Families of elliptic curves with genus 2 covers of degree 2

Claus Diem

November 8, 2005

Abstract

We study genus 2 covers of relative elliptic curves over an arbitrary base in which 2 is invertible. Particular emphasis lies on the case that the covering degree is 2. We show that the data in the "basic construction" of genus 2 covers of relative elliptic curves determine the cover in a unique way (up to isomorphism).

A classical theorem says that a genus 2 cover of an elliptic curve of degree 2 over a field of characteristic $\neq 2$ is birational to a product of two elliptic curves over the projective line. We formulate and prove a generalization of this theorem for the relative situation.

We also prove a Torelli theorem for genus 2 curves over an arbitrary base.

Key words: Elliptic curves, covers of curves, families of curves, curves of genus 2, curves with split Jacobian.

MSC2000: 14H45, 14H10, 14H30.

Introduction

The purpose of this article is to study covers $f : C \longrightarrow E$ where C/S is a (relative, smooth, proper) genus 2 curve, E/S is a (relative) elliptic curve and the base S is a locally noetherian scheme over $\mathbb{Z}[1/2]$. Particular emphasis lies on the case that the covering degree N is 2.

If one studies genus 2 covers of (relative) elliptic curves, it is convenient to restrict ones attention to so-called *minimal* covers. These are covers $C \longrightarrow E$ which do not factor over a non-trivial isogeny $\tilde{E} \longrightarrow E$. If now $f : C \longrightarrow E$ is a minimal cover and $x \in E(S)$, then $T_x \circ f$ is also one. This ambiguity motivates the notion of a *normalized* cover introduced in [10]: By definition, such a cover is minimal and satisfies a certain condition concerning the direct image of the Weierstraß divisor of C on E (for precise definition see below). Now for every minimal cover $f : C \longrightarrow E$ there is exactly one $x \in E(S)$ such that $T_x \circ f : C \longrightarrow E$ is normalized.

To every minimal cover $f : C \longrightarrow E$ one can associate in a canonical way an elliptic curve E'_f/S and an isomorphism of S-group schemes $\psi_f : E[N] \tilde{\longrightarrow} E'_f[N]$ which is anti-isometric with respect to the Weil pairing; see [10]. It is shown in [10] that for fixed S, E/S and $N \geq 3$, the assignment $f \mapsto (E_f, \psi_f)$ induces a monomorphism from the set of isomorphism classes of normalized genus 2 covers of degree N of E/S to the set of isomorphism classes of tuples (E', ψ) of elliptic curves E/S with an anti-isometric isomorphism $\psi : E[N] \tilde{\longrightarrow} E'[N]$. Explicit conditions are given when a tuple (E', ψ) corresponds to a normalized genus 2 cover $C \longrightarrow E$ of degree N over S – this is called "basic construction" in [10].

In this work, we show that the above assignment is in fact a monomorphism for all $N \geq 2$. Our starting point is a Torelli theorem (Theorem 1) for relative genus 2 curves which follows rather easily from the detailed appendix of [10]. With the help of this theorem, we prove a Torelli theorem for normalized genus 2 covers of (relative) elliptic curves; see Proposition 2.3. This result implies immediately that the "Torelli map" of [10] is a monomorphism for arbitrary $N \geq 2$. In [10], the corresponding statement is only proved for $N \geq 3$ and the proof is more involved; cf. [10, Proposition 5.12]. The injectivity of the above assignment then follows with other results of [10].

For $N = 2$ (and fixed S and E/S), tuples (E', ψ) as well as normalized covers $C \longrightarrow E$ have a non-trivial automorphism of order 2. This leads to a certain "non-rigidity" in the "basic construction": Any two covers corresponding to the same tuple (E', ψ) are isomorphic, but the isomorphism is not unique. We propose a "symmetric basic construction" which leads to a more rigid statement (and is more explicit than the "basic construction").

We then fully concentrate on the case that $N = 2$. We show in particular that for every normalized cover $f : C \longrightarrow E$ of degree 2, one has a canonical commutative diagram

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{f} & & \searrow & \\ E & & & & E'_f \ , \\ & \searrow & & \swarrow & \\ & & \mathbf{P} & & \end{array}$$

where $\mathbf{P} := E/\langle[-1]\rangle$ is a $\mathbb{P}^1$-bundle over S and all morphisms are covers of degree 2 such that the induced morphism $C \longrightarrow E \times_{\mathbf{P}} E'_f$ induces birational morphisms on the fibers over S; see Theorem 2 in Section 3 and Corollary 3.6. This generalizes a classical result on genus 2 curves with elliptic differentials of degree 2 over a field of characteristic $\neq 2$ which follows immediately from Kummer theory applied to the extension $\kappa(C)/\kappa(E/\langle[-1]\rangle)$.

Finally, we discuss a reinterpretation of this result and show that it is closely related to a general statement on $\mathbb{P}^1$-bundles which we prove in an appendix.

The study of genus 2 curves with split Jacobian has a long history which arguably started with the task of reducing hyperelliptic integrals of genus 2 of the first kind to sums of elliptic integrals. Here a substitution of variables gives rise to a genus 2 cover of an elliptic curve. The study for degree 2 dates back to Legendre who gave the first examples and Jacobi. More information on this classical material can be found in [11], pp.477-482.

It is now also classical that to every minimal cover $f : C \longrightarrow E$ one can in a canonical way associate a "complementary" minimal cover $C \longrightarrow E'_f$ of the same degree (unique up to translation on E); see e.g. [12]. The idea to describe genus 2 covers of a fixed elliptic curve E (over a field) by giving the complementary elliptic curve E'_f and a suitable anti-isometric isomorphism $E[N] \tilde{\longrightarrow} E'_f[N]$, where N is the covering degree, is due to G. Frey and E. Kani; see [4] and also [9]. The basic results for genus 2 covers of *relative* elliptic curves were obtained by E. Kani in [10].

An application of some results presented in this article can be found in [3]. In this work, examples of relative, non-isotrivial genus 2 curves C/S which possess an infinite tower of non-trivial étale covers $\cdots \longrightarrow C_i \longrightarrow \ldots C_0 = C$ such that for all i, $C_i \longrightarrow C$ is Galois and C_i/S is also a curve (in particular has *geometrically connected* fibers) are given. The genus 2 curves in question are covers of elliptic curves with covering degree 2, the base schemes are affine curves over finite fields of odd characteristic.

Terminology and notation

This work is closely related to [10]. With the exception of the following assumption, the following three definitions and Definition 2.7, all definitions and notations follow this work. We thus advise the reader to have [10] at hand when he goes through the details of this article. Note that although the primary emphasis of [10] lies on genus 2 covers of elliptic curves E_S, where E/K is an elliptic curve over a field K of characteristic $\neq 2$ and S is a K-scheme, as stated in various places of [10], the results of [10] hold for genus 2 covers of elliptic curves over arbitrary locally noetherian schemes over $\mathbb{Z}[1/2]$.

If not stated otherwise, all schemes we consider are assumed to be locally noetherian.

If $g \in \mathbb{N}_0$, then a *(relative) curve* of genus g over S is a smooth, proper morphism $C \longrightarrow S$ whose fibers are geometrically connected curves of genus g. (We thus do not assume that the genus is ≥ 1 or that for $g = 1$ C/S has a section.)

If C/S is a curve and $N \in \mathbb{N}$, $g \in \mathbb{N}_0$, then a genus g *cover* of degree N of C is an S-morphism $f : C' \longrightarrow C$, where C'/S is a genus g curve, which induces morphisms of the same degree N on the fibers over S. (Note that f is automatically finite, flat and surjective; cf. [10, Section 7, 7)].)

If C/S and C'/S are two curves of genus ≥ 2, we denote the scheme of S-isomorphisms from C to C' by $\mathbf{Iso}_S(C, C')$; cf. [2].

Following [14], a curve C/S is called *hyperelliptic* if it has a (by Lemma 1.1 necessarily unique) automorphism $\sigma_{C/S}$ which induces hyperelliptic involutions on the geometric fibers. For equivalent definitions of $\sigma_{C/S}$, see [14, Theorem 5.5].

We have used the following definition in the introduction; cf. [10]:

Let S be a scheme over $\mathbb{Z}[1/2]$, let C/S be a genus 2 curve and let E/S be an elliptic curve. Then a cover $f : C \longrightarrow E$ is *minimal* if it does not factor over a non-trivial isogeny $\tilde{E} \longrightarrow E$, and it is *normalized* if it is minimal and we have the equality of relative effective Cartier divisors

$$f_*(W_{C/S}) = 3\epsilon[0_{E/S}] + (2 - \epsilon)E[2]^{\#} ,$$

where $W_{C/S}$ is the Weierstraß divisor of C/S, $E[2]^{\#} := E[2] - [0_{E/S}]$ and $\epsilon = 0$ if $\deg(f)$ is even and $\epsilon = 1$ if $\deg(f)$ is odd.[1] Note that a normalized cover satisfies

$$f \circ \sigma_{C/S} = [-1] \circ f \; ; \tag{1}$$

cf. [10, Theorem 3.2 (c)].

We frequently use the following notation:

If $f : T \longrightarrow S$ is a morphism of schemes and $\varphi : X \longrightarrow Y$ is a morphism of S-schemes, we denote the morphism induced by base change via f by $f^*\varphi : f^*X \longrightarrow f^*Y$ or just $\varphi_T : X_T \longrightarrow Y_T$.

We use two different symbols to denote isomorphisms: If we just want to state that two objects X, Y in some category are isomorphic, we write $X \approx Y$. If X and Y are isomorphic with respect to a canonical isomorphism or with respect to a fixed isomorphism which is obvious from the context, we write $X \simeq Y$.

Acknowledgments. The author would like to thank G. Frey, E. Kani and E. Viehweg for various discussions related to this work.

1 A Torelli theorem for relative genus 2 curves

The purpose of this section is to prove the following theorem.

[1] There are misprints in the definitions in [10, Section 2] and [10, Section 3].

Theorem 1 *Let S be a scheme, let C/S and C'/S be two genus 2 curves. Then the map* $\mathrm{Iso}_S(C,C') \longrightarrow \mathrm{Iso}_S((J_C,\lambda_C),(J_{C'},\lambda_{C'}))$, $\varphi \mapsto \varphi_*$ *is an isomorphism.*

Here, by λ_C we denote the canonical polarization of the Jacobian J_C of a genus 2 curve C/S and for an isomorphism $\varphi : C \longrightarrow C'$ of two genus 2 curves over S, we define $\varphi_* := \lambda_{C'}^{-1} \circ \hat{(\varphi^*)} \circ \lambda_C = (\varphi^*)^{-1}$.

This Torelli theorem for (relative) genus 2 curves is well known in the case that S is the spectrum of an (algebraically closed) field; cf. e.g. [16, Theorem 12.1] where it is stated with a slightly different formulation for arbitrary hyperelliptic curves over algebraically closed fields.

Theorem 1 follows from Lemmata 1.2 and 1.6 which are proved below.

Let S be a scheme, and let C/S and C'/S be curves.

We will frequently use the fact that the formation of the Jacobian commutes with arbitrary base-change: Let $f : T \longrightarrow S$ be a morphism of schemes. Then we have canonical isomorphisms $(J_{C_T},\lambda_{C_T}) \simeq ((J_C)_T,(\lambda_C)_T)$, $(J_{C'_T},\lambda_{C'_T}) \simeq ((J_{C'})_T,(\lambda_{C'})_T)$. Moreover, under the obvious identification, we have

$$(\varphi_*)_T = (\varphi_T)_* : J_{C_T} \longrightarrow J_{C'_T} \text{ i.e. } f^*(\varphi_*) = (f^*\varphi)_* \,. \tag{2}$$

Lemma 1.1 *Let S be a connected scheme, let $s \in S$. Then the restriction map* $\mathrm{Iso}_S(C,C') \longrightarrow \mathrm{Iso}_{\kappa(s)}(C_s,C'_s)$ *is injective.*

Proof. The S-isomorphisms between C and C' correspond to sections of the S-scheme $\mathbf{Iso}_S(C,C')$. As this scheme is unramified over S (see [2, Theorem 1.11]), the result follows with [5, Exposé I, Corollaire 5.3.]. □

Lemma 1.2 *Let S be a connected scheme, let $s \in S$. Then the map* $\mathrm{Iso}_S(C,C') \longrightarrow \mathrm{Iso}_{\kappa(s)}((J_{C_s},\lambda_{C_s}),(J_{C'_s},\lambda_{C'_s}))$, $\varphi \mapsto (\varphi_s)_* = (\varphi_*)_s$ *is injective.*

Proof. This follows from the previous lemma and the classical Torelli Theorem (see [16, Theorem 12.1]). □

Lemma 1.3 *Let $S' \longrightarrow S$ be faithfully flat and quasi compact. Let $\varphi' : C_{S'} \longrightarrow C'_{S'}$ be an S'-isomorphism, and let $\alpha : J_C \longrightarrow J_{C'}$ be a homomorphism with $\alpha_{S'} = \varphi'_*$. Then there exists an S-isomorphism $\varphi : C \longrightarrow C'$ with $\varphi_{S'} = \varphi'$ and $\alpha = \varphi_*$.*

Proof. Let $S'' := S' \times_S S'$, let $p_1, p_2 : S'' \longrightarrow S'$ be the two projections. We want to show that $p_1^*\varphi' = p_2^*\varphi'$. Then the statement follows by faithfully flat descent; see [1, Section 6.1., Theorem 6].

By assumption we have $p_1^*(\varphi'_*) = p_2^*(\varphi'_*)$. Together with (2) this implies that $(p_1^*\varphi')_* = (p_2^*\varphi')_*$. Now the equality $p_1^*\varphi' = p_2^*\varphi'$ follows with the previous lemma. □

The following lemma is a special case of [17, Proposition 6.1], the "Rigidity Lemma".

Lemma 1.4 *Let S be a connected scheme, let $s \in S$. Let $A/S, A'/S$ be two abelian schemes. Then the map* $\mathrm{Hom}_S(A, A') \longrightarrow \mathrm{Hom}_{\kappa(s)}(A_s, A'_s)$ *is injective.*

Lemma 1.5 *Let C/S and C'/S be genus 2 curves, and assume that both curves have a section. Then the map* $\mathrm{Iso}_S(C, C') \longrightarrow \mathrm{Iso}_S((J_C, \lambda_C), (J_{C'}, \lambda_{C'}))$, $\varphi \mapsto \varphi_*$ *is surjective.*

Proof. Let $a : S \longrightarrow C$ be a section. Let $j_a : C \longrightarrow J_C$ be the immersion associated to a; cf. [10, Section 7, 6)]. Analogously, let $a' : S \longrightarrow C'$ be a section, and let $j_{a'} : C' \longrightarrow J_{C'}$ be the associated immersion. Now $j_a(C)$ is a Cartier divisor on J_C which defines the principal polarization λ_C. (Indeed, for all $s \in S$, we have $\lambda_{C_s} = \lambda_{\mathcal{O}(j_a(C)_s)} : J_{C_s} \longrightarrow J_{C'_s}$. The equality $\lambda_C = \lambda_{\mathcal{O}(j_a(C))}$ follows with Lemma 1.4.) Analogously, $j_{a'}(C')$ is an a Cartier divisor on $J_{C'}$ which defines the principal polarization $\lambda_{C'}$.

Let $\alpha : J_C \longrightarrow J_{C'}$ be an isomorphism which preserves the principal polarizations, i.e. which satisfies $\hat{\alpha} \circ \lambda_{C'} \circ \alpha = \lambda_C$.

Then λ_C is given by the divisor $\alpha^{-1}(j_{a'}(C'))$. It follows from [10, Lemma 7.1] that $\alpha^{-1}(j_{a'}(C')) = T_x^{-1}(j_a(C))$ for some $x \in J_C(S)$. This can be rewritten as $(\alpha^{-1} \circ j_{a'})(C') = (T_{-x} \circ j_a)(C)$. Note here that $\alpha^{-1} \circ j_{a'} : C' \longrightarrow J_C$ and $T_{-x} \circ j_a : C \longrightarrow J_C$ are closed immersions, and we have an equality of the associated closed subschemes of $J_{C'}$. This means that there exists an isomorphism of schemes $\varphi : C \longrightarrow C'$ such that $\alpha^{-1} \circ j_{a'} \circ \varphi = T_{-x} \circ j_a$, i.e. $j_{a'} \circ \varphi = \alpha \circ T_{-x} \circ j_a$. A short calculation shows that φ is in fact an S-isomorphism.

The equality $j_{a'} \circ \varphi = \alpha \circ T_{-x} \circ j_a$ immediately implies that $\varphi_* = \alpha$. □

Lemma 1.6 *Let C/S, C'/S be two genus 2 curves. Then the map* $\mathrm{Iso}_S(C, C') \longrightarrow \mathrm{Iso}_S((J_C, \lambda_C), (J_{C'}, \lambda_{C'}))$, $\varphi \mapsto \varphi_*$ *is surjective.*

Proof. Let $W_{C/S}$, $W_{C'/S}$ be the Weierstraß divisors of C/S and C'/S respectively and let $W := W_{C/S} \times_S W_{C'/S}$. Now the canonical map $W \longrightarrow S$ is faithfully flat and quasi compact (in fact it is finite flat of degree 36), and C_W/W as well as C'_W/W have sections (namely the sections induced by $W_{C/S} \hookrightarrow C$, $W_{C'/S} \hookrightarrow C'$). It follows by the above lemma that $\mathrm{Iso}_W(C_W, C'_W) \longrightarrow \mathrm{Iso}_W((J_{C_W}, \lambda_{C_W}), (J_{C_W}, \lambda_{C_W}))$, $\varphi \mapsto \varphi_*$ is surjective. The claim now follows with Lemma 1.3. □

The above considerations easily imply:

Corollary 1.7 *Let $C/S, C'/S$ be hyperelliptic curves, let $\varphi : C \longrightarrow C'$ be an S-isomorphism. Then*

$$\sigma_{C'/S} \circ \varphi = \varphi \circ \sigma_{C/S} \; .$$

Proof. We can assume that S is connected. Let $s \in S$. It is well known that $(\sigma_{C_s})_* = [-1], (\sigma_{C'_s})_* = [-1]$. This implies $(\sigma_{C'_s})_* \circ (\varphi_s)_* = -(\varphi_s)_* = (\varphi_s)_* \circ (\sigma_{C_s})_*$. The result now follows with Lemma 1.2. □

We also have:

Lemma 1.8 *Let C/S be a hyperelliptic curve. Then $(\sigma_{C/S})_* = [-1]$.*

Proof. This follows from the well known result over the spectrum of a field by Lemma 1.4. □

2 Review of the "basic construction"

Theorem 1 can be used to prove a Torelli theorem for normalized genus 2 covers of elliptic curves which in turn can be used to simplify some proofs in [10] as well as to strengthen the results for the case that the covering degree N is 2. This is done in the first half of this section. Throughout the section, we freely use results from [10].

Let S be a scheme over $\mathbb{Z}[1/2]$. The following definition is analogous to the "notation" in Section 3 of [10].

Definition 2.1 Let E/S be an elliptic curve. Let $f_1 : C_1 \longrightarrow E, f_2 : C_2 \longrightarrow E$ be two genus 2 covers. Then an *isomorphism* between f_1 and f_2 is an S-isomorphism $\varphi : C_1 \longrightarrow C_2$ such that $f_1 = f_2 \circ \varphi$.

The following lemma shows (in particular) that given two isomorphic genus 2 covers of the same elliptic curve, one of the covers is normalized if and only if the other is.

Lemma 2.2 *Let $E_1/S, E_2/S$ be an elliptic curves, let $C_1/S, C_2/S$ be genus 2 curves. Let $f : C_2 \longrightarrow E_2$ be a normalized cover, let $\varphi : C_1 \longrightarrow C_2$ be an S-isomorphism and $\alpha : E_2 \longrightarrow E_1$ an isomorphism of elliptic curves. Then $\alpha \circ f \circ \varphi : C_1 \longrightarrow E_1$ is normalized.*

Proof. We can assume that S is connected. Obviously, $\alpha \circ f \circ \varphi$ is minimal. By Corollary 1.7 and (1), we have $\alpha \circ f \circ \varphi \circ \sigma_{C_1/S} = \alpha \circ f \circ \sigma_{C_2/S} \circ \varphi = \alpha \circ [-1]_{E_2/S} \circ f \circ \varphi = [-1]_{E_1/S} \circ \alpha \circ f \circ \varphi : C_1 \longrightarrow E_1$. By [10, Theorem 3.2 (c)] we have to show that for some geometric point $s \in S$, $(\alpha \circ f \circ \varphi)_s : (C_1)_s \longrightarrow (E_1)_s$ is normalized.[2]

[2] In [10, Theorem 3.2 (c)], the condition that S be connected should be inserted.

Let $s \in S$. It is well-known that $\varphi_s^{-1}(W_{(C_2)_s}) = W_{(C_1)_s}$. We have $\#(f^{-1}([0_{(E_2)_s}]) \cap W_{(C_2)_s}) = \#(\varphi_s^{-1}\left(f^{-1}(\alpha^{-1}([0_{(E_1)_s}])) \cap W_{(C_2)_s}\right)) = \#(\varphi_s^{-1}(f^{-1}(\alpha^{-1}([0_{(E_1)_s}]))) \cap \varphi_s^{-1}(W_{(C_2)_s})) = \#((\alpha \circ f \circ \varphi_s)^{-1}([0_{(E_1)_s}]) \cap W_{(C_1)_s})$. Now with [10, Corollary 2.3], the result follows. □

The following proposition can be viewed as a Torelli theorem for normalized genus 2 covers of (relative) elliptic curves.

Proposition 2.3 *Let E/S be an elliptic curve, and let $f_1 : C_1 \longrightarrow E, f_2 : C_2 \longrightarrow E$ be two normalized genus 2 covers. Then the bijection $\mathrm{Iso}_S(C_1, C_2) \longrightarrow \mathrm{Iso}_S((J_{C_1}, \lambda_{C_1}), (J_{C_2}, \lambda_{C_2})), \varphi \mapsto \varphi_*$ of Theorem 1 induces a bijection between*

- *the set of isomorphisms between the normalized genus 2 covers f_1 and f_2*

  and

- *the set of isomorphisms α between the principally polarized abelian varieties (J_{C_1}, λ_{C_1}) and (J_{C_2}, λ_{C_2}) satisfying $(f_1)_* = (f_2)_* \circ \alpha$.*

Proof. We only have to show the surjectivity.

Let α be an isomorphism between (J_{C_1}, λ_{C_1}) and (J_{C_2}, λ_{C_2}) satisfying $(f_1)_* = (f_2)_* \circ \alpha : J_{C_1} \longrightarrow E$. Let φ be the unique S-isomorphism $C_1 \longrightarrow C_2$ with $\varphi_* = \alpha$. We thus have $(f_1)_* = (f_2 \circ \varphi)_*$. By [10, Lemma 7.2], there exists a unique $x \in E(S)$ such that $T_x \circ f_1 = f_2 \circ \varphi$. As by Lemma 2.2 both f_1 and $f_2 \circ \varphi$ are normalized, we have in fact $f_1 = f_2 \circ \varphi$. □

Remark 2.4 The equality $(f_1)_* = (f_2)_* \circ \alpha$ in the above proposition can be restated as $\alpha \circ f_1^* = f_2^*$; cf. the calculation in the proof of [10, Theorem 2.6].

Remark 2.5 If $\deg(f_1) \geq 3$ (or $\deg(f_2) \geq 3$), there is in fact at most one isomorphism between f_1 and f_2; cf. [10, Proposition 3.3].

Application to the study of the Hurwitz functor

As in [10], let E/K be an elliptic curve over a field of characteristic $\neq 2$ (or more generally over a ring in which 2 is invertible or even a scheme over $\mathbb{Z}[1/2]$). As always, we use the notation of [10].

Proposition 2.3 and Remark 2.4 immediately imply that the "Torelli map" $\tau : \mathcal{H}_{E/K,N} \longrightarrow \mathcal{A}_{E/K,N}$ of [10] is a monomorphism for arbitrary $N > 1$; cf. [10, Proposition 5.12].

The functor $\Psi : \mathcal{H}_{E/K,N} \longrightarrow \mathcal{X}_{E,N,-1}$ of [10, Corollary 5.13] is thus in fact a monomorphism for arbitrary $N > 1$. Furthermore, the functor $\mathcal{H}_{E/K,N} \longrightarrow \mathcal{J}_{E/K,N}$ of [10, Proposition 5.17] is an isomorphism for arbitrary $N > 1$, and $\tau : \mathcal{H}_{E/K,N} \longrightarrow \mathcal{A}_{E/K,N}$ is always an open immersion of functors. This of course shortens the proof of Theorem 1.1. at the end of Section 5 in [10].

It follows that the covers obtained with the "basic construction" ([10, Corollary 5.19]) are always unique up to isomorphism for any $N > 1$. For $N \geq 3$, one sees with [10, Proposition 5.4] that given two covers associated to the same anti-isometry $\psi : E[N] \longrightarrow E'[N]$, there is a *unique* isomorphism between them.

Let us again consider genus 2 covers $f : C \longrightarrow E$ of an elliptic curve E/S, where S is a scheme over $\mathbb{Z}[1/2]$. The "basic construction" now reads as follows (as always, we use the notations of [10], in particular $E'_f := \ker(f_*)$).

Proposition 2.6 (Basic construction) *Let $N > 1$ be a natural number. Let $E/S, E'/S$ be two elliptic curves, and let $\psi : E[N] \longrightarrow E'[N]$ be an anti-isometry which is "theta-smooth" (in the sense that the induced principal polarization λ_J on $J_\psi := (E \times E')/\mathrm{Graph}(\psi)$ is theta-smooth). Then there is a normalized genus 2 cover $f : C \longrightarrow E$ of degree N such that (E', ψ) is equivalent to (E'_f, ψ_f) (where $\psi_f : E[N] \longrightarrow E'_f[N]$ is the induced anti-isometry). The cover f is unique up to isomorphism (up to unique isomorphism if $N \geq 3$). Moreover, every normalized genus 2 cover of degree N arises in this way.*

We now give a more symmetric formulation of the "basic construction". This "symmetric basic construction" has the advantage that it is more rigid than the basic construction for $N = 2$.

For this "symmetric basic construction", we fix two elliptic curves E/S, E'/S.

Definition 2.7 A *symmetric pair* (with respect to E/S and E'/S) is a triple (C, f, f'), where C/S is a genus 2 curve and $f : C \longrightarrow E$, $f' : C \longrightarrow E'$ are minimal covers such that $\ker(f_*) = \mathrm{Im}((f')^*)$ and $\ker(f'_*) = \mathrm{Im}(f^*)$. We say that a symmetric pair is *normalized* if both f and f' are normalized. By an *isomorphism* of two symmetric pairs $(C_1, f_1, f'_1), (C_2, f_2, f'_2)$ we mean an S-isomorphism $\varphi : C_1 \longrightarrow C_2$ such that $f_1 = f_2 \circ \varphi$ and $f'_1 = f'_2 \circ \varphi$.

Remark 2.8 It follows from Lemma 2.2 that given two isomorphic symmetric pairs, one of the symmetric pairs is normalized if and only if the other is.

Remark 2.9 If C/S is a genus 2 curve and $f : C \longrightarrow E$, $f' : C \longrightarrow E'$ are minimal covers such that $\ker(f_*) = \mathrm{Im}((f')^*)$, then by dualization, one also has $\ker(f'_*) = \mathrm{Im}(f^*)$, i.e. (C, f, f') is a symmetric pair.

Remark 2.10 If (C, f, f') is a symmetric pair, then E' (with $(f')^* \circ \lambda_{E'} : E' \longrightarrow J_C$) is (canonically isomorphic to) $\ker(f_*) = E'_f$. (If E/S is some elliptic curve, we denote the canonical polarization $E \longrightarrow J_E = \hat{E}$ by λ_E.)

Lemma and Definition 2.11 *If (C, f, f') is a symmetric pair, then the degrees of f and f' are equal; this number is called the* degree *of the symmetric pair.*

Proof. Let $N := \deg(f)$. Then by [10, Theorem 3.2 (f)], f^* also has degree N. By [10, Corollary 5.3] and Remark 2.10, $(f')^* \circ \lambda_{E'} : E' \hookrightarrow J_C$ has also degree N, and it follows again with [10, Theorem 3.2 (f)] that $\deg(f') = \deg((f')^*) = N$. □

Lemma 2.12 *Let E/S be an elliptic curve, let C/S be a genus 2 curve, and let $f : C \longrightarrow E$ be a minimal cover. Then there exists a unique normalized cover $c_f : C \longrightarrow E'_f$ such that $(c_f)^* \circ \lambda_{E'_f}$ is the canonical immersion $E'_f \hookrightarrow J_C$.*[3] *In particular, if f is normalized, then (C, f, c_f) is a normalized symmetric pair.*

Proof. This is a special case of [10, Theorem 3.2 (f)]. □

Proposition 2.13 *Let $E/S, E'/S$ be two elliptic curves, let (C, f, f') be a symmetric pair of degree N associated to E/S and E'/S. Then there is a unique $\psi : E[N] \xrightarrow{\sim} E'[N]$ with $(f^*)_{|E[N]} = (f')^* \circ \psi$.*[4] *This ψ is an anti-isometry. Moreover, ψ only depends on the isomorphism class of (C, f, f').*

Proof. By Remark 2.10, the existence and uniqueness is [10, Proposition 5.2]. The fact that ψ only depends on the isomorphism class of (C, f, f') is straightforward. □

Proposition 2.14 *With the notation of the previous proposition, let*

$$\pi := f^* \circ \lambda_E \circ \mathrm{pr} + (f')^* \circ \lambda_{E'} \circ \mathrm{pr}' : E \times_S E' \longrightarrow J_C \ ,$$

where $\mathrm{pr} : E \times_S E' \longrightarrow E$ and $\mathrm{pr} : E \times_S E' \longrightarrow E'$ are the two projections. Then π has kernel $\mathrm{Graph}(-\psi)$. The pull-back to the canonical principal polarization of J_C under π is N-times the canonical product polarization. In particular, ψ is theta-smooth.

Proof. This is [10, Proposition 5.5]. □

The following "symmetric basic construction" can be viewed as a converse to Proposition 2.13.

Proposition 2.15 (Symmetric basic construction) *Let $N > 1$ be a natural number. Let $E/S, E'/S$ be two elliptic curves, and let $\psi : E[N] \longrightarrow E'[N]$ be an anti-isometry which is theta-smooth. Then there exists a normalized symmetric pair (C, f, f') with respect to E/S and E'/S with $(f^*)_{|E[N]}$*

[3] In [10, Corollary 5.13], $(c_f)^* \circ \lambda_{E'_f}$ is denoted by $(f')^*$.

[4] Note that just as in [10] we tacitly identify $E[N]$ with $J_E[N]$.

$= (f')^* \circ \psi$. *The normalized symmetric pair with these properties is essentially unique, i.e. it is unique up to unique isomorphism.*

Proof. Let N, $E/S, E'/S$ and $\psi : E[N] \longrightarrow E'[N]$ be as in the assertion.

To show the existence, one could use the "basic construction". There is however also the following more direct approach:

Consider the abelian variety $J_\psi := (E \times_S E')/\mathrm{Graph}(-\psi)$. By [10, Proposition 5.7] there exists a unique principal polarization λ_J on J_ψ whose pull-back to $E \times_S E'$ via the projection map is N-times the canonical product polarization. By assumption and [10, Proposition 5.14], (J_ψ, λ_J) is isomorphic to a Jacobian variety of a curve C/S. By [10, Theorem 3.2 (f)] there exist normalized covers $f : C \longrightarrow E$ and $f' : C \longrightarrow E'$ with $f^* \circ \lambda_E = h_\psi, (f')^* \circ \lambda_{E'} = h'_\psi$, where $h_\psi : E \longrightarrow J_\psi$ and $h'_\psi : E' \longrightarrow J_\psi$ are defined by inclusion into $E \times_S E'$ composed with the projection onto J_ψ; cf. [10, Corollary 5.9]. By the exact sequences (28) in [10, Corollary 5.9], the conditions $\ker(f_*) = \mathrm{Im}((f')^*)$ and $\ker(f'_*) = \mathrm{Im}(f^*)$ are fulfilled.

We now show the uniqueness. Let $(C_1, f_1, f'_1), (C_2, f_2, f'_2)$ be two normalized symmetric pairs associated to E, E' and ψ. We claim that there exists a unique isomorphism $\alpha : J_{C_1} \longrightarrow J_{C_2}$ of abelian varieties with $\alpha \circ f_1^* = f_2^*$ and $\alpha \circ (f'_1)^* = (f'_2)^*$.

Let

$$\begin{aligned} \pi_1 &:= f_1^* \circ \lambda_E \circ \mathrm{pr} + (f'_1)^* \circ \lambda_{E'} \circ \mathrm{pr}' : \quad E \times_S E' \longrightarrow J_{C_1/S} , \\ \pi_2 &:= f_2^* \circ \lambda_E \circ \mathrm{pr} + (f'_2)^* \circ \lambda_{E'} \circ \mathrm{pr}' : \quad E \times_S E' \longrightarrow J_{C_2/S} , \end{aligned}$$

where $\mathrm{pr} : E \times_S E' \longrightarrow E$ and $\mathrm{pr}' : E \times_S E' \longrightarrow E'$ are the two projections.

The two conditions on α are equivalent to $\alpha \circ \pi_1 = \pi_2 : E \times_S E' \longrightarrow J_{C_2/S}$. The assertion follows since by Proposition 2.14 $\pi_1 : E \times_S E' \longrightarrow J_{C_1/S}$ and $\pi_2 : E \times_S E' \longrightarrow J_{C_2/S}$ both have kernel $\mathrm{Graph}(-\psi)$.

The fact that f_1, f'_1, f_2 and f'_2 all have degree N implies that the pullbacks of λ_{C_1} and λ_{C_2} to $E \times_S E'$ via π_1 and π_2 respectively are N-times the canonical product polarizations. Together with the definition of α, this in turn implies that $\hat{\alpha} \circ \lambda_{C_2} \circ \alpha = \lambda_{C_1}$, i.e. α preserves the principal polarizations.

Let $\varphi : C_1 \longrightarrow C_2$ be the unique S-isomorphism such that $\varphi_* = \alpha$; cf. Theorem 1. By Proposition 2.3 and Remark 2.4, we have $f_1 = f_2 \circ \varphi$ and $f'_1 = f'_2 \circ \varphi$. The uniqueness of α implies that $\varphi : C_1 \longrightarrow C_2$ with these two properties is unique. □

Remark 2.16 Let S, E/S, E'/S and $\psi : E[N] \longrightarrow E'[N]$ be as in the "symmetric basic construction" but without the assumption that ψ is theta-smooth. Then by [10, Corollary 5.16] there exists a uniquely determined largest open subscheme U of S such that $\psi_{|U}$ is theta-smooth. Now U is the largest open subscheme of S over which a symmetric pair with respect to E_U/U and E'_U/U corresponding to ψ exists; this is obvious from Proposition 2.14 and the very definition of theta-smoothness.

3 Genus 2 covers of degree 2

We now concentrate on the case that the covering degree N is 2. As above, let S be a scheme over $\mathbb{Z}[1/2]$.

In the sequel, by an *isomorphism* $E[2] \longrightarrow E'[2]$, where E/S and E'/S are elliptic curves, we always mean an isomorphism of S-group schemes. Note that every such isomorphism is an anti-isogeny. The following proposition is a special case of [9, Theorem 3].

Proposition 3.1 *Let $E/S, E'/S$ be two elliptic curves, let $\psi : E[2] \longrightarrow E'[2]$ be an isomorphism. Then ψ is theta-smooth if and only if for no geometric point s of S, there exists an isomorphism $\alpha : E_s \longrightarrow E'_s$ such that $\alpha_{|E_s[2]} = \psi_s : E_s[2] \longrightarrow E'_s[2]$.*

Remark 3.2 Under the conditions of the proposition, let s be a geometric point of S. Assume that E_s has j-invariant $\neq 0, 1728$. Then if E'_s is isomorphic to E_s (i.e. if the j-invariants of the two curves are equal), there exist exactly two isomorphisms between E_s and E'_s. If α is one of these, $-\alpha$ is the other. This means that the isomorphisms between E_s and E'_s induce a *canonical* identification of $E_s[2]$ and $E'_s[2]$. Under the above assumption on the j-invariant of E_s, the following assertions are thus equivalent.

- *There does not exist an isomorphism $\alpha : E_s \longrightarrow E'_s$ such that $\alpha_{|E_s[2]} = \psi_s : E_s[2] \longrightarrow E'_s[2]$.*
- *$j(E_s) \neq j(E'_s)$ or $j(E_s) = j(E'_s)$ and, under the canonical identification of $E_s[2]$ and $E'_s[2]$, $\psi_s \neq \mathrm{id}_{E_s[2]}$.*

Proposition 3.3 *Let $E/S, E'/S$ be two elliptic curves with an isomorphism $\psi : E[2] \longrightarrow E'[2]$. Let C/S be a genus 2 curve, and let (C, f, f') be a normalized symmetric pair for E/S and E'/S. Then $(f^*)_{|E[2]} = (f')^* \circ \psi$ if and only if $\psi \circ f_{|W_{C/S}} = (f')_{|W_{C/S}}$.*

Proof. Let $E/S, E'/S, \psi, C, f$ and f' be as in the proposition. We only have to show the equivalence after a faithfully flat base change. We can thus assume that C/S has 6 distinct Weierstraß sections. Now by [10, Theorem 3.2 (d)], there exists an embedding $j : C \longrightarrow J_C$ which satisfies $j \circ \sigma_C = [-1] \circ j$, $[0_{J_C}] \cap j(C) = \emptyset$. This implies in particular that $j(W_{C/S}) \subset J_C[2]^{\#}$, where $J_C[2]^{\#} := J_C[2] - [0_{J/S}]$.

Assume that $f^*|_{E[2]} = (f')^* \circ \psi$. Then $f_{*|J_C[2]} = \lambda_E^{-1} \circ \widehat{(f^*)} \circ (\lambda_C)_{|J_C[2]} = \lambda_E^{-1} \circ \hat{\psi} \circ \widehat{((f')^*)} \circ (\lambda_C)_{|J_C[2]} = \psi^{-1} \circ f'_{*|J_C[2]} : J_C[2] \longrightarrow E[2]$. (We make the usual identification of $E[2]$ with $\hat{E}[2]$ and $J_C[2]$ with $\hat{J}_C[2]$.) Composition with $j_{|W_{C/S}}$ implies $f_{|W_{C/S}} = \psi^{-1} \circ (f')_{|W_{C/S}}$, i.e. $\psi \circ f_{|W_{C/S}} = (f')_{|W_{C/S}}$.

Let us now assume that $\psi \circ f_{|W_{C/S}} = (f')_{|W_{C/S}}$. We want to show that $\psi \circ f_{*|J_C[2]^\#} = f'_{*|J_C[2]^\#}$. As $J_C[2] = [0_{J/S}] \dot\cup J_C[2]^\#$ and clearly $\psi \circ f_{*|[0_{J/S}]} = f'_{*|[0_{J/S}]}$, this implies that $\psi \circ f_{*|J_C[2]} = f'_{*|J_C[2]} : J_C[2] \longrightarrow E[2]$. The equality $(f^*)_{|E[2]} = ((f')^*)_{E[2]} \circ \psi$ then follows by "dualization" similarly to above.

By the fact that (C, f, f') is a normalized symmetric pair, we have $\ker(f_*)[2] = \ker(f'_*)[2]$, i.e. $\ker(f_{*|J_C[2]}) = \ker(f'_{*|J_C[2]})$. Let these (equal) kernels be denoted by K. Then $f_{*|J_C[2]}$ and $f'_{*|J_C[2]}$ induce homomorphisms $\overline{f_{*|J_C[2]}} : J_C[2]/K \longrightarrow E[2], \overline{f'_{*|J_C[2]}} : J_C[2]/K \longrightarrow E'[2]$. Since these homomorphisms are surjective and $J_C[2]/K$, $E[2]$ and $E'[2]$ are étale over S of degree 4, they are in fact isomorphisms. Let $p : J_C[2] \longrightarrow J_C[2]/K$ be the canonical projection. Then the equality $\psi \circ f_{*|J[2]} = f'_{*|J[2]}$ implies

$$\psi \circ \overline{f_{*|J_C[2]}} \circ p \circ j_{|W_{C/S}} = \overline{f'_{*|J_C[2]}} \circ p \circ j_{|W_{C/S}} .$$

We claim that $p \circ j_{|W_{C/S}} : W_{C/S} \longrightarrow (J_C[2]/K)^\#$ is an étale cover.

We have $f_{|W_{C/S}} = \overline{f_{*|J_C[2]}} \circ p \circ j_{|W_{C/S}}$. Since $f_{|W_{C/S}}$ induces an étale cover $W_{C/S} \longrightarrow E[2]^\#$ of degree 2 and $\overline{f_{*|J_C[2]}}$ is an isomorphism, $p \circ j_{|W_{C/S}} : W_{C/S} \longrightarrow (J_C[2]/K)^\#$ is also an étale cover of degree 2.

As any surjective étale S-cover is an epimorphism in the category of S-schemes (see [5, Exposé V, Proposition 3.6.]), we can thus derive that $\psi \circ \overline{f_{*|J_C[2]}}_{|(J_C[2]/K)^\#} = \overline{f'_{*|J_C[2]}}_{|(J_C[2]/K)^\#}$, in particular $\psi \circ f_{*|J_C[2]^\#} = f'_{*|J_C[2]^\#} : J_C[2]^\# \longrightarrow E[2]^\#$. □

With the above two propositions, the "symmetric basic construction" can be restated as follows:

Proposition 3.4 (Symmetric basic construction for degree 2 – second form) *Let S be a scheme over $\mathbb{Z}[1/2]$. Let $E/S, E'/S$ be two elliptic curves, and let $\psi : E[2] \longrightarrow E'[2]$ be an isomorphism such that for no geometric point s of S, there exists an isomorphism $\alpha : E_s \longrightarrow E'_s$ such that $\alpha_{|E_s[2]} = \psi_s$. Then there exists an essentially unique (i.e. unique up to unique isomorphism) normalized symmetric pair (C, f, f') with $\psi \circ f_{|W_{C/S}} = (f')_{|W_{C/S}}$.*

Let $E/S, E'/S$ be elliptic curves, and let C/S be a genus 2 curve. Let (C, f, f') be a normalized symmetric pair with respect to E/S and E'/S.

Our goal is now to show that there exists a $\mathbb{P}^1$-bundle $\mathbf{P}$ and covers of degree 2 $E \longrightarrow \mathbf{P}, E' \longrightarrow \mathbf{P}$ such that the induced morphism $C \longrightarrow E \times_{\mathbf{P}} E'$ induces birational morphisms on the fibers over S.

Let $\tilde{q} : C \longrightarrow S, q : E \longrightarrow S, q' : E' \longrightarrow S$ be the structure morphisms. Let $\omega_{C/S} := \tilde{q}_* \Omega_{C/S}$. By Riemann-Roch and "cohomology and base change" ([18, §5, Corollary 3] and [7, Theorem 12.11]), this is a locally free sheaf of rank 2, and the canonical S-morphism $\tilde{\rho} : C \longrightarrow \mathbb{P}(\omega_{C/S})$ is a cover of degree 2.

By the same general theorems $q_*\mathcal{L}(2[0_E])$ is a locally free sheaf of rank 2, and the canonical S-morphism $\rho : E \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ is a cover of degree 2. Analogously, the canonical S-morphism $\rho' : E' \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ is a cover of degree 2.

Note that $(C, f, [-1]\circ f'), (C, [-1]\circ f, f')$ and $(C, [-1]\circ f, [-1]\circ f')$ are also normalized symmetric pairs with respect to E/S and E'/S corresponding to ψ.

There thus exist unique S-automorphisms $\tau, \tau', \tilde{\tau} : C \longrightarrow C$ with

$$\begin{array}{rclrcl} f \circ \tau & = & f\,, & f' \circ \tau & = & [-1] \circ f'\,, \\ f \circ \tau' & = & [-1] \circ f\,, & f' \circ \tau' & = & f'\,, \\ f \circ \tilde{\tau} & = & [-1] \circ f\,, & f' \circ \tilde{\tau} & = & [-1] \circ f'\,. \end{array}$$

Obviously, $\tau \circ \tau' = \tilde{\tau} = \tau' \circ \tau$ and $\tilde{\tau} = \sigma_{C/S}$.

The automorphisms τ and τ' are automorphisms of the covers f and f' respectively, and $\sigma_{C/S}$ is an automorphism of the cover $C \longrightarrow \mathbb{P}(\omega_{C/S})$. We need the following lemma which is a special case of [14, Lemma 5.6].

Lemma 3.5 *Let X and Y be connected schemes over $\mathbb{Z}[1/2]$. Let $h : X \longrightarrow Y$ be a finite and flat morphism of degree 2. Then the automorphism group of h is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and h is a geometric quotient of X under* $\mathrm{Aut}(h)$.

As a special case of this lemma we obtain: The cover $f : C \longrightarrow E$ is a geometric quotient of C under $\langle\tau\rangle$, and $f' : C \longrightarrow E'$ is a geometric quotient of C under $\langle\tau'\rangle$.

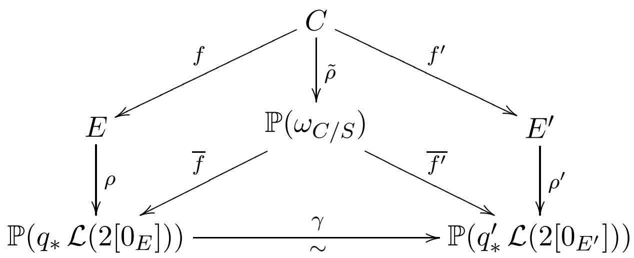
Furthermore, the canonical morphism $\tilde{\rho} : C \longrightarrow \mathbb{P}(\omega_{C/S})$ is a geometric quotient of C under $\langle\sigma_{C/S}\rangle$ (see also [10, Lemma 3.1] and [14, Theorem 5.5]), and the canonical morphisms $\rho : E \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$, $\rho' : E' \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ are geometric quotients of E and E' under $\langle[-1]\rangle$ respectively.

By (1), the automorphism $[-1]$ on E is induced by $\sigma_{C/S}$, and this implies that $\rho \circ f : C \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ is a geometric quotient of C under $\langle\tau, \tau'\rangle = \langle\tau, \sigma_{C/S}\rangle$. Similarly, $\rho' \circ f' : C \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ is also a geometric quotient of C under $\langle\tau, \tau'\rangle$. Keeping in mind that a geometric quotient is also a categorial quotient (see [5, Exposé V, Proposition 1.3.]), this implies the following theorem.

Theorem 2 *Let S be a scheme over $\mathbb{Z}[1/2]$. Let C/S be a genus 2 curve, $E/S, E'/S$ elliptic curves and $f : C \longrightarrow E, f' : C \longrightarrow E'$ normalized covers of degree 2 with $\ker(f_*) = \mathrm{Im}((f')^*)$, $\ker(f'_*) = \mathrm{Im}(f^*)$. Let $q : C \longrightarrow S, q : E \longrightarrow S, q' : E' \longrightarrow S$ be the structure morphisms, and let $\tilde{\rho} : C \longrightarrow \mathbb{P}(\omega_{C/S})$, $\rho : E \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E])), \rho' : E' \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ be the canonical covers of degree 2.*

Then f and f' have unique automorphisms τ and τ' respectively which operate non-trivially on all connected components of C. These automorphisms have order 2 and satisfy $\tau \circ \tau' = \tau' \circ \tau = \sigma_{C/S}$. The cover $f : C \longrightarrow E$ is a geometric quotient of C under $\langle \tau \rangle$, $f' : C \longrightarrow E'$ is a geometric quotient of C under $\langle \tau' \rangle$, and $\tilde{\rho} : C \longrightarrow \mathbb{P}(\omega_{C/S})$ is a geometric quotient of C under $\langle \sigma_{C/S} \rangle$.

Now $\rho \circ f : C \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ as well as $\rho' \circ f' : C \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ are geometric quotients of C under $\langle \tau, \tau' \rangle$. We thus have a unique isomorphism $\gamma : \mathbb{P}(q_* \mathcal{L}(2[0_E])) \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ such that $\gamma \circ \rho \circ f = \rho' \circ f$, and we have unique morphisms $\overline{f} : \mathbb{P}(\omega_{C/S}) \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ and $\overline{f'} : \mathbb{P}(\omega_{C/S}) \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ such that $\rho \circ f = \overline{f} \circ \tilde{\rho}$ and $\rho' \circ f' = \overline{f'} \circ \tilde{\rho}$. All these morphisms are S-morphisms, and $\overline{f}, \overline{f'}$ are covers of degree 2.



Corollary 3.6 *Let S be a scheme over $\mathbb{Z}[1/2]$, let C/S be a genus 2 curve, let E/S be an elliptic curve, and let $f : C \longrightarrow E$ be a normalized cover of degree 2. Let $\mathbf{P} := E/\langle [-1] \rangle = \mathbb{P}(q_* \mathcal{L}(2[0_E]))$, let $\rho : E \longrightarrow \mathbf{P}$ be the canonical cover of degree 2, and let $c_f : C \longrightarrow E'_f$ be the normalized cover of degree 2 associated to f by Lemma 2.12. Then there exists a unique S-morphism $\phi' : E'_f \longrightarrow \mathbf{P}$ such that $\rho \circ f = \phi' \circ c_f$. The morphism ϕ' is a cover of degree 2.*

The induced morphism $C \longrightarrow E \times_{\mathbf{P}} E'_f$ induces birational morphisms on the fibers over S.

Remark 3.7 Let S be a scheme over $\mathbb{Z}[1/2]$, let C/S be a genus 2 curve, let E/S be an elliptic curve and let $f : C \longrightarrow E$ be a normalized cover of some degree N. Let $\tilde{\rho} : C \longrightarrow \mathbb{P}(\omega_{C/S})$, $\rho : E \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ be as above. Then just as in the case that the covering degree is 2, there exists a unique morphism $\overline{f} : \mathbb{P}(\omega_{C/S}) \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ with

$$\overline{f} \circ \tilde{\rho} = \rho \circ f ,$$

and this morphism is a cover of degree N.

Indeed, the normalized cover f satisfies $f \circ \sigma_{C/S} = [-1] \circ f$ by (1). This implies that $\rho \circ f \circ \sigma_{C/S} = \rho \circ f$. Note that as above $\tilde{\rho}$ is a geometric quotient of C under $\sigma_{C/S}$. The existence and uniqueness of $\overline{f}$ is now immediate, and it is straightforward to check that f is in fact a cover of degree N.

Let us assume that we are in the situation of the theorem.

The canonical maps $\rho : E \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ and $\rho' : E' \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ are ramified at $E[2]$, $E'[2]$ respectively – these are étale covers of S of degree 4 –, and the canonical map $C \longrightarrow \mathbb{P}(\omega_{C/S})$ is ramified at $W_{C/S}$ – this is an étale cover of S of degree 6. (We use that S is a scheme over $\mathbb{Z}[1/2]$).

Let P and P' be the relative effective Cartier divisors of $\mathbb{P}(q_* \mathcal{L}(2[0_E]))/S$ and $\mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))/S$ associated to the sections $\rho \circ 0_E : S \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ and $\rho' \circ 0_{E'} : S \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$.

The maps $\rho_{|E[2]^\#} : E[2]^\# \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ and $(\rho')_{|E'[2]^\#} : E'[2]^\# \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ are closed immersions. Let D and D' be the corresponding relative effective Cartier divisors – they are étale covers of degree 3 of S.

Using the theorem, the isomorphism $\psi : E[2] \stackrel{\sim}{\longrightarrow} E'[2]$ corresponding to the isomorphism class of (C, f, f') can be determined in yet another way.

Proposition 3.8 *Let* $\psi : E[2] \stackrel{\sim}{\longrightarrow} E'[2]$. *Then* $\psi \circ f_{|W_{C/S}} = (f')_{|W_{C/S}}$ *if and only if* $\rho' \circ \psi_{|E[2]^\#} = \gamma \circ \rho_{|E[2]^\#}$.

Proof. The equality $\psi \circ f_{|W_{C/S}} = (f')_{|W_{C/S}}$ implies $\rho' \circ \psi \circ f_{|W_{C/S}} = \rho' \circ (f')_{|W_{C/S}}$, and this implies $\rho' \circ \psi \circ f_{|W_{C/S}} = \gamma \circ \rho \circ f_{|W_{C/S}}$. As $f_{|W_{C/S}} : W_{C/S} \longrightarrow E[2]^\#$ is an étale cover of degree 2 (thus in particular an epimorphism in the category of étale S-covers) and $\rho' \circ \psi_{|E[2]^\#} : E[2]^\# \longrightarrow D'$ as well as $\gamma \circ \rho_{|E[2]^\#} : E[2]^\# \longrightarrow D'$ are isomorphisms, we can conclude that $\rho' \circ \psi_{|E[2]} = \gamma \circ \rho_{|E[2]^\#}$.

Now let $\psi : E[2] \longrightarrow E'[2]$ satisfy $\rho' \circ \psi_{|E[2]^\#} = \gamma \circ \rho_{|E[2]^\#}$. We have $\rho' \circ \psi \circ f_{|W_{C/S}} = \gamma \circ \rho \circ f_{|W_{C/S}} = \rho' \circ (f')_{|W_{C/S}}$. As $(\rho')_{|E[2]^\#} : E'[2]^\# \longrightarrow D'$ is an isomorphism, this implies that $\psi \circ f_{|W_{C/S}} = (f')_{|W_{C/S}}$. □

Let V be the *Kähler different divisor* of f. By definition, this is the closed subscheme of C which is defined by the zero'th Fitting ideal $F^0(\Omega_{C/E})$ of $\Omega_{C/E} = \Omega_f$. (For further information on Kähler different divisors see [13], [14] or the appendix of [8].)

In Section 6 of [14], the Weierstraß divisor of a relative hyperelliptic curve H/S has been defined as the Kähler different divisor of the canonical map $H \longrightarrow \mathbb{P}(\omega_{H/S})$. Now the discussion starting at the exact sequence (6.2) until the end of section 6 in [14] carries over to our case (the only difference being that V has degree 2 and not $2g + 2$ over S). We thus have:

Lemma 3.9

- $F^0(\Omega_{C/E}) = \mathrm{Ann}(\Omega_{C/E})$.
- *V is a relative effective Cartier divisor of degree 2 over S.*
- *V is the fixed point subscheme of C under the action of τ, i.e. V is the largest subscheme of C with the property that τ restricts to V and* $\tau_{|V} = \mathrm{id}_V$.

- *V is étale over S.*

Proof. The first assertion, which is written in [14, Remark 6.4], follows from the exact sequence (6.2) in [14] and the definition of the Kähler different divisor. The second, third and forth assertion can be adopted from the text below (6.2) in [14], [14, Proposition 6.5] and [14, Proposition 6.8] respectively. □

Lemma 3.10 *If S is reduced, then V is equal to the ramification locus of f endowed with the reduced induced scheme structure.*

Proof. By the first assertion the previous lemma, the support of V is equal to the set of points where f is ramified, i.e. to the ramification locus of f. Now since S is reduced and by the previous lemma V is étale over S, V is reduced (see [5, Exposé I, Proposition 9.2.]), and so the assertion follows. □

Proposition 3.11 *Under the conditions of Theorem 2, let $\iota : V \hookrightarrow C$ be the canonical closed immersion. Then $(f')_{|V} = f' \circ \iota : V \longrightarrow E'$ is the zero-element in the abelian group $E'(V)$.*

Proof. Let $p : V \longrightarrow S$ be the canonical morphism. We have to show that $f' \circ \iota = 0_{E'} \circ p$.

The fact that $\tau_{|V} = \mathrm{id}_V$ implies that $[-1] \circ f' \circ \iota = f' \circ \tau \circ \iota = f' \circ \iota$. As $E'[2]$ is the largest closed subscheme X of E' with $[-1]_{|X} = \mathrm{id}_X$, this implies that $f' \circ \iota$ factors through $E'[2]$.

Let us now assume that S is connected and let s be some geometric point of S. As $E'[2]$ and V are étale over S, the map $E'[2](V) \longrightarrow E'_s[2](V_s)$ is injective. We thus only have to check that $(f' \circ \iota)_s = 0_{E'_s} \circ p_s : V_s \longrightarrow E'_s$, i.e. $f'_s(V_s) = [0_{E'_s}]$. This is equation (4) in Appendix A. □

Remark 3.12 Essentially the same statement as in the above proposition holds if V is replaced by the ramification locus endowed with the reduced induced scheme structure (independently of S being reduced). This follows immediately from the proposition because by definition the canonical immersion of this scheme into C factors through V.

Remark 3.13 Let $\Delta := f_*(V)$ be the discriminant divisor of f. Then Δ is a relative effective Cartier divisor of E/S of degree 2. As the geometric fibers over S consist of exactly 2 topological points, it is also étale of degree 2 over S. In particular, the map $f_{|V} : V \longrightarrow \Delta$ is an isomorphism. Furthermore, if S is reduced, Δ is equal to the branch locus of f endowed with the reduced induced scheme structure. This can be proved analogously to Lemma 3.10.

4 A reformulation of Theorem 2

Together with the "symmetric basic construction" (Proposition 2.15) and Proposition 3.8, a consequence of Theorem 2 is:

Let S be a scheme over $\mathbb{Z}[1/2]$, and let $E/S, E'/S$ be two elliptic curves and $\psi : E[2] \longrightarrow E'[2]$ a theta-smooth isomorphism. Then with the notations of the previous sections, there is an S-isomorphism $\gamma : \mathbb{P}(q_* \mathcal{L}(2[0_E])) \tilde{\longrightarrow} \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ such that $\rho' \circ \psi_{|E[2]^\#} = \gamma \circ \rho_{|E[2]^\#}$ holds.

The existence of this isomorphism, which is canonically attached to (E, E', ψ) maybe at first sight seems a little bit a mystery. In fact, it can easily be derived from a general statement on $\mathbb{P}^1$-bundles:

Let $E/S, E'/S$ be two elliptic curves with an isomorphism $\psi : E[2] \longrightarrow E'[2]$ (not necessarily theta-smooth). Let $\rho : E \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E])), \rho' : E' \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ be the corresponding canonical projections. The maps ρ and ρ' are ramified at $E[2]$ and $E'[2]$ respectively. In particular, $\rho_{|E[2]^\#} : E[2]^\# \hookrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$ and $(\rho')_{|E'[2]^\#} : E'[2]^\# \hookrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ are closed immersions. Let D and D' be the corresponding closed subschemes – these are étale covers of S of degree 3. (We use that S is a scheme over $\mathbb{Z}[1/2]$.) Now $\psi_{|E[2]^\#} : E[2]^\# \tilde{\longrightarrow} E'[2]^\#$ induces a canonical isomorphism between D and D'. With Proposition B.4, we conclude:

Proposition 4.1 *There is a unique S-isomorphism $\gamma : \mathbb{P}(q_* \mathcal{L}(2[0_E])) \tilde{\longrightarrow} \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ such that the equality $\rho' \circ \psi_{|E[2]^\#} = \gamma \circ \rho_{|E[2]^\#}$ holds.*

Let us again assume that $\psi : E[2] \longrightarrow E'[2]$ is theta-smooth, and let γ be as in the proposition. Then we have the following alternative criterion for a triple (C, f, f') to be a normalized symmetric pair.

Proposition 4.2 *Let C/S be a genus 2 curve, let $f : C \longrightarrow E, f' : C \longrightarrow E'$ be covers of degree 2. Then (C, f, f') is a normalized symmetric pair corresponding to ψ if and only if $\gamma \circ \rho \circ f = \rho' \circ f'$.*

Proof. By Theorem 2, Proposition 3.8 and the uniqueness of γ, it is immediate that a normalized symmetric pair (C, f, f') corresponding to γ satisfies $\gamma \circ \rho \circ f = \rho' \circ f' : C \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$.

Let this equality be satisfied. If S is the spectrum of an algebraically closed field, the statement is proved in Lemma A.2.

In the general case, we can assume that S is connected. As a morphism between (relative) elliptic curves over a connected base is either an isogeny or zero and we already know that $f_* \circ (f')^*$ is zero fiberwise, $f_* \circ (f')^*$ is zero. As f' is obviously minimal, this implies that $\ker(f_*) = \mathrm{Im}((f')^*)$. Similarly, we have $\ker(f'_*) = \mathrm{Im}(f^*)$.

We now want to show that f is normalized. Let τ be the unique non-trivial automorphism of f which exists by Lemma 3.5, similarly let τ' be the unique non-trivial automorphism of f'. Then $\tau \circ \tau' = \sigma_{C/S}, \tau' \circ \tau = \sigma_{C/S}$.

(It is not difficult to check these equalities fiberwise, and this suffices by [10, Lemma 3.1].)

We claim that $[-1] \circ f = f \circ \sigma_{C/S}$. Indeed, as $\tau \circ \tau' = \tau' \circ \tau$, τ' induces an automorphism on E over $\mathbb{P}(q_* \mathcal{L}(2[0_E]))$. By looking at the fibers, one sees that this is not the trivial automorphism. It follows that the induced automorphism is $[-1]$. We thus have $[-1] \circ f = f \circ \sigma_{C/S}$.

By [10, Theorem 3.2] to show that f is normalized it now suffices to check that for some $s \in S$, $f_s : C_s \longrightarrow E_s$ is normalized. For this statement, we again refer to Lemma A.2.

The proof that f' is normalized is analogous.

We have $\rho' \circ \psi \circ f_{|W_{C/S}} = \gamma \circ \rho \circ f_{|W_{C/S}} = \rho' \circ (f')_{|W_{C/S}}$. As $(\rho')_{|E'[2]^\#} : E'[2]^\# \longrightarrow D$ is an isomorphism, it follows that that $\psi \circ f_{|W_{C/S}} = (f')_{|W_{C/S}}$.

By Proposition 3.3, (C, f, f') is a normalized symmetric pair corresponding to ψ. □

With the help of Lemma A.1, we can give a third form of the "symmetric basic construction" for $N = 2$.

Proposition 4.3 (Symmetric basic construction for degree 2 – third form) *Let S be a scheme over $\mathbb{Z}[1/2]$, let E/S, E'/S be two elliptic curves, and let $\psi : E[2] \longrightarrow E'[2]$ be an isomorphism. Let $\rho : E \longrightarrow \mathbb{P}(q_* \mathcal{L}(2[0_E]))$, $\rho' : E' \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ be the canonical covers of degree 2. Let $\gamma : \mathbb{P}(q_* \mathcal{L}(2[0_E])) \longrightarrow \mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))$ be the unique S-isomorphism which satisfies $\rho' \circ \psi_{|E[2]^\#} = \gamma \circ \rho_{|E[2]^\#}$. Assume the following two equivalent conditions are satisfied:*

- *For no geometric point s of S, there exists an isomorphism $\alpha : E_s \longrightarrow E'_s$ with $\alpha_{|E_s[2]} = \psi_s$.*
- *The images of the sections $\rho' \circ 0_{E'}$ and $\gamma \circ \rho \circ 0_E$ of $\mathbb{P}(q'_* \mathcal{L}(2[0_{E'}])) \longrightarrow S$ are disjoint.*

Then there exists a curve C/S and covers $f : C \longrightarrow E$, $f' : C \longrightarrow E'$ of degree 2 such that $\gamma \circ \rho \circ f = f' \circ \rho'$. Any such triple (C, f, f') is a normalized symmetric pair corresponding to ψ, and it is unique up to unique isomorphism.

If one assumes that the base-scheme is regular, one can give a more concrete description of the curve C and the covers f, f' (as well as to prove its existence in an alternative way).

Proposition 4.4 *Under the conditions of the above proposition, let S be regular. Then $E \times_{\mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))} E'$ (where the product is with respect to $\gamma \circ \rho$ and ρ') is reduced with total quotient ring $\kappa(E) \times_{\kappa(\mathbb{P}(q'_* \mathcal{L}(2[0_{E'}])))} \kappa(E')$. The normalization C of $E \times_{\mathbb{P}(q'_* \mathcal{L}(2[0_{E'}]))} E'$ is a genus 2 curve, and the induced* *maps* $f : C \longrightarrow E, f' : C \longrightarrow E'$ *are degree 2 covers which satisfy* $\gamma \circ \rho \circ f = f' \circ \rho'$.

Proof. As S is regular, it is also locally integral, in particular, its connected components are integral; see [15, Theorem 14.3], [6, I (4.5.6)]. We can thus assume that S is integral.

Let $\mathcal{F} := \rho'_* \mathcal{L}(2[0_{E'}])$. We first show that $E \times_{\mathbb{P}(\mathcal{F})} E'$ is integral and that its function field is $\kappa(E) \otimes_{\kappa(\mathbb{P}(\mathcal{F}))} \kappa(E')$.

The ring $\kappa(E) \otimes_{\kappa(\mathbb{P}(\mathcal{F}))} \kappa(E')$ is a field because by assumption, the generic points of $\rho'([0_{E'}])$ and $\gamma(\rho([0_E]))$ are distinct.

Let A be the coordinate ring of an affine open part U of $\mathbb{P}(\mathcal{F})$, let B and C the corresponding rings of the preimages of U in E and E'. We claim that the canonical map $B \otimes_A C \longrightarrow \kappa(B) \otimes_{\kappa(A)} \kappa(C) \simeq \kappa(E) \times_{\kappa(\mathbb{P}(\mathcal{F}))} \kappa(E')$ is injective.

We have $\kappa(B) \otimes_{\kappa(A)} \kappa(C) \simeq (B \otimes_A C) \otimes_A \kappa(A)$ as B and C are finite over A. We thus have to show that the map $A \otimes_B C \longrightarrow (B \otimes_A C) \otimes_A \kappa(A)$ is injective. Now, $A \longrightarrow \kappa(A)$ is injective and $B \otimes_A C$ is flat over A (C is flat over A, thus $C \otimes_A B$ is flat over B, and as B is flat over A, $B \otimes_A C$ is flat over A). This implies that $B \otimes_A C \longrightarrow (B \otimes_A C) \otimes_A \kappa(A)$ is injective. It follows that $B \otimes_A C$ is reduced.

We have seen that $B \otimes_A C$ is contained in the field $(B \otimes_A C) \otimes_A \kappa(A)$, and obviously $(B \otimes_A C) \otimes_A \kappa(A)$ is contained in the function field of $B \otimes_A C$. This implies that $(B \otimes_A C) \otimes_A \kappa(A) \simeq \kappa(E) \otimes_{\kappa(\mathbb{P}(\mathcal{F}))} \kappa(E')$ is the function field of $B \otimes_A C$.

We have seen that $E \times_{\mathbb{P}(\mathcal{F})} E'$ is integral (in particular reduced) and its function field is indeed $\kappa(E) \otimes_{\kappa(\mathbb{P}(\mathcal{F}))} \kappa(E')$.

We now show the statements on C.

The field $\kappa(S)$ is algebraically closed in $\kappa(E) \times_{\mathbb{P}(\kappa(\mathcal{F}))} \kappa(E')$, and as S is regular, S is normal; see [15, Theorem 19.4]. This implies with [6, III (4.3.12)] that the geometric fibers of C over S are connected.

Let W be the different divisor of $E \times_{\mathbb{P}(\mathcal{F})} E' \longrightarrow \mathbb{P}(\mathcal{F})$. Then $(E \times_{\mathbb{P}^1} E') - W$ is normal, because the domain of an étale morphism mapping to a normal scheme is normal; see [5, Exposé I, Corollaire 9.11.]. It follows that $C \longrightarrow E \times_{\mathbb{P}(\mathcal{F})} E'$ induces an isomorphism between the complement of the preimage of W in C and $(E \times_{\mathbb{P}(\mathcal{F})} E') - W$. Since the restriction of W to the fibers over S is zero-dimensional, it follows that $C \longrightarrow E \times_{\mathbb{P}(\mathcal{F})} E'$ induces birational morphisms on the fibers over S.

By Abhyankar's Lemma ([5, Exposé X, Lemme 3.6]) and "purity of the branch locus" ([5, Exposé X, Théorème 3.1.]), f is étale outside $(f')^{-1}([0_{E'}]))$ and f' is étale outside $f^{-1}([0_E])$. Let x be a topological point of C. As by assumption $\gamma(\rho([0_E]))$ and $\rho'([0_{E'}])$ are disjoint, $x \notin (f')^{-1}([0_{E'}])$ or $x \notin f^{-1}([0_E])$. In the first case, the morphism f is étale at x, and since E is smooth over S, C over S is smooth at x. In the second case, the argument is analogous and the conclusion is the same. It follows that C is smooth over S.

Let s be a geometric point of S. We have already shown that C_s is connected, and by what we have just seen, C_s is non-singular. We have to show that the genus of this curve is 2. We already know that $C_s \longrightarrow E_s \times_{\mathbb{P}^1_{\kappa(s)}} E'_s$ is birational. It follows that $C_s \longrightarrow E_s$ has degree 2. Since $\gamma(\rho([0_E])) \neq \rho'([0_{E'}])$, the morphism $C_s \longrightarrow E_s$ is ramified exactly at the preimages of $\rho'([0_{E'}])$ in E_s (here we use again Abhyankar's Lemma). This preimage consists of exactly two closed points. It follows that the genus of C_s is 2. □

A Genus 2 covers of degree 2 over fields

In this part of the appendix, we provide some results on genus 2 covers of elliptic curves of degree 2 over algebraically closed fields of characteristic $\neq 2$.

In the following, let $\overline{\kappa}$ be an algebraically closed field of characteristic $\neq 2$. Let $E/\overline{\kappa}, E'/\overline{\kappa}$ be two elliptic curves, $\psi : E[2] \xrightarrow{\sim} E'[2]$. Let $\phi : E \longrightarrow \mathbb{P}^1_{\overline{\kappa}}, \phi' : E' \longrightarrow \mathbb{P}^1_{\overline{\kappa}}$ be two covers of degree 2 which are ramified at $E[2]$ and $E'[2]$ respectively such that $\phi' \circ \psi_{|E[2]^\#} = \phi_{|E[2]^\#}$. Let C be the normalization of $E \times_{\mathbb{P}^1_{\overline{\kappa}}} E'$.

Let $P := \phi([0_E])$, $P' := \phi'([0_{E'}])$. By assumption, $\rho(E[2]^\#) = \rho'(E'[2]^\#)$; let this divisor be denoted by D.

Lemma A.1 *The following assertions are equivalent.*

a) The points P and P' are distinct.

b) $E \times_{\mathbb{P}^1_{\overline{\kappa}}} E'$ is irreducible.

c) $C/\overline{\kappa}$ is a genus 2 curve.

d) The two covers $\phi : E \longrightarrow \mathbb{P}^1_{\overline{\kappa}}$ and $\phi' : E' \longrightarrow \mathbb{P}^1_{\overline{\kappa}}$ are not isomorphic (i.e. there does not exist a $\overline{\kappa}$-isomorphism $\alpha : E \longrightarrow E'$ with $\phi = \phi' \circ \alpha$).

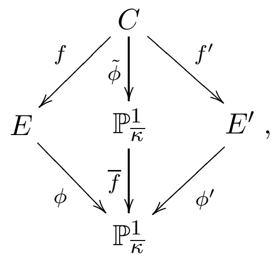
e) There does not exist an isomorphism of elliptic curves $\alpha : E \longrightarrow E'$ with $\alpha_{|E[2]} = \psi$.

Proof. Keeping in mind that C is regular, i.e. smooth over $\operatorname{Spec}(\overline{\kappa})$, the equivalence of the first four assertions is not difficult to show.

Assume that the covers are isomorphic via $\alpha : E \longrightarrow E'$. Then in particular $P = P'$. We have the isomorphisms $\phi_{|E[2]} : E[2] \longrightarrow D \cup P$, $(\phi')_{|E[2]} : E[2] \longrightarrow D \cup P$. It follows that $\alpha_{|E[2]} = (\phi'_{|D \cup P})^{-1} \circ \phi_{E[2]} = \psi$. In particular, α is an isomorphism of elliptic curves.

On the other hand, assume that there exists an isomorphism of elliptic curves $\alpha : E \longrightarrow E'$ with $\alpha_{|E[2]} = \psi$. Then $\phi_{|E[2]} = \phi' \circ \alpha_{|E[2]}$. It is well-known that this implies that $\phi = \phi' \circ \alpha$. □

Let us assume that the equivalent conditions of the lemma are satisfied. Then we have a commutative diagram



(3)

where all morphisms are covers of degree 2. We have that

- $\overline{f} : \mathbb{P}^1_{\overline{\kappa}} \longrightarrow \mathbb{P}^1_{\overline{\kappa}}$ is branched exactly at the set $P \cup P'$,
- $\tilde{\phi} : C \longrightarrow \mathbb{P}^1_{\overline{\kappa}}$ is branched exactly at the set $\overline{f}^{-1}(D)$,
- $f : C \longrightarrow E$ is branched exactly at the set $\phi^{-1}(P')$,
- $f' : C \longrightarrow E'$ is branched exactly at the set $(\phi')^{-1}(P')$.

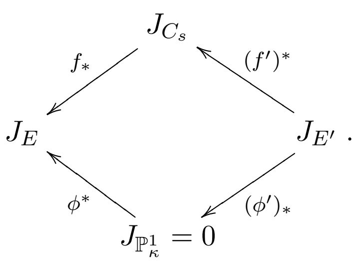
These statements can for example easily be proved with Abhyankar's Lemma.

Let $V \subset C$ be the ramification locus of f. Then $(\phi \circ f)(V) = P'$, i.e. $(\phi' \circ f')(V) = P'$, and this implies

$$f'(V) = [0_{E'}] \ . \tag{4}$$

Lemma A.2 *(C, f, f') is a normalized symmetric pair with respect to E and E' corresponding to ψ.*

Proof. It is not difficult to show that we have a commutative diagram



This implies that $f_* \circ (f')^*$ is zero. As f' is obviously minimal, this implies that $\ker(f_*) = \mathrm{Im}((f')^*)$. Similarly, we have $\ker(f'_*) = \mathrm{Im}(f^*)$.

By the above statements on the branching of $\overline{f}$ and $\tilde{\phi}$, over each point of D, there lie exactly 2 Weierstraß points. This implies that over each point of $E[2]^{\#}$ there also lie exactly 2 Weierstraß points. It follows that f is normalized.

The proof that f' is normalized is analogous.

We have $\phi' \circ \psi \circ f_{|W_{C/S}} = \phi \circ f_{|W_{C/S}} = \phi' \circ (f')_{|W_{C/S}}$. As $(\phi')_{|E[2]^{\#}} : E'[2]^{\#} \longrightarrow D$ is an isomorphism, we can conclude that $\psi \circ f_{|W_{C/S}} = (f')_{|W_{C/S}}$.

By Proposition 3.3, it follows that (C, f, f') is a normalized symmetric pair corresponding to ψ. □

Remark A.3 By Proposition 3.1, the last assertion of Lemma A.1 is equivalent to ψ being irreducible (i.e. theta-smooth).

Lemmata A.1 and A.2 can however also be used to prove Proposition 3.1 (i.e. [9, Theorem 3] in the special case that the covering degree is 2). By the definition of Theta-smoothness, we can thereby restrict ourselves to the case that $S = \overline{k}$.

If ψ satisfies the conditions of Lemma A.1, then by Lemma A.2 and Proposition 2.14, ψ is irreducible.

On the other hand, if ψ is irreducible and (C, f, f') is the corresponding symmetric pair, then we have degree 2 covers $\phi : E \longrightarrow \mathbb{P}^1_{\overline{k}}, E' \longrightarrow \mathbb{P}^1_{\overline{k}}$ which ramify at $E[2]$ and $E'[2]$ respectively with $\phi \circ f = \phi' \circ f'$ (for example by Theorem 2). Consequently, the equivalent conditions of Lemma A.1 hold.

Also Remark 2.16 can – for covering degree 2 – be derived from Lemma A.2: The open subset U of S where P and P' do not meet obviously has the correct properties.

B Some results on projective space bundles

In the following, let S be an arbitrary (not necessarily locally noetherian) scheme. Let $\mathbb{P}^1_S := \text{Proj}(\mathbb{Z}[X_0, X_1]) \times_{\text{Spec}(\mathbb{Z})} S$. Then $\mathcal{O}(1)$ on $\mathbb{P}^1_S$ has two canonical global generators, X_0 and X_1.

Lemma B.1 *Let $s_1, s_2, s_3, s'_1, s'_2, s'_3 : S \longrightarrow \mathbb{P}^1_S$ be six sections of $\mathbb{P}^1_S \longrightarrow S$ such that the images of s_1, s_2, s_3 as well as of s'_1, s'_2, s'_3 are pairwise disjoint. Then there exists a unique S-automorphism β of $\mathbb{P}^1_S$ with $\beta \circ s_i = s'_i$ for $i = 1, 2, 3$.*

Proof. By considering an open affine covering, we can restrict ourselves to the case that S is affine. The general case then follows by the uniqueness of α.

Each of the s_i, s'_i is given by an invertible sheaf with two global sections which generate it; cf. [7, II, Theorem 7.1.]. Let $U = \text{Spec}(A)$ be an affine open subset such that all these sheaves are trivial. We are going to show the result for $(s_i)_{|U}, (s'_i)_{|U}$ over U. Again the result in the lemma then follows by the uniqueness of α on U via the consideration of an open affine covering. Let us denote $(s_i)_{|U}$ by s_i, $(s_i')_{|U}$ by s_i'.

If $\beta : \mathbb{P}^1_A \longrightarrow \mathbb{P}^1_A$ is an automorphism, then $\beta^*(\mathcal{O}(1)) \approx \mathcal{O}(1) \otimes p^*(\mathcal{L})$, where $p : \mathbb{P}^1_A \longrightarrow \mathrm{Spec}(A)$ is the structure morphism and $\mathcal{L}$ is an invertible sheaf on $\mathrm{Spec}(A)$; see [17, 0. §5 b)].

Let us assume that $\beta \in \mathrm{Aut}_A(\mathbb{P}^1_A)$ satisfies $\beta \circ s_i = s_i'$ for some i, and let $\mathcal{L}$ be as above. Then $\mathcal{L} = (s_i)^* p^*(\mathcal{L}) = (s_i)^* \beta^*(\mathcal{O}(1)) = (s_i')^*(\mathcal{O}(1)) = \mathcal{O}_{\mathrm{Spec}(A)}$ by the above assumption on A.

We can thus restrict ourselves to automorphisms β with $\beta^*(\mathcal{O}(1)) \approx \mathcal{O}(1)$. Fixing an isomorphism of $\beta^*(\mathcal{O}(1))$ with $\mathcal{O}(1)$, $\beta^* X_0$ and $\beta^* X_1$ define two global sections of $\mathcal{O}(1)$. Thus β corresponds to two global section of $\mathcal{O}(1)$ which are unique up to multiplication by an element of A^*. Such elements can be written as $aX_0 + bX_1, cX_0 + dX_1$ $(a, b, c, d \in A)$ such that the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is invertible. The matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is thereby unique up to multiplication by an element of A^*.

By assumption on U, any of the sections s_i, s_i' is given by a tuple of two elements of A which generate the unit ideal. Furthermore, each of these tuples is unique up to multiplication by an element of A^*. We can thus uniquely represent any of the s_i, s_i' by an element in A^2/A^*.

Let $(f, g) \in A^2/A^*$ be such an element corresponding to s_i. Then $\beta \circ s_i$ is given by $(fa + gb, fc + gd) \in A^2/A^*$, i.e. it is given by the usual application of $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ on (f, g) from the right.

Note that the assumption on the images of the s_i and s_i' is equivalent to the condition that for all $t \in S$, the restrictions of s_1, s_2, s_3 to the fiber over t as well as the restrictions of the s_1', s_2', s_3' are distinct. This in turn is equivalent to the condition that for all prime ideals P of A, the tuples (f, g) as above stay distinct in $(A/P)^2/(A/P)^*$.

Now the result of this lemma follows from the following lemma which - for convenience - we formulate with the usual left operation. $\square$

We introduce the following notation: For $v \in A^2$, we write $\tilde{v}$ for the reduction of v modulo A^*.

Lemma B.2 *Let* $\begin{pmatrix} a_i \\ b_i \end{pmatrix}, \begin{pmatrix} a_i' \\ b_i' \end{pmatrix} \in A^2$ *for* $i = 1, 2, 3$ *be given such that for all prime ideals* P *of* A*, the* $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ *for* $i = 1, 2, 3$ *as well as the* $\begin{pmatrix} a_i' \\ b_i' \end{pmatrix}$ *for* $i = 1, 2, 3$ *define pairwise distinct elements in* $(A/P)^2/(A/P)^*$. *Then there exists an invertible matrix* $B \in M_{2\times 2}(A)$, *unique up to multiplication by an element of* A^*, *such that* $B\widetilde{\begin{pmatrix} a_i \\ b_i \end{pmatrix}} = \widetilde{\begin{pmatrix} a_i' \\ b_i' \end{pmatrix}} \in A^2/A^*$.

Proof. We show the existence first.

We only have to show the existence for $\widetilde{\begin{pmatrix} a_1' \\ b_1' \end{pmatrix}} = \widetilde{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \widetilde{\begin{pmatrix} a_2' \\ b_2' \end{pmatrix}} = \widetilde{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}, \widetilde{\begin{pmatrix} a_3' \\ a_3' \end{pmatrix}} = \widetilde{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

We claim that the matrix $M := \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in M_{2\times 2}(A)$ is invertible. Let d be the determinant of this matrix. By assumption, for all prime ideals P of A, the reduction of d modulo P is non-zero. It follows that d does not lie in any prime ideal, thus it is a unit (as otherwise it would lie in a maximal ideal).

Now M^{-1} maps $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $\begin{pmatrix} a \\ b \end{pmatrix}$ be the image of $\begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$. The assumption remains valid for the images of $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ under M^{-1}, and it says that a and b are not divisible by any prime ideal, i.e. they are units. The invertible matrix $M' := \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}$ fixes $\widetilde{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ and $\widetilde{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ and maps $\begin{pmatrix} a \\ b \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $B := M'M^{-1}$ has the desired properties.

Given what we have already shown, for the uniqueness it suffices to remark that only matrixes of the form aI ($a \in A^*$) fix $\widetilde{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \widetilde{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ and $\widetilde{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$. □

Lemma B.3 *Let D, D' be two subschemes of $\mathbb{P}^1_S$ such that $D \longrightarrow S, D' \longrightarrow S$ are étale covers of degree 3, let $\eta : D \longrightarrow D'$ be an S-isomorphism. Then there exists a unique S-automorphism of $\mathbb{P}^1_S$ such that $\alpha_{|D} = \eta$.*

Proof. As $D \longrightarrow S$ is an étale cover, there exists a Galois cover $T \longrightarrow S$ such that $D_T = D \times_S T \simeq T \cup T \cup T$ (isomorphism over T); cf. [5, Exposé V, 4 g)].

Let $t_1, t_2, t_3 : T \longrightarrow D_T$ be the three immersions. Then for any $\alpha \in \mathbb{P}^1_T$, the condition $\alpha_{|D_T} = \eta_T$ is equivalent to $\alpha \circ t_i = \eta_T \circ t_i$ for $i = 1, 2, 3$.

It follows from Lemma B.1 that there exists a unique automorphism α of $\mathbb{P}^1_T$ such that $\alpha_{|D_T} = \eta_T$.

This implies by Galois descent that there exists a unique automorphism α of $\mathbb{P}^1_S$ with $\alpha_{|D} = \eta$. □

Proposition B.4 *Let $\mathbf{P}, \mathbf{P}'$ be two $\mathbb{P}^1$-bundles over S. Let D be a subscheme of $\mathbf{P}$, D' a subscheme of $\mathbf{P}'$ such that $D \longrightarrow S$ and $D' \longrightarrow S$ are étale covers of degree 3. Let $\eta : D \longrightarrow D'$ be an S-isomorphism. Then there exists a unique S-isomorphism $\alpha : \mathbf{P} \longrightarrow \mathbf{P}'$ such that $\alpha_{|D} = \eta$.*

In particular, if $\mathbf{P}$ has three sections over S which do not meet, it is S-isomorphic to $\mathbb{P}^1_S$.

Proof. If $\mathbf{P}$ and $\mathbf{P}'$ are trivial bundles (i.e. S-isomorphic to $\mathbb{P}^1_S$), the result follows immediately from the previous lemma. The general case follows from the uniqueness of α by a glueing argument. □

Remark B.5 The subscheme D of $\mathbf{P}$ in the proposition is in fact a relative effective Cartier divisor of $\mathbf{P}$. This follows from [16, Corollary 3.9].

References

[1] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron Models.* Springer-Verlag, 1980.

[2] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Etudes Sci. Publ. Math.*, pages 75–109, 1969.

[3] C. Diem and G. Frey. Non-constant genus 2 curves with pro-Galois covers. To appear.

[4] G. Frey and E. Kani. Curves of genus 2 covering elliptic curves and an arithmetical application. In *Arithmetic Algebraic Geometry (Texel Conf., 1989)*, volume 89 of *Prog. Math.*, pages 153–176, Boston, 1991. Birkhäuser.

[5] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA I)*, volume 1960/61 of *Séminaire de Géométrie Algébrique.* Institut des Hautes Études Scientifiques, Paris.

[6] A. Grothendieck with J. Dieudonné. Eléments de Géométrie Algébrique (I-IV). ch. I: Springer-Verlag, Berlin, 1971; ch. II-IV: Publ. Math. Inst. Hautes Etud. Sci. 8,11, 17, 20,24, 28, 32, 1961-68.

[7] R. Hartshorne. *Algebraic Geometry.* Springer-Verlag, New York, 1977.

[8] E. Kani. The number of genus 2 covers of an elliptic curve. To appear.

[9] E. Kani. The number of curves of genus two with elliptic differentials. *J. reine angew. Math.*, 485:93–121, 1997.

[10] E. Kani. Hurwitz spaces of genus 2 covers of an elliptic curve. *Collect. Math.*, 54(1):1–51, 2003.

[11] A. Krazer. *Lehrbuch der Thetafunktionen.* Chelsea Publishing Company, 1970. Reprint of the 1903 edition.

[12] R. Kuhn. Curves of genus 2 with split Jacobian. *Trans. Am. Math. Soc.*, 307:41–49, 1988.

[13] E. Kunz. *Kähler Differentials.* Vieweg, Wiesbaden, 1986.

[14] K. Lønsted and S. Kleiman. Basics on families of hyperelliptic curves. *Compos. Math.*, 38:83–111, 1979.

[15] H. Matsumura. *Commutative ring theory.* Cambidge University Press, Cambridge, UK, 1986.

[16] J. Milne. Jacobian varieties. In G. Cornell and J. Silverman, editors, *Arithmetic Geometry*, pages 167–212. Springer-Verlag, 1986.

[17] D. Mumford. *Geometric Invariant Theory.* Springer-Verlag, 1965.

[18] D. Mumford. *Abelian Varieties.* Tata Institute for Fundamental Research, 1970.

Universität Leipzig, Fakultät für Mathematik und Informatik,
Augustusplatz 10, 04109 Leipzig, Germany.
email: diem@math.uni-leipzig.de