi \in \ker D_u$, $u \in K$ vanishes identically if it vanishes on an open subset. We recall that K and $\bar{K}$ were assumed to be path-connected. Besides, the smooth bundles R and $\ker D$ over $K \times \bar{K} \times [\rho_o,\infty)$,

$$\ker D = \bigcup_\chi \{\chi\} \times \ker D_{w_\chi} \subset H$$

are not identical but we will see below that they have the same rank.

Again, we prove the asserted estimate by contradiction. Let us assume that there is a sequence $(\chi_n) \subset K \times \bar{K} \times [\rho_o,\infty)$ for $n\to\infty$ such that $\rho_n \to \infty$ and without loss of generality in view of the compactness of K and $\bar{K}$

$$u_n \to u \quad\text{and}\quad v_n \to v,$$

and that there are $\xi_n \in H^\perp_{\chi_n}$ such that

$$\|\xi_n\|_{1,p} = 1 \quad\text{and}\quad \|D_{w_{\chi_n}}\xi_n\|_{0,p} \to 0 \tag{4.38}$$

as $n\to\infty$.

In order to be able to reduce the proof to the estimates over u_n and v_n given by Proposition 4.4.9, we verify that on compact intervals around the

gluing circle $\{\rho_n\} \times S^1 \subset Z^{\epsilon_i}$ the restricted $\|\cdot\|_{1,p}$-norm of ξ_n is negligible in the limit, that is, for all $r > 0$

$$\|\xi_{n|[\rho_n - r,\rho_n + r]}\|_{1,p} \to 0 \tag{4.39}$$

as $n\to\infty$. This means that ξ_n becomes a so-called multi-bump solution in the limit. We choose a cut-off function $\gamma \in C^\infty_o(\mathbb{R},[0,1])$ with

$$\operatorname{supp}\gamma \subset [-2,2] \quad\text{and}\quad \gamma_{|[-1,1]} \equiv 1,$$

and we define the rescaled versions of γ by

$$\gamma^r_n(s) = \gamma(\tfrac{s-\rho_n}{r})\,.$$

Hence we have

$$\operatorname{supp}\gamma^r_n \subset [\rho_n - 2r,\rho_n + 2r], \quad \gamma^r_{n|[\rho_n - r,\rho_n + r]} \equiv 1\,.$$

Considering the preglued mapping $w_{\chi_n}(s,t)$ on the cylindrical ends $\psi_i(Z^{\epsilon_i})$, we have $w_{\chi_n}(\rho_n,t) = y_i(t)$ for all $t \in S^1$, and setting $\bar{\rho}_n = \frac{1}{2}\rho_n$ implies

$$\left.\begin{aligned} w_{\chi_n}(\rho_n - \bar{\rho}_n,\cdot) &= (u_n)_i(\tfrac{\rho_n}{2},\cdot) \\ w_{\chi_n}(\rho_n + \bar{\rho}_n,\cdot) &= v_n(-\tfrac{\rho_n}{2},\cdot) \end{aligned}\right\} \overset{C^\infty(S^1)}{\longrightarrow} y_i\,.$$

Hence, by construction, the sequence

$$\big((w_{\chi_n})_{\rho_n}\big)_{|[-\bar{\rho}_n,\bar{\rho}_n]\times S^1} \overset{C^\infty_{\text{loc}}}{\longrightarrow} y_i$$

converges exponentially fast, i.e. of order $\mathrm{e}^{-\epsilon\rho_n}$. In particular, provided that ρ_n is large enough, $\zeta_n(s,t) \in T_{y_i(t)}M$ is well-defined for $|s| \leqslant \bar{\rho}_n$, $t \in S^1$, by

$$\zeta_n(s) = \phi_n(s)\xi_n(s+\rho_n),$$
$$\phi_n(s) = \nabla_2 \exp\big(\exp^{-1}_{y_i} w_{\chi_n}(s+\rho_n)\big)^{-1}\,.$$

Considering $\gamma^r(s) = \gamma(\frac{s}{r})$ for $r < \bar{\rho}_n$, the Finsler structures $\|\cdot\|_{1,p}$ and $\|\cdot\|_{0,p}$ lead to

$$\big|\|\gamma^r\zeta_n\|_{1,p} - \|\gamma^r_n\xi_n\|_{1,p}\big| \to 0$$

and

$$\big|\|\zeta_{n|[-\bar{\rho}_n,\bar{\rho}_n]}\|_{1,p} - \|\xi_{n|[\frac{1}{2}\rho_n,\frac{3}{2}\rho_n]}\|_{1,p}\big| \to 0\,.$$

Using now the isomorphism property for the translation invariant operator

$$D_{y_i}\colon H^{1,p}_\Sigma(y_i^* TM) \overset{\cong}{\longrightarrow} L^p(y_i^* X^J),$$

see Theorem 3.1.13, we obtain the estimate

$$\begin{aligned} \|\gamma^r\zeta_n\|_{1,p} &\leqslant c(y_i)\,\|D_{y_i}(\gamma^r\zeta_n)\|_{0,p} \\ &\leqslant \frac{\tilde{c}(y_i,\gamma)}{r}\,\|\zeta_{n|[-2r,2r]}\|_{1,p} + c(y_i)\,\|\gamma^r D_{y_i}\zeta_n\|_{0,p}\,. \end{aligned} \tag{4.40}$$

Considering the identity

$$\|D_{y_i}\zeta_n\|_{0,p}^{[-2r,2r]} = \|D_{y_i}(\phi_n(\xi_n)_{\rho_n})\|_{0,p}^{[-2r,2r]}$$

we verify that

$$\left|\|D_{y_i}\zeta_n\|_{0,p}^{[-2r,2r]} - \|D_{w_{\chi_n}}\xi_n\|_{0,p}^{[\rho_n-2r,\rho_n+2r]}\right| \to 0$$

for $n \to \infty$. Thus, the assumption (4.38) plugged in the right-hand side of (4.40) provides the estimate

$$\limsup_n \|\gamma^r\zeta_n\|_{1,p} = O(\tfrac{1}{r})$$

for all $r > 0$, hence (4.39) follows.

We now are prepared to reduce the proof of the estimate in the Theorem to Proposition 4.4.9. Recalling (4.37) in Definition 4.4.4, we consider

$$\begin{aligned} \xi_n^- &= \beta^-_{-\rho_n+1}\xi_n \in H^{1,p}_\Sigma(\tilde{u}_n), \quad & \tilde{u}_n &= u_n\#^o_{\rho_n}y_i, \\ \xi_n^+ &= \beta^+_{-\rho_n-1}\xi_n \in H^{1,p}_\Sigma(\tilde{v}_n), \quad & \tilde{v}_n &= y_i\#^o_{\rho_n}v_n, \end{aligned}$$

with $\tilde{u}_n \to u$ and $(\tilde{v}_n)_{2\rho_n} \to v$. Then, ξ_n^- satisfies

$$\langle \xi_n^-, h\rangle_{L^2,J} = \langle \xi_n, h\widehat{\#}_{\rho_n}0\rangle_{L^2,J} = 0$$

for all $h \in \ker D_{\tilde{u}_n}$ in view of the definition of $\widehat{\#}$. Hence, $\xi_n^- \in H^\perp_{\tilde{u}_n}$ and we can apply Proposition 4.4.9 which yields the estimate

$$\begin{aligned} \|\beta^-_{-\rho_n+1}\xi_n\|_{1,p} &= \|\xi_n^-\|_{1,p} \\ &\leqslant c(u)\,\|D_{\tilde{u}_n}\xi_n^-\|_{0,p} \\ &= c(u)\,\|D_{w_{\chi_n}}(\beta^-_{-\rho_n+1}\xi_n)\|_{0,p} \\ &\leqslant c(u,\beta^-)\,\|\xi_n\|_{1,p}^{[\rho_n-2,\rho_n]} + c(u)\,\|D_{w_{\chi_n}}\xi_n\|_{0,p}\,. \end{aligned}$$

Since the first term on the right-hand side converges to zero, due to (4.39) and the second term due to the assumption (4.38), it follows that

$$\|\beta^-_{-\rho_n+1}\xi_n\|_{1,p} \to 0$$

for $n \to \infty$. Analogously, we prove

$$\|\beta^+_{-\rho_n-1}\xi_n\|_{1,p} \to 0,$$

so that the combination with (4.39) gives rise to

$$\begin{aligned} \|\xi_n\|_{1,p} &\leqslant \left\|(1-\beta^-_{-\rho_n+1}-\beta^+_{-\rho_n-1})\xi_n\right\|_{1,p} \\ &\quad + \|\beta^-_{-\rho_n+1}\xi_n\|_{1,p} + \|\beta^+_{-\rho_n-1}\xi_n\|_{1,p} \\ &\to 0 \end{aligned}$$

contradicting the assumption that $\|\xi_n\|_{1,p} = 1$ for all $n \in \mathbb{N}$. ∎

We asserted in the above proof that the smooth bundles R and $\ker D$ over $K \times \bar{K} \times [\rho_o, \infty)$ have the same rank, namely

$$\begin{aligned} \dim R_\chi &= \dim\ker D_u + \dim\ker D_v \\ &= \dim\ker D_{w_\chi}\,. \end{aligned}$$

This will follow now from Proposition 4.4.5 which was obtained from the analysis of the linear gluing operation in Chapter 3. It helps us to deduce from Theorem 4.4.10 the existence of a right inverse associated to D_{w_χ}.

4.4.11 Corollary *The restriction of D to the smooth subbundle $H^\perp$*

$$D_{|H^\perp}\colon H^\perp \to L^p_\Sigma(\#^o(K\times\bar{K}\times[\rho_{K,\bar{K}},\infty))^*X^J)$$

is a smooth bundle isomorphism over $K\times\bar{K}\times[\rho_{K,\bar{K}},\infty)$, and its inverse

$$G = D^{-1}_{|H^\perp}\colon L^p_\Sigma \to H^\perp \subset H$$

is uniformly bounded with respect to the fibre-wise operator norm.

Proof. At first, Theorem 4.4.10 only implies that $(D_{w_\chi})_{|H^\perp_\chi}$ is injective with a uniform estimate and that it has a closed range. It remains to prove that it is surjective. We have the L^2_J-orthogonal splitting

$$H = H^\perp \perp_{L^2,J} R$$

and

$$H^\perp_\chi \cap \ker D_{w_\chi} = \{0\}$$

for all χ. Therefore, the L^2_J-orthogonal projection onto R in direction of $H^\perp$ yields an injection

$$\mathrm{Proj}^R_{L^2,J}\colon \ker D_{w_\chi} \to R_\chi\,.$$

Due to

$$\operatorname{ind} D_{w_\chi} = \operatorname{ind} D_u + \operatorname{ind} D_v$$

in view of Proposition 4.4.5 and the surjectivity of D_u and D_v, it follows that

$$\begin{aligned} \dim\ker D_u + \dim\ker D_v &\leqslant \dim\ker D_{w_\chi} \\ &\leqslant \dim R_\chi \\ &= \dim\ker D_u + \dim\ker D_v\,. \end{aligned}$$

Hence, it follows that $\mathrm{Proj}^R_{L^2,J}$ is an isomorphism, H splits into the smooth subbundles

$$H = H^\perp \oplus \ker D \to K\times\bar{K}\times[\rho_K,\infty)$$

and $D_{w_\chi|H^\perp_\chi}$ is an isomorphism for all χ. ∎

Let us sum up the results of this subsection about pregluing. Given compact path-connected subsets $K \subset \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J, k(H))$ and $\bar{K} \subset \widehat{\mathcal{M}}_{y_i,z}(J^{i+a}, H^{i+a})$, we obtain a smooth bundle homomorphism $D\colon H \to L$ over $K \times \bar{K} \times [\rho_o, \infty)$, where

$$H = \bigcup_\chi \{\chi\} \times H^{1,p}_\Sigma(w_\chi^* TM),$$
$$L = \bigcup_\chi \{\chi\} \times L^p_\Sigma(w_\chi^* X^J)\,.$$

There is a lower bound $\rho_{K,\bar{K}} > 0$ large enough such that D is fibre-wise surjective and the bundle H splits into closed smooth subbundles

$$H = H^\perp \oplus \ker D$$

so that

$$G = (D_{|H^\perp})^{-1}\colon L \to H$$

is a smooth right inverse. The bundles H and L are endowed with continuous Finsler structures $\|\cdot\|_{1,p}$ and $\|\cdot\|_{0,p}$ and G is uniformly bounded with respect to the induced operator-norm, that is, there is a $c_1 = c(K, \bar{K}) > 0$ such that

$$\|G\xi\|_{1,p} \leqslant c_1 \, \|\xi\|_{0,p}$$

for all $\xi \in L^p_\Sigma(w_\chi^* X^J)$, $\chi \in K \times \bar{K} \times [\rho_{K,\bar{K}}, \infty)$. In particular, the constant c_1 is independent of $\rho \in [\rho_{K,\bar{K}}, \infty)$.

4.4.2 The Contraction Mapping Principle

In the previous subsection we have constructed a suitable subbundle $H^\perp$ of H with $\operatorname{corank} H^\perp = \dim\ker D$, which plays the role of a normal bundle in which we intend to find a unique correction term $\gamma(\chi)$ satisfying

$$\overline{\partial}_{J,k}\big(\exp_{w_\chi} \gamma(\chi)\big) = 0\,.$$

In the first chapter we studied the linearization of $\overline{\partial}_{J,k}$ by means of special trivializations and local coordinates at smooth maps $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$. Using the Banach manifold analysis which is outlined in the appendix we can generalize this representation of $\overline{\partial}_{J,k}$ based on the J-Hermitian parallel translation P and the exponential map exp. We already generalized the notation $D_u = DF_u(0)$ for smooth maps which are no solutions of $\overline{\partial}_{J,k} u = 0$. In the appendix we prove the following result.

4.4.12 Theorem *The smooth section*

$$\overline{\partial}_{J,k}\colon \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) \to L^p_\Sigma(\mathcal{P}^* X^J)$$

associated to an admissible extension (J, k), the J-Hermitian parallel translation P from (2.1) in Section 2.2 and the exponential map exp give rise to a smooth bundle map defined on an open neighbourhood $\mathcal{D}$ of the zero section

$$F\colon H^{1,p}_\Sigma(\mathcal{P}^*\mathcal{D}) \to L^p_\Sigma(\mathcal{P}^* X^J),$$
$$F(u,\xi) = P(u,\xi)^{-1} \cdot \overline{\partial}_{J,k}(\exp_u \xi),$$

over the Banach manifold $\mathcal{P} = \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$. The fibre derivative at the zero section is given by

$$D_u\colon H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^* X^J),$$
$$D_u = \Lambda_J(u)(\nabla + \operatorname{Tor}(du,\cdot) + \nabla k)\,.$$

This is proved as Theorem A.0.8 in the appendix. We use the notation

$$D_u = DF_u(0)$$

and pull back this bundle map F over the space $K \times \bar{K} \times [\rho_{K,\bar{k}}, \infty)$ via the pregluing map.

4.4.13 Corollary *Given the nonlinear operator $\overline{\partial}_{J,k}$ as above, the parallel translation P and the exponential map exp induce a smooth bundle map over $K \times \bar{K} \times [\rho_{K,\bar{K}}, \infty)$, $f\colon \tilde{H} \to L$, $\tilde{H}_\chi = H^{1,p}_\Sigma(w_\chi^*\mathcal{D}) \subset H_\chi$,*

$$f(\xi) = P(w_\chi, \xi)^{-1} \cdot \overline{\partial}_{J,k}(\exp_{w_\chi} \xi)$$

with fibre derivative at χ, $Df_\chi(0) = D_{w_\chi}$ which possesses a uniformly bounded right inverse $G_\chi\colon L_\chi \to H_\chi$,

$$D_{w_\chi} \circ G_\chi = \mathrm{Id}_{L_\chi}, \quad \|G_\chi\|_{\mathcal{L}(L,H)} \leqslant c_1\,.$$

Moreover, for each $k \in \mathbb{N}$ there are constants $c_{o,k}, \epsilon_k > 0$ independent of $\chi \in K \times \bar{K} \times [\rho_{K,\bar{K}}, \infty)$ such that the k-th fibre derivatives $D^k f_\chi(\xi)\colon \oplus^k H_\chi \to L_\chi$, $\xi \in \tilde{H}_\chi$, are uniformly bounded as

$$\big\|D^k f_\chi(\xi)\big\|_{\mathcal{L}(\oplus^k H_\chi; L_\chi)} \leqslant c_{o,k}$$

for all $\|\xi\|_{H_\chi} < \epsilon_k$.

Proof. It remains to verify that the constants ϵ_k, $c_{o,k} > 0$ which are given for each χ by the smoothness of f_χ, are uniform with respect to χ. In view of the compactness of K, $\bar{K}$ and the smoothness of f, it suffices to consider the constants $\epsilon_k(\rho)$, $c_{o,k}(\rho)$ for $\rho \to \infty$. Since

$$w_\chi = u \#^o_\rho v \rightharpoonup (u, v)$$

converges geometrically as $\rho \to \infty$, we observe that the operator norms of $D^k f_\chi(\xi)$ depend on the limit solutions (u, v) which are localized in the compact sets K and $\bar{K}$ and the C^0-norm of $\xi\colon \Sigma \to TM$ as a map into the tangent bundle of M. Therefore, we find constants ϵ_k and $c_{o,k}$ only depending on K and $\bar{K}$, independent of ρ. ∎

The smoothness of $f\colon H \to L$ and the uniformly bounded right inverse allow us to prove the existence of a unique correction section

$$\gamma\colon K \times \bar{K} \times [\rho_{K,\bar{K}}, \infty) \to H^\perp,$$

such that we obtain correct solutions of

$$\overline{\partial}_{J,k}\big(\exp_{w_\chi} \gamma(\chi)\big) = 0\,.$$

4.4.14 Lemma (Contraction Lemma) *Let E, F be Banach spaces and $f \in C^3(E,F)$ with constants $c_o > 0$, $\epsilon_o > 0$ such that*

$$\|D^2 f(x)\|,\ \|D^3 f(x)\| < c_o$$

for all $|x| < \epsilon_o$, and let $G\colon F \to E$ be a bounded linear right inverse for $Df(0)$,

$$Df(0) \circ G = \mathrm{Id}_F, \quad \|G\|_{\mathcal{L}(F,E)} \leqslant c_1 .$$

Given $c_1 \geqslant \frac{1}{2}$ and $\epsilon = \min\left(\epsilon_o, (4c_0c_1)^{-1}\right)$, there exists a unique solution $x_o \in B_\epsilon(0) \cap G(F)$ of $f(x_o) = 0$ provided that $\left|f(0)\right| < \epsilon(2c_1)^{-1}$. This solution x_o then satisfies

$$|x_o| \leqslant 2c_1\left|f(0)\right|,$$

and

$$Df(x_o)_{|G(F)}\colon G(F) \xrightarrow{\cong} F$$

is an isomorphism.

Proof. Expressing $f\colon E \to F$ as

$$f(x) = f(0) + Df(0)\cdot x + N(x)$$

where the nonlinear part $N(x)$ can be written as

$$N(x) = \int_0^1 (1-s)D^2 f(sx)(x,x)ds,$$

we observe that, for the constants $\epsilon_o, c_o > 0$, $|x| < \epsilon_o$ implies $\left|N(x)\right| < \frac{c_o}{2}|x|^2$. Moreover, the assumption on $D^2 f$ and $D^3 f$ gives rise to an estimate

$$\left|N(x) - N(y)\right| \leqslant c_o\left(|x| + |y|\right)|x-y|$$

for all $|x|, |y| \leqslant \epsilon_o$ provided that $\epsilon_o > 0$ is not too large, for example $\epsilon \leqslant 6$. Hence, we have for $|x|, |y| < \epsilon_o$

$$\left|GN(x) - GN(y)\right| \leqslant c_oc_1\left(|x| + |y|\right)|x-y| . \tag{4.41}$$

We choose

$$\epsilon < \min\left(\epsilon_o, (4c_oc_1)^{-1}\right) \leqslant \frac{1}{2c_o}$$

and assume that $\left|f(0)\right| < \epsilon(2c_1)^{-1}$. Then the mapping

$$\begin{aligned} T\colon B_\epsilon(0) \cap G(F) &\to G(F), \\ Tx &= -Gf(0) - GN(x), \end{aligned}$$

maps $B_\epsilon(0) \cap G(F)$ into itself because

$$|Tx| \leqslant \left|Gf(0)\right| + \frac{c_oc_1}{2}|x|^2 \leqslant \epsilon$$

due to the assumptions. Moreover, the estimate (4.41) implies that

$$|Tx - Ty| \leqslant \frac{1}{2}|x-y|$$

for all $x, y \in B_\epsilon(0) \cap G(F)$, and $G(F)$ is closed because G is a right inverse of $Df(0)$. Hence, we obtain a unique fixed point $Tx_o = x_o$, that is, there exists a unique solution of

$$x_o \in B_\epsilon(0) \cap G(F), \quad f(x_o) = 0 .$$

This solution x_o satisfies

$$|x_o| \leqslant 2\left|Gf(0)\right| \leqslant 2c_1\left|f(0)\right| . \tag{4.42}$$

Considering now

$$A = Df(x_o) \circ G\colon F \to F,$$

we observe that $Df(x_o)_{|G(F)}$ is an isomorphism onto F if and only if A has this property. Thus, we have to verify that $\|A - \mathrm{Id}\,\| < 1$. But this follows from

$$\begin{aligned} \|A - \mathrm{Id}\,\| &= \left\|\left(Df(x_o) - Df(0)\right) \circ G\right\| \\ &\leqslant c_oc_1|x_o| \\ &\leqslant c_oc_1\epsilon \leqslant \frac{1}{4} . \end{aligned}$$

■

The last argument in the proof of the Contraction Lemma immediately implies the following consequence.

4.4.15 Corollary *If, in addition to the assumptions of Lemma 4.4.14, $Df(0)$ is Fredholm and $\left|f(0)\right| < \epsilon(2c_1)^{-1}$, then the surjective operator $Df(x_o)$ associated to the fixed point x_o is also Fredholm with the same index,*

$$\operatorname{ind} Df(x_o) = \operatorname{ind} Df(0) .$$

Proof. We replace $Df(x_o)$ by $Df(tx_o)$, $t \in [0,1]$, in the last argument of the proof of Lemma 4.4.14. ■

We now apply this local contraction lemma to the framework of a smooth bundle map, that is, the Banach spaces E, F and the map f vary smoothly in dependence of a variable in a manifold.

4.4.16 Proposition *Let $E, F \to B$ be smooth Banach space bundles with continuous fibre-wise norms over a smooth base manifold B, and let $f\colon E \to F$ be a smooth bundle map endowed with constants $\epsilon_o, c_o > 0$ such that the fibre derivatives $D^k f_b(x) \in \mathcal{L}(\oplus^k E_b; F_b)$ for $k = 2, 3$ satisfy*

$$\left\|D^2 f_b(x)\right\|,\ \left\|D^3 f_b(x)\right\| < c_o$$

for all $x \in E_b$, $b \in B$ *with* $|x|_E < \epsilon_o$. *Given a smooth right inverse bundle homomorphism* $G\colon F \to E$ *over* B *and a uniform constant* $c_1 \geqslant \frac{1}{2}$ *such that*

$$\|G_b\| \leqslant c_1, \quad Df_b(0) \circ G_b = \mathrm{Id}_b,$$

for all $b \in B$, *there exists a unique section* $\gamma\colon B \to E$ *satisfying*

$$f \circ \gamma = 0, \quad \gamma(b) \in G_b(F_b) \quad \text{and} \quad \big|\gamma(b)\big|_E < \epsilon$$

for all $b \in B$, *provided that*

$$\big|f_b(0)\big|_F < \frac{\epsilon}{2c_1}$$

for all $b \in B$, *where* $\epsilon = \min\big(\epsilon_o, (4c_oc_1)^{-1}\big)$. *Moreover, this section* γ *in* E *is smooth and satisfies*

$$\big|\gamma(b)\big|_E \leqslant 2c_1\big|f_b(0)\big|_F$$

for all $b \in B$.

Proof. The smooth right inverse G provides us with a splitting of E into smooth closed subbundles

$$E = \ker Df(0) \oplus G(F)\,.$$

Due to the Contraction Lemma applied to each fibre E_b, $b \in B$, we obtain the existence and uniqueness result for the section

$$\gamma\colon B \to G(F)$$

as a map such that all properties except the smoothness are guaranteed. The smoothness, in particular the continuity, is proven by means of the Implicit Function Theorem. We use that, in view of the Contraction Lemma, the fibre derivative restricted to $G(F)$,

$$Df_b(\gamma(b))_{|G(F_b)}\colon G(F_b) \xrightarrow{\cong} F_b \tag{4.43}$$

at a solution of $f(\gamma(b)) = 0$ is an isomorphism. Let us fix a $b_o \in B$ and choose a local trivialization of $G(F)$ and F over a neighbourhood $U(b_o)$. Now

$$f\colon U \times X \to Y$$

is a smooth map for suitable Banach spaces, $\gamma\colon U \to X$ is already given as a map solving

$$f(\gamma(b)) = 0$$

for all $b \in U(b_o)$ and it holds that

$$D_2 f(b_o, \gamma(b_o))\colon X \xrightarrow{\cong} Y$$

is an isomorphism. Then, the smoothness of γ follows from the Implicit Function Theorem. ∎

The latter result can now directly be applied to the smooth bundle map $f\colon H \to L$ from Corollary 4.4.13. The crucial point is that we obtain uniform constants ϵ_o, c_o, $c_1 > 0$ where ϵ_o is uniformly bounded from zero and c_o, c_1 from above. Thus, it remains to verify that the condition

$$\big\|f_\chi(0)\big\|_{1,p} < \frac{\epsilon}{2c_1}$$

is satisfied. This follows from the construction of the pregluing operation. Computing for $\chi = (u, v, \rho)$

$$f_\chi(0) = \overline{\partial}_{J,k}(w_\chi)$$

over the cylindrical end $\psi_i(Z^{\epsilon_i})$ amounts to

$$f_\chi(0)_i(s,t) = \begin{cases} 0, & s \leqslant \rho - 1, \\ \overline{\partial}_{J,k}(\tilde{u}^l_\rho)_i(s,t), & \rho - 1 < s \leqslant \rho, \\ \overline{\partial}_{J^{i+a},H^{i+a}}(\tilde{v}^r_\rho)(s,t), & \rho < s \leqslant \rho + 1, \\ 0, & \rho + 1 < s\,. \end{cases} \tag{4.44}$$

Since u and v converge exponentially fast towards y_i, (4.44) yields the estimate

$$\big\|f_\chi(0)\big\|_{1,p} \leqslant c(u,v)\,\mathrm{e}^{-\delta\rho} \tag{4.45}$$

with $\delta = \delta(y_i)$.

4.4.17 Definition *Given the compact sets* K *and* $\bar{K}$ *as above, we choose* $\rho_o = \rho_o(K, \bar{K}) \geqslant \rho_{K,\bar{K}}$ *as provided by Theorem 4.4.10 large enough such that (4.45) provides the condition for Proposition 4.4.16. Then we define the map*

$$\begin{gathered} \#\colon K \times \bar{K} \times [\rho_o, \infty) \to \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b}(J, k(H))\,, \\ u \#_\rho v = \exp_{w_\chi} \gamma(\chi), \end{gathered}$$

with $\chi = (u, v, \rho)$ *and* $w_\chi = u \#^o_\rho v$, *where*

$$\gamma\colon K \times \bar{K} \times [\rho_o, \infty) \to H^\perp$$

is the smooth section provided by Proposition 4.4.16 and Corollary 4.4.13.

We now have to verify all properties of $\#$ claimed in Theorem 4.4.1. We have already deduced the smoothness of $\#$ from Proposition 4.4.16. Next, let $\chi_n = (u_n, \hat{v}_n, \rho_n) \subset K \times \bar{K} \times [\rho_o, \infty)$ be a sequence with

$$u_n \to u, \quad \hat{v}_n \to \hat{v}, \quad \rho_n \to \infty\,.$$

Consequently, (4.45) holds for (χ_n) with constants $c = c(u, \hat{v})$ and $\delta = \delta(y_i)$ independent of $n \in \mathbb{N}$, and Proposition 4.4.16 yields

$$\big\|\gamma(\chi_n)\big\|_{1,p} \leqslant 2c_1 c(u,\hat{v})\,\mathrm{e}^{-\delta\rho_n}\,.$$

Hence, we obtain the asymptotic continuity property

$$u_n \#_{\rho_n} \hat{v}_n \rightharpoonup (u, \hat{v}),$$

because the pregluing $\#^o$ satisfies the geometrical convergence property by construction.

It remains to verify the reverse result that every geometrically convergent sequence finally lies in the range of $\#$ and that the embedding result holds. Let us first prove the local diffeomorphism property.

4.4.18 Lemma *Given (u^o, v^o) in the interior of $K \times \bar{K}$ and $\rho^o > \rho_o(K, \bar{K})$ large enough, there exist open neighbourhoods $U(u^o, v^o) \subset K^o \times \bar{K}^o$, $V(0) \subset H^{1,p}_\Sigma(w^*_{\chi^o}TM)$ and constants $\delta, \epsilon > 0$ such that*

$$\begin{aligned} \Phi\colon U(u^o, v^o) \times (\rho^o - \delta, \rho^o + \delta) \times B^\perp_{U,\rho^o,\delta}(\epsilon) &\to V(0), \\ ((u, v, \rho), \xi) &\mapsto \exp^{-1}_{w_{\chi^o}}(\exp_{w_\chi} \xi), \end{aligned}$$

with

$$B^\perp_{U,\rho^o,\delta}(\epsilon) = \{\, \xi \in H^\perp_\chi \mid \chi \in U(u^o, v^o) \times (\rho^o - \delta, \rho^o + \delta),\ \|\xi\|_{1,p} < \epsilon \,\}$$

is a diffeomorphism. Moreover, there exists an $r_o = r_o(u^o, v^o) > 0$ such that

$$B_{\chi^o}(r_o) = \{\, \xi \in H^{1,p}_\Sigma(w^*_{\chi^o}TM) \mid \|\xi\|_{1,p} < r_o \,\} \subset V(0),$$

r_o and δ as well as the neighbourhood $U(u^o, v^o)$ are independent of ρ_o for ρ_o large enough.

Proof. We compute the differential of Φ at $(\chi^o, 0) = (u^o, v^o, \rho^o, 0)$

$$D\Phi(\chi^o, 0)\colon T_{u^o}\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b} \times T_{v^o}\widehat{\mathcal{M}}_{y_i,z} \times \mathbb{R} \times H^\perp_{\chi^o} \to H^{1,p}_\Sigma(w^*_{\chi^o}TM),$$

in order to prove the local diffeomorphism property by means of the Inverse Function Theorem. Recalling the pregluing map

$$\begin{aligned} \#^o\colon K \times \bar{K} \times [\rho_o, \infty) &\to \mathcal{M}, \\ \chi &\mapsto w_\chi, \end{aligned}$$

from Definition 4.4.4 on page 139, we obtain

$$\begin{aligned} &D\Phi(\chi, \xi)(h, k, \tau, \zeta) \\ =&\nabla_2 \exp(\exp^{-1}_{w_{\chi^o}} w_\chi)^{-1}\Big[\nabla_1 \exp(\xi) \circ D\#^o(\chi)(h, k, \tau) + \nabla_2 \exp(\xi)\cdot\zeta\Big] \end{aligned}$$

and hence, for $(\chi, \xi) = (\chi^o, 0)$,

$$D\Phi(\chi^o, 0) = D\#^o(\chi^o) + \mathrm{Id}_{H^\perp_{\chi^o}}\,.$$

The invertibility of $D\Phi(\chi^o, 0)$ now follows from the fact that, as $\rho_o \to \infty$, the differential $D\#^o(\chi^o)$ can be treated like the linear pregluing map $\widehat{\#}_\chi$. First, we observe that, due to the construction of $\widehat{\mathcal{M}}_{y_i,z} = \mathcal{M}_{y_i,z}/\mathbb{R}$, the linear map

$$\begin{aligned} T_v\widehat{\mathcal{M}} \oplus \mathbb{R} &\xrightarrow{\cong} T_v\mathcal{M} = \ker D_v, \\ (k, \tau) &\mapsto k - 2\tau(\partial_s v), \end{aligned}$$

is an isomorphism and we compute with $T_u\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b} = \ker D_u$

$$D\#^o(\chi^o)(h, k, \tau) - \widehat{\#}_{\chi^o}(h, k - 2\tau(\partial_s v)) = a_{\rho^o}(h, k, \tau)$$

where $a_\rho(h, k, \tau)$ has the form

$$\begin{aligned} a_\rho\colon H^{1,p}_\Sigma(u^*TM) \times H^{1,p}(v^*TM) \times \mathbb{R} &\to H^{1,p}_\Sigma(w^*_\chi TM), \\ a_\rho(x, y, \tau) &= A(\chi)\cdot x + B(\chi)\cdot(y - 2\tau\partial_s v)_{-2\rho} + \tau C(\chi), \end{aligned}$$

with

$$\begin{aligned} A(\chi) &= (1 - \beta^-_{-\rho+1})\beta^-_{-\rho}\nabla_2 \exp(\beta^-_{-\rho}\exp^{-1}_{y_i} u_i) \circ \nabla_2 \exp(\exp^{-1}_{y_i} u_i)^{-1}, \\ B(\chi) &= \beta^+_{-\rho}(1 - \beta^+_{-\rho-1})\nabla_2 \exp\left(\beta^+_{-\rho}\exp^{-1}_{y_i}(v_{-2\rho})\right) \circ \nabla_2 \exp\left(\exp^{-1}_{y_i}(v_{-2\rho})\right)^{-1}, \\ C(\chi) &= -\partial_s\beta^-_{-\rho}\exp^{-1}_{y_i} u_i - \partial_s\beta^+_{-\rho}\exp^{-1}_{y_i}(v_{-2\rho})\,. \end{aligned}$$

Hence, for ρ^o large enough, it follows analogously to Theorem 4.4.10 and Corollary 4.4.11 that the L^2_J-orthogonal projection in direction of $H^\perp_{\chi^o}$ yields an isomorphism

$$\mathrm{Proj}^R_{L^2,J}\colon \ker D_{w_{\chi^o}} \xrightarrow{\cong} R(D\#^o(\chi^o))$$

and $D\#^o(\chi^o)$ is injective. Therefore, the differential $D\Phi(\chi^o, 0)$ is invertible. This invertibility holds for all $(\chi, \xi) \in W(0) \subset H^\perp_{U(\chi^o)}$ where $W(0)$ is an open neighbourhood of the zero section over an open neighbourhood

$$U(\chi^o) = U(u^o, v^o) \times (\rho^o - \delta, \rho^o + \delta) \subset K \times \bar{K} \times [\rho_o, \infty)\,.$$

We observe that all involved estimates and neighbourhoods, that is $U(u^o, v^o)$ and δ, are independent of ρ if $\rho \geqslant \rho_o$ is large enough. We can apply the Inverse Function Theorem, so that for $W(0)$ small enough, Φ turns out to be a local diffeomorphism

$$\Phi\colon H^\perp_{|U(\chi_o)} \supset W(0) \xrightarrow{\cong} V(0) \subset H^{1,p}_\Sigma(w^*_{\chi^o}TM),$$

where $V(0)$ contains a ball $\{\, \xi \in H^{1,p}_\Sigma(w^*_{\chi^o}TM \mid \|\xi\|_{1,p} < r_o \,\}$ of radius $r_o > 0$ independent of ρ^o large enough. Moreover, $W(0) \subset H^\perp_{U(\chi^o)}$ is uniformly bounded away from the zero section. It contains a $B^\perp_{U,\rho^o,\delta}(\epsilon)$ for some $\epsilon > 0$. ■

We now are able to prove the local diffeomorphism property of the gluing operation $\#$ and the asymptotic transitivity.

4.4.19 Proposition *Given the compact subsets K and $\bar{K}$ with non-void interior as above, there exists a lower parameter bound $\rho_o = \rho_o(K, \bar{K})$ such that the gluing operation*

$$\#\colon K \times \bar{K} \times [\rho_o, \infty) \to \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b}$$

is a local diffeomorphism. Moreover, every geometrically convergent sequence $(w_n) \subset \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b}$

$$w_n \rightharpoonup (u, \hat{v}) \subset K^o \times \bar{K}^o$$

finally lies in the image of $\#$.

PROOF. The first statement follows directly from Lemma 4.4.18, because the correction section γ yields an embedding

$$\tilde{\gamma}\colon K \times \bar{K} \times [\rho_o, \infty) \hookrightarrow H^{\perp}, \qquad \chi \mapsto (\chi, \gamma(\chi)),$$

so that $\# = \exp_{\#^o}(\Phi \circ \tilde{\gamma})$ is a local diffeomorphism on a neighbourhood $U(\chi^o)$ for every χ^o. Next, since (w_n) converges geometrically to the broken solution $(u, \hat{v})$, we can represent it for $n \geqslant n_o$ large enough as

$$w_n = \exp_{w_{\bar{\chi}_n}} \xi_n^o, \quad \xi_n^o \in H^{1,p}_{\Sigma}(w^*_{\bar{\chi}_n} TM)$$

with $w_{\bar{\chi}_n} = u \#^o_{\bar{\rho}_n} v$ for a suitable sequence $\bar{\rho}_n \to \infty$. In view of the geometrical convergence $w_{\bar{\chi}_n} \rightharpoonup (u, \hat{v})$ we have $\|\xi_n^o\|_{1,p} \to 0$ for $n \to \infty$, in particular

$$\|\xi_n^o\|_{1,p} < r_o \quad \text{for all } n \geqslant n_o(r_o)\,.$$

Hence, Lemma 4.4.18 implies that

$$\xi_n = \Phi^{-1} \xi_n^o \in H^{\perp}_{\chi_n}$$

for all $n \geqslant n_o$ for sequences $\chi_n = (u_n, v_n, \rho_n)$ with

$$u_n \to u, \quad v_n \to \hat{v}, \quad \rho_n \to \infty\,.$$

It holds

$$\xi_n \in H^{\perp}_{\chi_n}, \quad \|\xi_n\|_{1,p} \to 0, \quad w_n = \exp_{w_{\chi_n}} \xi_n\,.$$

We find an $n_o \in \mathbb{N}$ such that $\|\xi_n\|_{1,p}$ is small enough for all $n \geqslant n_o$ so that w_n equals the uniquely determined solution

$$u_n \#_{\rho_n} v_n = \exp_{w_{\chi_n}} \xi_n = w_n\,. \quad \blacksquare$$

Let us now finish the proof of the main gluing theorem. The remaining part which we have to verify is the embedding property. This has largely be proven by the last proposition. The crucial point is the fact that the open neighbourhoods which are diffeomorphically mapped onto each other contain balls of radii which are independent of the gluing parameter ρ. The essential estimates are stable in the geometric limit.

4.4.20 Corollary *The gluing operation*

$$\#\colon K \times \bar{K} \times [\rho_o, \infty) \to \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b}$$

is an embedding.

PROOF. The local diffeomorphism property has already be proven. It remains to verify the injectivity of the whole map $\#$. Denoting $V_R = K \times \bar{K} \times [\rho_o, R]$ for $R > \rho_o$ we consider the following sets of singular points,

$$X_R = \{\, \chi \in V_R \,|\, \text{ex. } \chi' \in V_R,\ \chi' \neq \chi \text{ with } \#(\chi') = \#(\chi) \,\}\,.$$

Thus, we have to prove that all X_R, $R > \rho_o$, are empty. The local diffeomorphism property immediately implies that X_R is relatively open within V_R. Assuming $(\chi_n) \subset X_R$ with $\chi_n \to \chi_o$ for $n \to \infty$, we find $(\chi_n') \subset X_R$ such that due to the compactness of V_R, after choosing a subsequence,

$$\chi_n' \to \chi_o' \in V_R \quad \text{and} \quad \#(\chi_n') = \#(\chi_n)$$

for all $n \in \mathbb{N}$. Hence, $\#(\chi_o') = \#(\chi_o)$, and this implies that $\chi_o \in X_R$, that is, X_R is closed. Since we assumed K and $\bar{K}$ to be path-connected, this leads to the alternative

$$X_R = \emptyset \quad \text{or} \quad X_R = V_R\,.$$

Let us assume that $X_R = V_R$ for some $R > \rho_o$. Then $\bigcup_{R > \rho_o} X_R = K \times \bar{K} \times [\rho_o, \infty)$ and we can pick $\chi_n = (u, \hat{v}, \rho_n)$ with $\rho_n \to \infty$ so that

$$\#(\chi_n') = \#(\chi_n) \rightharpoonup (u, \hat{v})$$

for some $\chi_n' = (u_n', \hat{v}_n', \rho_n') \neq \chi_n$, with with

$$u_n' \to u, \quad \hat{v}_n' \to \hat{v}, \quad \rho_n' \to \infty\,.$$

We recall that in the beginning of the section we identified component-wise $\widehat{\mathcal{M}}_{y_i,z}$ as a subset of $\mathcal{M}_{y_i,z}$ by fixing the shifting invariance. It follows for (χ_n') that $|\rho_n - \rho_n'| \to 0$. After eventually enlarging K and $\bar{K}$ to relatively compact, open neighbourhoods, Lemma 4.4.18 provides us with local diffeomorphisms on balls around u and $\hat{v}$ with radii independent of $\rho_n' \to \infty$. Therefore, for ρ_n, ρ_n' large enough, $\#(\chi_n') = \#(\chi_n)$ implies $\chi_n' = \chi_n$ contradicting $(\chi_n) \subset \bigcup_R X_R$. Hence, X_R is empty for all $R > \rho_o$, that is, $\#$ is injective. $\blacksquare$

The last Corollary 4.4.20 completes the proof of the main gluing theorem, Theorem 4.4.1. In view of Theorem 4.3.21, this gluing theorem provides us with an explicit, smooth compactification of the spherical solution sets $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b} \cap \mathcal{M}_{\mathcal{S}}$. This plays a crucial role in the case of dimension 1, as will be seen in the next chapter. Before applying the central results about the solution spaces $\mathcal{M}_{\mathcal{S}}$ to the homology theory we analyze further extensions of the gluing operation.

4.4.3 Gluing for General Model Surfaces

We already considered a gluing operation for general model surfaces Σ^1, Σ^2 in the Fredholm chapter. In order to prove additivity for the index we defined gluing of model surfaces

$$\Sigma^{1,2}_{(\alpha,R)} = \Sigma^1 \#_{(\alpha,R)} \Sigma^2$$

in Definition 3.2.1. We recall that the gluing data (α, R) consists of a collection

$$(\alpha, R) = ((i_1, j_1), \dots, (i_r, j_r), (\rho_1, \dots, \rho_r))$$

of indices of cylindrical ends $\psi^1_{i_k}(Z^{\epsilon^1_{i_k}}) \subset \Sigma^1$ and $\psi^2_{j_k}(Z^{\epsilon^2_{j_k}}) \subset \Sigma^2$ with opposite orientations $\epsilon^1_{i_k} = -\epsilon^2_{j_k}$, $k = 1, \dots, r$, and gluing parameters $\rho_1, \dots, \rho_r \geqslant 0$.

The resulting surface $\Sigma^{1,2}_{(\alpha,R)}$ is canonically supplied with compact cylindrical coordinates

$$\psi_k \colon [-\rho_k, \rho_k] \times S^1 \hookrightarrow \Sigma^1 \#_{(\alpha,R)} \Sigma^2$$

for $k = 1, \dots, r$, see Definition 3.2.5. We equivalently use the notation $\Sigma^{1,2}_{(\alpha,R)} = \Sigma^1 \#_{(\alpha,R)} \Sigma^2$.

Based on this gluing of model surfaces we extend the gluing operation $\#$ for broken solutions $(u, \hat{v})$ from pairs (Σ, Z) to (Σ^1, Σ^2). It is important to notice that we considered unparameterized solutions $\hat{v} \in \widehat{\mathcal{M}}(Z)$ in the last section. The missing degree of freedom due to taking the quotient of the $\mathbb{R}$-action was caught in the gluing parameter $\rho \in [\rho_o, \infty)$, so that the gluing

$$\# \colon K \times \bar{K} \times [\rho_o, \infty) \hookrightarrow \mathcal{M}^{x_1, \dots, x_a}_{y_1, \dots, z, \dots, y_b}$$

with $K \subset \mathcal{M}^{x_1, \dots, x_a}_{y_1, \dots, y_b}$, $\bar{K} \subset \widehat{\mathcal{M}}_{y_i, z}$ became a local diffeomorphism. Now, we keep the parameterization and fix the gluing parameter. Moreover, we cannot use the shifting $v \mapsto v_{-2\rho}$ anymore, in order to provide a uniform right inverse G of D_{w_χ} as above for ρ large enough. The proof worked by considering multi-bump solutions of $D_{w_\chi}\xi = 0$ in the limit $\rho \to \infty$. As we already carried out in the proof of 3.2.9, the linear gluing operation in the Fredholm chapter, we now achieve the analogue by deforming the underlying model surface $\Sigma^1 \#_{(\alpha,R)} \Sigma^2$ parameterized by ρ. The last new aspect of the generalized gluing operation is that we simultaneously glue at several ends. Since there are still finitely many ends to be glued, $i = 1, \dots, r$, and all estimates involved are localized over the cylindrical ends, the necessary steps in the proof of the gluing theorem in this new situation can be transferred from Sections 4.4.1 and 4.4.2. Before we state the adapted version of the gluing theorem we make the following definitions.

4.4.21 Definition *We shortly denote the tuples of periodic orbits $x_i \in \mathcal{P}_1(H^i)$, which are fixed as boundary loops for the solution spaces, by*

$$\mathbf{x^i} = (x^i_1, \dots, x^i_{a_i}), \quad \mathbf{y^i} = (y^i_1, \dots, y^i_{b_i})$$

with $i = 1, 2$ for each model surface Σ^1 and Σ^2. Given the gluing data $\alpha = ((i_1, j_1) \dots, (i_r, j_r))$ we call the ν_i-tuples of admissible pairs $(J^k_i, H^k_i)_{k=1,\dots,\nu_i}$, $i = 1, 2$, and boundary orbits $\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2}$ ***gluable*** *if it holds for $\epsilon_{i_k} = +1$ $x^1_{i_k} = y^2_{j_k - a_2}$ and $(J^{i_k}_1, H^{i_k}_1) = (J^{j_k}_2, H^{j_k}_2)$ and analogously for $\epsilon_{i_k} = -1$, for all $k = 1, \dots, r$. Then we define*

$$\mathbf{x} = \mathbf{x^1} \#_\alpha \mathbf{x^2}, \quad \mathbf{y} = \mathbf{y^1} \#_\alpha \mathbf{y^2},$$
$$J = J_1 \#_{(\alpha,R)} J_2, \quad k(H) = k(H_1) \#_{(\alpha,R)} k(H_2)$$

in the obvious way with $R = (\rho_1, \dots, \rho_r)$. We also adapt the notion of geometrical convergence. We say that a sequence of solutions

$$w_n \in \mathcal{M}^{\mathbf{x}}_{\mathbf{y}}(\Sigma^1 \#_{(\alpha,R_n)} \Sigma^2), \quad R_n = (\rho_{1,n}, \dots, \rho_{r,n}), \quad n \in \mathbb{N},$$

converges geometrically *of first degree towards a broken solution*

$$w_n \rightharpoonup (u, v) \in \mathcal{M}^{\mathbf{x^1}}_{\mathbf{y^1}}(\Sigma^1, (J_1, k(H_1))) \times \mathcal{M}^{\mathbf{x^2}}_{\mathbf{y^2}}(\Sigma^2, (J_2, k(H_2)))$$

for $n \to \infty$, if $\rho_{k,n} \to \infty$ for all $k = 1, \dots, r$, the truncated solutions[2]

$$w_n|_{\Sigma^1_{(\alpha,\tilde{R})}} \xrightarrow{C^\infty_{loc}} u|_{\Sigma^1_{(\alpha,\tilde{R})}},$$
$$w_n|_{\Sigma^2_{(\alpha,\tilde{R})}} \xrightarrow{C^\infty_{loc}} u|_{\Sigma^2_{(\alpha,\tilde{R})}}$$

converge for all $\tilde{R} = (\tilde{\rho}_1, \dots, \tilde{\rho}_r) \in (\mathbb{R}^+)^r$, and if on the glued cylindrical pieces[3]

$$w_n \circ \psi_k \xrightarrow{C^\infty_{loc}} x^1_{i_k} \text{ resp. } y^1_{i_k}$$

in $C^\infty_{loc}(\mathbb{R} \times S^1, M)$ for all $k = 1, \dots, r$.

4.4.22 Theorem *Given compact subsets of solutions*

$$K_1 \times K_2 \subset \mathcal{M}^{\mathbf{x^1}}_{\mathbf{y^1}}(\Sigma^1, (J_1, k(H_1))) \times \mathcal{M}^{\mathbf{x^2}}_{\mathbf{y^2}}(\Sigma^2, (J_2, k(H_2)))$$

such that $(\mathbf{x^i}, \mathbf{y^i})$ and $(J_i, k(H_i))$ are gluable, there exists a constant $\rho_o = \rho_o(K_1, K_2) > 0$ and a family of smooth embeddings

$$\#_{(\alpha,R)} \colon K_1 \times K_2 \to \mathcal{M}^{\mathbf{x}}_{\mathbf{y}}(\Sigma^{1,2}_{(\alpha,R)}, (J, k(H))),$$
$$(u, v) \mapsto u \#_{(\alpha,R)} v,$$

parameterized by $R = (\rho_1, \dots, \rho_r) \in [\rho_o, \infty)^r$ such that $(\#_{(\alpha,R)})$ are local diffeomorphisms and

$$u \#_{(\alpha,R_n)} v \rightharpoonup (u, v)$$

converges geometrically of the first degree if $\rho_{k,n} \to \infty$ for $n \to \infty$ for all $k = 1, \dots, r$. Moreover, every sequence converging geometrically of first degree,

$$w_n \in \mathcal{M}^{\mathbf{x}}_{\mathbf{y}}(\Sigma^1 \#_{(\alpha,R_n)} \Sigma^2), \quad w_n \rightharpoonup (u, v)$$

for $n \to \infty$, finally lies in the image of $\#$, that is, there are strongly convergent sequences $u_n \to u$ in $\mathcal{M}^{\mathbf{x^1}}_{\mathbf{y^1}}(\Sigma^1)$ and $v_n \to v$ in $\mathcal{M}^{\mathbf{x^2}}_{\mathbf{y^2}}(\Sigma^2)$ such that

$$w_n = u_n \#_{(\alpha,R_n)} v_n$$

for all $n \geqslant n_o$ for some n_o large enough.

We already indicated above which points have to be adapted so that the proof of this theorem can be carried out in straight analogy to the main gluing theorem, Theorem 4.4.1.

Concerning this original gluing theorem we already studied the associated compactness results describing the discrepancy between strong and geometrical convergence by splitting off of unparameterized connecting trajectories $\hat{v} \in \widehat{\mathcal{M}}$. Here, however, the counterpart of the gluing theorem by a compactness result associated to families of model surfaces $\Sigma^1 \#_{(\alpha,R_n)} \Sigma^2$, $R_n \to \infty$, is still missing. We now provide it for simplest case of local dimension 0.

[2]See Definition 3.2.1 on page 64.
[3]See Figure 3.2 on page 68.

4.4.23 Proposition *Let $\phi_\omega \equiv 0$ and a sequence of spherical solutions*

$$w_n \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma^1 \#_{(\alpha,R_n)} \Sigma^2, (J, k(H)))$$

with $\dim_{loc} w_n = 0$ be given, where $(J, k(H))$ is as above,

$$J = J^1 \#_{(\alpha,R_n)} J^2, \quad k(H) = k(H^1) \#_{(\alpha,R_n)} k(H^2),$$

and $R_n = (\rho_{1,n}, \dots, \rho_{r,n})$, $\lim_{n\to\infty} \rho_{i,n} = \infty$ for all $i = 1, \dots, r$. Then there exists a subsequence $(n_m)_{m\in\mathbb{N}}$ and additional 1-periodic solutions $\hat{x}^1_{i_k}, \hat{y}^1_{i_k-a_1} \in \mathcal{P}_1(H^1_{i_k})$ such that $\mathbf{x} = \hat{\mathbf{x}}^1 \#_\alpha \hat{\mathbf{x}}^2$, $\mathbf{y} = \hat{\mathbf{y}}^1 \#_\alpha \hat{\mathbf{y}}^2$ as above, and

$$w_{n_m} \rightharpoonup (u, v) \in \mathcal{M}^{\hat{\mathbf{x}}^1}_{\hat{\mathbf{y}}^1}(\Sigma^1, (J_1, k(H_1))) \times \mathcal{M}^{\hat{\mathbf{x}}^2}_{\hat{\mathbf{y}}^2}(\Sigma^2, (J_2, k(H_2)))$$

converges geometrically of first degree as $m \to \infty$.

Proof. Recalling the uniform energy estimate from Lemma 4.3.14, we observe that for the special extension $k^o(H_1)\#_{(\alpha,R_n)}k^o(H_2)$ and $k^o(H_1\#_\alpha H_2)$ only differ by terms of the form $\beta dt \otimes X_H$ over the compact cylindrical gluing parts $\psi_k([-\rho_{k,n}, \rho_{n,k}] \times S^1) \subset \Sigma^1 \#_{(\alpha,R_n)} \Sigma^2$ where the cut-off function β is of the type $\beta = \beta^+_{\rho_{k,n}} \cdot \beta^-_{-\rho_{k,n}}$. Thus, $\partial_s \beta$ is uniformly bounded, independently of $\rho_{k,n}$. We obtain a uniform estimate of the same form as in Lemma 4.3.14

$$\Phi(w_n) = \omega(\overline{w}_n) + \sum_{i=1}^{b} \mathcal{A}_{H^{i+a}}(\tilde{y}_i) - \sum_{i=1}^{a} \mathcal{A}_{H^i}(\tilde{x}_i) + c(w_n)$$

where $|c(w_n)| \leqslant K$ is uniformly bounded for all $n \in \mathbb{N}$. In the case of general extensions $k(H_i)$ the proof of a uniform energy estimate is carried out analogously.

Analogously to the compactness theorem in Section 4.3.2, Theorem 4.3.3 implies that the sequence (w_n) is relatively C^∞_{loc}-compact. That is, we find C^∞-convergent subsequences over all fixed compact subsets of $\Sigma^1 \#_{(\alpha,R_n)} \Sigma^2$. Here it is important to note that the subsets must be uniformly compact for all $n \in \mathbb{N}$, that is, either compact in the pieces $\Sigma^1_{(\alpha,R_o)}$ and $\Sigma^2_{(\alpha,R_o)}$ for R_o fixed, or compact within some $\psi_k([-\rho_o, \rho_o] \times S^1)$, $k = 1, \dots, r$, with $\rho_o \leqslant \rho_{k,n}$ independent of $n \in \mathbb{N}$. The crucial point is to study the relative compactness of (w_n) over the latter interior cylindrical gluing parts. We claim that there is a subsequence $(n_m)_{m\in\mathbb{N}}$ such that for[4]

$$\psi_k \colon [-\rho_{k,n}, \rho_{k,n}] \times S^1 \hookrightarrow \Sigma^1 \#_{(\alpha,R_n)} \Sigma^2,$$

$k = 1, \dots, r$, we have C^∞_{loc}-convergence towards a contractible 1-periodic solution $x_{i_k} \in \mathcal{P}_1(H^i_1)$ or respectively $y_{i_k} \in \mathcal{P}_1(H^{i+a_1}_1)$ depending on $\operatorname{sgn} \epsilon_{i_k}$. This means

$$w_{n_m} \circ \psi_{k|[-\rho_o,\rho_o]\times S^1} \to \overline{x}_{i_k} \quad \text{in } C^\infty([-\rho_o, \rho_o] \times S^1, M)$$

[4] See Definition 3.2.5 and Figure 3.2.

for all $\rho_o \geqslant 0$ as $m \to \infty$ so that $\rho_o \leqslant \rho_{n_m}$, with $\overline{x}_{i_k}(s,t) = x_{i_k}(t)$. At first we can only deduce from the C^∞_{loc}-compactness that

$$w_{n_m} \circ \psi_k \xrightarrow{C^\infty_{\text{loc}}} \tilde{w}_k \in \mathcal{M}(Z),$$

that is, a suitable subsequence converges in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1)$ towards a finite energy trajectory. But, in view of the index assumption $\dim_{\text{loc}} w_n = 0$ for all $n \in \mathbb{N}$, it follows that $\dim_{\text{loc}} \tilde{w}_k = 0$ so that $\tilde{w}_k$ is a constant trajectory, that is, $\tilde{w}_k = \overline{x}_{i_k}$ for some $x_{i_k} \in \mathcal{P}_1(H^{i_k}_1)$. Otherwise, the main gluing theorem, Theorem 4.4.1 would imply $\dim_{\text{loc}} w_n > 0$ for m large. The 1-periodic solution x_{i_k} is contractible because we assumed the (w_n) to be spherical. The rest of the proof now follows by analogy with Theorem 4.3.19. ■

We finish this section about gluing results with an extension of gluing to the contraction operation which was introduced in Section 3.2 in the Fredholm chapter. Namely, we now glue different cylindrical ends of the same model surface. Obviously, a necessary condition must be the identity of the respective asymptotic data (x_i, J^i, H^i) and different orientations ϵ_i. In view of the index formula we observe that this contraction of pairs of ends does not alter the index. In fact, we again are able to prove that gluing in this framework yields a local diffeomorphism between the respective solution spaces.

4.4.24 Theorem *Let $\mathbf{x} = (x_1, \dots, x_a)$ and $\mathbf{y} = (y_1, \dots, y_b)$ be given with gluing data $\alpha = ((i_1, j_1), \dots, (i_r, j_r))$ such that*

$$x_{i_k} = y_{j_k - a}, \quad (J^{i_k}, H^{i_k}) = (J^{j_k}, H^{j_k})$$

for $k = 1, \dots, r$. Then $\operatorname{tr}_\alpha \mathbf{x}$, $\operatorname{tr}_\alpha \mathbf{y}$ and $\operatorname{tr}_{(\alpha,R)}(J, k(H))$ are well-defined in the obvious way. Given a compact subset $K \subset \mathcal{M}^{\mathbf{x}}_{\mathbf{y}}(\Sigma, (J, k(H)))$ there exists a $\rho_o(K) > 0$ and a family of embeddings

$$\begin{aligned} \operatorname{tr}_{(\alpha,R)} \colon K &\hookrightarrow \mathcal{M}^{\operatorname{tr}_\alpha \mathbf{x}}_{\operatorname{tr}_\alpha \mathbf{y}}(\operatorname{tr}_{(\alpha,R)} \Sigma, \operatorname{tr}_{(\alpha,R)}(J, k(H))), \\ u &\mapsto \operatorname{tr}_{(\alpha,R)} u, \end{aligned}$$

parameterized by $R = (\rho_1, \dots, \rho_r) \in [\rho_o, \infty)^r$ such that $(\operatorname{tr}_{(\alpha,R)})$ are local diffeomorphisms and

$$\operatorname{tr}_{(\alpha,R_n)} u \rightharpoonup u$$

converges geometrically of first degree if $R_n \to \infty$ as in Theorem 4.4.22. Moreover, every sequence converging geometrically of first degree

$$w_n \in \mathcal{M}^{\operatorname{tr}_\alpha \mathbf{x}}_{\operatorname{tr}_\alpha \mathbf{y}}(\operatorname{tr}_{(\alpha,R_n)} \Sigma), \quad w_n \rightharpoonup u$$

for $n \to \infty$ finally lies in the image of tr.

Here we used the notion of geometrical convergence as given in Definition 4.4.21. The proof of the contraction theorem above again works in complete analogy to the original gluing theorem. Also Proposition 4.4.23 can be slightly reformulated in view of the contraction operation.

4.4.25 Proposition *Let $\phi_\omega \equiv 0$. Given a sequence of spherical solutions*

$$w_n \in \mathcal{M}_{\mathbf{y}}^{\mathbf{x}}\big(\operatorname{tr}_{(\alpha,R_n)} \Sigma, \operatorname{tr}_{(\alpha,R_n)}(J, k(H))\big)$$

with $R_n \to \infty$ and $\dim_{loc} w_n = 0$ as in Proposition 4.4.23, there exist additional periodic orbits $\hat{x}_{i_1}, \dots, \hat{x}_{i_r}, \hat{y}_{i_1}, \dots, \hat{y}_{i_r}$ and a subsequence $(n_m)_{m \in \mathbb{N}}$ such that $\mathbf{x} = \operatorname{tr}_\alpha \hat{\mathbf{x}}$, $\mathbf{y} = \operatorname{tr}_\alpha \hat{\mathbf{y}}$ and

$$w_{n_m} \rightharpoonup w \in \mathcal{M}_{\hat{\mathbf{y}}}^{\hat{\mathbf{x}}}(\Sigma, (J, k(H)))$$

converges geometrically of first order.

CHAPTER 5

The Cohomology Operations

In this final chapter we exploit the results about the solution spaces in dimension 0 and 1. We will construct a cohomology operation $Z(\Sigma)$ on a tensor product of Floer cohomology groups for each type (a,b,g) of connected model surfaces Σ. It turns out that this operation is already uniquely defined by the topological type and it fits into a concept of a functor Z which satisfies the axioms of a topological field theory in the sense of [3]. Since the S^1-cobordism functor Z yields a $1+1$-dimensional theory, we can describe it equivalently in terms of a graded algebra HF^* over $\mathbb{Z}_2$ with unit and a non-degenerate symmetric bilinear form. This will be proven by means of decompositions of each model surface into pair-of-pants, cylinders and disks. The final result is that the Floer (co-)homology $HF^*(M,\mathbb{Z}_2)$, which is known to be canonically isomorphic to the singular homology of M as a $\mathbb{Z}_2$-vector space, is endowed with a multiplicative structure associated to the pair of pants. However, within the scope of this work, we leave it open to identify the resulting structure of HF^* as a graded algebra.

Contents

5.1 The Construction of Cohomology Operations

In this section we carry out the explicit construction of the intended cohomology operations based on the given data. Let us lay down the necessary setup.

First, we are given a connected compact model surface Σ with boundary which is characterized by:

1. the topological data (a,b,g): the numbers a of inward oriented cylindrical ends, b of outward oriented ends, and the genus g of the closed surface $\hat{\Sigma} = \Sigma \#_{i=1}^{\nu} D^{-\epsilon_i}$.
2. the differentiable structure, in particular on the cylindrical ends, where we have the cylindrical coordinates $(\psi_i)_{i=1,\dots,\nu}$.
3. the conformal structure j on the differentiable surface Σ such that on the cylindrical ends $j \circ T\psi_k = T\psi_k \circ i$, $k=1,\dots,\nu$.

Second, we have to specify parameters determining the Floer homology groups on which the operation associated to Σ shall act. Let us choose a regular pair (J^i, H^i) for each end of Σ, $i=1,\dots,\nu$. We find a regular, admissible extension $(J, k(H))$ over $\Sigma \times M$, see Definitions 2.3.3, 2.3.4, 4.2.1 and 4.2.14.

Furthermore, we restrict the whole theory to homology groups with coefficients in $\mathbb{Z}_2$. This is primarily a technical restriction because we decide to leave aside considerations about coherent orientations for the solution spaces $\mathcal{M}^{\cdots}_{\cdots}$. For an exposition of the concept of coherent orientations in Floer homology, the reader is referred to [21]. Moreover, the restriction to $\mathbb{Z}_2$ allows us to avoid the discussion of torsion groups in Floer homology.

A severer restriction in the present work is imposed by the choice of the considered class of symplectic manifolds. As we shall analyze in the third section of this concluding chapter, the entire present theory of cohomology operations is based on solution sets of mappings $\Sigma \to M$ whose homotopy classes are spherical, recall Definition 4.3.17. Concerning the second homotopy group $\pi_2(M)$ we consider the simplest type of symplectic manifolds in this situation. Let (M,ω) be a closed symplectic manifold with $\omega_{|\pi_2} = c_{1|\pi_2} = 0$, that is, the fundamental homomorphisms vanish,

$$0 = \phi_\omega \colon \pi_2(M) \to \mathbb{R}, \quad 0 = \phi_c \colon \pi_2(M) \to \mathbb{Z}\,.$$

A generalization of this setup would be the class of weakly monotone (semi-positive) symplectic manifolds. This step would require an extension of the coefficient ring $\mathbb{Z}_2$ of the homology theory to a suitable completion of a group ring like the Novikov ring, see e.g. [29].

After fixing the fundamental framework we define the operation associated to the initial data $(\Sigma, j, J, k(H))$. From Theorem 4.3.19 we deduce that the following definition makes sense.

5.1.1 Definition *Given* 1*-periodic orbits* $x_i \in \mathcal{P}_1(H^i)$ *for* $i=1,\dots,a$ *and* $y_i \in \mathcal{P}_1(H^{i+a})$ *for* $i=1,\dots,b$*, we define the* $\mathbb{Z}_2$*-number*

$$\langle x_1,\dots,x_a; y_1,\dots,y_b\rangle_{(\Sigma,j,J,k)} = \\ \#\left\{\, u \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma,j,J,k) \,\middle|\, \{\bar{u}\} \in \mathcal{S},\ \dim_{loc} u = 0 \,\right\} \quad \text{modulo}\, 2\,.$$

Note that we merely count spherical solutions here, recall Definition 4.3.17. The regular admissible pairs $(J^i, H^i)_{i=1,\dots,\nu}$ generate Floer homology groups via cochain complexes, which consist here of $\mathbb{Z}_2$-vector spaces,

$$C_i^* = C^*(J^i, H^i), \quad i = 1, \dots, \nu,$$

and are graded by the integer-valued Conley-Zehnder index[1], $\mu_{CZ} \colon \mathcal{P}_1(H^i) \to \mathbb{Z}$,

$$C_i^k = \operatorname{span}_{\mathbb{Z}_2} \{ \, x_i \in \mathcal{P}_1(H^i) \, | \, \mu_{CZ}(x_i) = k \, \} \, .$$

We define the tensor complexes over $\mathbb{Z}_2$, $C_1^* \otimes \dots \otimes C_a^*$ and $C_{a+1}^* \otimes \dots \otimes C_{a+b}^*$ which are endowed with the induced grading

$$(C_1^* \otimes \dots \otimes C_a^*)^k = \bigoplus_{k_1 + \dots + k_a = k} C_1^{k_1} \otimes \dots \otimes C_a^{k_a} \, .$$

The coboundary operators $\delta_i \colon C_i^* \to C_i^{*+1}$ induce the coboundary operator δ on $C_1^* \otimes \dots \otimes C_a^*$,

$$\delta = \oplus_{i=1}^a \, \mathrm{id} \otimes \dots \otimes \mathrm{id} \otimes \delta_i \otimes \dots \otimes \mathrm{id} \, .$$

5.1.2 Definition *We define a linear operator with respect to these tensor complexes by specifying its values on the basis elements,*

$$\begin{aligned} &\mathcal{O}(\Sigma, j, J, k) \colon C_1^* \otimes \dots \otimes C_a^* \to C_{a+1}^* \otimes \dots \otimes C_{a+b}^*, \\ \mathcal{O}(x_1 \otimes \dots \otimes x_a) = &\sum_{\substack{\{(y_1, \dots, y_b) | \\ y_i \in \mathcal{P}_1(H^{a+i})\}}} \langle x_1, \dots, x_a; y_1, \dots, y_b \rangle_{(\Sigma, j, J, k)} \, y_1 \otimes \dots \otimes y_b \, . \end{aligned}$$

It follows from the index formula in Theorem 3.3.11 that, under the condition that $\phi_c \equiv 0$, the operator $\mathcal{O}(\Sigma) = \mathcal{O}(\Sigma, j, J, k)$ has a fixed degree uniquely defined by the topological data of the connected model surface Σ,

$$\deg \mathcal{O}(\Sigma) = n(a + b + 2g - 2), \tag{5.1}$$

where $2n = \dim M$. The main result is now

5.1.3 Theorem *The operator $\mathcal{O}(\Sigma, j, J, k)$ commutes with the coboundary operator δ,*

$$\mathcal{O} \circ \delta = \delta \circ \mathcal{O},$$

that is, $\mathcal{O}(\Sigma, j, J, k)$ is a cochain operator.

PROOF. The proof of this central theorem follows the scheme of Floer's proof of the relation $\delta \circ \delta = 0$, cf. [19]. It is based on the cobordism analysis

[1] Recall $\phi_c = 0$.

for the 1-dimensional components of the solution spaces. At first we compute both sides of the asserted equation.

$$\begin{aligned} &(\mathcal{O} \circ \delta)(x_1 \otimes \dots \otimes x_a) \\ =&\mathcal{O}\big(\sum_{i=1}^a x_1 \otimes \dots \otimes (\delta_i x_i) \otimes \dots \otimes x_a\big) \\ =&\mathcal{O}\big(\sum_{i=1}^a \sum_{\substack{z \in \mathcal{P}_1(H^i) \\ \mu(z) = \mu(x_i) + 1}} n(x_i, z) \, x_1 \otimes \dots \otimes x_{i-1} \otimes z \otimes x_{i+1} \otimes \dots \otimes x_a\big) \\ =&\sum_{i=1}^a \sum_{z \dots} \sum_{\substack{\{(y_1, \dots, y_b) | \\ y_i \in \mathcal{P}_1(H^{i+a})\}}} n(x_i, z) \, \langle x_1, \dots, z, \dots, x_a; y_1, \dots, y_b \rangle \, y_1 \otimes \dots \otimes y_b, \end{aligned}$$

with $n(x_i, z) = \#_2 \widehat{\mathcal{M}}_{x_i, z}(J^i, H^i)$ where $\#_2$ denotes the number modulo 2. It is important to notice that, due to $\phi_c \equiv 0$, the third sum only takes into account b-tuples $(y_1, \dots, y_b)$ with

$$\sum_{i=1}^b \mu_{CZ}(y_i) = \sum_{i=1}^a \mu_{CZ}(x_i) + 1 \, . \tag{5.2}$$

Summing up we obtain

$$(\mathcal{O} \circ \delta)(x_1 \otimes \dots \otimes x_a) = \sum_{\{(y_1, \dots, y_b)\}} n_l \, y_1 \otimes \dots \otimes y_b$$

with $n_l \in \mathbb{Z}_2$ given by

$$n_l = \sum_{i=1}^a \sum_{\substack{z \in \mathcal{P}_1(H^i) \\ \mu(z) = \mu(x_i) + 1}} \#_2 \widehat{\mathcal{M}}_{x_i, z}(J^i, H^i) \cdot \#_2 \mathcal{M}_{y_1, \dots, y_b}^{x_1, \dots, z, \dots, x_a}(\Sigma, j, J, k) \, .$$

Analogously, we compute the right hand side

$$(\delta \circ \mathcal{O})(x_1 \otimes \dots \otimes x_a) = \sum_{\{(y_1, \dots, y_b)\}} n_r \, y_1 \otimes \dots \otimes y_b$$

with

$$n_r = \sum_{i=1}^b \sum_{\substack{z \in \mathcal{P}_1(H^{i+a}) \\ \mu(z) = \mu(y_i) - 1}} \#_2 \mathcal{M}_{y_1, \dots, z, \dots, y_b}^{x_1, \dots, x_a}(\Sigma, j, J, k) \cdot \#_2 \widehat{\mathcal{M}}_{z, y_i}(J^{i+a}, H^{i+a})$$

in the case of (5.2) and $n_r = 0$ otherwise. In order to prove the asserted commutativity of $\mathcal{O}$ and δ we have to verify that

$$n_l + n_r \equiv 0 \quad \text{modulo } 2 \, .$$

This is now proven by the cobordism concept for the solution spaces $\mathcal{M}_{y_1,\dots,y_b}^{x_1,\dots,x_a}$.

It is the outcome of the compactness result, Theorem 4.3.21, and the gluing result, Theorem 4.4.1, that the 1-dimensional components of $\mathcal{M}(\Sigma, j, J, k)$ which are not strongly compact provide a unique equivalence relation for broken solutions of degree 1, see Definition 4.3.20. Namely, let I be such a 1-dimensional non-compact component of $\mathcal{M}$. It is diffeomorphic to the 1-manifold $(-\infty, \infty)$,

$$\phi\colon (-\infty,\infty) \xrightarrow{\sim} I \subset \mathcal{M}_{y_1,\dots,y_b}^{x_1,\dots,x_a}\,.$$

If $s_n \to \infty$ we have geometrical convergence for some subsequence (s_{n_k}) in view of Theorem 4.3.21, without loss of generality

$$\phi(s_{n_k}) \rightharpoonup (u, \hat{v})\,.$$

By Theorem 4.3.19, the set of all broken solutions of degree 1 which might occur as such geometrical limits is finite. Thus, the gluing result, Theorem 4.4.1 provides us with pairwise disjoint embeddings

$$\#^i\colon [\rho_o, \infty) \hookrightarrow I, \quad i = 1, \dots, m,$$

for each broken solution as geometrical limit point of I. However, the interval $\phi^{-1}(\#^i([\rho_o,\infty)))$ cannot be bounded in $(-\infty,\infty)$. In that case every sequence $\rho_n \to \infty$ would have a subsequence (n_k) such that $\#^i(\rho_{n_k})$ converges strongly to a solution in I, contradicting the construction of the gluing operation. Hence, every interval $\#^i([\rho_o,\infty))$ describes exactly one end of I, that is, $m = 2$ and

$$(-\infty,\infty) \setminus \phi^{-1}(\#^1((\rho_o,\infty)) \cup \#^2((\rho_o,\infty))) = [a, b]$$

for some $-\infty < a < b < \infty$. The geometric limits of $\phi(s)$ for $s \to \pm\infty$ are therefore uniquely determined. Moreover, Theorem 4.4.1 states that the broken solutions $\phi(\pm\infty)$ associated to $\#^1$ and $\#^2$ are different. Summing up, exactly two different broken solutions of degree 1 are equivalent in this sense of cobordism and they belong to the following classes,

$$(\hat{u}, v) \in \widehat{\mathcal{M}}_{x_i,z}(J^i, H^i) \times \mathcal{M}_{y_1,\dots,y_b}^{x_1,\dots,z,\dots,x_a}(\Sigma, j, J, k)$$

or

$$(u, \hat{v}) \in \mathcal{M}_{y_1,\dots,z,\dots,y_b}^{x_1,\dots,x_a} \times \widehat{\mathcal{M}}_{z,y_i}(J^{i+a}, H^{i+a}),$$

where $(x_1,\dots,x_a)$ and $(y_1,\dots,y_b)$ are fixed, satisfying (5.2), and z ranges through $\mathcal{P}_1(H^i)$, $i = 1,\dots,a$, with $\mu_{CZ}(z) = \mu_{CZ}(x_i) + 1$ or through $\mathcal{P}_1(H^{i+a})$, $i = 1,\dots,b$, with $\mu_{CZ}(z) = \mu_{CZ}(y_i) - 1$. Since this cobordism relation by intervals interrelates unique pairs of different broken solutions of these types, it follows that the sum $n_l + n_r$ is even. ■

Before we finish this section with the conclusion from Theorem 5.1.3, we point out that, in view of the simplifying choice of coefficients in $\mathbb{Z}_2$, we immediately may identify the $\mathbb{Z}_2$-vector spaces

$$H^*(C_1^* \otimes \dots \otimes C_a^*; \delta) \cong H^*(C_1^*; \delta_1) \otimes \dots \otimes H^*(C_a^*; \delta_a)\,.$$

This follows from the Künneth formula. In case of $\mathbb{Z}$-coefficients we would have to take into consideration additional torsion groups derived from the Ext-functor. Recall that we denote the Floer cohomology groups by

$$H^k(C_i^*; \delta_i) = H^k(J^i, H^i; \mathbb{Z}_2),$$

for $k \in \mathbb{Z}$, $i = 1, \dots, a+b$. Theorem 5.1.3 now implies that the operator $\mathcal{O}$ induces an operation on the level of cohomology.

5.1.4 Definition *We define the cohomology operation $Z = Z(\Sigma, j, J, k)$ associated to the connected model surface Σ and the triple $(j, J, k(H))$ by*

$$Z = \mathcal{O}^*\colon H^*(C_1^* \otimes \dots \otimes C_a^*) \to H^{*+\deg \mathcal{O}}(C_{a+1}^* \otimes \dots \otimes C_{a+b}^*)\,.$$

In view of the Künneth formula, this is a cohomology operation

$$Z\colon H^*(H^1, J^1) \otimes \dots \otimes H^*(H^a, J^a) \to H^*(H^{a+1}, J^{a+1}) \otimes \dots \otimes H^*(H^\nu, J^\nu)$$

of degree

$$\boxed{\deg Z = n(a + b + 2g - 2)}\,.$$

Our following aim is to analyze how this operation Z actually depends on the given uncanonical data $(\psi_i)_{i=1,\dots,\nu}$, j, J and $k(H)$. The crucial question is whether we deal, for example with a conformal theory or with a topological one (cf. [51]).

The first observation concerns the dependence on the admissible extensions $(J, k(H))$. Let us consider the simple model surface

$$(\Sigma_{1,1;0}, i) = (\mathbb{R} \times S^1, i) = \mathbb{C}/i\mathbb{Z}$$

with $(a, b, g) = (1, 1, 0)$. Then we immediately observe that the operation $\mathcal{O}(\Sigma, j, J, k(H))$ corresponds to a homotopy

$$(J^1, H^1) \simeq (J^2, H^2)\,.$$

We know already from Floer's original theory that the induced operation $Z(\Sigma, j, J, k(H))$ is in fact independent of the extension $(J, k(H))$ and that it represents the natural isomorphism Φ_{21}^* used in order to identify the isomorphic groups

$$\Phi_{21}^*\colon H^*(J^1, H^1) \xrightarrow{\cong} H^*(J^2, H^2),$$

in a canonical way, see [48], Section 6., pp. 1332. These canonical isomorphisms[2] satisfy the relations of a connected simple system, (6.1) in [48],

$$\Phi_{31}^* = \Phi_{32}^* \circ \Phi_{21}^*, \quad \Phi_{11}^* = \mathrm{id}\,. \tag{5.3}$$

We intend to extend this invariance result for admissible extensions $(J, k(H))$ to general model surfaces. The generalization of (5.3) for the cohomology operation Z leads to the operation

$$Z(\Sigma, j)\colon \bigotimes_{i=1}^{a} HF^*(M, \mathbb{Z}_2) \to \bigotimes_{i=1}^{b} HF^*(M, \mathbb{Z}_2)$$

on the Floer cohomology groups which are independent from the pairs (J^i, H^i) as we will finally show in Section 5.4.3.

[2] Recall (1.2) in the introduction.

5.2 Homotopy Invariance

In this section we carry out the formal reduction of the operation $Z(\Sigma, j, J, k)$ to an operator $Z(\Sigma)$ by showing that, intrinsically, it maximally depends on the differentiable structure of the model surface $(\Sigma, (\psi_i)_{i=1,\dots,\nu})$. The underlying principle is the homotopy invariance for the entire Floer homology theory. This was already used for the proof of the natural properties of the canonical isomorphisms $\Phi^*_{\beta\alpha}$ which identify $H^*(J^\beta, H^\beta) \cong H^*(J^\alpha, H^\alpha)$.

The idea is to verify that the cohomology operation $Z(\Sigma, j, J, k(H))$ does not depend on data which can be varied continuously but leave the asymptotic conditions invariant. This is the case

- for the set of conformal structures j on Σ extending the cylindrical structures $(\psi_k^* i)_{k=1,\dots,\nu}$,
- for the set of T-admissible extensions J of $(J^i)_{i=1,\dots,\nu}$
- and likewise for the T-admissible extensions $k = k(H)$ of (H^i).

Our aim is to derive algebraic homotopy operators associated to homotopies

$$(j_0, J_0, k_0) \simeq (j_1, J_1, k_1)$$

through a connecting smooth arc $(j_\lambda, J_\lambda, k_\lambda)$, $\lambda \in [0,1]$. Analytically, we have to attach the additional parameter $\lambda \in [0,1]$ to the framework of manifolds of solutions.

Note that, if we could find the connecting arc $\gamma = (j_\lambda, J_\lambda, k_\lambda)$ such that all transversality and compactness conditions are met uniformly for $\lambda \in [0,1]$ we consequently obtained immediately the identity

$$\langle x_1, \dots, x_a; y_1, \dots, y_b \rangle_{(\Sigma,\gamma_0)} = \langle x_1, \dots, x_a; y_1, \dots, y_b \rangle_{(\Sigma,\gamma_1)} \tag{5.4}$$

by inspection of the 1-dimensional arcs

$$\{\, (\lambda, u_\lambda) \,|\, \lambda \in [0,1],\, u_\lambda \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, \gamma_\lambda) \,\}\,.$$

However, in general, not only the uniform transversality fails, but also strong compactness. With strong compactness we still could prove (5.4) with numbers in $\mathbb{Z}_2$. This follows from a simple cobordism argument, see figure 5.1. But it turns out that compactness holds in the sense of geometric convergence. Thus we will be able to define algebraic homotopy operators which establish the identity of $Z(\Sigma, \gamma_0)$ and $Z(\Sigma, \gamma_1)$ not on the level of the cochain groups but on the level of cohomology.

5.2.1 Definition *Given 1-periodic orbits $x_i \in \mathcal{P}_1(H^i)$, $i = 1, \dots, a$, $y_i \in \mathcal{P}_1(H^{i+a})$, $i = 1, \dots, b$, and a smooth 1-parameter family*

$$\gamma\colon \lambda \mapsto (j_\lambda, J_\lambda, k_\lambda(H)), \quad \lambda \in [0,1],$$

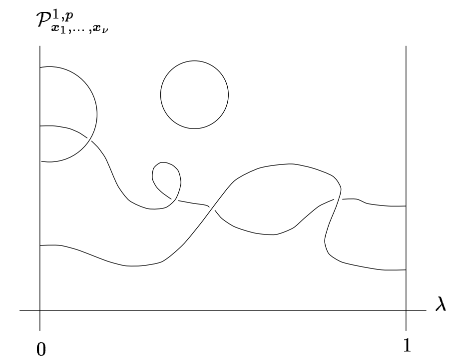


Figure 5.1: 1-dimensional compact cobordism

of conformal structures extending $(\psi_k^* i)_{k=1,\dots,\nu}$ and T-admissible pairs extending $(J^i, H^i)_{i=1,\dots,\nu}$ such that (j_0, J_0, k_0) and (j_1, J_1, k_1) are regular, we define the extended solution space of A-solutions

$$\widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, A, \gamma) = \{\, (\lambda, u) \in [0,1] \times C^\infty(\Sigma, M) \,|\, u \in \mathcal{M}_A \cap \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, \gamma_\lambda) \,\}\,.$$

for a fixed class $A \in H_2(M, \mathbb{Z})/\mathcal{S}$.

We now briefly carry out the necessary analytic program for this extended solution set, the discussion of transversality and of compactness for the 0-dimensional and 1-dimensional components.

5.2.2 Proposition (Transversality) *Given fixed regular triples (j_0, J_0, k_0) and (j_1, J_1, k_1) as above, there exists a smooth connecting arc γ such that all solution spaces for any $x_1, \dots, y_b$ are smooth submanifolds of $[0,1] \times \mathcal{P}^{1,p}_{x_1,\dots,y_b}$ with boundary. Each component is finite dimensional and, in view of the standard condition $\phi_c \equiv 0$, its dimension is given by*

$$\begin{aligned}\dim_{(\lambda,u_\lambda)} \widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, \gamma) &= \operatorname{ind} D_{u_\lambda} + 1 \\ &= 1 + \sum_{i=1}^{b} \mu_{CZ}(y_i) - \sum_{i=1}^{a} \mu_{CZ}(x_i) - n(a+b+2g-2),\end{aligned}$$

where D_{u_λ} is the linearization of $\overline{\partial}_{j_\lambda, J_\lambda, k_\lambda}$ at u_λ.

We point out that 0-dimensional solutions (λ, u_λ) necessarily satisfy $\operatorname{ind} D_{u_\lambda} = -1$ so that u_λ cannot be a regular solution of $\overline{\partial}_{j_\lambda, J_\lambda, k_\lambda}(u_\lambda) = 0$. These singular parameters λ are confined to the interior $(0,1)$ by assumption.

Proof. The proof has already been carried out in the transversality section 4.2.3. We merely have to consider the extended solution set $\widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, A, \gamma)$

as the set of vanishing points for the smooth section

$$F\colon [0,1]\times \mathcal{P}^{1,p}_{x_1,\dots,y_b}(\Sigma,M)\to\mathcal{E},\quad (\lambda,u)\mapsto\overline{\partial}_\lambda(u)$$

analyzed in Proposition 4.2.21. Then the assertion is contained in Theorem 4.2.23. ■

From now on we assume the connecting arc γ to be regular and fixed.

5.2.3 Proposition (Compactness) *Let us recall the condition $\phi_\omega\equiv 0$. Then the following holds:*

(a) Each 0-dimensional extended A-solution set $\widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma,A,\gamma)$ is a finite set $\{\,(\lambda_1,u_1),\dots,(\lambda_N,u_N)\,\}$ of cardinality $N_A(x_1,\dots,x_a;y_1,\dots,y_b;\gamma)$.

(b) The only obstruction to strong compactness of the 1-dimensional components is geometrical convergence towards broken solutions of degree 1. That is, the number of 1-dimensional components is finite and for each sequence $(\lambda_n,u_n)_{n\in\mathbb{N}}\subset\widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma,A,\gamma)$ which is not relatively compact there exists a broken solution

$$((\lambda,u),\hat v)\in\widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b}\times\widehat{\mathcal{M}}_{z,y_i}(J^{i+a},H^{i+a})$$

(or respectively $(\hat u,(\lambda,v))\in\widehat{\mathcal{M}}\times\widetilde{\mathcal{M}}$) with $\dim_{loc}(\lambda,u)=0$ and $\dim_{loc}\hat v=1$ [3] and a subsequence (n_k) such that

$$u_{n_k}\rightharpoonup(u,\hat v),\quad \lambda_{n_k}\to\lambda$$

in the sense of Definition 4.3.20.

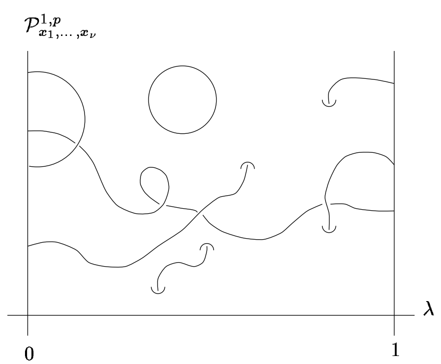


Figure 5.2: 1-dimensional cobordism with geometric convergence

[3]Here, the local dimension refers to the space $\mathcal{M}_{z,y_i}$ of parametrized trajectories.

PROOF. The proof is based on the compactness result for the C^∞_{loc}-topology, Theorem 4.3.22. Consequently (a) and (b) follow analogously to Theorems 4.3.19 and 4.3.21.

Let (λ_n,u_n) be a sequence of 0-dimensional solutions. We find a subsequence such that $\lambda_{n_k}\to\lambda_o\in(0,1)$ and hence

$$(j_{\lambda_{n_k}},J_{\lambda_{n_k}},k_{\lambda_{n_k}})\to(j_{\lambda_o},J_{\lambda_o},k_{\lambda_o})\,.$$

We are in the situation of Theorem 4.3.22 and conclude that, after choosing again a suitable subsequence, (u_{n_k}) converges in C^∞_{loc}. Now we argue exactly as in the proof of Theorem 4.3.19. The C^∞_{loc}-limit u of (u_{n_k}) is a solution of $\overline{\partial}_{\lambda_o}(u)=0$ and must fit the same boundary condition $u|\partial\Sigma=(x_1,\dots,x_a,y_1,\dots,y_b)$ because otherwise its index $\operatorname{ind}D_u$ would have decreased in contradiction to the assumption $\dim_{\text{loc}}(\lambda_{n_k},u_{n_k})=0$.

In the case (b) of 1-dimensional solutions (λ_n,u_n) we find again a subsequence (n_k), such that (λ_{n_k}) converges towards a singular parameter λ_o and (u_{n_k}) in C^∞_{loc} towards a solution u of $\overline{\partial}_{\lambda_o}(u)=0$. We consider suitable reparametrizations of the restrictions to the cylindrical ends and conclude that we obtain a broken solution $(u,\hat v)$ (or $(\hat u,v)$) where the index of $\hat v$ as a parametrized trajectory cannot be higher than 1 due to the index additivity $\operatorname{ind}D_u+\operatorname{ind}D_{\hat v}=\dim_{\text{loc}}(\lambda_{n_k},u_{n_k})-1=0$. ■

Strictly speaking, the index additivity used above has to be proven analogously to Corollary 4.4.2 by an appropriate gluing construction. This has to take into consideration the variation of the nonlinear operator $\overline{\partial}_{j_\lambda,J_\lambda,k_\lambda}$ due to the additional parameter λ. But also with respect to a complete description of the topology of the 1-dimensional components of the extended solution spaces, we have to provide a gluing result corresponding to the geometric convergence of Proposition 5.2.3 (b).

5.2.4 Proposition (Gluing) *Let*

$$((\lambda,u),\hat v)\in\widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b}(\Sigma,A,\gamma)\times\widehat{\mathcal{M}}_{z,y_i}(J^{i+a},H^{i+a})$$

be a broken solution of degree 1 with $\dim_{loc}(\lambda,u)=0$ and $\dim_{loc}\hat v=1$, then there exists a unique 1-dimensional component $K\subset\widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma,A,\gamma)$ and a smooth embedding $\#\colon[\rho_o,\infty)\hookrightarrow K$ satisfying

$$\#(\rho_n)\rightharpoonup((\lambda,u),\hat v)$$

for every sequence $\rho_n\to\infty$. Moreover, $\#$ is such that every sequence $(\lambda_n,u_n)\subset K$ converging geometrically, $(\lambda_n,u_n)\rightharpoonup((\lambda,u),\hat v)$ finally lies within $\#([\rho_o,\infty))$. The same result holds for broken solutions of the type

$$(\hat u,(\lambda,v))\in\widehat{\mathcal{M}}_{x_i,z}(H^i,J^i)\times\widetilde{\mathcal{M}}^{x_1,\dots,z,\dots,x_a}_{y_1,\dots,y_b}(\Sigma,A,\gamma)\,.$$

PROOF. The proof follows in straight analogy to the proof of Theorem 4.4.1. Here, the additional dependence on the parameter has to be taken into consideration. This is crucial also because the linearization of $\overline{\partial}_{j_\lambda,J_\lambda,k_\lambda}$ at u alone cannot be surjective due to $\operatorname{ind} D_u = \dim_{\text{loc}}(\lambda, u) - 1 = -1$. But the regularity of γ implies that the linearization of $F(\lambda, u) = \overline{\partial}_\lambda(u)$ is an isomorphism. Thus, based on the pregluing operation

$$((\lambda, u), \hat{v}, \rho) \mapsto (\lambda, u\#^o_\rho \hat{v})$$

where $\#^o_\rho$ is the original pregluing from Section 4.4.1, we analogously obtain a gluing map by means of the contraction mapping principle. ∎

Propositions 5.2.3 and 5.2.4 now provide us with the information needed to define the algebraic homotopy operator. Let us first sum up the different possibilities for boundaries of 1-dimensional components of $\widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, A, \gamma)$. We have the solutions for the original data $\gamma_0 = (j_0, J_0, k_0)$ and $\gamma_1 = (j_1, J_1, k_1)$ which form exactly the boundary points in the strong sense,

$$X_i = \{\, u \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, \gamma_i) \cap \mathcal{M}_A \,\}, \quad i = 0, 1\,.$$

On the other hand, we might also have boundary points in the weak, geometrical sense, namely the broken solutions of degree 1,

$$\begin{aligned} X_r = \{\, (u, \hat{v}) \mid\, & u \in \widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,z,\dots,y_b}(\Sigma, A, \gamma_\lambda),\ \hat{v} \in \widehat{\mathcal{M}}_{z,y_i}(J^{i+a}, H^{i+a}),\ \lambda \in (0,1) \\ & z \in \mathcal{P}_1(H^{i+a}),\ i = 1, \dots, b,\ \operatorname{ind} D_u = -1,\ \operatorname{ind} D_{\hat{v}} = 1 \,\} \end{aligned}$$

and

$$\begin{aligned} X_l = \{\, (\hat{u}, v) \mid\, & \hat{u} \in \widehat{\mathcal{M}}_{x_i,z}(J^i, H^i),\ v \in \widetilde{\mathcal{M}}^{x_1,\dots,z,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, A, \gamma_\lambda),\ \lambda \in (0,1) \\ & z \in \mathcal{P}_1(H^i),\ i = 1, \dots, a,\ \operatorname{ind} D_{\hat{u}} = 1,\ \operatorname{ind} D_v = -1 \,\}\,. \end{aligned}$$

From the cobordism argument with respect to both strong and geometrical boundary points we obtain

5.2.5 Corollary *Given a regular arc γ, the number of 1-dimensional components of $\widetilde{\mathcal{M}}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, A, \gamma)$ is finite and the set of these components defines an equivalence relation on the set $X = X_1 \cup X_2 \cup X_l \cup X_r$ such that exactly two different elements are equivalent. In particular, X contains an even number of elements.*

The variety of possible cobordisms is illustrated in Figure 5.2, where –) indicates an end with geometrical convergence towards a broken solution.

5.2.6 Definition *Given a regular arc γ we define the operator of degree -1*

$$\begin{aligned} \Psi(\gamma)\colon C^*_1 \otimes \dots \otimes C^*_a \to C^*_{a+1} \otimes \dots \otimes C^*_{a+b}, \\ \Psi(\gamma)(x_1 \otimes \dots \otimes x_a) = \\ \sum_{\substack{\{(y_1,\dots,y_b) \mid \\ y_i \in \mathcal{P}_1(H^{a+i})\}}} (N_A(x_1, \dots, x_a; y_1, \dots, y_b; \gamma) \mod 2)\ y_1 \otimes \dots \otimes y_b, \end{aligned}$$

where $N_A(\dots)$ is the number of 0-dimensional A-solutions from Proposition 5.2.3 (a).

5.2.7 Corollary *The operator $\Psi(\gamma)$ acts as an algebraic homotopy operator with respect to $Z(\Sigma, \gamma_0)$ and $Z(\Sigma, \gamma_1)$, that is,*

$$Z(\gamma_0) + Z(\gamma_1) \equiv \delta \circ \Psi(\gamma) + \Psi(\gamma) \circ \delta \mod 2\,.$$

Proof. The proof is a straight forward computation from the definitions of $Z(\gamma_i)$, δ and $\Phi(\gamma)$ exactly as in the proof of Theorem 5.1.3. We finally obtain within the sum over all generators $y_1 \otimes \dots \otimes y_b$ a sum over all boundary elements from X. Due to Corollary 5.2.5 the latter vanishes. ∎

We consequently have proven the independence of the operation $Z(\Sigma)$ from the data $(j, J, k(H))$.

5.2.8 Theorem *Given regular admissible extensions $\gamma_0 = (j_0, J_0, k_0(H))$ and $\gamma_1 = (j_1, J_1, k_1(H))$ of $(\psi^*_k i)_{k=1,\dots,\nu}$ and $(J^i, H^i)_{i=1,\dots,\nu}$, the associated operators on the tensor space of cohomology groups are identical,*

$$\begin{aligned} Z(\Sigma, \gamma_0) \equiv Z(\Sigma, \gamma_1)\colon & H^*(H^1, J^1) \otimes \dots \otimes H^*(H^a, J^a) \\ & \to H^*(H^{a+1}, J^{a+1}) \otimes \dots \otimes H^*(H^{a+b}, J^{a+b})\,. \end{aligned}$$

This means that this theory of cohomology operations associated to the model surfaces certainly depends on the topological data (a, b, g) but maximally on the differentiable structure and the fixed cylindrical coordinates. In the next section we will finally reduce this dependence to (a, b, g) and draw further important conclusions from the homotopy invariance which now is available.

5.3 The Topological Theory

In this section we prove that the cohomology operation $Z(\Sigma)$ is uniquely determined by the topological class of the compact connected oriented surface Σ with oriented boundary, i.e. by the topological data (a, b, g). We view such a topological surface as an S^1-cobordism.

It is a classical fact that two connected compact orientable topological surfaces are homeomorphic exactly if they have the same genus and number of boundary components. The same holds true in the category of differentiable surfaces, see e.g. [28]. Hence, connected compact oriented surfaces with oriented boundary are classified by the topological data (a, b, g). This can be passed on to the induced operators $Z(\Sigma)$.

5.3.1 Theorem *The operator $Z(\Sigma)$ is independent of the differentiable structure on the model surface Σ, in particular of the choice of the cylindrical coordinates $(\psi_i)_{i=1,\dots,\nu}$. In other terms, given two model surfaces $(\Sigma_j, (\psi^j_i)_{i=1,\dots,\nu})$, $j = 1, 2$, of the same type (a, b, g) and a fixed family of regular pairs $((J^i, H^i))$, the operators $Z(\Sigma_1)$ and $Z(\Sigma_2)$ are identical.*

Proof. The main part of the proof is due to the following fundamental proposition.

5.3.2 Proposition *Given model surfaces* $(\Sigma^i, (\psi^i_k)_{k=1,\dots,\nu})$, $i = 1, 2$, *of the same topological type* (a, b, g), *there exists an orientation preserving diffeomorphism* $\Phi\colon \Sigma^1 \xrightarrow{\cong} \Sigma^2$ *such that*

$$\Phi(\psi^1_k(Z^{\epsilon_k})) = \psi^2_k(Z^{\epsilon_k}), \quad \epsilon_k = \epsilon^1_k = \epsilon^2_k,$$
$$(\psi^2_k)^{-1} \circ \Phi \circ \psi^1_k = \mathrm{id}_{Z^{\epsilon_k}},$$

for all $k = 1, \dots, \nu$.

PROOF. In view of the original definition of a model surface we go over to the compact cylindrical coordinates with uniform orientation $\phi^i_k\colon [0,1] \times S^1 \to \overline{\Sigma}^i$, $i = 1, 2$,

$$\phi^i_k(s,t) = \psi^i_k\Big(\frac{\epsilon_k s}{\sqrt{1-s^2}}, t\Big).$$

Moreover, we glue disks into these compact ends by means of $\varphi^i_k\colon \{\, z \in \mathbb{C} \mid \frac{1}{2} \leqslant |z| \leqslant 1 \,\} \to \overline{\Sigma}^i$ with

$$\varphi^i_k(r\, \mathrm{e}^{i2\pi t}) = \phi^i_k\Big(-\frac{\ln r}{\ln 2}, t\Big).$$

Thus we obtain the closed surfaces

$$\hat{\Sigma}^i = \overline{\Sigma}^i \cup_{\varphi^i_k, k=1,\dots,\nu} D^2(0)$$

of genus g with smooth embeddings $\varphi^i_k\colon D^2(0) \hookrightarrow \hat{\Sigma}^i$ of the unit disk. Since these embedded disks are pairwise disjoint and compact we may assume without loss of generality that the φ^i_k are restrictions to $D^2(0)$ of smooth coordinate charts $\varphi^i_k\colon \mathbb{R}^2 \to \hat{\Sigma}^i$ with pairwise disjoint image. We have to prove the existence of a diffeomorphism $\Phi\colon \hat{\Sigma}^1 \to \hat{\Sigma}^2$ satisfying

$$\Phi(\varphi^1_k(D^2(0))) = \varphi^2_k(D^2(0)) \quad \text{and} \quad \Phi \circ \varphi^1_{k|D^2(0)} = \varphi^2_{k|D^2(0)} \tag{5.5}$$

for all $k = 1, \dots, \nu$.

The first step is to find a diffeomorphism Φ_o which maps the centers according to the indices, $\varphi^1_k(0) \mapsto \varphi^2_k(0)$. This follows from the classical fact that closed surfaces are classified by their genus, see for example [28], Theorem 9.3.5. We obtain a smooth diffeomorphism $\Psi\colon \hat{\Sigma}^1 \xrightarrow{\sim} \hat{\Sigma}^2$, which maps the $\varphi^1_k(0)$ onto ν disjoint points x_k. We find smooth compact arcs $\gamma_k\colon [0,1] \to \hat{\Sigma}^2$ connecting each x_k with $\varphi^2(0)$ such that the $(\gamma_k(t))$ are pairwise disjoint for each $t \in [0,1]$. This isotopy $\gamma\colon \{x_1, \dots, x_\nu\} \times [0,1] \to \hat{\Sigma}^2$ can be extended to a diffeotopy $F\colon \hat{\Sigma}^2 \times [0,1] \to \hat{\Sigma}^2$, see for instance [28], Theorem 8.1.3. Thus, we obtain the required diffeomorphism as $\Phi_o = F(\cdot, 1) \circ \Psi$.

The second step is to find a diffeotopy from Φ_o to Φ such that (5.5) is satisfied. Considering the orientation preserving diffeomorphism on a suitably small disk of radius ϵ

$$f_0 = (\varphi^2_k)^{-1} \circ \Phi_o \circ \varphi^1_k\colon B_\epsilon(0) \to \mathbb{R}^2,$$

satisfying $f_0(0) = 0$, we find an isotopy f_t such that $f_{1|B_\epsilon(0)} = \mathrm{id}$ for ϵ sufficiently small. This follows from the homotopy $t \mapsto \frac{1}{t} f_0(t\cdot)$ to $Df_0(0)$ and linear

algebra, $Df(0) \simeq \mathrm{id}$. Eventually choosing ϵ smaller we can extend the isotopies $\varphi^2_k \circ f_t \circ (\varphi^1_{k|B_\epsilon(0)})^{-1}$, $k = 1, \dots, \nu$, to a diffeotopy $\Phi_t\colon \hat{\Sigma}^1 \to \hat{\Sigma}^2$ so that the diffeomorphism Φ_1 satisfies

$$(\varphi^2_k)^{-1} \circ \Phi_1 \circ \varphi^1_{k|B_\epsilon(0)} = \mathrm{id}_{|B_\epsilon(0)}$$

for all $k = 1, \dots, \nu$. Finally, given $\delta > 0$, we choose a diffeomorphism between the disks of radius $1 + \delta$ and ϵ

$$h\colon B_{1+\delta}(0) \to B_\epsilon(0), \quad h(r\, \mathrm{e}^{i\phi}) = \rho(r) r\, \mathrm{e}^{i\phi},$$

with

$$\rho(r) = \begin{cases} \epsilon, & r \leqslant 1, \\ 1, & r \geqslant 1 + \frac{\delta}{2}. \end{cases}$$

For $\delta > 0$ small enough, this radial deformation applied to the coordinate charts φ^i_k leads to the diffeomorphism Φ satisfying (5.5). ∎

5.3.3 Corollary *Given a connected model surface* $(\Sigma, (\psi_k))$ *and a permutation* σ *of the cylindrical ends preserving the orientations,* $\sigma(\epsilon_k) = \epsilon_k$, *there exists a diffeomorphism* $\Phi\colon \Sigma \xrightarrow{\sim} \Sigma$ *extending* σ *such that*

$$\psi^{-1}_{\sigma(k)} \circ \Phi \circ \psi_k = \mathrm{id}_{Z^{\epsilon_k}}$$

for all $k = 1, \dots, \nu$.

Let us now prove the statement of Theorem 5.3.1. Given the two model surfaces $(\Sigma^i, (\psi^i_k)_{k=1,\dots,\nu})$ we choose for Σ^2 a regular admissible triple (j_2, J_2, k_2) extending $(\psi^2_k)^* i$ and $(J^k, H^k)_{k=1,\dots,\nu}$. Using the diffeomorphism Φ from Proposition 5.3.2 we pull these structures back to the model surface Σ^1,

$$j_1 = \Phi^* j_2 = T\Phi^{-1} j_2 T\Phi,$$
$$J_1(z,m) = J_2(\Phi(z), m),$$
$$k_1(z,m) = k_2(\Phi(z), m) T\Phi(z).$$

This triple $(j_1, J_1, k_1) = \Phi^*(j_2, J_2, k_2)$ is a regular admissible extension of $(\psi^1_k)^* i$ and (J^k, H^k), $k = 1, \dots$. Moreover, we obtain a one-to-one correspondence of solutions such that the boundary loops are identical,

$$\Phi^*\colon \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(j_2, J_2, k_2) \xrightarrow{\sim} \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(j_1, J_1, k_1),$$
$$u \mapsto u \circ \Phi.$$

Note that the diffeomorphism property of Φ implies that $\Phi^* J_2$ is again regular, that is, that transversality holds. This proves the identity

$$\begin{aligned} &\langle x_1, \dots, x_a; y_1, \dots, y_b \rangle_{(\Sigma^1, j_1, J_1, k_1)} \\ =&\langle x_1, \dots, x_a; y_1, \dots, y_b \rangle_{(\Sigma^2, j_2, J_2, k_2)}. \end{aligned} \tag{5.6}$$

The homotopy invariance from Theorem 5.2.8 then proves that the cohomology operations $Z(\Sigma^1)$ and $Z(\Sigma^2)$ are identical. ∎

We have proved that the cohomology operations are uniquely associated to labeled S^1-cobordisms. By this we mean topological classes of compact connected surfaces with distinguished oriented ends. Before we advance towards the abstraction from the concretely given regular pairs (J^k, H^k) for $k = 1, \ldots, a+b$, we point out that the operator

$$Z(\Sigma) = Z(a, b, g)\colon \bigotimes_{k=1}^{a} H^*(J^k, H^k) \to \bigotimes_{k=1}^{b} H^*(J^{a+k}, H^{a+k})$$

carries a further inherent symmetry in the case of identical pairs. Assume without loss of generality that the first $r \leqslant a$ pairs (J^k, H^k) are identical. Then Corollary 5.3.3 combined with the same pulling-back technique for the regular triple (j, J, k) as in the above proof imply that

$$Z(a, b, g) \circ \left(\sigma \otimes (\otimes^{a-r}\,\mathrm{id})\right) \;=\; Z(a, b, g) \tag{5.7}$$

In particular, the multiplication $Z(2, 1, g)$ in the case of $(H^1, J^1) = (H^2, J^2)$ is commutative. This symmetry is certainly linked with the choice of $\mathbb{Z}_2$-coefficients. Carrying out a cohomology theory with coefficients in $\mathbb{Z}$, based on the concept of coherent orientations, one should expect skew-commutativity.

5.4 Deformations and Compositions

In this section we deduce further fundamental relations for the cohomology operations $Z(\Sigma)$ from the homotopy invariance and the gluing operations. We will observe that the 'cobordism functor' Z behaves very naturally in relation with sewing topological model surfaces at their ends. This will be obtained from the compactness-gluing dualism analyzed in Chapter 4. Let us first describe the process of stretching the model surface at interior compact cylindrical parts. This corresponds to representing a model surface as a glued surface $\Sigma^1 \#_{(\alpha,R)} \Sigma^2$.

5.4.1 Stretching

Let $(\Sigma, (\psi_i)_{i=1,\ldots,\nu})$ be a fixed model surface. We consider an r-tuple of smooth embeddings of the annulus $A = [-1, 1] \times S^1$ into Σ,

$$\phi_1, \ldots, \phi_r\colon A \hookrightarrow \Sigma$$

such that the images are pairwise disjoint and lie within the complement of the cylindrical ends,,

$$\phi_i(A) \cap \phi_j(A) = \emptyset, \quad i \neq j \quad \text{and} \quad \phi_i(A) \subset \mathrm{int}(\Sigma_0)$$

for $i = 1, \ldots, r$. Observe that the genus g of Σ is the maximal r such that there exists ϕ_i, $i = 1, \ldots, r$ such that $\Sigma \setminus \bigcup_{i=1}^{r} \phi_i(A)$ is connected. Let the conformal structure j be an admissible extension of $(\psi_k^* i)_{k=1,\ldots,\nu}$ satisfying

$$T\phi_k \circ i = j \circ T\phi_k$$

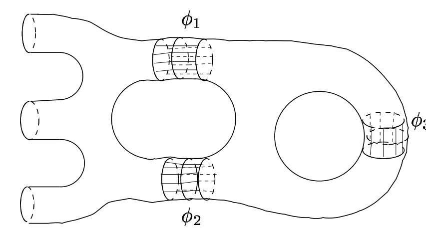


Figure 5.3: A decomposition of a model surface

for all $k = 1, \ldots, r$. We now deform j as follows. We choose a 1-parameter family of diffeomorphisms $h_\rho\colon [-\rho, \rho] \xrightarrow{\sim} [-1, 1]$ for $\rho \geqslant \rho_o > 1$ such that $h_\rho(0) = 0$, $h_\rho(\pm\rho) = \pm 1$ and for $0 < 3\epsilon < \rho_o - 1$

$$h'_\rho(s) = \begin{cases} 1, & |s| \geqslant \rho - \epsilon, \\ \frac{1-2\epsilon}{\rho-2\epsilon}, & |s| \leqslant \rho - 3\epsilon, \end{cases}$$

$$h''_\rho \neq 0 \quad \text{for} \quad \rho - 3\epsilon < |s| < \rho - \epsilon\,,$$

see Figure 5.4. Then we define the parametrized embeddings $\phi_k^\rho\colon [-\rho, \rho] \times S^1 \hookrightarrow$

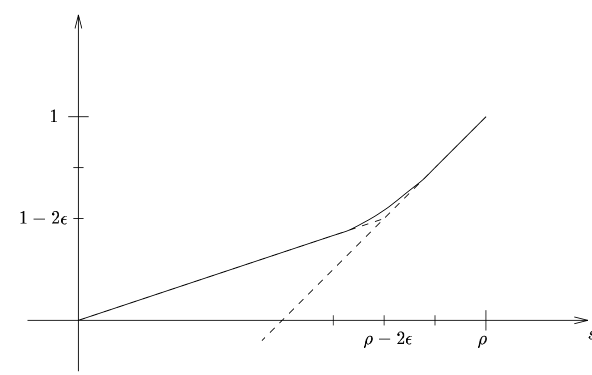


Figure 5.4: The stretching map h_ρ.

Σ by

$$\phi_k^\rho = \phi_k \circ (h_\rho \times \mathrm{id}_{S^1})$$

for all $k = 1, \ldots, r$. Let us denote the r-tuples of deformation parameters by $R = (\rho_1, \ldots, \rho_k)$ and $\phi_k^R = \phi_k^{\rho_k}$. We obtain a family of conformal structures

$[\rho_o, \infty)^r \ni R \mapsto j_R$ given by

$$j_{R|\Sigma \setminus \bigcup_k \phi_k(A)} = j \quad \text{and} \quad T\phi_k^R \circ i = j_R \circ T\phi_k^R .$$

5.4.1 Definition *Given (j_R) as above, let us consider an r-parameter family $(J_R, k_R)_{R\in[\rho_o,\infty)^r}$ of admissible extensions of $(J^i, H^i)_{i=1,\dots,\nu}$ such that (J_R, k_R) restricted to $\Sigma \setminus \bigcup_k \phi_k(A)$ is independent of R and has the form*

$$J_R(\phi_k^R(s,t),x) = \begin{cases} J_1(s,t,x), & s \leqslant T - \rho_k, \\ J(t,x), & |s| \leqslant \rho_k - T, , \\ J_2(s,t,x), & s \geqslant \rho_k - T, \end{cases}$$

$$k_R(\rho_k^R(s,t),x) \circ T\rho_k^R(s,t) = \begin{cases} k_1(s,t,x), & s \leqslant T - \rho_k, \\ -dt \otimes X_H(t,x), & |s| \leqslant T - \rho_k, \\ k_2(s,t,x), & s \geqslant \rho_k - T \end{cases}$$

on the cylindrical deformation regions, with fixed $0 < T < \rho_o$ and for some J_1, J_2, J, k_1, k_2 and H. We call such a family (j_R, J_R, k_R) a ***stretching family*** *associated to the pair (J, H).*

Obviously, the notion of such a stretching family is equivalent to representing (Σ, j_R, J_R, k_R) as a model surface obtained from a gluing- respectively contraction-operation as considered in Sections 3.2 and 4.4.3. See Definitions 3.2.1 and 4.4.21. For example, let $r = 1$ and $\phi = \phi_1 \colon A \hookrightarrow \operatorname{int}\Sigma_0$ such that $\Sigma \setminus \phi(A)$ is connected. Then we have

$$(\Sigma, j_\rho) = (\operatorname{tr}_\rho \check\Sigma, \operatorname{tr}_\rho \check j)$$

where $\check\Sigma$ is a model surface of the type $(a+1, b+1, g-1)$ with cylindrical coordinates given by

$$\check\psi_i(s,t) = \begin{cases} \psi_i(s,t), & i = 1,\dots,a, \\ \phi^\rho(\rho + s, t), & -\rho \leqslant s \leqslant 0,\ i = a+1, \end{cases}$$

$$\check\psi_{i+a+1}(s,t) = \begin{cases} \psi_i(s,t) & i = 1,\dots,b, \\ \phi^\rho(s-\rho, t), & 0 \leqslant s \leqslant \rho,\ i = b+1 \end{cases}$$

with the according $\check j$.

In view of this equivalence of deforming j with representing Σ as a glued surface, the following lemma generalizes the uniform energy estimate used in the proof of Proposition 4.4.23.

5.4.2 Lemma *Given a stretching family (j_R, J_R, k_R) as defined above, the flow energy $\Phi_{j_R,J_r,k_R}(u)$ of solutions $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(\Sigma, j_R, J_R, k_R) \cap \mathcal{M}_A$ for fixed ends $x_1, \dots, x_\nu$ and a class $A \in H_2(M,\mathbb{Z})/\mathcal{S}$ is independent of the deformation R.*

Proof. Using a computation as in Lemma 4.3.14 on page 127, this follows directly from the definition of the stretching family. ∎

Note that we consider a more general situation than in the gluing case, because we do not assume that (J, H) in Definition 5.4.1 is a regular pair. In particular, the case $H = 0$ allows us to finally prove that counting A-solutions for classes $0 \neq A \in H_2(M,\mathbb{Z})/\mathcal{S}$ does not yield a nontrivial contribution to a cohomology operation.

5.4.3 Theorem *Let $A \in H_2(M,\mathbb{Z})/\mathcal{S}$ and $Z_A(a,b,g)$ be the cohomology operator defined by counting 0-dimensional A-solutions,*

$$\langle x_1,\dots,x_a; y_1,\dots,y_b\rangle^A_{\Sigma,j,J,k} = \\ \#\{\, u \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b} \mid u \in \mathcal{M}_A,\ \dim_{loc} = 0 \,\} \mod 2 .$$

Then $Z_A \equiv 0$ if $A \neq 0$.

Proof. $g \geqslant 1$ is a necessary condition for the existence of solutions u with $0 \neq \{\bar u\} + \mathcal{S} = A \in H_2(M,\mathbb{Z})/\mathcal{S}$. Recall that the genus of a surface with boundary is equal to the unique number g such that any $g+1$ disjoint embedded circles $S^1 \hookrightarrow \Sigma$ disconnect Σ but there exists g disjoint circles not disconnecting the surface. Thus we can choose pairwise disjoint embeddings of the annulus $(-1,1)\times S^1$ into $\operatorname{int}(\Sigma_0)$, $\phi_1,\dots,\phi_g$ as above, such that $\hat\Sigma \setminus \bigcup_{i=1}^g \phi_i((-1,1)\times S^1)$ is a surface of genus 0. This implies that for any map $u \in C^0(\hat\Sigma, M)$ with $u_*[\hat\Sigma] + \mathcal{S} = A \neq 0$ there is at least one loop $u \circ \phi_i(0,\cdot) \in C^0(S^1, M)$ that is not contractible. Let us assume that there exist $\alpha_k \in H^*(J^k, H^k)$, $k = 1,\dots,a$ with

$$Z_A(\alpha_1 \otimes \ldots \otimes \alpha_a) \neq 0 .$$

We choose a stretching family $(j_{R_n}, J_{R_n}, k_{R_n})$ for $R_n \to \infty$ satisfying

$$k_{R_n|\phi_k([-1,1]\times S^1)} = 0$$

for all $k = 1,\dots,g$. Moreover, we may choose J_{R_n} such that all triples $(j_{R_n}, J_{R_n}, k_{R_n})$ are regular. This is possible because, in view of the remark following Theorem 4.2.20, we deal with a countable family. Due to the homotopy invariance for Z_A, we find after eventually taking a subsequence (n_k) fixed 1-periodic solutions $x_i \in \mathcal{P}_1(H^i)$, $i = 1,\dots,a$, $y_i \in \mathcal{P}_1(H^{i+a})$, $i = 1,\dots,b$, and A-solutions

$$u_n \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(j_{R_n}, J_{R_n}, k_{R_n}) \cap \mathcal{M}_A$$

for all $n \in \mathbb{N}$. In view of Proposition 5.4.4, the flow energies $\Phi(u_n)$ are uniformly bounded, the local dimensions are 0 for all u_n and $\phi_\omega \equiv 0$ by general assumption. Thus we obtain the compactness result that for a suitable subsequence

$$u_{n_k} \xrightarrow{C^\infty_{\mathrm{loc}}} u \in C^\infty(\overline{\Sigma}, M),$$

and thus $\{\bar u\} + \mathcal{S} = A$. If we consider especially the cylindrical parts

$$u_n \circ \phi_k^{R_n} \colon [-R_n, R_n] \times S^1 \to M$$

we deduce from $\phi_\omega \equiv 0$ after choosing again a suitable subsequence of (u_{n_k}) that for all $k = 1, \ldots, g$

$$u_{n_l} \circ \phi_k^{R_{n_l}} \xrightarrow{C^\infty_{\mathrm{loc}}} v_k \in C^\infty(\mathbb{R} \times S^1, M) \quad \text{and} \quad v_k = \text{const} .$$

The convergence towards a bounded energy solution on the infinite cylinder follows from Theorem 4.3.3. The fact that v_k is necessarily constant is due to the assumption that $\phi_\omega = 0$. In general, due to the so-called removal-of-singularities theorem it follows that v_k extends to a pseudo-holomorphic sphere. For the special case of ω vanishing on $\pi_2(M)$ see also the proof of Theorem 8 in [31], pp. 239–240. Hence, by choice of ϕ_k, $k = 1, \ldots, g$, it follows that $\{\bar{u}\} \in \mathcal{S}$ contradicting the assumption $A \neq 0$. ■

5.4.2 Gluing and Composition

This section is essentially based on the gluing results from Section 4.4.3. We merely have to reorganize the analytical results in order to extract the algebraic information about the cohomology operations. We show that the gluing operation on the solution spaces for two model surfaces corresponds to the composition of the respective algebraic operations.

Let us first consider the special case that two surfaces Σ^1 and Σ^2 are glued along all exits of Σ^1 and all entries of Σ^2, see Figure 5.5.

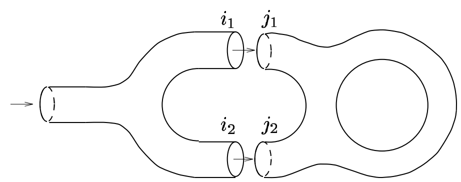


Figure 5.5: Gluing model surfaces

5.4.4 Proposition *Let Σ^i, $i = 1, 2$ be two connected model surfaces of type (a_i, b_i, g_i) with $b_1 = a_2 = c$ and endowed with regular pairs $(J_i^k, H_i^k)_{k=1,\ldots,a_i+b_i}$ such that*

$$(J_1^{k+a_1}, H_1^{k+a_1}) = (J_2^k, H_2^k), \quad k = 1, \ldots, c .$$

Denoting by $\Sigma^1 \#_\alpha \Sigma^2$ the class of model surfaces obtained by gluing with $\alpha = ((a_1+1, 1), \ldots, (a_1+c, c))$ [4], it holds

$$Z(\Sigma^1 \#_\alpha \Sigma^2) = Z(\Sigma^2) \circ Z(\Sigma^1) .$$

[4] see Definition 3.2.1

Proof. Let (J_i, k_i), $i = 1, 2$, be admissible regular extensions of (J_i^k, H_i^k) and $R_n = (\rho_{1,n}, \ldots, \rho_{c,n})$ be a sequence of gluing parameters with $\rho_{k,n} \to \infty$ for all $k = 1, \ldots, c$ as $n \to \infty$. We obtain the glued admissible extensions $J_n^\# = J_1 \#_{(\alpha,R_n)} J_2$ and $k_n^\# = k_1 \#_{(\alpha,R_n)} k_2$ on $\Sigma^1 \#_{(\alpha,R_n)} \Sigma^2$ as used in Definition 4.4.21. Since we deal with a countable family of deformations we can assume (J_i, k_i) such that all $(J_n^\#, k_n^\#)_{n \in \mathbb{N}}$ are regular. In order to compute the left hand side of the asserted identity we consider an arbitrary fixed cohomology class $\{\sum_{\mathbf{x}} a_{\mathbf{x}} x_1 \otimes \ldots \otimes x_{a_1}\} \in H^*(H^1, J^1) \otimes \ldots \otimes H^*(H^{a_1}, J^{a_1})$. Here we use the notation $\mathbf{x} = (x_1, \ldots, x_{a_1})$ as in Section 4.4.3. We obtain

$$Z(\Sigma^1 \#_\alpha \Sigma^2)(\{\sum_{\mathbf{x}} a_{\mathbf{x}} x_1 \otimes \ldots \otimes x_{a_1}\})$$
$$= \Big\{ \sum_{\mathbf{z_n}} \#_2(\mathcal{M}^{\mathbf{x}}_{\mathbf{z_n}}(\Sigma^1 \#_{(\alpha,R_n)} \Sigma^2, J_n^\#, k_n^\#) \cap \mathcal{M}_{\mathcal{S}}) z_{1,n} \otimes \ldots \otimes z_{b_2,n} \Big\} .$$

Again $\#_2$ denotes the cardinality modulo 2. We observe that the cohomology class on the right hand side is independent of the deformation R_n, but in general not the representing 1-periodic solutions $z_{k,n} \in \mathcal{P}_1(H_2^{k+a_2})$. However we can choose a subsequence (n_l) without loss of generality again denoted by (n) such that $\mathbf{z}$ remains fixed. Thus it suffices to consider the numbers as $n \to \infty$

$$\langle \mathbf{x}, \mathbf{z} \rangle_n = \#_2(\mathcal{M}^{\mathbf{x}}_{\mathbf{z}}(\Sigma^1_{(\alpha,R_n)} \Sigma^2, J_n^\#, k_n^\#) \cap \mathcal{M}_{\mathcal{S}})$$

for $\mathbf{x}, \mathbf{z}$ fixed with $\deg(\mathbf{z}) - \deg(\mathbf{x}) = \frac{1}{2} \dim M (a_1 + b_2 + 2(g_1 + g_2 + c - 1) - 2) = d$ where g_i is the genus of Σ^i. Observe that $g = g_1 + g_2 + c - 1$ is the entire genus of the glued surface $\Sigma^1 \#_\alpha \Sigma^2$. Due to the restriction to spherical solutions we are able to apply Proposition 4.4.23 and Theorem 4.4.22. These provide that the above numbers $\langle \mathbf{x}, \mathbf{z} \rangle_n$ are independent for n large and satisfy the identity

$$\langle \mathbf{x}, \mathbf{z} \rangle_n \equiv \sum_{\deg(\mathbf{y}) = \deg(\mathbf{x}) + d} \#(\mathcal{M}^{\mathbf{x}}_{\mathbf{y}}(J_1, k_1) \cap \mathcal{M}_{\mathcal{S}}) \#(\mathcal{M}^{\mathbf{y}}_{\mathbf{z}}(J_2, k_2) \cap \mathcal{M}_{\mathcal{S}}) \mod 2 .$$

This completes the proof. ■

The last proposition deals with the situation that a given model surface is decomposed by c pairwise non-intersecting embedded circles $\phi_k(S^1)$ into exactly two connected model surfaces. Before we consider the most general case of decomposition we consider the analogous contraction-operation tr_α which corresponds to embedded decomposition circles $\phi_k(S^1)$ such that the complement surface is still connected. Using Theorem 4.4.24 and Proposition 4.4.25 we obtain in complete analogy to above the following algebraic contraction operator. Given the regular pairs $(J^i, H^i)_{i=1,\ldots,a+b}$ and a connected model surface Σ of type (a, b, g) we use the short hand notation

$$\mathcal{A}_i^* = H^*(J^i, H^i), \quad i = 1, \ldots, a+b,$$

for the multi-linear operator

$$Z(\Sigma) \colon \bigotimes_{i=1}^{a} \mathcal{A}_i^* \to \bigotimes_{i=1}^{b} \mathcal{A}_i^* .$$

Assuming that $(J^{i_k}, H^{i_k}) = (J^{j_k}, H^{j_k})$, for a c-tuple $\alpha = ((i_1, j_1), \dots, (i_c, j_c))$ with $i_k \leqslant a$ and $j_k > a$ we define the contracted operator

$$\operatorname{tr}_\alpha Z(\Sigma) \colon \bigotimes_{i \in \operatorname{tr}_\alpha(1,\dots,a)} \mathcal{A}_i^* \to \bigotimes_{i \in \operatorname{tr}_\alpha(a+1,\dots,a+b)} \mathcal{A}_i^*$$

by successively taking the $\mathbb{Z}_2$-trace for each pair of algebraic entries (i_k, j_k), $k = 1, \dots, c$. The following proposition states that the algebraic contraction tr_α corresponds geometrically to the process of gluing handles into each entrance-exit pair occuring in the gluing data α.

5.4.5 Proposition *Let Σ be a connected model surface of type (a, b, g) and the ends labeled by regular pairs which are admissible for the contraction operation tr_α as above. Then*

$$\operatorname{tr}_\alpha Z(\Sigma) = Z(\operatorname{tr}_\alpha \Sigma)$$

and the contracted operator has the same degree as $Z(\Sigma)$,

$$\deg\big(\operatorname{tr}_\alpha Z(\Sigma)\big) = \deg\big(Z(\Sigma)\big) = n(a + b + 2g - 2)\,.$$

As we noticed already in Section 3.2 gluing and contracting is associative. Moreover, both can be viewed as equivalent processes. Contracting corresponds to successive gluing with handles of the type $(1,1,0)$. Vice versa, gluing can be considered to be a contracting process for a model surface consisting of more than one connected component. In fact, considering only connected model surfaces is unnecessarily restrictive. The entire analysis for the solution spaces is valid for every compact oriented surface with oriented ends except that the fundamental Proposition 5.3.2 which is based on the classification of surfaces only holds component-wise. From now on we allow the model surface to be disconnected.

5.4.6 Definition *By the topological type of a (disconnected) model surface we understand the collection $(k, (a_i, b_i, g_i))$ of the number k of components and the (connected) topological type of each component. Given two model surfaces Σ^j of the types $(k^j, (a_i^j, b_i^j, g_i^j))$, $j = 1, 2$ we denote the disjoint union by $\Sigma^1 \dot\cup \Sigma^2$.*

Let us assume without loss of generality that the surfaces Σ^i are connected and supplied with regular pairs $(J_i^k, H_i^k)_{k=1,\dots,a_i+b_i}$. Then we apply a suitable renumbering to these pairs and the cylindrical ends of $\Sigma^1 \dot\cup \Sigma^2$ yielding $(J^k, H^k)_{k=1,\dots,a+b}$ for $a = a_1 + a_2$ and $b = b_1 + b_2$ such that the orientations are $\epsilon_i = -1$ if $i \leqslant a$ and $\epsilon_i = +1$ if $i > a$. We obtain the obvious identity

$$Z(\Sigma^1 \dot\cup \Sigma^2) = Z(\Sigma^1) \otimes Z(\Sigma^2), \tag{5.8}$$

for the operators

$$\begin{aligned} &H^*(J^1, H^1) \otimes \ldots \otimes H^*(J^{a_1+a_2}, H^{a_1+a_2}) \\ &\to H^*(J^{a_1+a_2+1}, H^{a_1+a_2+1}) \otimes \ldots \otimes H^*(J^{a+b}, H^{a+b}), \end{aligned}$$

see Figure 5.6. This allows us now to unify the whole concept of gluing, contraction and composition.

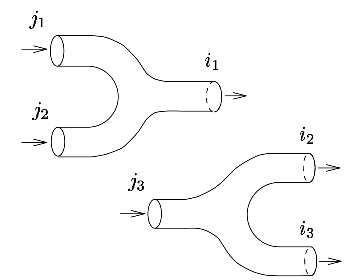


Figure 5.6: Operator associated to a disjoint union of model surfaces.

5.4.7 Definition *Let Σ^i be as above. Given the gluing data $\alpha = ((i_1, j_1), \dots, (i_c, j_c))$ with $2c$ different labels $i_1, \dots, i_c \leqslant a$ and $j_1, \dots, j_c > a$, the class of model surfaces $\Sigma^1 \#_\alpha \Sigma^2$ is uniquely determined. We define the composition*

$$Z(\Sigma^2) \circ_\alpha Z(\Sigma^1) = Z(\Sigma^1 \#_\alpha \Sigma^2)\,.$$

By Proposition 5.4.4 this definition is compatible with the standard composition where all exits of Σ^1 are glued with all entrances of Σ^2. In view of Proposition 5.4.5 and (5.8) composing and contracting can now be stated to be equivalent as follows.

5.4.8 Corollary *Under the above conditions we have*

$$Z(\Sigma^2) \circ_\alpha Z(\Sigma^1) = Z(\Sigma^1 \#_\alpha \Sigma^2) = Z\big(\operatorname{tr}_\alpha(\Sigma^1 \dot\cup \Sigma^2)\big) = \operatorname{tr}_\alpha\big(Z(\Sigma^1) \otimes Z(\Sigma^2)\big)\,.$$

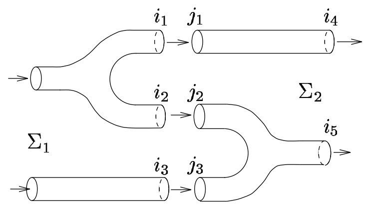
Note that the degree of $Z(\Sigma^1 \#_\alpha \Sigma^2)$ is well-defined independently of α. Obviously, it has to be

$$\deg Z(\Sigma^1 \#_\alpha \Sigma^2) = Z(\Sigma^1) + Z(\Sigma^2)$$

because transformations of α act by permutations of the cylindrical ends. In general, the degree of $Z(\Sigma)$ for a model surface of type $(k, (a_i, b_i, g_i))$, that is, number of components k, sum of entrances $a = \sum_i a_i$ and analogously $b = \sum_i b_i$, $g = \sum_i g_i$ is

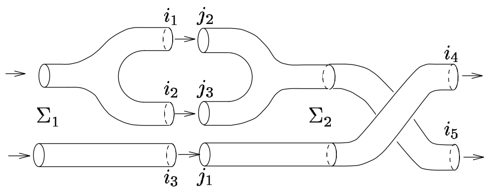
$$\boxed{\deg(Z) = n(a + b + 2g - 2k)} \tag{5.9}$$

in view of (5.8). The difference $g - k$ for the surface $\Sigma^1 \#_\alpha \Sigma^2$ is independent of α. However, without further analysis, the associated operator $Z(\Sigma^1 \# \alpha \Sigma^2)$ still depends on the gluing data α as is illustrated by the example in Figure 5.7.



$$\alpha = ((i_1, j_1), (i_2, j_2), (i_3, j_3))$$

$$\overset{?}{=}$$



$$\alpha' = ((i_1, j_2), (i_2, j_3), (i_3, j_1))$$

Figure 5.7: Permutation of gluing data.

5.4.3 The Canonical Isomorphisms

The principle used in the proof of the homotopy invariance in Section 5.2 goes back to the standard Floer theory which states that every regular homotopy between two regular pairs (J^α, H^α) and (J^β, H^β) induces the same, therefore canonical homomorphism

$$\Phi^*_{\beta\alpha} \colon H^*(J^\alpha, H^\alpha) \to H^*(J^\beta, H^\beta)$$

of degree 0. This is equivalent to the special case of the model surface $Z = \Sigma_{1,1,0}$ in the present S^1-cobordism theory. Using the gluing techniques and the constant homotopy it is proven that

$$\Phi^*_{\gamma\beta} \circ \Phi^*_{\beta\alpha} = \Phi^*_{\gamma\alpha}, \qquad \Phi^*_{\alpha\alpha} = \mathrm{id} \tag{5.10}$$

for all $\alpha, \beta, \gamma \in \mathcal{I}$, where $\mathcal{I}$ denotes the set of all regular pairs (J^α, H^α). In the terms of Conley, $\mathcal{I}$ forms a **connected simple system**. This allows to define Floer cohomology groups, which are independent from the regular pair, together with natural isomorphisms,

$$HF^k(M, \mathbb{Z}_2) = \Big\{ (x_\alpha) \in \prod_{\alpha \in \mathcal{I}} H^k(J^\alpha, H^\alpha) \,\big|\, x_\beta = \Phi^*_{\beta\alpha} x_\alpha \text{ f.a. } \alpha, \beta \in \mathcal{I} \Big\},$$

$$\pi_\alpha \colon HF^k(M, \mathbb{Z}_2) \xrightarrow{\cong} H^k(J^\alpha, H^\alpha), \quad (x_\gamma)_{\gamma \in \mathcal{I}} \mapsto x_\alpha\,.$$

We can now generalize the relation (5.10) to arbitrary model surfaces.

5.4.9 Corollary *Let a given model surface Σ be endowed with two different labelings by regular pairs $(\alpha_i)_{i=1,\dots,a+b}$ and $(\beta_i)_{i=1,\dots,a+b}$, $\alpha_i, \beta_i \in \mathcal{I}$. Then, denoting the associated cohomology operations by*

$$Z(\alpha) \colon \bigotimes_{i=1}^{a} H^*(\alpha_i) \to \bigotimes_{i=a+1}^{a+b} H^*(\alpha_i),$$

and respectively $Z(\beta)$, and

$$\Phi^{\otimes^a}_{\beta\alpha} = \Phi_{\beta_1\alpha_1} \otimes \ldots \otimes \Phi_{\beta_a\alpha_a}$$

respectively $\Phi^{\otimes^b}_{\beta\alpha}$ with $\Phi^{\otimes}_{\alpha\beta} = (\Phi^{\otimes}_{\beta\alpha})^{-1}$, it holds

$$Z(\beta) \circ \Phi^{\otimes^a}_{\beta\alpha} = \Phi^{\otimes^b}_{\beta\alpha} \circ Z(\alpha)\,.$$

This relation implies that the cohomology operations $Z(\Sigma)$ are well-defined for the Floer cohomology groups $HF^*(M, \mathbb{Z}_2)$. Let us choose the fixed short hand notation

$$\mathcal{A}^* = HF^*(M, \mathbb{Z}_2)$$

for the $\mathbb{Z}$-graded $\mathbb{Z}_2$-vector space. Thus, we may finally define the ultimate version of cohomology operations.

5.4.10 Definition *Let Σ be a given topological class of a (disconnected) model surface of type $(k, (a_i, b_i, g_i)_{i=1,\dots,k})$ with labeled ends. Then the cohomology operation $Z(\Sigma)$ is defined by*

$$Z(\Sigma) = \otimes_{i=a+1}^{a+b} \pi^{-1}_{\alpha_i} \circ Z(\alpha) \circ \otimes_{i=1}^{a} \pi_{\alpha_i}$$

where $\alpha = (\alpha_i)_{i=1,\dots,a+b}$ is a fixed family of regular pairs associated to the labeling of Σ which satisfies $\epsilon_i = -1$ for $i \leqslant a = a_1 + \ldots + a_k$ and $\epsilon_i = +1$ for $a < i \leqslant b = b_1 + \ldots + b_k$.

In view of Corollary 5.4.9, this definition of $Z(\Sigma)$ does not depend on the choice of α.

We now give an alternative presentation for the operators $Z(\Sigma)$. In the next chapter we will reconsider the natural representation of Poincaré duality in Floer homology. We can view the reversal of the orientation of an end of the surface Σ, that is of S^1, as interchanging $\mathcal{A}^* = HF^*(M, \mathbb{Z}_2)$ with its dual vector space. We recapture the grading by defining

$$\overline{\mathcal{A}}^k = \mathrm{Hom}(\mathcal{A}^{-k}, \mathbb{Z}_2)\,.$$

This is compatible with the grading given by the Conley-Zehnder index μ_{CZ} because the reversal of the parametrization $t \mapsto -t$ of a 1-periodic solution changes the index by factor -1. This grading gives rise to the identification

$$\begin{aligned} \mathcal{A}^d_{a,b} &= \big(\bigotimes\nolimits^a \overline{\mathcal{A}}^* \otimes \bigotimes\nolimits^b \mathcal{A}^*\big)^d \\ &= \big\{\, Z \in \mathrm{Hom}\big(\bigotimes\nolimits^a \mathcal{A}^*, \bigotimes\nolimits^b \mathcal{A}^*\big) \,\big|\, \deg Z = d \,\big\}\,. \end{aligned}$$

Let us conclude this section by summing up the above results in a form which fits in Atiyah's concept of a topological quantum field theory, see [3] or [4].

5.4.11 Theorem *Given a closed symplectic manifold (M,ω) satisfying the conditions $\phi_\omega = 0$ and $\phi_c = 0$, there is an associated functor Z which assigns*

(1) a finite-dimensional $\mathbb{Z}$-graded $\mathbb{Z}_2$-vector space to each compact oriented 1-dimensional manifold S,

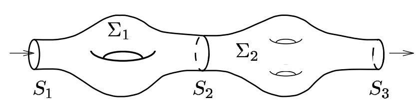
(2) a vector $Z(\Sigma) \in Z(S)$ for each compact oriented surface Σ with boundary[5] $\partial\Sigma = S$.

This functor satisfies the axioms

A1 (Involutory) $Z(S^*) = \overline{Z(S)}$, where S^* denotes S with opposite orientation and $\overline{Z(S)}$ is the dual $\mathbb{Z}_2$-space,

A2 (Multiplicativity) $Z(S_1 \cup S_2) = Z(S_1) \otimes Z(S_2)$, where $\cup$ is the disjoint union.

A3 (Associativity) For a composite cobordism $\Sigma = \Sigma_1 \cup_{S_2} \Sigma_2$ it holds



$$Z(\Sigma) = Z(\Sigma_2) \circ Z(\Sigma_1) \in \mathrm{Hom}\,(Z(S_1), Z(S_3)).$$

A4 (Non-triviality I) $Z(\emptyset) = \mathbb{Z}_2$ for the empty 1-manifold, and

A5 (Non-triviality II) $Z(S \times I)$ is the identity endomorphism of $Z(S)$, where $I = [0,1]$.

This abstract scheme is realized in Floer homology as follows. Given a compact oriented surface Σ with oriented boundary $S = \partial\Sigma$ of type (a,b), we associate to S the graded $\mathbb{Z}_2$-vector space

$$Z(S) = \mathcal{A}^*_{a,b} = \mathrm{Hom}(\otimes^a HF^*, \otimes^b HF^*),$$

where $HF^* = HF^*(M,\mathbb{Z}_2)$ is the $\mathbb{Z}$ graded Floer (co-)homology with coefficients in $\mathbb{Z}_2$. The vector $Z(\Sigma)$ is the cohomology operation from Definition 5.4.10. Functoriality for Z is given in its simplest way. Since Z is homotopy invariant the group of orientation preserving homeomorphisms of S acts on $Z(S)$ via its group of components, that is, by permutations of the entrances and the

[5] which might, of course, be empty

exits, respectively. The vector $Z(\Sigma)$ is invariant under orientation preserving diffeomorphisms of Σ with respect to the oriented ends. This has been shown in Section 5.3. Corollary 5.3.3 stated that $Z(\Sigma)$ is invariant under orientation preserving permutations σ, τ of the boundary if Σ is connected, recall (5.7),

$$\sigma(\partial\Sigma^+) \circ Z(\Sigma) \circ \tau(\partial\Sigma^-) = Z(\Sigma)\,. \tag{5.11}$$

The main part in the deduction of the S^1-cobordism functor Z is the associativity property (A3). This was shown by Corollary 5.4.8. The non-triviality property (A5) is already developed within the original Floer homology theory. It follows from the canonical isomorphisms, see (5.10). The ingredient of the theory which is new with respect to the formulation as a topological field theory, is the grading of the vector space $Z(S)$ and the related degree of the vector $Z(\Sigma)$. The formula (5.9) expresses the compatibility of the degree with the associativity property (A3).

As a test for the consistency of the cobordism functor Z we ask for the simplest invariant $Z(S^1 \times S^1)$ associated to M. An immediate consequence of the axiomatic presentation of this cobordism theory is

5.4.12 Corollary *The number $Z(S^1 \times S^1)$ reproduces the $\mathbb{Z}_2$-Euler characteristic of M,*

$$Z(S^1 \times S^1) \equiv \dim \mathcal{A}^* \equiv \chi(M) \mod 2\,.$$

This follows more generally for disjoint unions of circles as

$$\dim Z(S) \equiv Z(S \times S^1) \mod 2$$

from the fact that the mapping torus

$$\Sigma_f = S \times [0,1]/(S \times \{1\} \sim f(S) \times \{0\})$$

associated to $f \in \mathrm{Aut}^+(S)$ yields the topological invariant

$$\mathrm{Trace}_{\mathbb{Z}_2} Z(f) = Z(\Sigma_f),$$

see [3] or [4].

Let us prove this identity for the torus $\Sigma = S^1 \times S^1$ directly. This is easily carried out because counting generic parametrized tori, that is, for the regular data (J,k) (!), amounts exactly to counting contractible periodic solutions. (This also holds for more general situations like for instance monotone symplectic manifolds.)

Proof of Corollary 5.4.12. Let $H \in C^\infty(S^1 \times M, \mathbb{R})$ be a regular Hamiltonian and $J \in C^\infty(S^1 \times M)$ be an ω-compatible almost complex structure. In order to compute the number $Z(S^1 \times S^1)$ we have to consider the smooth solutions of

$$\begin{aligned} &u\colon S^1 \times S^1 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to M,\\ &\partial_s u + J(t,u)\partial_t u + \nabla H_t(u) = 0\,. \end{aligned} \tag{5.12}$$

We immediately compute the energy of such a solution as

$$\Phi_{J,H}(u) = \int_{S^1\times S^1} u^*\omega\,.$$

On the other side, we know that $Z(S^1\times S^1)$ is merely based on the solutions of spherical type. Thus, the condition $\phi_\omega = 0$ implies that

$$\partial_s u \equiv 0, \quad u(s,\cdot) = x, \quad \dot{x} = X_{H_t}(x),$$

for all $s\in\mathbb{R}$. This is also clear from the real-valued action functional which has to increase strictly along non-constant trajectories u. But u from (5.12) is also 1-periodic in s-direction, $u(s+1,t) = u(s,t)$. That is, u remains s-constantly within a non-degenerate 1-periodic solution for H. Due to the independence of s, the linearization of $\overline{\partial}_{J,H}$ at u has the form

$$D_u = \partial_s + J_o\partial_t + S(t)$$

where it is considered as an operator

$$D_u \colon H^{1,p}(S^1\times S^1,\mathbb{C}^n) \to L^p(S^1\times S^1,\mathbb{C}^n),$$

without loss of generality for $p=2$. Since $A_s = J_o\partial_t + S(t)$ is invariant under s-translation and the loop $S(t)$ is admissible in the sense of Definition 3.1.1, we can prove analogously to Proposition 3.1.12 that D_u is an isomorphism. Thus the pair (J,H) is regular for the closed model surface $S^1\times S^1$. Counting the solutions of (5.12) amounts exactly to counting the 1-periodic solutions of the Hamiltonian equation associated to H. Consequently the restriction by counting only spherical solutions confines us to the contractible 1-periodic solutions, that is, $\mathcal{P}_1(H)$. Finally, since

$$\chi(H^*(H,J)) \equiv \dim C^*(H,J) \mod 2$$

and by definition $\dim C^*(H,J) = \#\mathcal{P}_1(H)$, counting the torus solutions of (5.12) modulo 2 yields the Euler characteristic of $HF^*(M,\mathbb{Z}_2)$ modulo 2 and this is by Floer's classical result, Theorem 1.1.2, the Euler characteristic of M modulo 2. ∎

In the more general case of monotone symplectic manifolds, for example $\mathbb{C}P^n$ with the standard structure, counting 0-dimensional solutions again restricts to the s-constant solutions with zero energy. However, in general, one has to show that there is a regular pair (J,H) such that also non-s-constant solutions are regular. This can be carried out exactly like the proof of the existence of regular pairs (J,H) for "gradient flow" trajectories, when (J,H) have to be s-independent, see e.g. [23].

5.5 The Floer Algebra, Results and Outlook

In the last section we observed that the functor Z which assigns cohomology operations on Floer homology to each S^1-cobordism fits into the concept of an

axiomatic topological field theory in dimensions $1+1$. It appears however to be too general a concept for the occuring cobordism relations. In fact, by a simple decomposition argument and the transitivity relation (A3) for the operators we can reduce the whole theory to the study of elementary building blocks for all oriented cobordisms.

In the beginning of the last section we analyzed the decomposition of given model surfaces by properly embedded loops. Stretching over tubular neighbourhoods of the type of annuli yielded the process which is complementary to sewing of surfaces along their ends. The only condition which has to be guaranteed is that the images of the loops under the solution mappings u into the symplectic manifold have to be contractible. This secures that the periodic solutions of Hamiltonian equations which appear in the stretching limit are contractible and therefore generators of Floer homology. But the contractibility is always true because the cohomology operations are built only upon the spherical solutions. This fact proved as Theorem 5.4.3 is the key of a reduction of the 'topological field theory' Z to a $\mathbb{Z}$-graded algebra $\mathcal{A}^*$ over $\mathbb{Z}_2$.

In the theory of Riemann surfaces one considers decompositions of surfaces of genus $g \geqslant 2$ into building blocks of genus 0 with three boundary components. More precisely, one always finds a decomposition of a closed connected oriented surface of genus $g\geqslant 2$ into pairs of pants. In fact, by identifying each such Riemann surface with a cubic graph, one argues that it admits a **pair-of-pants decomposition** by $3g-3$ embedded loops, see for example [7]. A model for the pair of pants is the sphere with three disks removed.

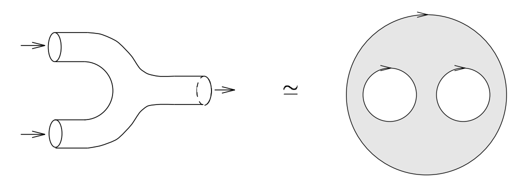


Figure 5.8: The pair of pants.

In the class of oriented surfaces with boundary we have to add the compact disk as an elementary building block. Since we consider oriented S^1-cobordisms, that is, the boundaries of the surfaces are additionally oriented, we have to decide for orientations on the boundaries of the basic disk and the pair of pants. We also have to add both handles with two entries respectively exits in order to be able to reverse boundary orientations by gluing. We complete our minimal set of building blocks by the following list:

1. The standard disk $D^2 = \Sigma_{0,1,0}$ with one exit,
2. the pair of pants $\Sigma_{2,1,0}$ with two entries and one exit and
3. the two handles $\Sigma_{2,0,0}$ and $\Sigma_{0,2,0}$ with two entries respectively two exits.

Thus, every operator $Z(\Sigma)$ associated to a model surface $\Sigma_{a,b,g}$ is decomposable into the above building blocks. The entire theory is uniquely determined by these basic operators.

In the following section we discuss the basic operators in detail. In particular we study the relation with the standard cohomology of the manifold M. Recall that the choice of a regular time-independent Hamiltonian H which is C^2-small enough together with a suitably regular J yields a natural isomorphism

$$HF^*(M,\mathbb{Z}_2) \cong H^{\mathrm{sing}}_{n-*}(M,\mathbb{Z}_2)\,. \tag{5.13}$$

The identification of Floer cohomology with standard homology but with reversed and shifted grading is due to the definition of the action functional and the grading given by the Conley-Zehnder index. We have chosen sign conventions such that the negative gradient flow for the Hamiltonian corresponds to positive gradient flow for the action. Let us again sum up these conventions.

Given the symplectic form ω on M we associate a compatible almost complex structure J by demanding that $\omega\circ(\mathrm{Id}\times J) = \langle\cdot,\cdot\rangle_J$ is a Riemannian metric on TM. The Hamiltonian vector field X_H for a function $H \in C^\infty(S^1\times M,\mathbb{R})$ is determined by $dH_t = -\omega(X_{H_t},\cdot)$. In the case of the standard symplectic vector space $(\mathbb{R}^{2n},\omega_o)$ with $\omega_o = \sum_{j=1}^n dx_j\wedge dy_j$ we obtain the standard structure J_o compatible with ω_o as $J_o = i\oplus\ldots\oplus i$ with $i\partial_{x_j} = \partial_{y_j}$ and $i\partial_{y_j} = -\partial_{x_j}$. The symplectic action $\mathcal{A}$ (without Hamiltonian term) is defined for a contractible loop $x\in\Omega^o(M)$ equipped with a fixed homotopy class of an extension u_x to the disk, $\tilde{x} = [x,u_x]$, by

$$\mathcal{A}(\tilde{x}) = \int_{D^2} u_x^*\omega\,.$$

In the case of $(\mathbb{R}^2,\omega_o)$, this yields exactly the algebraic area enclosed by the loop x. In order to obtain the 1-periodic contractible solutions of the Hamiltonian equation, $\dot{x}(t) = X_{H_t}(x(t))$, as critical points for the Hamiltonian action $\mathcal{A}_H$, we have to set

$$\mathcal{A}_H(\tilde{x}) = \mathcal{A}(\tilde{x}) - \int_0^1 H_t(x(t))dt\,.$$

We now recall the conventions for the Conley-Zehnder index μ_{CZ}, which provides the $\mathbb{Z}$-grading on $\mathrm{Crit}\,\mathcal{A}_H$ if $\phi_c = 0$. In order to have an analogy to the finite dimensional Morse theory, we demand that μ_{CZ} acts as a relative Morse index for the action functional $\mathcal{A}_H$. A relative increase of $\mathcal{A}_H$ has to correspond to an increase in μ_{CZ}. Recalling Theorem 3.3.7, the index μ_{CZ} for symplectic arcs in the linear situation is defined such that $\mu_{CZ}([t\mapsto \mathrm{e}^{(2k+1)\pi it}]) = 2k+1$ for the symplectic vector space $(\mathbb{R}^2,\omega_o)$. This definition is related to the standard structure $J_o = i\oplus\ldots\oplus i$.

Having fixed these natural choices for the action $\mathcal{A}_H$ and the index μ_{CZ}, it only remains to decide whether we want to consider a homology theory by using the negative "relative gradient flow" for $\mathcal{A}_H$, or a cohomology theory induced by the positive "relative gradient flow". The latter is realized by the bounded energy solutions of the partial differential equation

$$(\partial_s + J\partial_t + \nabla H)(u) = 0,$$

where $\nabla H = -JX_H$. In the present work we have opted for the formulation as a cohomology theory. Such a decision turns out to be crucial in the definition of a symplectic homology theory for open subsets of $\mathbb{R}^{2n}$, where relative homology groups with respect to a filtration by the symplectic action are considered, see [31] and [22]. In our situation of a compact symplectic manifold, the existence of the inherent symmetry by Poincaré duality implies that both versions, cohomology and homology, are equivalent. In any case, due to the above conventions, a Floer cohomology theory corresponds to the classical homology theory shifted by $n = \frac{1}{2}\dim M$ as indicated above. For the converse, the classical cohomology of M would be represented by the Floer homology groups, as long as it is considered as a graded $\mathbb{Z}_2$-vector space.

5.5.1 The Basic Operators

Let us now discuss in detail the basic elements for the cobordism functor Z. Table 5.1 on page 195 lists the simplest operators $Z(\Sigma)$.

5.5.1.1 The Disk

The operator $Z(\Sigma_{0,1,0})$ represents a fixed element in $\mathcal{A}^{-n} = HF^{-n}$ of order $-n$. The isomorphism (5.13) identifies it with an element in $H_{2n}(M,\mathbb{Z}_2)$ which must be the fundamental class of M. It cannot be zero because it acts as the neutral element for the pair-of-pants product as we will see below. If we considered a theory with $\mathbb{Z}$-coefficients it could be a priori any nonzero integer multiple. We denote this operator by

$$e = Z(\Sigma_{0,1,0})\,.$$

5.5.1.2 The Handles

We have already deduced from classical Floer theory that the model surface $\Sigma_{1,1,0}$ gives rise to the identity endomorphism on HF^*. Before we present a similar deduction that the handles with equally oriented ends correspond under (5.13) to the Poincaré duality, we argue by simply using the axiomatic properties of Z that

$$\beta = Z(\Sigma_{2,0,0}) \quad\text{and}\quad \bar{\beta} = Z(\Sigma_{0,2,0})$$

are regular 2-tensors of degree 0, see Figure 5.9. Due to the invariance under permutation of cylindrical ends of equal orientation, recall (5.11), both 2-tensors are symmetric.

Just homotopy invariance and the associativity (A3) imply that

$$(\beta\otimes\mathrm{id})\circ(\mathrm{id}\otimes\bar{\beta}) = (\mathrm{id}\otimes\beta)\circ(\bar{\beta}\otimes\mathrm{id}) = \mathrm{id}\,. \tag{5.14}$$

Therefore both the symmetric bilinear form $\beta\colon \mathcal{A}^*\otimes\mathcal{A}^*\to\mathbb{Z}_2$ and the dual 2-tensor $\bar{\beta}\in\mathcal{A}^*\otimes\mathcal{A}^*$ must be non-degenerate. We can view them as isomorphisms

$$\beta\colon \mathcal{A}^k \xrightarrow{\cong} \bar{\mathcal{A}}^k,\quad \bar{\beta} = \beta^{-1}$$

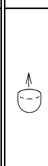
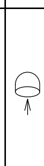
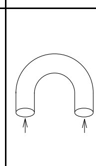
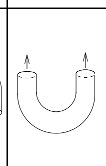
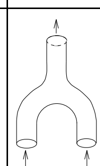
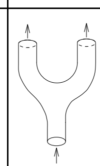
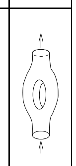
(a,b,g)	symbol	degree	algebraic operator $Z(\Sigma)$	corresponding operator on singular (co-)homology
$(0,1,0)$		$-n$	$e\colon \mathbb{Z}_2 \to HF^{-n}$	$[M] \in H_{2n}(M,\mathbb{Z}_2)$
$(1,0,0)$		$-n$	$\bar{e}\colon HF^{n} \to \mathbb{Z}_2$	$1 \in H^0(M,\mathbb{Z}_2) = \mathbb{Z}_2$
$(1,1,0)$		0	id_{HF^*}	$\mathrm{id}_{H_*(M,\mathbb{Z}_2)}$
$(2,0,0)$		0	$\beta\colon HF^k \otimes HF^{-k} \to \mathbb{Z}_2,\ k \in \mathbb{Z}$	$\mathrm{PD}\colon H_k \xrightarrow{\cong} H^{2n-k},\ k = 0,\dots,2n$
$(0,2,0)$		0	$\bar{\beta} \in \bigoplus_{k\in\mathbb{Z}} HF^k \otimes HF^{-k}$	$\mathrm{PD}^{-1}\colon H^k \xrightarrow{\cong} H_{2n-k},\ k = 0,\dots,2n$
$(2,1,0)$		n	$m\colon HF^k \otimes HF^l \to HF^{k+l+n}$, $k,l \in \mathbb{Z}$	$m\colon H_k \otimes H_l \to H_{k+l-2n}$
$(1,2,0)$		n	$\bar{m}\colon HF^m \to \bigoplus_{k+l=m+n} HF^k \otimes HF^l$	$\bar{m}\colon H^k \otimes H^l \to H^{k+l}$
$(1,1,1)$		$2n$	$m \circ \bar{m}\colon HF^{-n} \to HF^{n}$	$m \circ \bar{m}\colon H_{2n}(M,\mathbb{Z}_2) \to H_0(M,\mathbb{Z}_2)$

Table 5.1: Comparison of cohomology operations

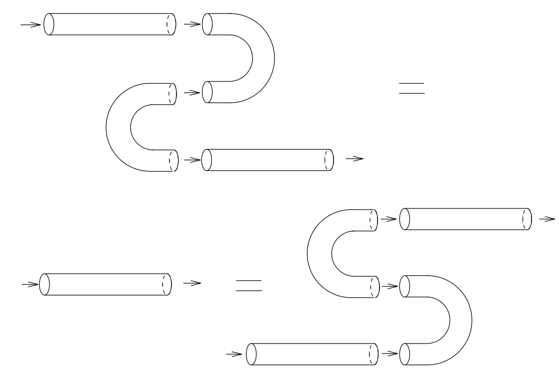


Figure 5.9: The duality and regularity of the bilinear form β.

for $k \in \mathbb{Z}$, which are inverse to each other. Note that, due to Corollary 5.4.12, it holds

$$\beta \circ \bar{\beta} = Z(\Sigma_{0,0,1}) \equiv \chi(M) \mod 2\,.$$

In view of (5.13), β yields an isomorphism

$$H_{n-k} \cong HF^k \xrightarrow{\beta} \mathrm{Hom}(HF^{-k},\mathbb{Z}_2) \cong H^{n+k}\,.$$

Considering the Cauchy-Riemann type equation $\overline{\partial}_{J,k}u = 0$ on the handle $\Sigma_{2,0,0}$ where both ends are equipped with the same regular pair (H,J) we may transfer the computation to the standard cylinder $\mathbb{R} \times S^1$ after conformally reparametrizing one of the ends by $(s,t) \to (-s,-t)$. Thus we are able to consider k as a homotopy of Hamiltonians just like for the identity operator. But the conformal coordinate change, without loss of generality on the positive end, leads to the change of sign $H \to -H$,

$$\begin{aligned} \overline{\partial}_{J,k}u &= \partial_s u + J\partial_t u + \nabla H, \quad \text{for } s \leqslant -T,\\ \overline{\partial}_{J,k}u &= \partial_s u + J\partial_t u - \nabla H, \quad \text{for } s \geqslant T\,. \end{aligned}$$

In Morse homology, the change of the sign of the Morse function provides the Poincaré duality[6]

$$H_k(M,\mathbb{Z}) \xrightarrow{\cong} H^{2n-k}(M,\mathbb{Z}),$$

see [50], Section 5.2. This relation between Poincaré duality and the symmetry operation of reversing the orientation of S^1 justifies the difference n between the grading of HF^* and H_{sing}.

[6] Recall that $\dim M = 2n$.

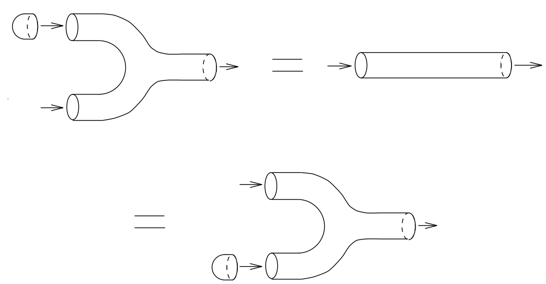


Figure 5.10: The unit element for the pair-of-pants multiplication.

5.5.1.3 The Pair of Pants

The most important element of this theory is obviously the multiplication

$$m = Z(\Sigma_{2,1,0})\colon (\mathcal{A}\otimes\mathcal{A})^* \to \mathcal{A}^{*+n}\,,$$

an operator of degree n. Analogously to (5.14) we deduce from the 'axiomatic' properties in Theorem 5.4.11 that e is the neutral element for m,

$$m\circ(e\otimes \mathrm{id}) = m\circ(\mathrm{id}\otimes e) = \mathrm{id}\,,$$

see Figure 5.10.

Besides, this fact implies the existence of pair-of-pants solutions. Choose a properly embedded loop on the standard model surface $\mathbb{R}\times S^1$ and carry out the stretching process on a tubular neighbourhood with a fixed regular pair (J,H) on this cylindrical region. Then, we obtain a pair of pants with an adherent disk in the limit.

Moreover, let us state again the associativity for the multiplication

$$m\circ(m\otimes\mathrm{id}) = Z(\Sigma_{3,1,0}) = m\circ(\mathrm{id}\otimes m)$$

which follows from homotopy invariance and the associativity (A3), see Figure 5.11. The permutation invariance (5.11) for connected model surfaces accounts for commutativity,

$$m(a,b) = m(b,a)$$

for all $a,b\in\mathcal{A}^*$. Altogether we obtain the following result.

5.5.1 Theorem *Let (M,ω) be a closed symplectic manifold satisfying the condition that $\phi_\omega = \phi_c = 0$. Then the $\mathbb{Z}$-graded Floer cohomology with coefficients in $\mathbb{Z}_2$, $HF^*(M,\mathbb{Z}_2)$, carries the structure of a graded algebra $(\mathcal{A}^*, m, e, \beta)$ with an associative and commutative multiplication m of degree $n = \frac{1}{2}\dim M$, unit*

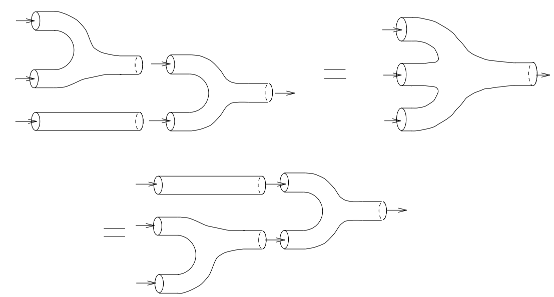


Figure 5.11: The associativity of the pair-of-pants product.

e of degree $-n$ and non-degenerate symmetric bilinear form β of degree 0 which is compatible with m in the sense that

$$\beta(m(a,b),c) = \beta(a,m(b,c))$$

for all $a,b,c\in HF^*$. Moreover, β has an 'adjoint' $\bar\beta\in(HF^*\otimes HF^*)^0$ such that (5.14) holds.

The algebra structure described in this theorem neglecting the grading is also called a Frobenius algebra. Such a Frobenius algebra generally follows from a 1+1-dimensional topological field theory satisfying the axioms in Theorem 5.4.11. For the compatibility of the non-degenerate bilinear form and the multiplication see Figure 5.12.

However, in the concrete case of Floer homology, relating the multiplication m back to the standard homology $H_*^{\mathrm{sing}}(M,\mathbb{Z}_2)$ via (5.13) requires a far more involved analysis. At this stage, we only state the result:

> *Under the condition that $\phi_c = \phi_\omega = 0$, the pair-of-pants multiplication m on the Floer cohomology $HF^*(M,\mathbb{Z}_2)$ is canonically isomorphic to the cup-product. The canonical vector space isomorphisms from Theorem 1.1.2*
>
> $$\Phi^*\colon HF^k(M,\mathbb{Z}_2) \xrightarrow{\cong} H_{n-k}^{sing}(M,\mathbb{Z}_2)$$
>
> *composed with the Poincaré duality isomorphism provide a canonical algebra isomorphism from $(\mathcal{A},m,e,\beta)$ to the $\mathbb{Z}_2$-cohomology ring of the closed oriented manifold M.*

Note that, strictly speaking, the isomorphism Φ^* leads to a product structure on the standard homology of M,

$$m\colon H_{n-k}\otimes H_{n-l}\to H_{-(k+l)}\,.$$

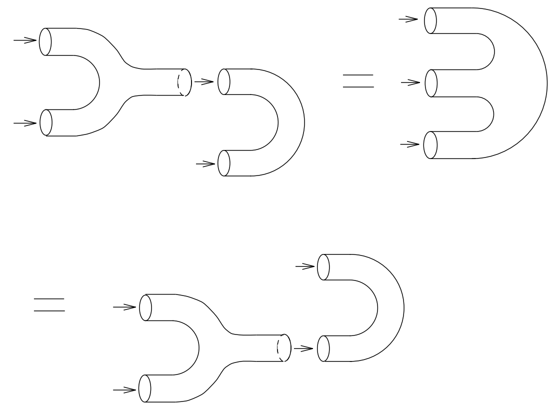


Figure 5.12: The compatibility of the pair-of-pants product with the Poincaré duality.

But applying Poincaré duality yields a cohomology product of the right degree. Thus we should rather expect the intrinsic cup product for the operation $Z(\Sigma_{1,2,0})$. The fundamental class e is then replaced by the unit element $\bar{e} = Z(\Sigma_{1,0,0}) \cong 1 \in H^0(M, \mathbb{Z}_2)$.

5.5.2 Outlook

In this exposition we constructed the analytical basis for a cobordism functor Z in the Floer homology of a symplectic manifold. However, this was executed under several restrictions.

First, we restricted the counting scheme for 0-dimensional solutions to coefficients in $\mathbb{Z}_2$. In full generality, one has to carry out a complete coherent orientation programme, as used in [21]. This means compatible orientations for all determinant bundles associated to the spaces of Fredholm operators which occur as linearizations of $\overline{\partial}_{J,k}$ at solutions.

Second, we restricted the analysis to the case of $\phi_\omega = \phi_c = 0$. This provided us with a well-defined action functional on the loop space component $\Omega^\circ(M)$ and a well-defined integer-valued index for 1-periodic solutions. The latter is responsible for the $\mathbb{Z}$-grading on $\mathcal{A}^*$. Floer however already considered a more general situation for the construction of his homology theory. He admitted the far wider class of monotone symplectic manifolds, i.e.

$$\phi_\omega = \lambda \phi_c, \quad \lambda \geqslant 0\,.$$

This condition is sufficient to exclude the bubbling-off phenomenon which can obstruct the necessary compactness for the moduli spaces of bounded energy

solutions. In this more general situation in which for example the consideration of $\mathbb{C}P^n$ with the Fubini-Study structure is allowed, the grading of HF^* is only given in $\mathbb{Z}_{2N}$, where N is the so-called minimal Chern number, defined by $N\mathbb{Z} = \mathrm{Image}(\phi_c)$. Also the action functional is no longer real valued. It can only be well-defined on a multiple cover of $\Omega^\circ(M)$ with covering group $\pi_2(M)/(\ker \phi_\omega)$.

Analogously to Floer's proof for the homology theory, in the case of monotone symplectic manifolds, the 0-dimensional components of the solution spaces $\mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}(J,k)$ can be proven to be compact and therefore finite. This requires a more general, relative energy estimate combined with the condition of monotonicity of (M,ω). In this work we simply used the well-defined action functional. But by using a purely relative formulation, we see that for two solutions $u, u' \in \mathcal{M}^{x_1,\dots,x_a}_{y_1,\dots,y_b}$ the homology class relative to the fixed boundary, $([u]-[u'])_{\mathrm{rel}\,\partial\Sigma} = A \in H_2(M,\mathbb{Z})$, is well-defined. Then, the restriction of merely counting solutions of a fixed local dimension implies that $c_1(A) = 0$ for the relative class A. This follows from the index formula in Theorem 3.3.11 on page 88. Thus, the monotonicity condition also yields $\{\omega\}(A) = 0$. Using the analogous energy formula from Lemma 4.3.14 on page 127, we obtain a uniform energy estimate for solutions of local dimension 0 and 1. However, we still need a more refined version of Theorem 4.3.12 excluding bubbling-off of pseudo-holomorphic spheres. Such a compactness proof implies that the functor Z is still well-defined. In fact, it is possible to generalize the S^1-cobordism functor Z to the class of weakly monotone symplectic manifolds as considered in [29]. Such generality requires the choice of a different coefficient ring than $\mathbb{Z}_2$ or $\mathbb{Z}$, namely a Novikov ring with respect to the abelian group $\pi_2(M)/(\ker \phi_\omega \cap \ker \phi_c)$, cf. [29], [38].

A detailed concrete discussion of the multiplicative structure given by the pair-of-pants product on Floer (co-)homology will be carried out in [49]. As already mentioned above, the pair-of-pants product under the condition that $\phi_\omega = 0$ turns out to be canonically isomorphic to the cup product. In that case, this ring structure still gives no means to distinguish Floer homology from the standard cohomology ring. But that changes drastically in the presence of J-holomorphic spheres. The pair-of-pants product, which can be defined for the more general class of weakly monotone symplectic manifolds[7], generally provides a different ring structure from the standard cohomology ring of M. It turns out that the pair-of-pants product represents a deformation of the cup product due to the presence of J-holomorphic spheres which is already known as the so-called quantum cohomology, cf. [57], [54], [44], [45], [38]. For example[8], on $\mathbb{C}P^n$ with its standard structure, where the minimal Chern number is $N = n+1$, quantum cohomology is isomorphic to the quotient of the polynomial ring over $\mathbb{Z}$ with the generators p of degree 2 and q, q^{-1} of degree $2N$ and $-2N$ and the relations $p^{n+1} = q$ and $qq^{-1} = 1$,

$$QH^*(\mathbb{C}P^n) = \frac{\mathbb{Z}[p,q,q^{-1}]}{\langle\, p^{n+1} = q,\ q^{-1}q = 1\,\rangle}\,.$$

[7]For a definition of weak monotonicity we refer to [29] or [38].

[8]cf. Example 8.1.6 in [38]

The crucial point for the product structure in Floer homology is that allowing $\phi_\omega \neq 0$ leads to homologically richer classes of possible pair-of-pants solutions counted for the operator $Z(\Sigma_{2,1,0})$. In [49] we will show that a slightly more refined definition of $Z(\Sigma_{2,1,0})$ with respect to the spherical classes (compare Definition 5.1.1) is equivalent to counting J-holomorphic spheres under the additional condition of three further incidence relations. Namely, three prescribed points of the spherical domain, say $\{0, 1, \infty\} \in \mathbb{C}P^1$ have to be mapped into given homology cycles which are geometrically represented by unstable manifolds of critical points for the negative gradient flow of three given Morse functions[9]. This amounts to the definition of quantum cohomology.

Summing up, it turns out that the cobordism functor Z developed in this thesis provides a ring structure on Floer homology HF^* which goes beyond the classical module isomorphism $HF^* \cong_{\text{module}} H_*(M)$ and detects differences due to Gromov's theory of J-holomorphic spheres in comparison with the standard ring structure on $H^*(M)$.

[9] For a sketch of these arguments compare also [43].

CHAPTER A

Banach Manifold Analysis

In this appendix we intend to give the analytical details for the proof of the Banach manifold property of $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ and the associated Banach bundles. We begin with studying the functorial properties of the spaces $H^{1,p}_\Sigma(\xi)$, $L^p_\Sigma(\xi)$ for $\xi \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$ which in Eliasson's terminology are defined as section functors, see [13] and also [50].

A.0.2 Proposition *The mappings $\mathfrak{S} = H^{1,p}_\Sigma$ and L^p_Σ which associate to each $\xi \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$ a vector space with Banach space topology,*

$$\mathfrak{S}\colon \mathrm{Vec}^*_{C^\infty}(\Sigma) \to \mathrm{Ban},$$

have the following functorial properties.

(a) $\mathfrak{S}$ associates to each C^∞-bundle homomorphism $\varphi\colon \xi \to \eta$ a linear map

$$\mathfrak{S}_*\varphi \in \mathcal{L}(\mathfrak{S}(\xi);\mathfrak{S}(\eta)), \quad (\mathfrak{S}_*\varphi)\cdot s = \varphi \cdot s$$

such that

$$\mathfrak{S}_*\colon C^\infty_\Sigma(\mathrm{Hom}(\xi,\eta)) \to \mathcal{L}(\mathfrak{S}(\xi);\mathfrak{S}(\eta))$$

is continuous.

(b) For $\mathfrak{S} = H^{1,p}_\Sigma$ with $p > 2$, $\mathfrak{S}(\xi) \hookrightarrow C^0(\xi)$ is a continuous embedding for each finite-dimensional vector bundle $\xi \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$, and the section functor $\mathfrak{S}$ induces a continuous mapping

$$\mathfrak{S}(\mathrm{Hom}(\xi,\eta)) \to \mathcal{L}(\mathfrak{S}(\xi);\mathfrak{S}(\eta)), \quad A \mapsto (A_*\colon s \mapsto As),$$

for all $\xi, \eta \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$.

The functor $\mathfrak{S}$ is called a **section functor** if it satisfies part (a) of this statement.

Proof. In connection to Definition 2.1.5 we have already argued that the Banach space topologies of $H^{1,p}_\Sigma(\xi)$ and $L^p_\Sigma(\xi)$ are well-defined in a way which is independent of the choice of the asymptotic trivialization of $\xi \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$. The crucial point for the following estimates is that in the definition of these functors we combined the standard Lebesgue measure of $\mathbb{R} \times S^1$ over the cylindrical ends of Σ with a special differentiable structure on the compact surface $\overline{\Sigma}$. Without loss of generality we assume that the bundle ξ is trivial, so that it is sufficient

to compute for $H^{1,p}(Z^\pm, \mathbb{R}^{2n})$ and $L^p(Z^\pm, \mathbb{R}^{2n})$. Given $A \in C^\infty_\Sigma(\mathrm{Hom}(\xi,\eta))$ we obtain the estimates

$$\|As\|_{0,p} \leqslant \|A\|_\infty \|s\|_{0,p}$$

and

$$\begin{aligned}\big\|(As)'\big\|_{0,p} &= \|A's + As'\|_{0,p} \leqslant \|A's\|_{0,p} + \|As'\|_{0,p}\\ &\leqslant \|A'\|_{0,p}\|s\|_\infty + \|A\|_\infty \|s'\|_{0,p}\,.\end{aligned}$$

Lemma 2.1.3 guarantees that the norms $\|A'\|_{0,p}$ and $\|A\|_\infty$ are finite. Furthermore, over the infinite cylinder we have the continuous Sobolev embedding $H^{1,p}(Z^\pm, \mathbb{R}^n) \hookrightarrow C^0(Z^\pm, \mathbb{R}^n)$, i.e. there is a uniform constant $c(p) > 0$ such that

$$\|v\|_\infty \leqslant c(p)\,\|v\|_{1,p} \quad \text{f.a. } v \in H^{1,p}(Z^\pm, \mathbb{R}^{2n})\,. \tag{A.1}$$

Hence,

$$\begin{aligned}\|(As)'\|_{0,p} &\leqslant c(p)\,\|A'\|_{0,p}\|s\|_{1,p} + \|A\|_\infty \|s\|_{1,p}\\ &\leqslant \big(c(p)\|A'\|_{0,p} + \|A\|_\infty\big)\,\|s\|_{1,p}\end{aligned}$$

leads to the estimate

$$\|As\|_{1,p} \leqslant \tilde{c}(\|A'\|_{0,p}, \|A\|_\infty)\,\|s\|_{1,p}\,. \tag{A.2}$$

From Lemma 2.1.3 we know that this continuity constant $\tilde{c}$ depends continuously on $\|A\|_{C^1_\Sigma}$. Therefore $\mathfrak{S} = H^{1,p}_\Sigma$ and $\mathfrak{S} = L^p_\Sigma$ are continuous section functors. Moreover, (A.2) for $A \in H^{1,p}_\Sigma(\mathrm{Hom}(\xi,\eta))$ yields

$$\|A\|_{\mathcal{L}(H^{1,p};H^{1,p})} \leqslant \bar{c}(p)\|A\|_{1,p}, \tag{A.3}$$

proving (b). ∎

A.0.3 Proposition *Let $\xi, \eta \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$ be given together with an open subset $\mathcal{O} \subset \xi$ such that there is a section $\gamma \in C^0(\xi)$ with compact support in Σ^o and $\gamma(\overline{\Sigma}) \subset \mathcal{O}$. Then every smooth bundle map $f \in C^\infty_\Sigma(\mathcal{O}, \eta)$ satisfying $f(0_{|\partial\Sigma}) \subset 0_{|\partial\Sigma}$ induces a well-defined and continuous map*

$$f_*\colon \mathfrak{S}(\mathcal{O}) \to \mathfrak{S}(\eta), \quad s \mapsto f \circ s,$$

on the open subset

$$\mathfrak{S}(\mathcal{O}) = \{\, s \in \mathfrak{S}(\xi) \mid s(\overline{\Sigma}) \subset \mathcal{O}\,\}\,.$$

Proof. It follows immediately from the continuous embedding $\mathfrak{S}(\xi) \hookrightarrow C^0(\xi)$ that $\mathfrak{S}(\mathcal{O})$ is open within $\mathfrak{S}(\xi)$. Since f is a C^∞_Σ-smooth bundle map on the compact base $\overline{\Sigma}$, its fibre restrictions f_z are in particular uniformly Lipschitz continuous. Thus, any finite family of local trivializations yields the estimates

$$\begin{aligned}\big|f_z(x_z) - f_z(y_z)\big| &\leqslant \mathrm{const}\ |x_z - y_z| \quad \text{for all } x_z, y_z \in \xi_z,\\ \big|f_z(s(z))\big| &\leqslant \mathrm{const}\ |s(z)| + |f_z(0)| \quad \text{for all } z \in \overline{\Sigma}\ .\end{aligned} \tag{A.4}$$

From $f(0) \in C^\infty_\Sigma(\eta)$ together with Corollary 2.1.4 we deduce that the map

$$f_* : H^{1,p}_\Sigma(\mathcal{O}) \to H^{1,p}_\Sigma(\eta)$$

is well-defined. An analogous uniform estimate involving the first derivative of f and the inequalities (A.4) yields the continuity of f_*. ■

These properties of the section functor $\mathfrak{S} = H^{1,p}_\Sigma$ listed in Propositions A.0.2 and A.0.3 lead to the following fundamental lemma.

A.0.4 Lemma (Fundamental Lemma) *Let $\mathfrak{S}$, $\mathcal{O} \subset \xi$, η and the bundle map $f \in C^\infty_\Sigma(\mathcal{O}, \eta)$ be as above. Then, the map $f_* \colon \mathfrak{S}(\mathcal{O}) \to \mathfrak{S}(\eta)$ is smooth and its k-th derivative at s is given by*

$$D^k f_*(s) = \mathfrak{S}_*(F^k f \circ s),$$

where $F^k f \colon \mathcal{O} \to \mathrm{Hom}(\oplus^k \xi; \eta)$ denotes the k-th fibre derivative of the bundle map f.

PROOF. This lemma is proven exactly like Fundamental Lemma A.5 in [50]. One proceeds by induction on k. ■

Let us present a slight generalization of this crucial result. Due to the non-compactness of Σ on the cylindrical ends, any section in $H^{1,p}_\Sigma(\xi)$ in a smooth bundle ξ over Σ has to decay to zero. For some aspects, this turns out to be to strong a restriction. For example, we will have to consider sections in homomorphism bundles whose asymptotic decay is of $H^{1,p}_\Sigma$-quality but which do not converge to the zero operator. Here we use the following trick.

Let $\xi, \eta \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$ with $\xi_{|\partial\Sigma} = \eta_{|\partial\Sigma}$. Then we define $\mathrm{Hom}(\xi, \eta) \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$ by

$$\mathrm{Hom}(\xi, \eta)_z = \mathrm{Hom}(\xi_z, \eta_z)$$

for $z \in \overline{\Sigma}$.

A.0.5 Lemma *Given a smooth bundle $\zeta \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$, an open subset $\mathcal{O} \subset \zeta$ satisfying the conditions of Proposition A.0.3 and a smooth bundle map F satisfying*

$$F \in C^\infty(\mathcal{O}, \mathrm{Hom}(\xi, \eta)), \quad F(0_{|\partial\Sigma}) \subset \{\mathrm{Id}_{\xi_{|\partial\Sigma}}\}$$

where ξ, η are given as above, then it induces a smooth map

$$F_* \colon \mathfrak{S}(\mathcal{O}) \to \mathcal{L}(\mathfrak{S}(\xi); \mathfrak{S}(\eta)) .$$

PROOF. We choose a smooth section $s_o \in C^\infty_\Sigma(\mathcal{O})$ with $s_{o|\partial\Sigma} = 0$. Thus, $F \circ s_o$ is a smooth section in $\mathrm{Hom}(\xi, \eta)$ and due to Proposition A.0.2 (a) it holds that

$$(F \circ s_o)_* = \mathfrak{S}_*(F \circ s_o) \in \mathcal{L}(\mathfrak{S}(\xi); \mathfrak{S}(\eta)) .$$

Since $(F \circ s_o)_{|\partial\Sigma} = \mathrm{Id}_{\xi_{|\partial\Sigma}}$, the map

$$f \colon \mathcal{O} \to \mathrm{Hom}(\xi, \eta),$$
$$f(z, v) = F(z, v) - F(z, s_o(z)),$$

is a smooth bundle map which satisfies the assumptions of the Fundamental Lemma above. Hence, the induced map

$$f_* \colon \mathfrak{S}(\mathcal{O}) \to \mathfrak{S}(\mathrm{Hom}(\xi, \eta))$$

is smooth and, due to Proposition A.0.2 (b), its range is smoothly mapped into $\mathcal{L}(\mathfrak{S}(\xi); \mathfrak{S}(\eta))$. This leads to the smooth map

$$F_* = f_* + (F \circ s_o)_* \colon \mathfrak{S}(\mathcal{O}) \to \mathcal{L}(\mathfrak{S}(\xi); \mathfrak{S}(\eta)) . \quad ■$$

At this stage we point out that the properties (a) and (b) in Proposition A.0.2 of the section functor $\mathfrak{S}$ were axiomatized by Eliasson in [13]. A section functor with these properties is called a manifold model. The above Fundamental Lemma provides the key for the smoothness of the change of charts on the asserted Banach manifold. We now present the proof of Theorem 2.1.7. Note that we cannot directly apply Eliasson's results, because his manifold models for mapping spaces are confined to compact domains. Here, we want to consider model surfaces with cylindrical ends which, however, are endowed with suitable structures such that all essential statements still hold.

PROOF OF THEOREM 2.1.7 AND COROLLARY 2.1.8. Let us recall the definitions

$$\begin{aligned}
&\mathcal{V}_h = H^{1,p}_\Sigma(h^*\mathcal{D}) \subset H^{1,p}_\Sigma(h^*TM),\\
&\Phi_h(z, v) = (z, \exp_{h(z)} v),\\
&\mathcal{U}_h = \{\, g \in C^0(\overline{\Sigma}, M) \mid \operatorname{graph} g \subset \Phi_h(h^*\mathcal{D}),\ \Phi_h^{-1} \circ (\mathrm{id}, g) \in \mathcal{V}_h \,\},\\
&\Psi_h \colon \mathcal{U}_h \to \mathcal{V}_h, \quad g \mapsto \Phi_h^{-1} \circ (\mathrm{id}, g) .
\end{aligned}$$

Provided that we have already proven that the chart transformation maps

$$\Psi_{hk} = \Psi_h \circ \Psi_k^{-1} \colon \mathcal{V}_k \supset \Psi_k(\mathcal{U}_k \cap \mathcal{U}_h) \to \mathcal{V}_h$$

are diffeomorphisms, this family of bijections uniquely fixes the topology on $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$. Then this subset of $C^0(\overline{\Sigma}, M)$ is locally modeled on $H^{1,p}_\Sigma(\mathbb{R}^{2n})$.

Let us consider $h, k \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ be such that $\mathcal{U}_k \cap \mathcal{U}_h \neq \emptyset$. Then

$$\begin{aligned}
\mathcal{O}_h =\{\, &(z, \xi) \in h^*\mathcal{D} \subset h^*TM \mid \\
&\exp_{h(z)} \xi = \exp_{k(z)} \zeta \text{ for some } \zeta \in k^*\mathcal{D} \,\}
\end{aligned} \tag{A.5}$$

and respectively $\mathcal{O}_k$ are open subsets of h^*TM and k^*TM satisfying the assumption of Proposition A.0.3. Furthermore, the mapping

$$\Phi_{kh} = \Phi_k^{-1} \circ \Phi_h \colon h^*TM \supset \mathcal{O}_h \to \mathcal{O}_k \subset k^*TM$$

is a smooth bundle map with $\Phi_{kh}(0_{|\partial\Sigma}) \subset 0_{|\partial\Sigma}$, because $h_{|\partial\Sigma} \equiv k_{|\partial\Sigma}$. Thus the Fundamental Lemma implies that

$$(\Phi_{kh})_* \colon \mathfrak{S}(\mathcal{O}_h) \to \mathfrak{S}(\mathcal{O}_k)$$

is a smooth map. Since $(\Phi_{kh})_*^{-1} = (\Phi_{hk})_*$, it is a diffeomorphism between open subsets of Banach spaces. Hence, the smoothness of the chart transformation maps follows from the observation that

$$\Psi_h(\mathcal{U}_h \cap \mathcal{U}_k) \subset \mathcal{O}_h \quad \text{and} \quad \Psi_k \circ \Psi_h^{-1} = (\Phi_{kh})_* \,.$$

We observe that $C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ lies dense within $C^0_{x_1,\dots,x_\nu}(\overline{\Sigma}, M)$ and any exponential map defined by a smooth spray satisfies the condition that there is a neighbourhood $\mathcal{D}$ of the zero section in the tangent bundle such that

$$\mathcal{D} \overset{\approx}{\hookrightarrow} M \times M, \quad v \mapsto (\tau(v), \exp(v)),$$

is a diffeomorphism onto a neighbourhood of the diagonal. Thus, any two such exponential mappings, for example associated to different Riemannian metrics by the Levi-Civita connections, give rise to compatible atlases of charts. Consequently, the differentiable structure of $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ is uniquely determined and independent of the Riemannian metric. The separability follows from that of $C^0_{x_1,\dots,x_\nu}(\overline{\Sigma}, M)$ and of $H^{1,p}_\Sigma$.

Let us now consider a smooth map $f \in C^\infty(M, N)$. Since each $\gamma \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ is mapped to $f \circ \gamma \in C^0_{f(x_1),\dots,f(x_\nu)}(\overline{\Sigma}, N)$ and

$$C^\infty_{f(x_1),\dots,f(x_\nu)}(\Sigma, N) \subset C^0_{f(x_1),\dots,f(x_\nu)}(\overline{\Sigma}, N)$$

is a dense subset, we find a representation

$$f \circ \gamma = \exp_k w, \quad k \in C^\infty_{f(x_1),\dots,f(x_\nu)}(\Sigma, N), \quad w \in C^0(k^*\mathcal{D}_N) \,.$$

Similarly, γ is represented by

$$\gamma = \exp_h v, \quad h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M), \quad v \in H^{1,p}_\Sigma(h^*\mathcal{D}_M) \,.$$

Choosing an open neighbourhood $\tilde{\mathcal{O}}_h \subset h^*\mathcal{D}_M$ such that the condition for Proposition A.0.3 is satisfied for the fibre respecting map

$$\psi^f_{kh} = \exp_k^{-1} \circ f \circ \exp_h \colon \tilde{\mathcal{O}}_h \to k^*\mathcal{D}_N,$$

we obtain the smooth map $(\psi^f_{kh})_* \colon H^{1,p}_\Sigma(\tilde{\mathcal{O}}_h) \to H^{1,p}_\Sigma(k^*\mathcal{D}_N)$ from Fundamental Lemma A.0.4. ∎

After the proof of the manifold property for $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$, we now consider the bundles $H^{1,p}_\Sigma(\mathcal{P}^*TM)$ and $L^p_\Sigma(\mathcal{P}^*TM)$ over this space as stated in Theorem 2.2.1. Let us first recall the following mappings associated to any arbitrary smooth connection K on the tangent bundle TM. Denoting by

$$K \colon T(TM) \to TM \quad \text{and} \quad D\tau \colon T(TM) \to TM$$

the connection map K and the differential of the canonical tangent bundle projection τ onto M, we consequently obtain a decomposition of $T(TM)$ into a horizontal and a vertical sub-bundle,

$$T(TM) = T_h(TM) \oplus T_v(TM),$$

where

$$T_{\xi,v}(TM) = \ker D\tau(\xi), \quad T_{\xi,h}(TM) = \ker K(\xi)$$

for $\xi \in TM$. Recalling the exponential map $\exp \colon TM \supset \mathcal{D} \to M$ defined by the geodesics associated to K,

$$c(t) = \exp_p(tv), \quad c(0) = p = \tau(v), \quad \dot{c}(0) = v, \quad \nabla_t \dot{c} = 0,$$

where $\mathcal{D}$ again denotes the injectivity neighbourhood, we obtain the following isomorphisms:

$$\begin{aligned} \nabla_1 \exp(\xi) &= D\exp(\xi) \circ (D\tau_{|T_{\xi,h}(TM)})^{-1} \colon T_{\tau(\xi)}M \overset{\approx}{\to} T_{\exp(\xi)}M, \\ \nabla_2 \exp(\xi) &= D\exp(\xi) \circ (K_{|T_{\xi,v}(TM)})^{-1} \colon T_{\tau(\xi)}M \overset{\approx}{\to} T_{\exp(\xi)}M \,. \end{aligned} \tag{A.6}$$

It is important to mention that the identity

$$\nabla_1 \exp(0) = \nabla_2 \exp(0) = \mathrm{id}_{T_{\tau(0)}M} \tag{A.7}$$

holds for any arbitrary smooth connection K map.

As an aside, we recall the definition of the covariant derivation associated to the connection map K. Given any $v \in T_pM$ and $X \in C^\infty(TM)$, we set

$$\nabla_v X(p) = K \circ D_pX \cdot v \,.$$

For a vector field $v(t)$ along a curve $\gamma(t)$, $v(t) \in T_{\gamma(t)}M$, we have

$$\nabla_t v(t_o) = \nabla_{\dot{\gamma}(t_o)} X, \quad X \in C^\infty(TM), \quad X(\gamma(t)) = v(t)$$

well-defined independently of the extension X of γ.

We return to the map Φ_{kh} on which we based the change of charts $\Psi_{kh} = (\Phi_{kh})_*$ in the above proof of the manifold property. We now consider the fibre-wise derivative

$$\begin{aligned} F\Phi_{kh} &\colon \mathcal{O}_h \to \mathrm{Hom}(h^*TM, k^*TM) \\ F\Phi_{kh}(z,\xi) &= \nabla_2 \exp(\exp^{-1}_{k(z)} \exp_{h(z)} \cdot \xi)^{-1} \circ \nabla_2 \exp(\xi) \,. \end{aligned} \tag{A.8}$$

Let $\mathfrak{T} \colon \mathrm{Vec}^*_{C^\infty}(\Sigma) \to \mathrm{Ban}$ be a section functor satisfying the condition that

$$\begin{aligned} \mathfrak{T}_* \colon \mathfrak{S}(\mathrm{Hom}(\xi,\eta)) &\to \mathcal{L}(\mathfrak{T}(\xi); \mathfrak{T}(\eta)), \\ A &\mapsto (s \mapsto A \cdot s) \end{aligned} \tag{A.9}$$

is continuous for each pair $\xi, \eta \in \mathrm{Vec}^*_{C^\infty}(\Sigma)$. Here we work with $\mathfrak{T} = L^p_\Sigma$ and $H^{1,p}_\Sigma$ respectively. Considering the proof of Lemma A.0.5 we observe that the result is also true for the section functor $\mathfrak{T}$ if (A.9) holds true. Hence, we only have to verify that the bundle map $F = F\Phi_{kh}$ satisfies

$$\begin{aligned} F(0_{|\partial\Sigma}) &\subset \{\, \mathrm{Id}_{h^*TM|\partial\Sigma} \} \\ &= \{\, \mathrm{Id}_{x_i^*TM} \mid i = 1, \dots, \nu \,\} \end{aligned}$$

which follows from (A.7). We consequently obtain the smooth maps

$$\begin{aligned} \mathfrak{S}(\mathcal{O}_h) &\to \mathcal{L}(\mathfrak{T}(h^*TM);\, \mathfrak{T}(k^*TM)), \\ s &\mapsto \mathfrak{T}_*(F\Phi_{kh} \circ s), \end{aligned} \tag{A.10}$$

and

$$\begin{aligned} \mathfrak{S}(\mathcal{O}_h) &\to \mathcal{L}(\mathfrak{S}(h^*TM); \mathfrak{S}(k^*TM)), \\ s &\mapsto \mathfrak{S}_*(F\Phi_{kh} \circ s) = D\Psi_{kh}(s)\,. \end{aligned} \tag{A.11}$$

These preparations allow us to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. We give the same proof as in [50]. It holds for both section functors $\mathfrak{T} = L^p_\Sigma$ and $H^{1,p}_\Sigma$. Each two curves $\alpha, \beta \in C^0_{x_1,\dots,x_\nu}(\overline{\Sigma}, M)$ satisfying

$$\beta = \exp \circ \xi \quad \text{for some} \quad \xi \in C^0(\alpha^*\mathcal{D}) \subset C^0(\alpha^*TM)$$

give rise to the homomorphism section $G_{\beta\alpha} \in C^0(\mathrm{Hom}(\alpha^*TM, \beta^*TM))$ by

$$G_{\beta\alpha}(z) \cdot v(z) = \nabla_2 \exp(\xi(z)) \cdot v(z), \quad z \in \overline{\Sigma}\,.$$

$G_{\beta\alpha}(z)$ is an isomorphism for each $z \in \overline{\Sigma}$. Considering charts based on curves $k, h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ like in the proof of Theorem 2.1.7 we obtain the identity

$$\nabla_2 \exp(\Phi_{kh} \circ s) \circ (F\Phi_{kh} \circ s) \cdot v = \nabla_2 \exp(s) \cdot v$$

from the formula (A.8) applied to $s \in \mathfrak{S}(\mathcal{O}_h)$ and $v \in \mathfrak{S}(h^*TM)$, that is

$$\mathfrak{S}_*(F\Phi_{kh} \circ s) = \mathfrak{S}_*\big((G_{gk})^{-1} \circ G_{gh}\big) \tag{A.12}$$

for $g = \exp_h s \in \mathcal{P}$. We define the fibre at g by

$$\mathfrak{T}(g^*TM) = \{\, G_{gh} \cdot v \mid v \in \mathfrak{T}(h^*TM) \,\}\,.$$

Since (A.12) implies the identity

$$G_{gk} \circ \mathfrak{S}_*(F\Phi_{kh} \circ s) \cdot v = G_{gh} \cdot v,$$

this definition does not depend on the choice of h and (A.10) guarantees that the Banach space topology on $\mathfrak{T}(g^*TM)$ is well-defined. The smooth map (A.10) also yields the smoothness of the transition maps

$$G_{gh} \colon \mathfrak{T}(h^*TM) \to \mathfrak{T}(g^*TM)$$

providing a local trivialization of the bundle $\mathfrak{T}(\mathcal{P}(x_1, \dots, x_\nu)^*TM)$. Referring to (A.11) we conclude by stating that the latter transition maps satisfy the identity

$$(G_{gk})^{-1} \circ G_{gh} = \mathfrak{S}_*(F\Phi_{kh} \circ s) = D\Psi_{kh}(s),$$

when we consider the manifold model $\mathfrak{T} = \mathfrak{S} = H^{1,p}_\Sigma$ as the section functor. Hence the Banach bundle

$$H^{1,p}_\Sigma(\mathcal{P}(x_1, \dots, x_\nu)^*TM) = \bigcup_{h \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)} H^{1,p}_\Sigma(h^*TM)$$

represents the tangent bundle of the Banach manifold $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$. ■

At this stage we add some remarks. First, we have seen in the proof that the vector spaces

$$\mathfrak{T}(u^*TM), \quad u \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M),$$

which are already well-defined as vector spaces of sections of certain continuous vector bundles, carry a unique Banach space topology given by the section functor $\mathfrak{T} = H^{1,p}_\Sigma$ or $\mathfrak{T} = L^p_\Sigma$. At the end of this appendix we shall derive continuous Finsler structures for the Banach space bundles. Here we point out that the property (b) in Proposition A.0.2 continues to hold in case of $\mathfrak{T} = \mathfrak{S} = H^{1,p}_\Sigma$ also for the bundles

$$\xi = u^*TM, \quad u \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)\,.$$

That is,

$$\mathfrak{S}\big(\mathrm{Hom}(u^*TM, v^*TM)\big) \to \mathcal{L}(\mathfrak{S}(u^*TM); \mathfrak{S}(v^*TM)) \tag{A.13}$$

is a continuous linear mapping for all $u, v \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$. This is proven analogously to the construction of $\mathfrak{T}(u^*TM)$ above.

Secondly, we prove that the maps

$$\exp_g \colon H^{1,p}_\Sigma(g^*\mathcal{D}) \to \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$$

can be extended from $g \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ to any $u \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$. Note first that the open subset $\mathcal{D} \subset TM$ gives rise to an open neighbourhood of the zero section in $H^{1,p}_\Sigma(\mathcal{P}^*TM)$,

$$H^{1,p}_\Sigma(\mathcal{P}^*\mathcal{D}) = \bigcup_{u \in \mathcal{P}} \{u\} \times H^{1,p}_\Sigma(u^*\mathcal{D})\,.$$

A.0.6 Proposition *The smooth map* $\exp\colon \mathcal{D} \to M$ *induces a smooth map*

$$\begin{aligned} \widetilde{\exp}\colon H^{1,p}_\Sigma(\mathcal{P}^*\mathcal{D}) &\to \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M), \\ (u, \xi) &\mapsto \exp_u \xi\,. \end{aligned}$$

Proof. We represent $\widetilde{\exp}$ with respect to a trivialization of $H^{1,p}_\Sigma(\mathcal{P}^*TM)$ and local coordinates of $\mathcal{P}$. In order to prove the smoothness of $\widetilde{\exp}$ at (u,ξ) we choose $g \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ close enough to u such that

$$u = \exp_g s, \quad \xi = \nabla_2 \exp(s) \cdot a$$

for some $s, a \in H^{1,p}_\Sigma(g^*\mathcal{D})$. Analogously, we pick an $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ such that

$$\exp_u \xi = \exp_h \sigma$$

for some $\sigma \in H^{1,p}_\Sigma(h^*\mathcal{D})$. Now we have to prove that

$$\begin{aligned} f_*\colon H^{1,p}_\Sigma(g^*\mathcal{D}) \times H^{1,p}_\Sigma(g^*\mathcal{D}) &\to H^{1,p}_\Sigma(h^*TM), \\ (s,a) &\mapsto \sigma = \sigma(s,a), \end{aligned}$$

is a smooth mapping. We observe that f_* is induced by the following smooth bundle map over Σ,

$$\begin{aligned} f\colon g^*\mathcal{D} \oplus g^*\mathcal{D} &\to h^*TM, \\ f(z,x,y) &= \exp^{-1}_{h(z)} \Big(\exp_{\exp_{g(z)} x} \big(\nabla_2 \exp(x) \cdot y\big)\Big), \end{aligned}$$

where we eventually have to choose a smaller neighbourhood $\widetilde{\mathcal{D}} \subset \mathcal{D}$ of the zero section. Since

$$\begin{aligned} f_{|\partial\Sigma_i}(0,y) &= \exp^{-1}_{x_i} \Big(\exp_{\exp_{x_i} 0} \big(\nabla_2 \exp(0) \cdot y\big)\Big) \\ &= y \end{aligned}$$

for all $y \in x_i^* TM \subset g^*TM_{|\partial\Sigma_i}$, $i = 1, \dots, \nu$, in particular

$$f_{|\partial\Sigma}(0,0) = 0,$$

f satisfies the condition for the Fundamental Lemma A.0.4. Hence, the assertion follows from the identification

$$H^{1,p}_\Sigma(g^*\mathcal{D} \oplus g^*\mathcal{D}) = H^{1,p}_\Sigma(g^*\mathcal{D}) \times H^{1,p}_\Sigma(g^*\mathcal{D}). \quad \blacksquare$$

Let us now attack the last theorem about the required Banach bundle analysis. We consider the bundles

$$\mathfrak{T}(\mathcal{P}^*E) \to \mathcal{P}(x_1, \dots, x_\nu)$$

where $\mathfrak{T}$ denotes the section functor L^p_Σ and E stands for $L = T^*\Sigma \otimes TM$ and $X^J = T^{0,1}\Sigma \otimes_J TM$ respectively. Moreover, we will now prove that

$$\Lambda_J\colon C^\infty_\Sigma(L) \to C^\infty_\Sigma(X^J), \quad \phi \mapsto \phi + J \circ \phi \circ j,$$

induces the bundle homomorphism

$$\mathfrak{T}(\Lambda_J)\colon \mathfrak{T}(\mathcal{P}^*L) \to \mathfrak{T}(\mathcal{P}^*X^J),$$

which in Theorem 2.2.5 is again denoted by Λ_J.

Proof of Theorem 2.2.5. We first consider the case $E = L$. Here we can use the same techniques as for the bundle $\mathfrak{T}(\mathcal{P}^*TM)$. Similar to the proof of Theorem 2.2.1 we have to study the maps $G_{\beta\alpha} \in C^0\big(\mathrm{Hom}(\alpha^*L, \beta^*L)\big)$, where $\alpha, \beta \in C^0_{x_1,\dots,x_\nu}(\overline{\Sigma}, M)$ satisfy

$$\beta = \exp_\alpha s, \quad s \in C^0(\alpha^*\mathcal{D}) \subset C^0(\alpha^*TM).$$

We use again the linearization of the exponential charts as introduced above,

$$G_{\beta\alpha}(z) \cdot \phi_\alpha(z) = \nabla_2 \exp(s(z)) \circ \phi_\alpha(z) \in \mathrm{Hom}(T_z\Sigma, T_{\beta(z)}M).$$

Analogous to Theorem 2.2.1 we represent $g \in \mathcal{P}$ with respect to the smooth mappings $h, k \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$, i.e. $g = \exp_h s^h = \exp_k s^k$ and we compute

$$(G^{-1}_{gk} \circ G_{gh}) \cdot \phi_h = (F\Phi_{kh}(s^h)) \circ \phi_h.$$

Thus, the identity

$$G_{gk} \cdot (\mathfrak{T}_*(F\Phi_{kh} \circ s^h) \circ \phi_h) = G_{gh} \cdot \phi_h$$

implies that

$$\mathfrak{T}(g^*L) = \{\, G_{gh} \cdot \phi_h \,|\, \phi_h \in \mathfrak{T}(h^*L) \,\}$$

inherits a well-defined Banach space topology. By the section functor property (A.9) we again obtain the smoothness of

$$\begin{aligned} \mathfrak{S}(h^*TM) &\to \mathcal{L}(\mathfrak{T}(h^*L); \mathfrak{T}(k^*L)), \\ s^h &\mapsto \big(\phi^h \mapsto \mathfrak{T}_*(F\Phi_{kh} \circ s^h) \cdot \phi^h\big). \end{aligned}$$

This proves the smoothness of the local trivialization of $\mathfrak{T}(\mathcal{P}^*L)$ for both $\mathfrak{T} = H^{1,p}_\Sigma$ and L^p_Σ.

In the case of the sub-bundle X^J parameterized by $J \in \mathcal{J}$ we encounter the difficulty that the previously defined transition maps $G_{\beta\alpha} \in \mathrm{Hom}(\alpha^*L, \beta^*L)$ do not respect the sub-bundle structure $X^J \subset L$. In general, we do not have

$$\nabla_2 \exp(s(z)) \circ J(z, \alpha(z)) \stackrel{?}{=} J(z, \beta(z)) \circ \nabla_2 \exp(s(z)).$$

Thus, we have to find a different appropriate structure which is compatible with the almost complex structure J of the bundle $\tilde{\tau}\colon \pi^*TM \to \overline{\Sigma} \times M$. As we already explained in Section 2.2 we now use a Hermitian connection K^J for the construction of the local trivialization morphisms $G_{\beta\alpha}$. We employ the Hermitian parallel translation P along geodesics for the Levi-Civita connection defining the exponential map exp, that is along $t \mapsto \exp_x(tv)$, $t \in [0,1]$, for $(x,v) \in \mathcal{D}$. Using the identification

$$\mathrm{Hom}(TM, TM) = T^*M \otimes TM \to M \times M$$

for the smooth bundle over $M \times M$ with fibre

$$\mathrm{Hom}(TM, TM)_{(m,\bar{m})} = \mathrm{Hom}(T_m M, T_{\bar{m}} M),$$

we obtain P as a smooth map

$$\begin{aligned} P\colon\ \mathrm{pr}^*\mathcal{D} &\to \mathrm{pr}^*\,\mathrm{Hom}(TM,TM),\\ (z,v) &\mapsto \big(P(z,v)\colon T_{\tau(v)}M \xrightarrow{\cong} T_{\exp v}M\big), \end{aligned}$$

similar to the map $\nabla_2 \exp$. Recall the projection $\mathrm{pr}\colon \overline{\Sigma}\times M \to M$, and that J may depend explicitly on the variable $z \in \overline{\Sigma}$. We define

$$\tilde{G}_{\beta\alpha}(z)\cdot\phi_\alpha(z) = P(z,s(z))\circ\phi_\alpha(z),$$

for $z \in \overline{\Sigma}$ as in (2.3). Note that we introduce a different notation in order to draw the distinction between L and X^J where we use different connection maps. Due to $P\circ J = J\circ P$ we have for $\phi_\alpha \in \alpha^* X^J$ the identity

$$(\tilde{G}_{\beta\alpha}\cdot\phi_\alpha)(z)\circ j(z) = -J(z,\beta(z))\circ(\tilde{G}_{\beta\alpha}\cdot\phi_\alpha)(z)$$

for all $z\in\overline{\Sigma}$, i.e. $\tilde{G}_{\beta\alpha}\cdot\phi_\alpha \in \beta^* X^J$. The next step is to verify that for $\mathfrak{T} = H^{1,p}_\Sigma$ and $\mathfrak{T} = L^p_\Sigma$

$$\mathfrak{T}(g^*X^J) = \{\, \tilde{G}_{gh}\cdot\phi_h \,|\, \phi_h \in \mathfrak{T}(h^*X^J)\,\}$$

for $g = \exp_h s \in \mathcal{P}$, $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)$, is a well-defined Banach space, and that

$$\begin{aligned} \mathfrak{S}(h^*\mathcal{D}) &\to \mathcal{L}(\mathfrak{T}(h^*X^J);\mathfrak{T}(k^*X^J)),\\ s^h &\mapsto (\phi_h \mapsto \tilde{G}^{-1}_{\exp_k s^k,k}\circ\tilde{G}_{\exp_h s^h,h}\cdot\phi_h) \end{aligned}$$

is smooth. We define

$$F_{kh}(s) = \tilde{G}^{-1}_{gk}\circ\tilde{G}_{gh}$$

for $g = \exp_h s^h = \exp_k s^k \in \mathcal{P}$, $h,k \in C^\infty_{x_1,\dots,x_\nu}(\Sigma,M)$, with

$$\begin{aligned} F_{kh}\colon \mathcal{O}_h &\to \mathrm{Hom}(h^*TM;k^*TM),\\ F_{kh}(z,v)\cdot w &= P(z,\exp^{-1}_{k(z)}\exp_{h(z)} v)^{-1}\circ P(z,v)\cdot w \end{aligned}$$

for $w\in T_{h(z)}M$. Since P is smooth, F_{kh} is a smooth bundle map between the bundles over $\overline{\Sigma}$, and we verify that $F_{kh}(z,0) = \mathrm{id}_{T_{h(z)}M}$ for all $z\in\partial\Sigma$. Hence, F_{kh} satisfies the condition of Lemma A.0.5, and together with the property (A.9) of the section functor $\mathfrak{T}$ we obtain the smoothness of the well-defined map

$$\begin{aligned} \mathfrak{T}_*(F_{kh})\colon \mathfrak{S}(\mathcal{O}_h) &\to \mathcal{L}(\mathfrak{T}(h^*X^J);\mathfrak{T}(k^*X^J)),\\ s &\mapsto \mathfrak{T}_*(F_{kh}\circ s), \end{aligned}$$

analogously to (A.10). Hence, $\mathfrak{T}(g^*X^J)$ is well-defined for $g\in\mathcal{P}$. $\mathfrak{T}_*(F_{kh})$ provides the transition maps for the bundle $\mathfrak{T}(\mathcal{P}^*X^J)$ compatible with the given atlas for $\mathcal{P}(x_1,\dots,x_\nu)$ due to (2.2), because we chose the parallel translation along the exp-geodesics.

In order to prove the smoothness of

$$\begin{aligned} \mathfrak{T}(\Lambda_J)\colon \mathfrak{T}(\mathcal{P}^*L) &\to \mathfrak{T}(\mathcal{P}^*X^J),\\ \mathfrak{T}(\Lambda_J)_u\phi &= \phi + J(\cdot,u)\circ\phi\circ j, \quad u\in\mathcal{P}(x_1,\dots,x_\nu), \end{aligned}$$

it is sufficient to show that the trivialization maps $\tilde{G}_{\beta\alpha}$ on $\mathfrak{T}(\mathcal{P}^*X^J)$ are compatible with $G_{\beta\alpha}$ on $\mathfrak{T}(\mathcal{P}^*L)$, because

$$\tilde{G}^{-1}_{gh}\mathfrak{T}(\Lambda_J)_g\phi = \tilde{G}^{-1}_{gh}\phi + J(\cdot,h)(\tilde{G}^{-1}_{gh}\cdot\phi)\,j = \Lambda_J(\tilde{G}^{-1}_{gh}\phi)$$

for all $\phi\in\mathfrak{T}(g^*L)$. Note that we can prove the smoothness of

$$\mathcal{P}\to H^{1,p}_\Sigma(\mathcal{P}^*\,\mathrm{End}(TM)), \quad u\mapsto J(\cdot,u),$$

for any $J\in C^\infty_\Sigma(\pi^*\,\mathrm{End}(TM))$ analogously to (A.9) – (A.10). This only requires a suitable variation of Lemma A.0.5 where we have to replace the asymptotic condition $\mathrm{Id}_{\xi_{|\partial\Sigma}}$ by J. It remains to show that

$$\mathcal{O}_h \to \mathcal{L}(\mathfrak{T}(h^*L);\mathfrak{T}(h^*L)), \quad s\mapsto \tilde{G}^{-1}_{\exp_h s,h}\circ G_{\exp_h s,h},$$

is smooth. Since

$$\begin{aligned} H\colon\ \mathrm{pr}^*\mathcal{D} &\to \mathrm{pr}^*\,\mathrm{End}(TM),\\ (z,v) &\mapsto \Big(P(z,v)^{-1}\circ\nabla_2\exp(v)\colon T_{\tau(v)}M \xrightarrow{\cong} T_{\tau(v)}M\Big), \end{aligned}$$

with $H(z,0)=\mathrm{id}$ is a smooth bundle map over $\overline{\Sigma}\times M$, we obtain the induced smooth map from Lemma A.0.5,

$$\mathfrak{T}_*(H)\colon \mathfrak{S}(\mathcal{O}_h)\to\mathcal{L}(\mathfrak{T}(h^*F);\mathfrak{T}(h^*L)), \quad s\mapsto H\circ s\,.$$

We have $\mathfrak{T}_*(H)(s) = \tilde{G}^{-1}_{\exp_h s,h}\circ G_{\exp_h s,h}$ because of (2.2). ■

Analogously to Proposition A.0.6 we can prove that also P can be smoothly extended to $H^{1,p}_\Sigma(\mathcal{P}^*\mathcal{D})$.

A.0.7 Proposition *The parallel translation $P\colon\ \mathrm{pr}^*\mathcal{D}\to\mathrm{pr}^*\,\mathrm{Hom}(TM,TM)$ induces a smooth map into the homomorphism bundle over $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma,M)$,*

$$\begin{aligned} P_*\colon H^{1,p}_\Sigma(\mathcal{P}^*\mathcal{D}) &\to \mathrm{Hom}\,(L^p_\Sigma(\mathcal{P}^*X^J),L^p_\Sigma(\mathcal{P}^*X^J)),\\ (u,\xi) &\mapsto \Big(P(\xi)\colon L^p_\Sigma(u^*X^J)\xrightarrow{\cong} L^p_\Sigma((\exp_u\xi)^*X^J)\Big)\,. \end{aligned}$$

The proof is carried out by means of Lemma A.0.5 in straight analogy to the proof of the smoothness of $\widetilde{\exp}$.

In view of the foundational Section 2.3 it remains to provide the proof of Proposition 2.3.1.

Proof of Proposition 2.3.1. We recall that in Section 2.4 we already computed the linearization of $d+k$ with respect to certain trivializations. However, for this fixed trivialization at an $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$, the differential of $(d+k)_{\mathrm{triv}}$ was only formally computed at h itself. Then, we can explicitly express it in terms which are easy to handle. In order to prove the smoothness we employ the local trivialization of $L^p_\Sigma(\mathcal{P}^*L)$ by means of

$$G_{u\alpha} = \nabla_2 \exp(s) \colon L^p_\Sigma(\alpha^*L) \xrightarrow{\cong} L^p_\Sigma(u^*L)$$

for $u = \exp_\alpha s$, $s \in H^{1,p}_\Sigma(\alpha^*\mathcal{D})$ and $\alpha \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ as introduced in the proof of Theorem 2.2.5.

At first, we consider $s \in C^\infty_\Sigma(\alpha^*\mathcal{D})$ with $s_{|\partial\Sigma} = 0$. Using the identities (A.6) and (A.7) we obtain for

$$d \colon u \mapsto du \in C^\infty_\Sigma(u^*L)$$

the representation

$$\begin{aligned}&\nabla_2 \exp(s)^{-1} d(\exp_\alpha s)\\ =&\nabla_2 \exp(s)^{-1}(\nabla_1 \exp(s) \circ d\alpha + \nabla_2 \exp(s) \circ \nabla s)\\ =&\Theta(s) \circ d\alpha + \nabla s,\end{aligned}$$

with $\Theta = \nabla_2 \exp^{-1} \cdot \nabla_1 \exp$, so that $\Theta(0_x) = \mathrm{Id}_{T_xM}$. We emphasize that, here, ∇ denotes the Levi-Civita connection. The crucial point is that this local representation $G^{-1}_{u\alpha} \circ d(\exp_\alpha s)$ is not an element in $L^p_\Sigma(\alpha^*L)$ in general. Namely,

$$\lim_{\epsilon_i s \to \infty} d\alpha(\psi_i(s,t)) \tfrac{\partial}{\partial t} = \tfrac{d}{dt} x_i(t)\,.$$

But if we add the zero order term k, i.e.

$$\begin{aligned}&G^{-1}_{u\alpha}(d+k)(\exp_\alpha s)\\ &= \Theta(s) \circ d\alpha + \nabla_2 \exp(s)^{-1} k(\exp_\alpha s) \nabla s,\end{aligned} \tag{A.14}$$

we profit from the assumption that

$$\lim_{\epsilon_i s \to \infty} k(\psi_i(s,t), x_i(t)) = -dt \otimes \frac{d}{dt} x_i(t)$$

for all $t \in S^1$, $i = 1, \dots, \nu$. Observing that in view of $s_{|\partial\Sigma} = 0$ we have $\Theta(0)$, $\nabla_2 \exp(0) = \mathrm{id}$ on $\partial\Sigma$, we conclude that (A.14) can be rewritten as

$$G^{-1}_{u\alpha}(d+k)(\exp_\alpha s) = f(s) + \nabla s,$$

where $f \colon \alpha^*\mathcal{D} \to \alpha^*L$ is a smooth bundle map with $f(0_{|\partial\Sigma}) = 0$. Thus the Fundamental Lemma A.0.4 implies that f induces a smooth map

$$f_* \colon H^{1,p}_\Sigma(\alpha^*\mathcal{D}) \to H^{1,p}_\Sigma(\alpha^*L) \subset L^p_\Sigma((\alpha^*L)\,.$$

Since

$$\nabla \colon H^{1,p}_\Sigma(\alpha^*\mathcal{D}) \to L^p_\Sigma(\alpha^*L)$$

is a continuous linear operator, we deduce that the local representation of $d+k$ at $\alpha \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ allows a smooth extension to

$$G^{-1}_{u\alpha} \cdot (d+k) \circ \exp_\alpha \colon H^{1,p}_\Sigma(\alpha^*\mathcal{D}) \to L^p_\Sigma(\alpha^*L)\,. \quad \blacksquare$$

Let us sum up the last results by the following combination.

A.0.8 Theorem *The smooth section*

$$\overline{\partial}_{J,k} \colon \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M) \to L^p_\Sigma(\mathcal{P}^*X^J)$$

associated to an admissible extension (J,k), the J-Hermitian parallel translation P and the exponential map exp give rise to a smooth bundle map defined on an open neighbourhood of the zero section

$$\begin{aligned}F \colon H^{1,p}_\Sigma(\mathcal{P}^*\mathcal{D}) &\to L^p_\Sigma(\mathcal{P}^*X^J),\\ F(u,\xi) &= P_*(u,\xi)^{-1} \cdot \overline{\partial}_{J,k}(\exp_u \xi),\end{aligned}$$

over the Banach manifold $\mathcal{P} = \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$. The fibre derivative at the zero section is given by

$$\begin{aligned}D_u \colon H^{1,p}_\Sigma(u^*TM) &\to L^p_\Sigma(u^*X^J),\\ D_u &= \Lambda_J(u)(\nabla + \mathrm{Tor}(du,\cdot) + \nabla k)\,.\end{aligned}$$

Proof. It only remains to justify the representation of the fibre derivative of F at 0. This follows immediately from Proposition 2.4.1 where this derivative was computed at curves $h \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ which lie dense in $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$. By the same methods as above we observe that

$$\mathrm{Tor}(du,\cdot) \colon H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*L)$$

is smoothly extendible over the entire manifold $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$. $\blacksquare$

A.0.9 Proposition *The above family of operators*

$$D_u \colon H^{1,p}_\Sigma(u^*TM) \to L^p_\Sigma(u^*X^J), \quad u \in \overline{\partial}^{-1}_{J,k}(0),$$

can be extended as a smooth bundle homomorphism

$$D \colon H^{1,p}_\Sigma(\mathcal{P}^*TM) \to L^p_\Sigma(\mathcal{P}^*X^J)$$

over $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$.

Proof. The proof is again based on Lemma A.0.5. The main part is to show that the Hermitian covariant derivative ∇ yields a smooth bundle homomorphism. We already dealt with $\mathrm{Tor}(du,\cdot)$ above and $\Lambda(u)$ within the proof of Theorem 2.2.5. The covariant derivative $\nabla\xi$ of a smooth section $\xi \in C^\infty_\Sigma(\alpha^*TM)$ over $\alpha \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ is defined as $\nabla\xi = K^J \circ T\xi$, where $K^J \colon T(TM) \to TM$ is the Hermitian connection map. Considering this with respect to the local trivialization maps $G_{u\alpha}$ used above, we can prove that ∇ is smoothly extendible over the bundle $H^{1,p}_\Sigma(\mathcal{P}^*TM) \to \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$. The same holds for the map $\xi \mapsto \nabla_\xi k \in H^{1,p}_\Sigma(\mathcal{P}^*L)$. $\blacksquare$

Let us now provide the proofs for the additional non-intrinsic structures on the bundles $H^{1,p}_\Sigma(\mathcal{P}^*TM)$ and $L^p_\Sigma(\mathcal{P}^*X^J)$ which were required in Section 4.4.1. Given the j-compatible 2-form σ with $(\psi_i)^*\sigma = ds \wedge dt$ on the cylindrical ends we defined the L^2_J-scalar product

$$\langle \xi, \zeta \rangle_{L^2,J} = \int_\Sigma \langle \xi, \zeta \rangle_J \sigma$$

for $\xi \in L^p_\Sigma(u^*TM)$, $\zeta \in L^q_\Sigma(u^*TM)$, $p^{-1} + q^{-1} = 1$, $u \in \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$.

We consider the local trivialization of $L^p_\Sigma(\mathcal{P}^*TM)$ for $p \geqslant 1$ which is induced by the parallel translation map P associated to a fixed J-Hermitian connection. For fixed $u_o \in \mathcal{P}$ together with a suitably small neighbourhood $U(u_o) \subset \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ we obtain the smooth map

$$\begin{aligned} \Phi\colon U(u_o) \times L^p_\Sigma(u_o^*TM) &\to L^p_\Sigma(U(u_o)^*TM), \\ (u, \xi) &\mapsto P(\cdot, \exp^{-1}_{u_o} u) \cdot \xi \,. \end{aligned}$$

Here we use Propositions A.0.6 and A.0.7 which stated that P and exp can be smoothly extended over the whole manifold $\mathcal{P}$. Since

$$\langle P(z,v) \cdot w, P(z,v) \cdot w' \rangle_{J(z, \exp_{u_o(z)} v)} = \langle w, w' \rangle_{J(z, u_o(z))}$$

for all $(z,v) \in u_o^*\mathcal{D}$, $w, w' \in T_{u_o(z)}M$, we obtain

$$\langle \Phi(u)\xi, \Phi(u)\zeta \rangle_{L^2,J} = \langle \xi, \zeta \rangle_{L^2,J}$$

for all $(\xi, \zeta) \in L^p_\Sigma(u_o^*TM) \times L^q_\Sigma(u_o^*TM)$, $u \in U(u_o)$.

Analogously, using the parallel translation in order to construct a local trivialization of $L^p_\Sigma(\mathcal{P}^*X^J)$,

$$\begin{aligned} \Phi\colon U(u_o) \times L^p_\Sigma(u_o^*X^J) &\to L^p_\Sigma(U(u_o)^*X^J), \\ (u, \phi) &\mapsto \tilde{G}_{u,u_o} \circ \phi, \end{aligned}$$

where $\tilde{G}$ is defined as above, we observe that for $\|\cdot\|_{0,p;u} = \|\Phi(u) \cdot \|_{0,p}$ it holds

$$\begin{aligned} \|\phi\|_{0,p;u} &= \Big(\int_\Sigma (\langle \Phi(u)\phi, \Phi(u)\phi \rangle_{\bar{j}})^{\frac{p}{2}} \sigma \Big)^{\frac{1}{p}} \\ &= \|\phi\|_{0,p;u_o} \,. \end{aligned}$$

Hence, we immediately obtain the continuity of $\|\cdot\|_{0,p}$ on $L^p_\Sigma(\mathcal{P}^*X^J)$ and the Finsler structure property as stated in Proposition 4.4.8.

In order to analyze the norm $\|\cdot\|_{1,p}$ on $H^{1,p}_\Sigma(\mathcal{P}^*TM)$ we use the local trivialization by means of $\nabla_2 \exp$. However, here we also have to prove that $\|\cdot\|_{1,p}$ is uniquely extendible from $C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ to $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$ as a continuous map. Given a fixed $u_o \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ and an open neighbourhood $U(u_o) = \mathcal{U}_{u_o}$ as used for the differentiable atlas of $\mathcal{P}$, we consider

$$\Phi(u) = \nabla_2 \exp(\exp^{-1}_{u_o} u)\colon u_o^*TM \xrightarrow{\cong} u^*TM$$

for $u \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M) \cap U(u_o)$. Analogously to Proposition A.0.7 we can prove that $u \mapsto \Phi(u)$ is a smooth map, that is

$$\Phi(u) \in \mathcal{L}(H^{1,p}_\Sigma(u_o^*TM); H^{1,p}_\Sigma(u^*TM))$$

is well-defined for all $u \in U(u_o)$ and depends smoothly on u_o and $u \in \mathcal{P}$. Now, the key point is to prove the following estimate for the given $u_o \in C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$. We claim that for every $\epsilon > 0$ there is an open neighbourhood $V_\epsilon(u_o) \subset U(u_o)$ of u_o such that

$$(1-\epsilon)\|\xi\|_{1,p} \leqslant \big\|\Phi(u)\xi\big\|_{1,p} \leqslant (1+\epsilon)\|\xi\|_{1,p} \tag{A.15}$$

holds for all $\xi \in H^{1,p}_\Sigma(u_o^*TM)$, $u \in C^\infty_\Sigma \cap V_\epsilon(u_o)$. Then we argue by means of the denseness of $C^\infty_{x_1,\dots,x_\nu}(\Sigma, M)$ in $\mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$, that $\|\cdot\|_{1,p}$ can be uniquely extended to $\mathcal{P}$ such that (A.15) holds for all $u \in V_\epsilon(u_o)$. Hence, $\|\cdot\|_{1,p}$ becomes a continuous Finsler structure on the bundle $H^{1,p}_\Sigma(\mathcal{P}^*TM) \to \mathcal{P}^{1,p}_{x_1,\dots,x_\nu}(\Sigma, M)$.

Recalling the definition of $\|\cdot\|_{1,p}$, we have for $u \in C^\infty_\Sigma \cap U(u_o)$ and $\xi \in H^{1,p}_\Sigma(u_o^*TM)$,

$$\big\|\Phi(u)\xi\big\|^p_{1,p} = \int_\Sigma \Big(|\Phi(u)\xi|^p_J + |\nabla(\Phi(u)\xi)|^p_J \Big) \sigma \,.$$

By definition $|v|^p_J = \langle v, v \rangle^{\frac{p}{2}}_{J(m)}$ for $v \in T_mM$, and $|\cdot|_J$ on $T^*\Sigma \otimes TM$ is defined by means of a partition of unity on Σ as described in Section 4.4.1. Since these Riemannian metrics $\langle \cdot, \cdot \rangle_J$ on TM and $T^*\Sigma \otimes TM \to \Sigma \times M$ are smooth as well as the maps

$$\exp\colon \mathcal{D} \to M \times M, \quad \nabla_2 \exp\colon \mathcal{D} \to \mathrm{Hom}(TM, TM),$$

we can estimate

$$\int_\Sigma \Big(|\Phi(u)\xi|^p_J - |\xi|^p_J \Big) \sigma \leqslant c(u_o) \| \exp^{-1}_{u_o} u \|^p_\infty \cdot \|\xi\|^p_{0,p}$$

and

$$\begin{aligned} &\int_\Sigma \Big(|\nabla(\Phi(u)\xi)|^p_J - |\nabla\xi|^p_J \Big) \sigma \\ \leqslant &\tilde{c}(u_o) \Big(\| \exp^{-1}_{u_o} u \|^p_\infty \cdot \|\nabla\xi\|^p_{0,p} + \| \exp^{-1}_{u_o} u \|^p_{1,p} \cdot \|\xi\|^p_\infty \Big) \end{aligned}$$

for some constants $c(u_o), \tilde{c}(u_o) > 0$. This proves the inequality (A.15).

Altogether, we have proven that the norm $\|\cdot\|_{1,p}$ is fibre-wise well-defined on $H^{1,p}_\Sigma(\mathcal{P}^*TM)$, that it is a continuous Finsler structure as asserted in Proposition 4.4.8, and that $\Phi(u) \in \mathcal{L}(H^{1,p}_\Sigma(u_o^*TM); H^{1,p}_\Sigma(u^*TM))$ is continuous with respect to the associated operator-norm.

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Curriculum Vitae

Am 16. Oktober 1967 wurde ich als Sohn von Hans-Ulrich und Gisela Schwarz in Tübingen (Deutschland) geboren. Die Grundschule besuchte ich von 1974-77 in Kiel und Bochum. In Bochum besuchte ich anschließend das mathematisch-naturwissenschaftliche Albert-Einstein-Gymnasium und legte 1986 das Abitur mit den Schwerpunktfächern Mathematik, Physik, Französisch und Geschichte ab.

Nach dem Grundwehrdienst begann ich 1987 zunächst das Studium der Physik an der Ruhr-Univeristät Bochum, schrieb mich jedoch im Februar 1988 auch für Mathematik als ersten Studiengang ein. Nach 9 Semestern Studium in Bochum erwarb ich im Frühjahr 1992 das Diplom in Mathematik. Während der Studienzeit wurde ich von der Studienstiftung des deutschen Volkes gefördert. Für die erste Hälfte meines Studiums bin ich besonders Prof. Alan T. Huckleberry zu Dank verpflichtet. Durch ihn und Prof. Helmut Hofer wurde meine Begeisterung für symplektische Geometrie geweckt. Ich profitierte zudem besonders von dem Unterricht bei Prof. Ralph Stöcker. Die Diplomarbeit schrieb ich unter der Betreuung von Prof. Hofer über die Konstruktion einer axiomatischen Homologietheorie basierend auf Morse-Theorie. Prof. Hofer und Prof. Eduard Zehnder ermöglichten, daß diese Arbeit nach einer Übersetzung ins Englische publiziert werden konnte. Sie erschien 1993 im Birkhäuser Verlag in der Reihe "Progress in Mathematics" unter dem Titel "Morse homology", PM 111.

Nach der Diplomarbeit erhielt ich im April 1992 ein Doktoranden-Stipendium des Graduiertenkollegs "Mathematische Physik und Geometrie" am Mathematischen Institut der Ruhr-Universität Bochum und begann mit der vorliegenden Dissertation unter der Betreuung von Prof. Hofer. Im April 1993 wechselte ich als Assistent an die ETH Zürich, wo ich zunächst als Doktorand von Prof. Zehnder und seit September 1993 wieder bei Prof. Hofer die Arbeit an der Dissertation fortsetzte.