NOTES ON FLOER HOMOLOGY AND LOOP SPACE HOMOLOGY

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ABSTRACT. Given the cotangent bundle T^*Q of a smooth manifold with its canonical symplectic structure, and a Hamiltonian function on T^*Q which is fiberwise asymptotically quadratic, its well-defined Floer homology with the pair-of-pants ring structure is ring-isomorphic to the singular homology of the free loop space of Q endowed with its loop product. The analogous statement is true for the based loop space versions and the Pontrjagin product. This article gives an overview of the construction of this ring isomorphism which is based on Legendre duality and moduli spaces of flow trajectories of hybrid type, which are half Floer trajectories for the Hamiltonian problem and half Morse trajectories for the Lagrangian problem.

1. INTRODUCTION

Initially, Floer homology had been invented in order to provide a proof of the non-degenerate Arnold conjecture for symplectic fixed points, [Flo88a, Flo88b, Flo89]. Given a closed symplectic manifold (M^{2n}, ω) and a non-degenerate, time-periodic Hamiltonian $H\colon \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$, the Arnold conjecture claims that there should be at least a minimal number of 1-periodic solutions $x \in \mathcal{P}_1(H)$ of the Hamiltonian differential equation $\dot{x}(t) = X_H(t, x(t))$, that is, equivalently, fixed points of the time-1-map of the flow generated by the Hamiltonian vector field X_H of H. This minimal number should be not smaller than the sum of the Betti numbers of M, that is $\operatorname{rk} H_*(M)$. Floer's key idea was to build formally a chain complex generated by the 1-periodic solutions $\mathcal{P}_1(H)$, a finite subset of the free loop space of M, and to use a Cauchy-Riemann type PDE for cylindrical maps $u\colon \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ connecting $x, y \in \mathcal{P}_1(H)$, in order to select and count particular trajectories in the free loop space in order to define a chain boundary operator. This Cauchy-Riemann type PDE makes use of Gromov's theory of pseudoholomorphic curves, [Gro]. The homology of the Floer chain complex turns out to be an invariant of the manifold M, independent of the Hamiltonian function H. In fact, this Floer homology HF_* was proven to be isomorphic as a module over some suitably chosen coefficient ring to the standard homology of M, or in view of Poincaré duality, to the cohomology of M (e.g. [HS]).

Subsequently, further algebraic structures were constructed on this Floer homology, in particular a natural ring structure, the so-called pair-of-pants ring structure, [Sch95]. In view of the above isomorphism to $H^*(M)$, it was shown in [PSS] that, in general, the pair-of-pants ring structure is a deformation of the cohomology ring of M known as its quantum cohomology.

However, all these results did not expose any nontrivial topological structure concerning the free loop space of M. This turns out to be essentially due to the compactness assumption on the symplectic manifold M. As soon as this is dropped and as also the

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Hamiltonian function H deviates essentially from being compactly supported, its associated Floer homology, if well-defined, tends to be a truly infinite-dimensional homology theory.

In this article, we focus on the case of cotangent bundles $(M,\omega) = (T^*Q, \omega_{\mathsf{Liouv}})$ with their canonical symplectic structure and their canonical Lagrangian fibration by the cotangent spaces. A result first proven by Viterbo [Vit] shows that Floer homology for the fiberwise quadratic Hamiltonian on T^*Q given by a Riemannian metric g on Q is isomorphic as a graded Abelian group to the singular homology of the free loop space of the manifold Q. Another proof of this result was given by Salamon and Weber [SW]. Whereas Viterbo uses a method based on generating functions, Salamon and Weber use an approach relating the Cauchy-Riemann type PDE of Floer theory to the parabolic heat equation for the L^2-gradient of the energy functional on the free loop space of Q.

The present article gives an outline of again another approach, [AS04, AS05]. Our aim was to find a more immediate construction on the level of the Floer chain complex which could also be used to analyze the pair-of-pants ring structure of Floer homology, which clearly must correspond to some ring structure on the homology of the free loop space of the manifold Q. It is worth mentioning that here, in the context of noncompact symplectic manifolds T^*Q, the symmetry between a cohomology and a homology theory is finally broken. In fact, it turns out that the pair-of-pants ring structure of Floer homology corresponds to the loop product known from the work [CS] by Chas and Sullivan[1]. In our approach, the concrete tool to represent the homology of the free loop space of Q is Morse homology, [Sch93, AM04]. The philosophy of our construction of the chain complex isomorphism between the Floer chain complex for H and the Morse chain complex of the free loop space of Q is to build so-called hybrid moduli spaces of connecting trajectories which are half Morse trajectories for the negative gradient flow of a Lagrangian action functional on the free loop space of Q, parametrized by "flow-time" from minus infinity to zero, and half Floer trajectories in the free loop space of T^*Q parametrized from time zero to infinity. Coupling these two different types of half-trajectories is established via a Lagrangian boundary condition which uses crucially the fact that the Hamiltonian function on T^*Q on the Floer chain complex side is Legendre dual to the Lagrangian function on the tangent bundle TQ providing the Morse theory for the free loop space of Q. Counting such hybrid trajectories defines the chain complex homomorphism from the Morse chain complex into the Floer chain complex. It is intriguing to see that Legendre duality only allows such a construction in one direction, namely from the Lagrangian picture towards the Hamiltonian picture if using the negative gradient flow which corresponds to homology theories. Nevertheless, one can still show that this chain complex homomorphism has to be invertible. The result that Floer homology is isomorphic to the free loop space homology as a graded Abelian group already on the chain complex level has been proven for a large, fairly general class of Hamiltonian functions with fiberwise asymptotic quadratic upper and lower bound in [AS04]. The proof that this isomorphism of homologies identifies the pair-of-pants product with the loop product of Chas-Sullivan is given in [AS05]. Moreover, also the analogous versions for the based loop space are considered in both papers. Related results have been also achieved by Cohen [Coh] and Ramirez [Ram]. In this article, we present a condensed outline of our results from [AS04, AS05]. The main result is given in Theorem 2.5 below.

[1]Here, we thank Octav Cornea for pointing out to us the presence of this preprint in November 2002 when already our first version of the ring isomorphism had been established.

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2. Main Result

2.1. **Loop Space Homology.** Let us briefly recollect the classical and well-known facts about loop space homology which will be extended by the loop product ring structure in the setting of manifolds.

Let (Q, q_o) be a pointed topological space and denote its free and based loop space respectively by

$$\Lambda Q = C^0(\mathbb{T}, Q), \quad \mathbb{T} = \mathbb{R}/\mathbb{Z},$$
$$\Omega Q = \left\{ c \in \Lambda Q \mid c(0) = q_0 \right\}.$$

We have the classical Serre fibration

$$\Omega Q \overset{i}{\hookrightarrow} \Lambda Q \xrightarrow{\text{ev}} Q, \quad \text{ev}(c) = c(0).$$

Without further remarks we will denote in the following H_* for singular homology with integer coefficients. We have the continuous concatenation injection

$$\#\colon \Omega Q \times \Omega Q \to \Omega Q, \quad (c, c') \mapsto c\#c',$$
$$c\#c'(t) = \begin{cases} c(2t), & 0 \le t \le \frac{1}{2}, \\ c'(2t-1), & \frac{1}{2} \le t \le 1, \end{cases} \tag{1}$$

which induces the homomorphism

$$\#_*\colon H_*(\Omega Q) \otimes H_*(\Omega Q) \to H_*(\Omega Q) \tag{2}$$

of graded groups and for which we have the classical result in the case of coefficients in a field $\mathbf{k}$ (see for instance [FHT]),

THEOREM 2.1. *$\big(H_*(\Omega Q; \mathbf{k}), \#_*\big)$ is a graded Hopf algebra.*

From now on, let us assume that Q is a smooth, oriented, n-dimensional manifold. Moreover, for simplicity we will restrict our perspective throughout this article to compact manifolds without boundary.

For analytical reasons we restrict ΛQ and ΩQ to dense subspaces which carry the structure of smooth Hilbert manifolds. In fact, given any smooth embedding $Q \subset \mathbb{R}^N$, the Hilbert space $W^{1,2}(\mathbb{T}, \mathbb{R}^N)$ of square integrable functions with weak derivatives in L^2 defines a smooth structure of a Hilbert manifold on

$$W^{1,2}(\mathbb{T}, Q) = \left\{ c \in W^{1,2}(\mathbb{T}, \mathbb{R}^N) \mid c(\mathbb{T}) \subset Q \right\}.$$

Using an auxiliary Riemannian structure g on Q, we obtain a well-defined Riemannian structure on $W^{1,2}(\mathbb{T}, Q)$ as a Hilbert manifold by

$$\langle v, w \rangle = \int_{\mathbb{T}} \big(g_{c(t)}(v(t), w(t)) + g_{c(t)}(\nabla_{\dot c} v, \nabla_{\dot c} w) \big) dt.$$

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Clearly, $\mathrm{ev}^{-1}(q_o) = \{\, c \in W^{1,2}(\mathbb{T}, Q) \mid c(0) = q_o \,\}$ is a smooth submanifold of codimension n and it is well-known that the dense inclusions

$$\begin{aligned} &W^{1,2}(\mathbb{T}, Q) \hookrightarrow C^0(\mathbb{T}, Q) = \Lambda Q, \\ &\mathrm{ev}^{-1}(q_o) \hookrightarrow \Omega Q, \end{aligned}$$

are homotopy equivalences. We now use the notations ΛQ and ΩQ for the smooth Hilbert manifolds.

Summing up, we observe

LEMMA 2.1. *$\Omega Q \hookrightarrow \Lambda Q$ is a smooth embedding of codimension n with oriented normal bundle $\nu \to \Omega Q$, and $\Omega Q \hookrightarrow \Lambda Q \xrightarrow{\mathrm{ev}} Q$ is a locally trivial fiber bundle.*

Proof. The first assertion follows from the fact that the map ev is a submersion onto an n-dimensional oriented manifold. Find open coverings $Q = \bigcup U_\alpha = \bigcup V_\alpha$, $U_\alpha \subset\subset V_\alpha$, with marked points $x_\alpha \in U_\alpha$ and smooth maps $h_\alpha \colon U_\alpha \to \mathrm{Diff}\, Q$ in the sense that $h_\alpha \colon U_\alpha \times Q \to Q$ is smooth and $h_\alpha(x, \cdot)$ is a diffeomorphism for all $x \in U_\alpha$, with

$$\begin{aligned} &h_\alpha(x, x_\alpha) = x \text{ and } h_\alpha(x, U_\alpha) \subset V_\alpha \quad \text{f.a. } x \in U_\alpha, \\ &h_\alpha(x, y) = y \quad \text{f.a. } x \in Q \setminus V_\alpha \,. \end{aligned}$$

Then consider $\Lambda Q_{|U_\alpha} = \mathrm{ev}^{-1}(U_\alpha) = \{\, c \in \Lambda Q \mid c(0) \in U_\alpha \,\}$ and define

$$\begin{aligned} &\phi_\alpha \colon \Lambda Q_{|U_\alpha} \to \mathrm{ev}^{-1}(x_\alpha) \times U_\alpha, \\ &\phi_\alpha(c) = \big(h_\alpha(c(0))^{-1} \circ c, c(0)\big) \,. \end{aligned}$$

We have the fiber diffeomorphism

$$\phi_\alpha^{-1}(\cdot, x) \colon\ \mathrm{ev}^{-1}(x_\alpha) \to \mathrm{ev}^{-1}(x)$$

for all $x \in U_\alpha$, and $\mathrm{ev} \colon \Lambda Q \to Q$ is a locally trivial fiber bundle. □

Note that the normal bundle ν of $\Omega Q = \mathrm{ev}^{-1}(q_o)$ can be identified with $\mathrm{ev}^* TQ_{|\Omega Q}$. Since ν is an oriented rank-n vector bundle, we have a well-defined Thom class

$$u \in H^n\big(\nu(\Omega Q), \nu(\Omega Q) \setminus \mathbb{O}_{\Omega Q}\big)$$

such that the Thom isomorphism

$$\cap\, u \colon H_*(\nu, \nu \setminus \mathbb{O}) \xrightarrow{\cong} H_{*-n}(\Omega Q)$$

yields the definition of

$$\cap \colon H_*(\Lambda Q) \to H_{*-n}(\Omega Q) \tag{3}$$

by composing

$$H_*(\Lambda Q) \xrightarrow{j_*} H_*(\Lambda Q, \Lambda Q \setminus \Omega Q) \xrightarrow{\cong} H_*(\nu, \nu \setminus \mathbb{O}) \xrightarrow{\cong} H_{*-n}(\Omega Q),$$

where j_* is taken from the long exact sequence and the middle isomorphism is excision. Note that this definition of $\cap \colon H_*(\Lambda Q) \to H_{*-n}(\Omega Q)$ uses the manifold property of Q and its orientability in an essential way.

Let us now consider in an analogous manner the codimension-n-submanifold

$$X = \big\{\, (c, d) \in \Lambda Q \times \Lambda Q \mid c(0) = d(0) \,\big\},$$

consisting of figure-eight-loops. Similarly, the normal bundle is oriented and we can define

$$\cap \colon H_*(\Lambda Q \times \Lambda Q) \to H_{*-n}(X) \,. \tag{4}$$

This allows now the following

DEFINITION 2.1. *The* **loop product**

$$\circ\colon H_*(\Lambda Q)\otimes H_*(\Lambda Q)\to H_{*-n}(\Lambda Q)$$

is given by the composition

$$H_*(\Lambda Q)\otimes H_*(\Lambda Q)\xrightarrow{\times} H_*(\Lambda Q\times\Lambda Q)\xrightarrow{\cap} H_{*-n}(X)\xrightarrow{\#_*} H_{*-n}(\Lambda Q)\,.$$

In view of the intersection product

$$\bullet\colon H_*(Q)\otimes H_*(Q)\to H_{*-n}(Q)$$

which is well-defined for the oriented manifold Q, we have

PROPOSITION 2.2. *The following are ring homomorphisms,*

$$\begin{aligned}\cap\colon \big(H_*(\Lambda Q),\circ\big)&\to\big(H_{*-n}(\Omega Q),\#_*\big),\\ \mathrm{ev}_*\colon \big(H_*(\Lambda Q),\circ\big)&\to\big(H_*(Q),\bullet\big)\,.\end{aligned}$$

Our aim is to find precisely this triple of homological ring structures $\circ$,$\#_*$ and $\bullet$ with ring homomorphisms isomorphically within the framework of Floer homology. We mention here again, that the loop product was already constructed in [CS], considered in a similar context in [CJ, CV], and that similar results relating the ring structures to Floer theory are also contained in [Coh, Ram].

2.2. **Floer Homology for the Cotangent Bundle.** Here we recall briefly the general setup of Floer homology for Hamiltonian systems before we apply it to the situation of cotangent bundles.

Let (P^{2n},ω) be a symplectic manifold, $H\colon\mathbb{T}\times P\to\mathbb{R}$ a smooth Hamiltonian, explicitly time-dependent and 1-periodic in time t. Let X_H be the associated non-autonomous vector field defined by $\omega(X_H,\cdot)=-dH$ and denote the set of 1-periodic solutions by

$$\mathcal{P}^\Lambda(H)=\{\,x\in\Lambda P\mid \dot x(t)=X_H(t,x(t))\text{ f.a. }t\in\mathbb{T}\,\}\,.$$

Assume that H is non-degenerate, i.e. we have transverse intersection $\operatorname{graph}\phi_H^1\pitchfork\triangle_P\subset P\times P$ for the time-1-map of the non-autonomous flow ϕ_H^t of X_H, see also $(H_0)_\Lambda$ below in Section 4. Let us also assume that we have an integral grading $\mu^\Lambda\colon\mathcal{P}^\Lambda(H)\to\mathbb{Z}$ by the Conley-Zehnder index. Given a t-dependent almost complex structure $J\colon\mathbb{T}\times P\to\operatorname{End}(TP)$ on P, compatible with ω (i.e. $\omega(\cdot,J(t)\cdot)$ Riem. metric f.a. $t\in\mathbb{T}$) we define for $x,y\in\mathcal{P}^\Lambda(H)$ the space of parametrized Floer trajectories

$$\begin{aligned}\mathcal{M}_{x,y}(J,H)\;=\;\big\{\,u\colon\mathbb{R}\times\mathbb{T}\to P\;\big|\;&\lim_{s\to-\infty}u(s,\cdot)=x,\;\lim_{s\to+\infty}u(s,\cdot)=y,\\ &u_s-J(t,u)\big(u_t-X_H(t,u)\big)=0\,\big\}\,.\end{aligned}$$

Througout this article we make frequent use in an ad hoc manner of the symbol $\mathcal{M}$ denoting varying kinds of solution spaces. Note the choice of the sign in the (anti-)CR-operator which implies that $u\in\mathcal{M}_{x,y}$ can be viewed as a negative flow line for the unregularized L^2-gradient of the Hamiltonian action functional.

The first essential compactness result of Floer theory states that for generic J, the space $\widehat{\mathcal{M}}_{x,y}=\mathcal{M}_{x,y}(J,H)\big/\mathbb{R}$ of unparametrized trajectories divided by the $\mathbb{R}$-shift-action, for $x\neq y$, is a manifold of dimension $\mu^\Lambda(x)-\mu^\Lambda(y)-1$, and that it is compact in dimension 0 and compact up to simple splitting of trajectories in dimension 1, provided that

a) P is a closed symplectic manifold and the cohomology classes $\{\omega\},c_1(TM,J)\in H^2(M)$ satisfy suitable conditions with respect to $\pi_2(M)$, or

b) P is noncompact, $\{\omega\}, c_1$ satisfy suitable conditions and J, H have a suitable asymptotic behaviour on the end of P.

Below, the latter case will be made precise for cotangent bundles $P = T^*Q$.

Next, we set for $x, y \in \mathcal{P}^\Lambda(H)$

$$\langle x, y\rangle \;=\; \begin{cases} \#_{\mathsf{alg}}\widehat{\mathcal{M}}_{x,y}, & \mu^\Lambda(x) - \mu^\Lambda(y) = 1, \\ 0, & \text{else,} \end{cases}$$

where $\#_{\mathsf{alg}}$ denotes counting of trajectories with signs according to their coherent orientation, [FH].

We define the finitely generated, graded, free Abelian group $CF_*^\Lambda(H) = \mathcal{P}^\Lambda(H) \otimes \mathbb{Z}$ and the boundary operator

$$\begin{aligned} &\partial\colon CF_*^\Lambda(H) \to CF_{*-1}^\Lambda(H), \\ &\partial\big(\textstyle\sum_x a_x x\big) \;=\; \displaystyle\sum_{\mu^\Lambda(y)=\mu^\Lambda(x)-1} a_x \langle x, y\rangle\, y\,. \end{aligned}$$

Note that this is the simplest possible version of the Floer chain complex without the use of Novikov rings (see [HS]), which will be sufficient for our framework of cotangent bundles.

The main theorem of Floer theory is

THEOREM 2.3 (Floer et al.). *a) $\big(CF_*(H), \partial = \partial(J, H)\big)$ is a chain complex, i.e. $\partial^2 = 0$ and thus $HF_*(J, H) = \ker\partial/\operatorname{im}\partial$ is well-defined.*
b) If P is a closed symplectic manifold (with suitable conditions on $\omega_{|\pi_2}$ and $c_{1|\pi_2}$), then

$$HF_*(H, J) \;\cong\; H^{n-*}(P)$$

are isomorphic in a natural way.

Note the shift in the grading for the isomorphism of Floer homology with singular homology of P. This is due to the chosen normalizations for the Conley-Zehnder index. The choices are such that the grading μ^Λ on $\mathcal{P}^\Lambda(H)$ and the Hamiltonian action functional (see below) are positively proportional on iteration towers $\big(x^{(m)}\big)_{m\in\mathbb{Z}}$, $x^{(m)}(t) = x(mt)$ and $\mu^\Lambda(x^{(-1)}) = -\mu^\Lambda(x)$ under time-reversal which in view of b) corresponds to Poincaré duality $H^{n-*}(P) \cong H_{n+*}(P)$.

Let us now consider this concept of Floer homology for cotangent bundles of smooth n-dimensional manifolds Q which we assume (mainly for reasons of simplicity) to be closed and oriented. $P = T^*Q$ carries the canonical symplectic structure $\omega = d\theta_{\mathsf{Liouv}}$ where the Liouville-1-form is canonically given by

$$\theta_{\mathsf{Liouv}} = \sum_{i=1}^n p_i\, dq_i$$

in local coordinates $(q_1, \ldots, q_n) \in Q$. Let again $H\colon \mathbb{T} \times T^*Q \to \mathbb{R}$ be a non-degenerate Hamiltonian and consider $x, y \in \mathcal{P}^\Lambda(H)$ and $\mathcal{M}_{x,y}^\Lambda = \mathcal{M}_{x,y}(J, H)$ as above. Note that, for the exact symplectic form $\omega = d\theta_{\mathsf{Liouv}}$ we have the Hamiltonian action functional for paths or closed curves x in T^*Q defined by

$$\mathcal{A}_H(x) = \int_0^1 \big(x^*\theta_{\mathsf{Liouv}} - H(t, x(t))dt\big)\,.$$

There is also a version of Floer homology for the case of the based loop space ΩQ. Consider $q_o \in Q$ and the Lagrangian submanifold $T^*_{q_o}Q \subset (T^*Q, \omega)$. Then, let

$$\begin{aligned} x, y \in \mathcal{P}^\Omega(H) \;&=\; \big\{\, x\colon [0,1] \to T^*Q \mid \dot{x}(t) = X_H(t, x(t)),\ x(0), x(1) \in T^*_{q_o}Q \,\big\}, \\ \mathcal{M}^\Omega_{x,y}(J,H) \;&=\; \big\{\, u\colon \mathbb{R}\times[0,1] \to T^*Q \mid u_s = J(t,u)\big(u_t - X_H(t,u)\big), \\ &\qquad\qquad u(s,0), u(s,1) \in T^*_{q_o}Q \text{ f.a. } s \in \mathbb{R}, \\ &\qquad\qquad \lim_{s\to-\infty} u(s,\cdot) = x,\ \lim_{s\to+\infty} u(s,\cdot) = y \,\big\}\,. \end{aligned}$$

Here, we have to assume H to be non-degenerate with respect to q_o, that is for $x \in \mathcal{P}^\Omega(H)$ the linearized time-1-map $D\phi^1_H(x(0))$ maps $T^*_{q_o}Q = T_{x(0)}(T^*_{q_o}Q)$ transverse to $T^*_{q_o}Q = T_{x(1)}(T^*_{q_o}Q)$. We also have a $\mathbb{Z}$-grading by the Maslov index μ^Ω for paths of Lagrangian subspaces, see $(H_0)_\Omega$ below in Section 4.

We now try to define

$$\partial\colon CF^{\Lambda \text{ resp.}\Omega}_*(J,H) \to CF^{\Lambda \text{ resp.}\Omega}_{*-1}(J,H)$$

for non-degenerate H and generic J. The crucial new problem is to guarantee the necessary compactness properties of the spaces $\widehat{\mathcal{M}}_{x,y}$ of unparametrized trajectories. Since $\omega = d\theta$ is exact, there is no obstruction by bubbling-off, but we need suitable asymptotic conditions on H and J in order to grant a priori C^0-estimates for trajectories $u \in \mathcal{M}_{x,y}(J,H)$.

Pick any Riemannian metric g on Q. The associated Levi-Civita connection provides a canonical ω-compatible almost complex structure J_{LC} by mapping vertical tangent spaces onto horizontal ones and reversely. The main assumption on the **admissible class of Hamiltonian functions** $H\colon \mathbb{T}\times T^*Q \to \mathbb{R}$ is the following: For any coordinate chart $\big(U, (q_1, \dots, q_n)\big)$ on Q

(H_1): there exist constants $h_0 > 0$ and $h_1 \geq 0$ such that

$$dH(t,q,p)[p\frac{\partial}{\partial p}] - H(t,q,p) \;\geq\; h_0|p|^2 - h_1,$$

and

(H_2): there exists a constant $h_2 \geq 0$ such that

$$\begin{aligned} \big|\partial_q H(t,q,p)\big| \;&\leq\; h_2\big(1 + |p|^2\big), \\ \big|\partial_p H(t,q,p)\big| \;&\leq\; h_2\big(1 + |p|\big), \end{aligned}$$

for all $t \in \mathbb{T}$, $p \in T^*_qQ$, $q \in U$. Here, $p\frac{\partial}{\partial p}$ is the canonical Liouville vector field, $\omega(p\frac{\partial}{\partial p}, \cdot) = \theta_{\mathsf{Liouv}}$. (H_1) essentially states that H grows at least fiberwise quadratically like $\frac{1}{2}h_0|p|^2 - \mathrm{const}$ and (H_2) implies that H grows at most fiberwise quadratically like $\frac{1}{2}h_2|p|^2 + \mathrm{const}$, and it is radially convex in p for $|p|$ large. Note that apart from the size of the concrete constants h_0, h_1, h_2, the form of the estimates (H_1) and (H_2) is invariant under coordinate changes on Q and independent of the choice of g. A typical example is given by physical Hamiltonians such as

$$H(t,q,p) = \frac{1}{2}\big|p - A(t,q)\big|^2 + V(t,q)\,.$$

The asymptotic condition on the ω-compatible almost complex structure J is that for a suitable constant j_o depending on g

(J) $$\|J - J_{LC}\|_\infty < j_o\,.$$

The first result now states that we have well-defined Floer homology for the given class of Hamiltonian functions on the cotangent bundle.

THEOREM 2.4. [AS04] *Given H and a generic J on $P = T^*Q$ satisfying (H_1), (H_2), (J) above, the Floer chain complexes $\big(CF_*^{\Lambda}(H), \partial(J,H)\big)$ and $\big(CF_*^{\Omega}(H), \partial(J,H)\big)$ are well-defined and for any generic J, J' satisfying (J) there exist quasi-isomorphisms providing $HF_*^{\Lambda}(H)$ and $HF_*^{\Omega}(H)$ independent from J. Moreover, given two Hamiltonians H, H' satisfying (H_1), (H_2) there exist natural isomorphisms $HF_*^{\Lambda}(H) \xrightarrow{\cong} HF_*^{\Lambda}(H')$ and $HF_*^{\Omega}(H) \xrightarrow{\cong} HF_*^{\Omega}(H')$ satisfying the cocycle condition.*

The latter isomorphisms are called the *continuation isomorphisms*. The central result of the present article formulates the full context of the ring isomorphisms of the above Floer homology with loop space homology.

THEOREM 2.5. [AS05] *Given a Hamiltonian H on the cotangent bundle T^*Q satisfying above conditions (H_1) and (H_2), we have the following additional structures and relations for its Floer homology.*

(a): *HF_*^{Λ} and HF_*^{Ω} carry natural bilinear operations*

$$m^{\Omega}\colon HF_k^{\Omega}(H^{(r)}) \otimes HF_l^{\Omega}(H^{(s)}) \to HF_{k+l}^{\Omega}(H^{(r+s)}),$$
$$m^{\Lambda}\colon HF_k^{\Lambda}(H^{(r)}) \otimes HF_l^{\Lambda}(H^{(s)}) \to HF_{k+l-n}^{\Lambda}(H^{(r+s)}),$$

for all $k, l \in \mathbb{Z}$, $r, s \in \mathbb{N}$ with $H^{(r)}(t,\cdot) = rH(rt,\cdot)$.

(b): *There exist natural homomorphisms compatible with m^{Ω} and m^{Λ},*

$$\mathbf{h}\colon \big(HF_*^{\Lambda}(H), m^{\Lambda}\big) \to \big(HF_{*-n}^{\Omega}(H), m^{\Omega}\big),$$
$$\mathbf{e}\colon \big(HF_*^{\Lambda}(H), m^{\Lambda}\big) \to \big(H_*(Q), \bullet\big).$$

(c): *There exist natural $\mathbb{Z}$-module isomorphisms compatible with the ring structures m^{Ω} and $\circ$, respectively m^{Λ} and $\#_*$,*

$$\Phi^{\Lambda}\colon \big(HF_*^{\Lambda}(H), m^{\Lambda}\big) \xrightarrow{\cong} \big(H_*(\Lambda Q), \circ\big),$$
$$\Phi^{\Omega}\colon \big(HF_*^{\Omega}(H), m^{\Omega}\big) \xrightarrow{\cong} \big(H_*(\Omega Q), \#_*\big)$$

such that the diagram

$$\begin{array}{ccccc}
\big(H_*(Q), \bullet\big) & \xleftarrow{\ \mathbf{e}\ } & \big(HF_*^{\Lambda}, m^{\Lambda}\big) & \xrightarrow{\ \mathbf{h}\ } & \big(HF_{*-n}^{\Omega}, m^{\Omega}\big) \\
\Big\| & & \Phi^{\Lambda}\big\downarrow\cong & & \Phi^{\Omega}\big\downarrow\cong \\
\big(H_*(Q), \bullet\big) & \xleftarrow{\ \mathrm{ev}_*\ } & \big(H_*(\Lambda Q), \circ\big) & \xrightarrow{\ \cap\ } & \big(H_{*-n}(\Omega Q), \#_*\big)
\end{array}$$

commutes.

The isomorphisms Φ^{Λ} and Φ^{Ω} as $\mathbb{Z}$-module-isomorphisms are already given in [AS04]. All statements about the ring structures are proven in [AS05]. The following remarks should be made:

(i) By naturality in (a)-(c) we mean two aspects: One is the compatibility with the canonical continuation isomorphism $HF_*^{\Lambda/\Omega}(H) \xrightarrow{\cong} HF_*^{\Lambda/\Omega}(H')$ associated to changes of the Hamiltonians H, H' both satisfying (H_1) and (H_2). The other important aspect is that these homological structures are already defined on chain level.

(ii) The homomorphisms $\mathbf{e}$ and ev_* are endowed with natural right inverses

$$\mathbf{i}\colon \big(H_*(Q), \bullet\big) \to \big(HF_*^{\Lambda}(H), m^{\Lambda}\big),$$
$$i_*\colon \big(H_*(Q), \bullet\big) \to \big(H_*(\Lambda Q), \circ\big),$$

i.e. $\mathbf{e} \circ \mathbf{i} = \mathrm{id}_{H_*(Q)} = \mathrm{ev}_* \circ i_*$, where i_* is induced by the inclusion of constant loops, $Q \hookrightarrow \Lambda Q$, $c(t) = q$, $t \in \mathbb{T}$. The latter right-inverse property is seen in an analogous manner as Proposition 2.2 by standard algebraic topology.

(iii) Not explicitly mentioned above is a natural $\mathbb{R}$-filtration on HF^Λ, HF^Ω by the values of the Hamiltonian action functional. The "ring structures" m^Λ, m^Ω can be shown to be fully compatible with that filtration. (See (9) below.)

(iv) It is an intriguing question, open to us, whether the Hopf algebra structure of (HF_*^Ω, m^Ω) implied by the isomorphism Φ^Ω can be seen immediately on the side of Floer homology.

3. Ring Structures and Ring-Homomorphisms

In this section we sketch the definitions of m^Λ, m^Ω, $\mathbf{h}$, $\mathbf{e}$, $\mathbf{i}$ and $\Phi^{\Lambda\,\text{resp.}\Omega}$. Full details will be contained in [AS05].

3.1. **The pair-of-pants product.**

3.1.1. *Based Loop Space.* From now on, the symbol $\#$ will denote concatenation in time without rescaling. We first construct m^Ω for the based loop space case. Let

$$\Sigma = \mathbb{R} \times [0,1] \,\dot\cup\, \mathbb{R} \times [1,2] \big/ (s,1-) \sim (s,1+) \text{ for } s \geq 0$$

i.e. the interior $\overset{\circ}{\Sigma} = \mathbb{R} \times (0,2) \setminus (-\infty, 0) \times \{1\}$ carries the standard complex structure as an open subset of $\mathbb{C}$, and Σ is a noncompact Riemann surface with boundary where local conformal boundary coordinates at $(0,0) \in \Sigma$ are given by $z \mapsto \sqrt{z} \in \{\,\Re z \geq 0\,\}$. For $s \leq 0$ we distinguish the different boundary points of Σ by the notation $(s, 1+)$ and $(s, 1-)$.

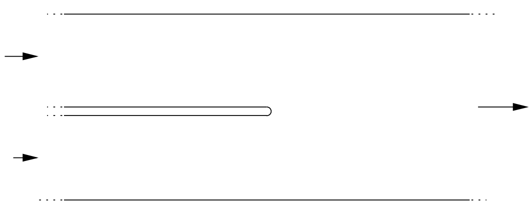

FIGURE 1. Σ, Strip with a slit

Let $H\colon \mathbb{T} \times T^*Q \to \mathbb{R}$ and $q_o \in Q$ be such that both

$$\begin{aligned} \mathcal{P}_1^\Omega(H) &= \big\{\, x\colon [0,1] \to T^*Q \,\big|\, x(0), x(1) \in T^*_{q_o}Q,\ \dot x = X_H(t,x) \,\big\}, \text{ and} \\ \mathcal{P}_2^\Omega(H) &= \big\{\, x\colon [0,2] \to T^*Q \,\big|\, x(0), x(2) \in T^*_{q_o}Q,\ \dot x = X_H(t,x) \,\big\} \end{aligned}$$

consist only of non-degenerate solutions. Observe that we have the canonical identification

$$\mathcal{P}_2(H) = \mathcal{P}_1(H^{(2)}), \quad H^{(2)}(t,x) = 2H(2t,x)$$

by rescaling. Moreover, we add the further, generic non-degeneracy assumption that $x(1) \notin T^*_{q_o}Q$ for all $x \in \mathcal{P}_2^\Omega(H)$. Then, given $x, y \in \mathcal{P}_1^\Omega(H)$ and $z \in \mathcal{P}_2^\Omega(H)$, we define the solution set

$$
\begin{aligned}
\mathcal{M}^{\Omega}_{x,y;z} \;=\; \Big\{ & u\colon \Sigma \to T^*Q \in C^\infty(\mathring{\Sigma}) \cap C^0(\Sigma) \;\Big| \\
& u_s - J(t,u)\big(u_t - X_H(t,u)\big) = 0 \quad \text{f.a. } (s,t) \in \mathring{\Sigma}, \\
& \lim_{s\to-\infty} u(s,t) = \begin{cases} x(t), & 0 \le t \le 1-, \\ y(t-1), & 1+ \le t \le 2, \end{cases} \\
& \lim_{s\to\infty} u(s,t) = z(t),\; 0 \le t \le 2, \\
& u(s,t) \in T^*_{q_o}Q \text{ f.a. } [s,t] \in \partial\Sigma = \Sigma \setminus \mathring{\Sigma} \Big\}\,.
\end{aligned}
$$

Note that J in general has to be allowed to be dependent on $t \in [0,2]$ and $J(t+1,\cdot) \neq J(t,\cdot)$, $0 \le t \le 1$.

PROPOSITION 3.1. *For a generic choice of $\big(J(t,\cdot)\big)_{t\in[0,2]}$ and if $x\#y \neq z$, the solution set $\mathcal{M}^{\Omega}_{x,y;z}$ is a manifold of dimension $\mu^{\Omega}(x) + \mu^{\Omega}(y) - \mu^{\Omega}(z)$.*

An important feature is the sharp energy estimate

$$
\int_{-\infty}^{\infty}\int_0^2 |u_s|_J^2 \, ds dt \;=\; \mathcal{A}_H(x) + \mathcal{A}_H(y) - \mathcal{A}_H(z) \ge 0\,. \tag{5}
$$

To allow for more general situations, we consider non-degenerate $H\colon [0,1]\times T^*Q \to \mathbb{R}$ and $K\colon [1,2]\times T^*Q \to \mathbb{R}$ with $H(1,\cdot) = K(1,\cdot)$ and we let $H\#K\colon [0,2]\times T^*Q$ be the concatenation which generically is again non-degenerate. Then, more generally, we will construct the operation

$$
m^{\Omega}\colon HF^{\Omega}_*(H) \otimes HF^{\Omega}_*(K) \to HF^{\Omega}_*(H\#K), \tag{6}
$$

where $H\#K$ is again viewed as rescaled over $[0,1]$ like above. By compatibility with the continuation isomorphisms we obtain also the operation m^{Ω} for $K = H$ and $H\#K = H^{(2)}$.

PROPOSITION 3.2. *Under the assumptions (H_1) and (H_2) for H, K and generic J satisfying (J) we have:*

(a) *$\mathcal{M}^{\Omega}_{x,y;z}$ is compact in dimension 0, and*
(b) *in dimension 1, $\mathcal{M}^{\Omega}_{x,y;z}$ is compact up to simple splitting of Floer trajectories of relative index 1 at at most one end of Σ.*

As a consequence, we can define the operation m^{Ω} on chain level

$$
\begin{aligned}
& m^{\Omega}\colon CF^{\Omega}_k(H) \otimes CF^{\Omega}_l(K) \to CF^{\Omega}_{k+l}(H\#K), \\
& m^{\Omega}(x,y) = \sum_{\mu(z)=\mu(x)+\mu(y)} \#_{\mathsf{alg}} \mathcal{M}^{\Omega}_{x,y;z}(H\#K, J)\, z,
\end{aligned} \tag{7}
$$

and it satisfies

$$
\partial_{H\#K} \circ m^{\Omega}(x,y) = m^{\Omega}(\partial_H x, y) \pm m^{\Omega}(x, \partial_K y)\,. \tag{8}
$$

Due to the energy estimate (5), m^{Ω} is compatible with the $\mathbb{R}$-filtration by the Hamiltonian action,

$$
\begin{aligned}
& CF^a_*(H) \;=\; \Big\{ \sum_{x\in\mathcal{P}(H)} a_x\, x \in CF_*(H) \;\Big|\; a_x = 0 \text{ for } \mathcal{A}_H(x) > a \Big\}, \\
& m^{\Omega}\colon CF^a(H) \otimes CF^b(K) \to CF^{a+b}(H\#K)\,.
\end{aligned} \tag{9}
$$

3.1.2. *Free Loop Space.* The construction of m^Λ for HF^Λ is carried out in analogous manner. Let

$$\begin{aligned}\Sigma \,=\, \mathbb{R}\times[0,1]\,\dot\cup\,\mathbb{R}\times[1,2]/_\sim \text{ with}&\\ (s,0)\sim(s,1-)\quad\text{and}\quad(s,1+)\sim(s,2)\quad\text{for } s\le 0,&\\ \text{and }\ (s,1-)\sim(s,1+)\quad\text{and}\quad(s,0)\sim(s,2)\quad\text{for } s\ge 0\,.&\end{aligned}$$

This endows Σ with the structure of a smooth, noncompact Riemann surface with an obvious holomorphic 2-sheeted branched cover $p\colon \Sigma\to\mathbb{R}\times\mathbb{T}$, $[s,t]\mapsto(s,t \mod 1)$, with a single branch point on $(0,0)\sim(0,1)\sim(0,2)$. As before, holomorphic coordinates near $[0,1]$ are provided by the $\sqrt{z}$-map.

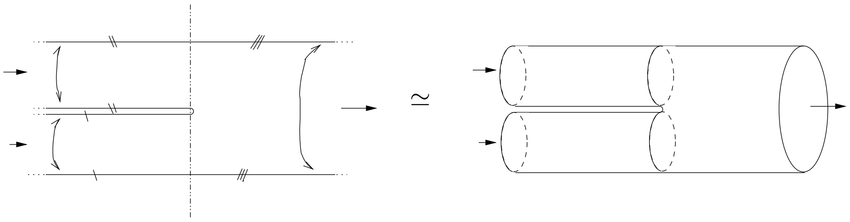

FIGURE 2. The Pair of Pants

Choose generic non-degenerate perturbations of $H^{(r)}$ and $H^{(s)}$, again denoted by H, K, such that $H(1)=K(0)$ and such that

$$x\#y\notin\mathcal{P}^\Lambda(H\#K)\quad\text{for any } x\in\mathcal{P}(H),\ y\in\mathcal{P}(K)\text{ with } x(1)=y(0)\,.$$

Again, after rescaling we identify $\mathcal{P}_2^\Lambda(H\#K)\subset C^\infty(\mathbb{R}/2\mathbb{Z},T^*Q)$ with a set of 1-periodic loops. Given $x\in\mathcal{P}_1^\Lambda(H)$, $y\in\mathcal{P}_1^\Lambda(K)$ and $z\in\mathcal{P}_2^\Lambda(H\#K)$ we set

$$\begin{aligned}\mathcal{M}^\Lambda_{x,y;z} \,=\, \big\{\, u\colon\Sigma\to T^*Q \text{ smooth}\,\big|\,&\\ u_s - J(t,u)\big(u_t - X_H(t,u)\big) = 0\quad \text{f.a. } (s,t)\in\Sigma,&\\ \lim_{s\to-\infty} u(s,t) = \begin{cases} x(t), & 0\le t\le 1-,\\ y(t), & 1+\le t\le 2,\end{cases}&\\ \lim_{s\to\infty} u(s,t) = z(t),\ 0\le t\le 2\,\big\}\,.&\end{aligned}$$

Analogously to Proposition 3.1, for generic J, $\mathcal{M}^\Lambda_{x,y;z}$ is a manifold of dimension

$$\dim\mathcal{M}^\Lambda_{x,y;z} = \mu^\Lambda(x)+\mu^\Lambda(y)-\mu^\Lambda(z)-n\,. \tag{10}$$

We have also the analogous compactness result as in Proposition 3.2, and thus we can define

$$\begin{aligned} m^\Lambda\colon CF_k^\Lambda(H)\otimes CF_l^\Lambda(K)\to CF^\Lambda_{k+l-n}(H\#K),&\\ \text{with }\ \partial^\Lambda_{H\#K}\circ m^\Lambda = m^\Lambda\circ\big(\partial^\Lambda_H\otimes\mathrm{id}_K \pm \mathrm{id}_H\otimes\partial^\Lambda_K\big)\,.&\end{aligned}\tag{11}$$

Here, we point out that the above branched-cover definition of the pair-of-pants domain Σ is essential for the sharp energy estimate

$$\mathcal{A}_H(x)+\mathcal{A}_K(y)-\mathcal{A}_{H\#K}(z) \,=\, \iint_\Sigma |\partial_s u|_J^2\,ds\,dt \,\ge\, 0,$$

and it is crucial for the C^0-estimate for solutions in the noncompact space T^*Q, where H and K have to satisfy (H_1) and (H_2). Also again, m^Λ is compatible with the $\mathbb{R}$-filtration of HF_*^Λ.

3.2. **The Ring Homomorphisms between free loop space Floer homology and based loop space Floer homology and classical homology.**

3.2.1. *The Homomorphism* $\mathbf{h}$. Let us now give the construction of the homomorphism

$$\mathbf{h}\colon \big(HF_*^\Lambda, m^\Lambda\big) \to \big(HF_{*-n}^\Omega, m^\Omega\big)\,.$$

We define the underlying domain as

$$\Sigma = \mathbb{R}\times[0,1]/\sim \qquad \text{with} \quad (s,0)\sim(s,1) \text{ for } s\le 0,$$

i.e. the interior $\overset{\circ}{\Sigma}$ is given by the open subset $\mathbb{R}\times\mathbb{T}\setminus[0,\infty)\times\{0\}$ of the cylinder $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$ with the standard complex structure. Again, Σ carries the structure of a Riemann surface with boundary, the holomorphic boundary chart at $(0,0)$ again given by the $\sqrt{z}$-map.

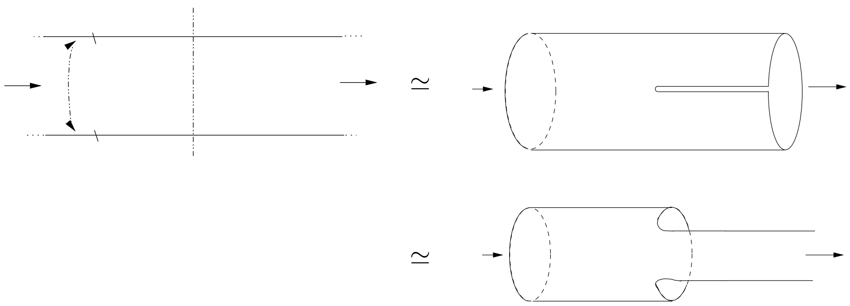

FIGURE 3. The Cylinder with Slit

We now assume $H\colon \mathbb{T}\times T^*Q \to \mathbb{R}$ to be simultaneously non-degenerate for closed orbits $\mathcal{P}_1^\Lambda(H)$ as well as for the open orbits $\mathcal{P}_1^\Omega(H)$ based at $q_o\in Q$. Moreover, we assume that $x(0)\notin T_{q_o}^*Q$ for all $x\in\mathcal{P}_1^\Lambda(H)$. The corresponding Floer-trajectory solution set is defined as

$$\begin{aligned}\mathcal{M}_{x;y}^{\Lambda\to\Omega}(J,H) \;=\; \Big\{\, & u\in C^\infty(\overset{\circ}{\Sigma},T^*Q)\cap C^0(\Sigma,T^*Q)\;\Big|\\ & u_s - J\big(u_t - X_H(t,u)\big) = 0,\\ & \lim_{s\to-\infty} u(s,\cdot) = x,\ \lim_{s\to\infty} u(s,\cdot) = y,\\ & u(s,0),u(s,1)\in T_{q_o}^*Q \text{ f.a. } s\ge 0\,\Big\}\,.\end{aligned}$$

PROPOSITION 3.3. *In view of the non-degeneracy assumption on H and q_o, it holds that*

(a) *for generic J, $\mathcal{M}_{x;y}^{\Lambda\to\Omega}$ is a smooth manifold of dimension $\mu^\Lambda(x)-\mu^\Omega(y)-n$,*
(b) *it is compact in dimension 0 and $\mathcal{A}_H(x)\ge\mathcal{A}_H(y)$ whenever it is nonempty, and*
(c) *it is compact in dimension 1 up to simple splitting of Floer trajectories of type Λ respectively Ω at the respective end of Σ.*

As before in (7) and (11) we define $\mathbf{h}$ on the chain level by

$$\mathbf{h}\colon CF_*^\Lambda \to CF_{*-n}^\Omega, \qquad \mathbf{h}(x) = \sum_y \langle x, y\rangle^{\Lambda\to\Omega} y, \tag{12}$$

with

$$\langle x, y\rangle^{\Lambda\to\Omega} = \begin{cases} \#_{\mathsf{alg}}\mathcal{M}^{\Lambda\to\Omega}_{x;y}\,, & \mu^\Lambda(x) - \mu^\Omega(y) = n, \\ 0\,, & \text{else}, \end{cases}$$

for $x \in \mathcal{P}_1^\Lambda(H)$, $y \in \mathcal{P}_1^\Omega(H)$. By 3.3 (c) we have $\partial^\Omega \circ \mathbf{h} = \mathbf{h} \circ \partial^\Lambda$ and hence, $\mathbf{h}$ is well-defined on homology level. We also have a sharp energy estimate which implies that $\mathcal{A}_H^\Omega(y) \leq \mathcal{A}_H^\Lambda(x)$ whenever $\mathcal{M}^{\Lambda\to\Omega}_{x;y} \neq \emptyset$, so that compatibility with the $\mathbb{R}$-filtration follows,

$$\mathbf{h}\colon HF_*^{\Lambda,\leq a}(H) \to HF_{*-n}^{\Omega,\leq a}(H)\,. \tag{13}$$

PROPOSITION 3.4. *The two operators*

$$\mathbf{h}\circ m^\Lambda \text{ and } m^\Omega \circ (\mathbf{h}\otimes\mathbf{h})\colon CF_k^{\Lambda,\leq a}(H)\otimes CF_l^{\Lambda,\leq b}(K) \to CF_{k+l-2n}^{\Omega,\leq a+b}(H\#K)$$

for generic H, K satisfying the conditions for Proposition 3.2, (11) and Proposition 3.3 are chain-homotopy equivalent. Consequently,

$$\mathbf{h}\colon \big(HF_*^\Lambda, m^\Lambda\big) \to \big(HF_{*-n}^\Omega, m^\Omega\big)$$

is a homomorphism compatible with m^Λ and m^Ω.

We briefly sketch the key ideas of the proof. Composing $\mathbf{h}$ with m^Λ corresponds to a gluing result for for the solutions spaces $\mathcal{M}^\Lambda_{x,y;z}(H\#K)$ and $\mathcal{M}^{\Lambda\to\Omega}_{z;z'}(H\#K)$ for $x \in \mathcal{P}_1^\Lambda(H)$, $y \in \mathcal{P}_1^\Lambda(K)$, $z \in \mathcal{P}_2^\Lambda(H\#K)$ and $z' \in \mathcal{P}_2^\Omega(H\#K)$ where the domain Σ for $\mathcal{M}^{\Lambda\to\Omega}_{z;z'}$ is rescaled to width 2. This defines a 1-1 correspondence of

$$\dot\cup_{\mu^\Lambda(z)=k+l-n}\mathcal{M}^\Lambda_{x,y;z}\times\mathcal{M}^\Omega_{z,z'}$$

with a set $\mathcal{M}^I_{x,y;z}(R)$ of solutions

$$u\colon \Sigma_R^I \to T^*Q, \quad u_s - J(u)\big(u_t - X_H(t,u)\big) = 0,$$

on a domain

$$\Sigma_R^I = \mathbb{R}\times[0,1]\,\dot\cup\,\mathbb{R}\times[1,2]/\sim$$

with

$$\begin{aligned} (s,0)\sim(s,1-) \text{ and } (s,1+)\sim(s,2) &\qquad \text{for } s \leq -R, \\ (s,0)\sim(s,2) \text{ and } (s,1-)\sim(s,1+) &\qquad \text{for } -R \leq s \leq R, \text{ and} \\ (s,1-)\sim(s,1+) &\qquad \text{for } s \geq R, \end{aligned}$$

satisfying $u(s,0), u(s,2) \in T^*_{q_o}Q$ for $s \geq R$ and

$$\lim_{s\to-\infty} u(s,t) = \begin{cases} x(t), & 0 \leq t \leq 1-, \\ y(t), & 1+ \leq t \leq 2, \end{cases} \qquad \text{and } \lim_{s\to\infty} u(s,t) = z(t),\ 0 \leq t \leq 2,$$

provided that $R \geq R(x,y,z)$ is large enough.

This latter solution set $\mathcal{M}^I_{x,y;z}(R)$ is again a 0-dimensional compact, oriented manifold, and, up to simple splitting of Λ- and Ω-Floer trajectories, it is compactly cobordant to a set $\mathcal{M}^{II}x, y; z(R')$ of solutions

$$u\colon \Sigma_{R'}^{II} \to T^*Q, \quad u_s - J(u)\big(u_t - X_H(t,u)\big) = 0,$$

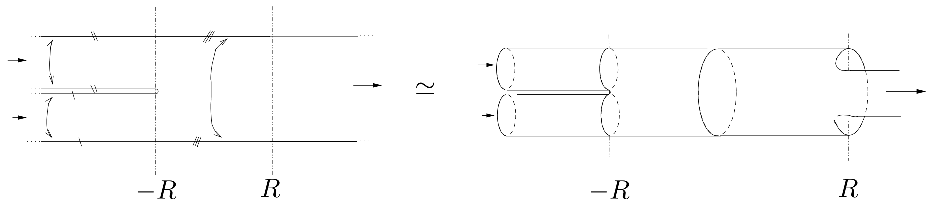

FIGURE 4. The glued domain Σ^I_R

on a domain

$$\Sigma^{II}_{R'} = \mathbb{R}\times[0,1]\,\dot\cup\,\mathbb{R}\times[1,2]/\sim$$

with

$$\begin{aligned} (s,0)\sim(s,1-) \text{ and } (s,1+)\sim(s,2) &\quad \text{for } s\le -R',\ \text{and}\\ (s,1-)\sim(s,1+) &\quad \text{for } s\ge R', \end{aligned}$$

satisfying

$$\begin{aligned} u(s,0),u(s,2)\in T^*_{q_o}Q \text{ for } s\ge -R'\\ \text{and } u(s,1-),u(s,1+)\in T^*_{q_o}Q \text{ for } -R\le s\le R'\,. \end{aligned}$$

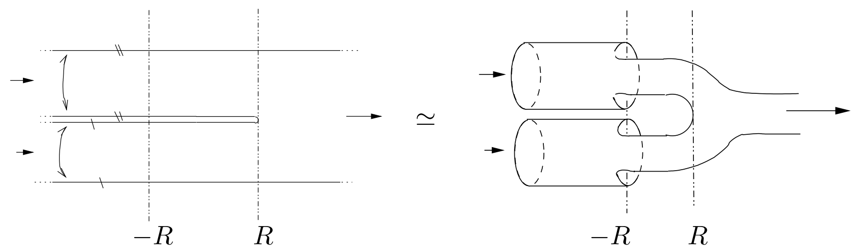

FIGURE 5. The glued domain Σ^{II}_R

The cobordism between $\mathcal{M}^I$ and $\mathcal{M}^{II}$ is obtained by a family of domains realizing a continuous deformation of Σ^I_R into $\Sigma^{II}_{R'}$. Again, for $R'\ge R'(x,y,z)$ large enough, the solutions in $\mathcal{M}^{II}_{x,y;z}(R')$ are in 1-1 correspondence with the disjoint union

$$\dot\bigcup\Big\{\big(\mathcal{M}^{\Lambda\to\Omega}_{x;x'}(H)\times\mathcal{M}^{\Lambda\to\Omega}_{y;y'}(K)\big)\times\mathcal{M}^{\Omega}_{x',y';z}(H\#K)\,\Big|\,\mu^\Lambda(x')=k-n,\\ \mu^\Lambda(y')=l-n\Big\}$$

by a second gluing argument. The fact, that the cobordism between $\mathcal{M}^I_{x,y;z}(R)$ and $\mathcal{M}^{II}_{x,y;z}(R')$ is only compact up to simple splitting of Floer trajectories, is responsible for $\mathbf{h}\circ m^\Lambda$ and $m^\Omega\circ(\mathbf{h}\otimes\mathbf{h})$ being only chain-homotopy equivalent instead of being equal.

3.2.2. *Construction of* **e** *with right-inverse* **i**. In order to carry out all constructions on the chain level, we represent the classical homology of Q by means of Morse homology with auxiliary Morse functions $f \in C^\infty(Q,\mathbb{R})$. Also the intersection product $\bullet$ is defined on chain level, namely by means of the trivalent-graph construction. This mean that we define a chain complex $\big(CM_*(f),\partial\big)$ by

$$
\begin{aligned}
CM_*(f) \;&=\; \mathrm{Crit}_* f \otimes \mathbb{Z},\\
\partial x \;&=\; \sum_{i(y)=i(x)-1} \#_{\mathsf{alg}}\big(\mathcal{M}_{x,y}(f,g)/\mathbb{R}\big)\, y,
\end{aligned}
$$

where the set of non-degenerate critical points $\mathrm{Crit}_* f$ is $\mathbb{Z}$-graded by the Morse index $i(\cdot)$, and (f,g) is a Morse-Smale pair of Morse function f and Riemannian metric g such that transversality holds, and thus

$$
\begin{aligned}
\mathcal{M}_{x,y}(f,g) \;&=\big\{\, \gamma\colon \mathbb{R}\to Q \,\big|\, \dot\gamma(s)+\nabla_g f(\gamma(s))=0,\ \gamma(-\infty)=x,\ \gamma(+\infty)=y \,\big\}\\
&= W^u(x) \pitchfork W^s(y)\,.
\end{aligned}
$$

is a manifold of dimension $i(x)-i(y)$. We have $\partial^2=0$ and

$$
H_*(Q)\cong HM_*(f) \overset{\mathsf{def}}{=} \big(CM_*(f),\partial\big) \;=\; \ker\partial/\operatorname{im}\partial\,.
$$

A homotopy of Morse functions $f\simeq f'$ provides a canonical continuation isomorphism $H_*(f)\cong H_*(f')$.

For the definition of the intersection product, we choose a generic triple of Morse functions f,f',f'' such that $\mathrm{Crit}\, f\cap \mathrm{Crit}\, f' = \emptyset$. For $x\in \mathrm{Crit}_k f$, $y\in \mathrm{Crit}_l f'$ and $z\in \mathrm{Crit}_m f''$ we set

$$
\begin{aligned}
\mathcal{M}_{x,y;z}(f,f',f'',g) \;=\; \Big\{\, &\gamma,\gamma'\in C^\infty((-\infty,0],Q),\ \gamma''\in C^\infty([0,\infty),Q) \,\Big|\\
&\dot\gamma+\nabla_g f(\gamma)=0,\ \dot\gamma'+\nabla_g f'(\gamma')=0,\ \dot\gamma''+\nabla_g f''(\gamma'')=0,\\
&\gamma(-\infty)=x,\ \gamma'(-\infty)=y,\ \gamma''(+\infty)=z,\\
&\gamma(0)=\gamma'(0)=\gamma''(0)\,\Big\}\,.
\end{aligned}
$$

For a generic Riemannian metric g, $\mathcal{M}_{x,y;z}(f,f',f'',g)$ is a manifold of dimension $i(x)+i(y)-i(z)-n$, compact in dimension 0 and compact up to simple splitting in dimension 1, such that counting elements of $\mathcal{M}_{x,y;z}$ with orientation defines

$$
\bullet\colon CM_k(f)\otimes CM_l(f')\to CM_{k+l-n}(f''), \tag{14}
$$

which induces the intersection product of Q on homology level. Note that here, the orientability of Q is required for the existence of a coherent orientation of the solution sets $\mathcal{M}_{x,y;z}$.

Let us now consider the half-cylinders

$$
Z^+=[0,\infty)\times\mathbb{T},\qquad Z^-=(-\infty,0]\times\mathbb{T},
$$

and define for $x\in\mathcal{P}_1^\Lambda(H)$ the space

$$
\begin{aligned}
\mathcal{M}_x^-(J,H) \;=\; \big\{\, &u\in C^\infty(Z^-,T^*Q) \,\big|\, u_s - J(u)\big(u_t - X_H(t,u)\big)=0,\\
&\lim_{s\to-\infty} u(s,t)=x(t),\ u(0,t)\in \mathbb{O}_Q\subset T^*Q \text{ f.a. } t\in\mathbb{T}\,\big\}
\end{aligned}
\tag{15}
$$

for generic J as above under the generic assumption on H that no $x\in\mathcal{P}_1^\Lambda(H)$ lies completely within the zero section $\mathbb{O}_Q$. Since the zero section is a Lagrangian submanifold with vanishing Maslov class, $\mu_{\mathsf{Maslov}}(c) = 0$ for all $c \in \Lambda Q$, we obtain that $\mathcal{M}_x^-$ is a smooth manifold of dimension

$$\dim \mathcal{M}_x^-(J,H) \,=\, \mu^\Lambda(x)\,.$$

Similarly, we define for any $q \in Q$ and $x \in \mathcal{P}_1^\Lambda(H)$ the set

$$\begin{aligned}\mathcal{M}^+_{q;x}(J,H) \,=\, \big\{\, u \in C^\infty(Z^+,T^*Q) \,\big|\, u_s - J(u)\big(u_t - X_H(t,u)\big) = 0,\\ \lim_{s\to+\infty} u(s,t) = x(t),\; u(0,t) \in T_q^*Q \text{ f.a. } t \in \mathbb{T}\,\big\}\end{aligned} \tag{16}$$

which, if nonempty, represents a manifold of dimension $\mu^\Lambda(x)$ for J generically chosen for the fixed $q \in Q$, provided that no $x \in \mathcal{P}_1^\Lambda(H)$ lies entirely within the Lagrangian fiber T_q^*Q.

Consider now $y \in \mathrm{Crit}\, f$ with $W^u(y)$, $W^s(y)$ the unstable and stable manifolds of the negative gradient flow of $-\nabla_g f$ on Q. We set

$$\begin{aligned}\mathcal{M}^-_{x;y}(H;f) \,&=\, \big\{\, u \in \mathcal{M}^-_x(H) \,|\, u(0,0) \in W^s(y)\,\big\},\\ \text{and } \mathcal{M}^+_{y;x}(f;H) \,&=\, \big\{\,(q,u) \,|\, q \in W^u(y),\; u \in \mathcal{M}^+_{q;x}(H)\,\big\}\,.\end{aligned} \tag{17}$$

For generic J and g, these spaces are manifolds of dimensions

$$\dim \mathcal{M}^-_{x;y} = \mu^\Lambda(x) - i(y), \quad \dim \mathcal{M}^+_{y;x} = i(y) - \mu^\Lambda(x)\,.$$

They are compact in dimension 0 and compact, up to simple splitting of either Λ-Floer trajectories on one end or Morse-trajectories on the other end, in dimension 1. Hence, we obtain on chain level well-defined group homomorphisms

$$\begin{aligned}\mathbf{e}\colon CF_k^\Lambda(H) \to CM_k(f), &\quad \mathbf{e}(x) = \sum_{i(y)=k} \big(\#_{\mathsf{alg}}\mathcal{M}^-_{x;y}(H,f)\big)\, y,\\ \mathbf{i}\colon CM_k(f) \to CF_k^\Lambda(H), &\quad \mathbf{i}(y) = \sum_{\mu^\Lambda(x)=k} \big(\#_{\mathsf{alg}}\mathcal{M}^+_{y;x}(f,H)\big)\, x\,.\end{aligned} \tag{18}$$

It can now be shown that $\mathbf{e}\circ\mathbf{i}$ is chain-homotopy equivalent to $\mathrm{id}_{CM_*(f)}$. The main argument is that the intermediate 0-dimensional solution space

$$\begin{aligned}\widetilde{\mathcal{M}}_{y,y'}(R) \,=\, \big\{\, u\colon [-R,R]\times\mathbb{T} \to T^*Q \,\big|\, & u_s - J(u)\big(u_t - X_H(t,u)\big) = 0,\\ & u\big(\{-R\}\times\mathbb{T}\big) \subset T_q^*Q,\; q \in W^u(y),\\ & u\big(\{+R\}\times\mathbb{T}\big) \subset \mathbb{O}_Q,\; u(R,0) \in W^s(y')\,\big\}\end{aligned}$$

is cobordant to $W^u(y)\cap W^s(y')$ via a homotopy $H \simeq 0$. Namely, by Stokes' theorem we have for $(-J)$-holomorphic annuli

$$\begin{aligned} u\colon [-R,R]\times\mathbb{T} \to T^*Q, &\quad u_s - J(u)u_t = 0,\\ u\big(\{-R\}\times\mathbb{T}\big) \subset T_q^*Q, &\quad u\big(\{+R\}\times\mathbb{T}\big) \subset \mathbb{O}_Q\end{aligned}$$

by compatibility of J with ω that

$$\begin{aligned} E(u) &= \tfrac12 \int_{-R}^R\int_0^1 \big(|u_s|_J^2 + |u_t|_J^2\big)\,ds dt\\ &= -\iint_{[-R,R]\times\mathbb{T}} u^*\omega \,=\, \int_{\mathbb{T}} u(-R,\cdot)^*\theta_{\mathsf{Liouv}} - \int_{\mathbb{T}} u(+R,\cdot)^*\theta_{\mathsf{Liouv}}\\ &=\, 0,\end{aligned} \tag{19}$$

because $\theta_{\mathsf{Liouv}}|_{\mathbb{O}_Q} = 0 = \theta_{\mathsf{Liouv}}|_{T_q^*Q}$, and hence $u = \mathrm{const}$. This proves that $\mathbf{e}\circ\mathbf{i} \simeq \mathrm{id}_{CM_*(f)}$.

Similarly, one can establish a suitable cobordism between the solution space

$$\begin{aligned}\widetilde{\mathcal{M}}^{I}_{x,x';y}(R) \;=\; \big\{\, u\colon \Sigma^-_R \to T^*Q \,\big|\, & u_s - J(u)\big(u_t - X_H(t,u)\big) = 0,\\ & u(-\infty,\cdot)_{|[0,1-]} = x,\; u(-\infty,\cdot)_{|[1+,2]} = x',\\ & u\big(\{R\}\times\mathbb{R}/2\mathbb{Z}\big) \subset \mathbb{O}_Q,\; u(R,0)\in W^s(y;f'')\,\big\},\end{aligned}$$

where $\Sigma^-_R = \big\{\,(s,t)\in\Sigma \,|\, s\le R\,\big\}$, $R>0$, for the pair-of-pants domain Σ for m^Λ from Section 3.1.2, and the solution space

$$\begin{aligned}\widetilde{\mathcal{M}}^{II}_{x,x';y}(R) \;=\; \big\{\,(u,u',\gamma,\gamma') \,\big|\, & u\in\mathcal{M}^-_x(J,H),\; u'\in\mathcal{M}^-_{x'}(J,H'),\\ & \gamma,\gamma'\colon[-R,0]\to Q,\; \dot\gamma = -\nabla_g f(\gamma),\; \dot\gamma' = -\nabla_g f'(\gamma'),\\ & \gamma(-R) = u(0,0),\; \gamma'(-R) = u'(0,0),\; \gamma(0)=\gamma'(0)\in W^s(y;f'')\,\big\}\end{aligned}$$

for the Morse functions f, f', f'' from (14). This is the main argument for proving that on chain level, the compositions

$$\mathbf{e}\circ m^\Lambda \;\simeq\; \bullet\circ(\mathbf{e}\otimes\mathbf{e})\colon CF^\Lambda_k(H)\otimes CF^\Lambda_l(K) \to CF^\Lambda_{k+l-n}(f'') \tag{20}$$

are chain homotopy equivalent. From these considerations we obtain

PROPOSITION 3.5. *For the linear operators* $\mathbf{e}\colon HF^\Lambda_* \to H_*(Q)$ *and* $\mathbf{i}\colon H_*(Q)\to HF^\Lambda_*$, *and the bilinear operators* $m^\Lambda\colon HF^\Lambda_*\otimes HF^\Lambda_* \to HF^\Lambda_{*-n}$ *and* $\bullet\colon H_*(Q)\otimes H_*(Q)\to H_{*-n}(Q)$ *we have*

(a) $\mathbf{e}\circ\mathbf{i} = \mathrm{id}_{H_*(Q)}$, *and*
(b) $\mathbf{e}(m^\Lambda(\alpha,\beta)) = e(\alpha)\bullet e(\beta)$, *for all* $\alpha,\beta\in HF^\Lambda_*$.

This completes the exposition of the arguments for statements (a) and (b) in Theorem 2.5 and the upper row in the commutative diagram. In the following section, we give details on the remaining vertical isomorphisms in that diagram which gives the main statement of the Theorem, namely that Floer homology in the cotangent bundle encodes the well-known loop space homology.

4. MORSE-HOMOLOGY ON THE LOOP SPACES ΛQ AND ΩQ, AND THE ISOMORPHISM

In this section, we give a short account of the construction of the explicit isomorphisms Φ^Λ and Φ^Ω from Theorem 2.5. These isomorphisms have been established in [AS04], by representing the classical homology of the loop space in terms of Morse homology for the Lagrangian action functional. The central idea of the isomorphism is the Legendre transform as a duality between the Lagrangian and the Hamiltonian picture. Different approaches to the same isomorphism of homology groups had been established first by Viterbo in [Vit] and by Salamon and Weber in [SW].

Let us consider the connected, compact, oriented smooth manifold Q with an auxiliary Riemannian metric g and an **admissible Lagrangian function** $L\colon\mathbb{T}\times TQ\to\mathbb{R}$ such that the following assumption hold: There exist constants $l_0>0$, $l_1\ge 0$ such that for any coordinate chart $\big(U,(q_1,\dots,q_n)\big)$ on Q and the associated local trivialization of $TQ_{|U}$ we have in these coordinates $(q_1,\dots,q_n,v_1,\dots,v_n)$

(L_1): $\partial^2_{vv}L(t,q,v)\ge l_0$ as a quadratic form, and

(L_2):
$$\begin{aligned}\big|\partial^2_{qq}L(t,q,v)\big| &\le l_1\big(1+|v|^2\big),\\ \big|\partial^2_{qv}L(t,q,v)\big| &\le l_1\big(1+|v|\big),\\ \big|\partial^2_{vv}L(t,q,v)\big| &\le l_1\end{aligned}$$

for all $t \in \mathbb{T}$, $q \in U$, $v \in T_qQ$.

These assumptions are formally similar to (H_1) and (H_2). They imply that the Lagrangian L grows fiberwise at least quadratically and at most quadratically. By the strict convexity condition (L_1) we have the **Legendre transform**

$$\begin{aligned} \mathfrak{L}_L\colon \mathbb{T}\times TQ &\xrightarrow{\cong} \mathbb{T}\times T^*Q, \\ (t,q,v) &\mapsto \big((t,q,dL(t,q,v)|T^v_{(q,v)}TQ\big), \end{aligned} \tag{21}$$

where $T^v_{(q,v)}TQ = \ker D\tau = T_v(T_qQ) = T_qQ$ can be canonically identified, and $\tau\colon TQ \to Q$ is the bundle projection. Using the inverse transform

$$\begin{aligned} \mathfrak{L}_L^{-1}\colon \mathbb{T}\times T^*Q &\to \mathbb{T}\times TQ, \\ (t,q,p) &\mapsto (t,q,v(t,q,p)), \end{aligned}$$

we have the Hamiltonian function $H(t,q,p)$ associated to $L(t,q,v)$ by

$$\begin{aligned} H(t,q,p) &= \max_{v\in T_qQ}\big(\langle p,v\rangle - L(t,q,v)\big) \\ &= \langle p, v(t,q,p)\rangle - L(t,q,v(t,q,p))\,. \end{aligned} \tag{22}$$

The Legendre transform $\mathfrak{L}_L$ gives a diffeomorphism by which we can also pull back the symplectic structure ω_{Liouv} from T^*Q to TQ, and the Hamiltonian vector field X_H is mapped to the vector field Y_L on TQ whose integral curves are characterized by the second order ODE in the q-variables

$$\nabla_t\big(\nabla_v L(t,q(t),\dot q(t)\big) = \nabla_q L\big(t,q(t),\dot q(t)\big), \tag{23}$$

where ∇_t denotes the covariant derivation along q, and ∇_v, ∇_q denote the vertical and the horizontal part of the gradient of L with respect to the induced Riemannian metric from g and its Levi-Civita connection. (23) is also the Euler-Lagrange equation for the Lagrangian action functional

$$\mathcal{E}_L(q) = \int_0^1 L\big(t,q(t),\dot q(t)\big)\,dt$$

viewed on ΛQ and, respectively, ΩQ. Its critical sets are the solution spaces

$$\begin{aligned} \mathcal{P}^\Lambda(L) &= \{\, q\in\Lambda Q \mid q \text{ solves (23)}\,\} \quad \text{and} \\ \mathcal{P}^\Omega(L) &= \{\, q\in\Omega Q \mid q \text{ solves (23)}\,\}\,. \end{aligned}$$

We use the same notation as in the Hamiltonian picture since by Legendre transform we have the canonical one-to-one correspondence

$$\begin{aligned} \pi\colon \mathcal{P}^{\Lambda\,\text{resp.}\Omega}(H) &\xrightarrow{1:1} \mathcal{P}^{\Lambda\,\text{resp.}\Omega}(L), \\ x=(q,p) &\mapsto q\,. \end{aligned}$$

The non-degeneracy conditions for the Legendre picture read:

- **$(L0)_\Lambda$:** For every solution $q\in\mathcal{P}^\Lambda(L)$ there exists no 1-periodic Jacobi vector field along q,
- **$(L0)_\Omega$:** for every solution $q\in\mathcal{P}^\Omega(L)$ there exists no Jacobi vector field along q vanishing for $t=0$ and $t=1$.

These conditions are equivalent to the non-degeneracy conditions for $x=\pi^{-1}(q)\in\mathcal{P}^{\Lambda\,\text{resp.}\Omega}(H)$ as solutions of the Legendre dual Hamiltonian problem, i.e.

- **$(H0)_\Lambda$:** the differential of the time-one integral map of X_H at $x(0)$ does not have the eigenvalue 1,

$(H0)_\Omega$**:** the differential of the time-one integral map of X_H, $T_{x(0)}T^*Q \to T_{x(1)}T^*Q$, maps the vertical subspace of $x(0)$ into a subspace having intersection $\{0\}$ with the vertical subspace at $x(1)$.

The crucial observation for the isomorphism between Morse homology for $\mathcal{E}_L$ in the Lagrangian picture and Floer homology for $\mathcal{A}_H$ in the Hamiltonian picture is the

MAIN LEMMA 4.1. *If $x = (q,p)\colon [0,1] \to T^*Q$ is continuous, with q of class $W^{1,2}$, then*

$$\mathcal{A}_H(x) \le \mathcal{E}_L(q),$$

the equality holding if and only if $p(t) = dL(t,q(t),\dot q(t))|_{T^v_{\dot q(t)}TQ}$, that is if and only if $(t,q(t),p(t)) = \mathfrak{L}_L(t,q(t),\dot q(t))$ for every $t \in [0,1]$. In particular, the Hamiltonian and the Lagrangian action functionals coincide on the solutions of the two systems.

Proof. From the definitions we have for $x(t) = \big(q(t),p(t)\big) \in T^*_{q(t)}Q$ that

$$\begin{aligned}\mathcal{A}_H(x) &= \int_0^1 \big(\langle p(t),\dot q(t)\rangle - H(t,q(t),p(t))\big)\,dt\\ &= \int_0^1 \Big(\langle p(t),\dot q(t)\rangle - \max_v \big[\langle p(t),v\rangle - L(t,q(t),v)\big]\Big)\,dt\\ &\le \int_0^1 L(t,q(t)\dot q(t))\,dt \;=\; \mathcal{E}_L(q),\end{aligned}$$

by choosing pointwise $v = \dot q(t)$, with equality exactly if $(t,q,p) = \mathfrak{L}_L(t,q,\dot q)$. □

On the canonically identified solution sets $\mathcal{P}^{\Lambda\,\mathrm{resp.}\Omega}(H) = \operatorname{Crit}\mathcal{A}_H$ and $\mathcal{P}^{\Lambda\,\mathrm{resp.}\Omega}(L) = \operatorname{Crit}\mathcal{E}_L$ we have not only the same critical values for $\mathcal{A}_H$ and $\mathcal{E}_L$ by the previous lemma, we also can identify the Conley-Zehnder index μ^Λ and the Maslov index μ^Ω on the Hamiltonian side with the Morse indices i_Λ and i_Ω of $\mathcal{E}_L$ on the Lagrangian side.

THEOREM 4.1 ([Dui, RS93, Web]). *Assuming that Q is orientable, let $q \in \mathcal{P}^\Lambda(L)$, and let $x \in \mathcal{P}^\Lambda(H)$, $(t,x(t)) = \mathfrak{L}_L(t,q(t),\dot q(t))$, be the corresponding 1-periodic solution of the Hamiltonian system on T^*Q. Then*

$$i_\Lambda(q) = \mu^\Lambda(x).$$

Let $q \in \mathcal{P}^\Omega(L)$ and let $x \in \mathcal{P}^\Omega(H)$, $(t,x(t)) = \mathfrak{L}_L(t,q(t),\dot q(t))$, be the corresponding solution of the Hamiltonian system on T^*Q. Then

$$i_\Omega(q) = \mu^\Omega(x).$$

The crucial property allowing the construction of Morse homology for $\mathcal{E}_L$ is the Palais-Smale property which had been proven by V. Benci,

THEOREM 4.2 ([Ben]). *Consider standard $W^{1,2}$-metrics on ΛQ and ΩQ, and let L satisfy the conditions (L_1) and (L_2). Then $\mathcal{E}_L$ has the Palais-Smale property, i.e. for any sequence $(q_n) \subset \Lambda Q$ respectively ΩQ with $(\mathcal{E}_L(q_n))$ bounded and $\operatorname{grad}_{W^{1,2}}\mathcal{E}_L(q_n) \to 0$ there exists a convergent subsequence of (q_n).*

As a consequence, see for instance[AM04] in this volume, we have well-defined Morse homology for $\mathcal{E}_L$. That is, we have the

DEFINITION 4.1. *Given $a, b \in \operatorname{Crit}\mathcal{E}_L \subset \Lambda Q$, we have*

$$\begin{aligned}W^u(a) &= \big\{\, \gamma \in C^\infty((-\infty,0],\Lambda Q) \mid \gamma(-\infty) = a,\ \dot\gamma = -\nabla\mathcal{E}_L(\gamma)\,\big\},\\ W^s(b) &= \big\{\, \gamma \in C^\infty([0,\infty),\Lambda Q) \mid \gamma(+\infty) = b,\ \dot\gamma = -\nabla\mathcal{E}_L(\gamma)\,\big\},\end{aligned}$$

where ΛQ is endowed with a $W^{1,2}$-structure and the gradient $\nabla\mathcal{E}_L$ is taken with respect to a Hilbert-metric on ΛQ as $W^{1,2}$-Hilbert manifold. We have

$$\begin{aligned} W^u(a)\cap W^s(b) &= \mathcal{M}^{\Lambda}_{a,b}(\mathcal{E}_L) \\ &= \big\{\,\gamma\colon\mathbb{R}\to\Lambda \mid \dot\gamma+\nabla\mathcal{E}_L(\gamma)=0,\ \gamma(-\infty)=a,\ \gamma(+\infty)=b\,\big\}, \end{aligned}$$

and completely analogously for the corresponding negative gradient flow on ΩQ.

Again, one can show that for a generic perturbation of the Hilbert metric on ΛQ respectively ΩQ, we have transversality $W^u(a)\pitchfork W^s(b)$, such that this trajectory space is a smooth, finite-dimensional manifold of dimension

$$\dim\mathcal{M}^{\Lambda\,\text{resp.}\Omega}_{a,b}(\mathcal{E}_L)\;=\;i_{\Lambda\,\text{resp.}\Omega}(a)-i_{\Lambda\,\text{resp.}\Omega}(b)\,.$$

Moreover, due to the Palais-Smale property of $\mathcal{E}_L$, the trajectory spaces $\mathcal{M}_{a,b}$ have the required compactness properties up to splitting of trajectories. We then define

$$CM_*(\mathcal{E}^{\Lambda\,\text{resp.}\Omega})\;=\;\mathcal{P}^{\Lambda\,\text{resp.}\Omega}(\mathcal{E}_L)\otimes\mathbb{Z},$$

which are $\mathbb{Z}$-graded by i_Λ respectively i_Ω and endowed with the $\mathbb{R}$-filtration

$$CM^a_*(\mathcal{E}^{\Lambda\,\text{resp.}\Omega})\;=\;\big\{\,x\in\mathcal{P}^{\Lambda\,\text{resp.}\Omega}(\mathcal{E}_L)\mid\mathcal{E}_L(x)\le a\,\big\}\otimes\mathbb{Z}\,.$$

Using the concept of Morse homology for $\mathcal{E}_L$, we define chain boundary operators ∂ on $CM_*(\mathcal{E}_L^{\Lambda\,\text{resp.}\Omega})$ by counting $\mathcal{M}_{a,b}(\mathcal{E}_l)$ as before and we obtain the Morse homology

$$HM_*(\mathcal{E}^{\Lambda\,\text{resp.}\Omega})\;=\;H_*\big(CM_*(\mathcal{E}^{\Lambda\,\text{resp.}\Omega}),\partial\big)$$

and likewise for the filtered complex $\big(CM^a_*,\partial\big)$.

Recalling from Section 2.1 that the inclusions of the Hilbert manifolds $W^{1,2}(\mathbb{T},Q)$ and $\{\,c\in W^{1,2}([0,1],Q)\mid c(0)=c(1)=q_o\,\}$ into $\Lambda^0Q=C^0(\mathbb{T},Q)$ and $\Omega^0=C^0\big(([0,1],\{0,1\}),(Q,q_o)\big)$ are homotopy equivalences, we obtain

THEOREM 4.3. *For all $a\in\mathbb{R}$ there exist canonical isomorphisms*

$$\begin{aligned} HM^a_*(\mathcal{E}_L^{\Lambda\,\textit{resp.}\Omega})\;&\cong\;H_*\big(\{\mathcal{E}_L^{\Lambda\,\textit{resp.}\Omega}\le a\}\big),\quad \textit{and}\\ HM_*(\mathcal{E}_L^{\Lambda\,\textit{resp.}\Omega})\;&\cong\;H_*\big(\Lambda^0Q\ \textit{resp.}\ \Omega^0Q\big), \end{aligned}$$

compatible with the induced homomorphisms from $\{\mathcal{E}\le a\}\subset\{\mathcal{E}\le b\}$ for $a\le b$, and $\lim_{\to}HM^a_*(\mathcal{E}_L)=HM_*(\mathcal{E}_L)$.

We now show that this Morse homological representation by the Lagrangian action functional is directly isomorphic to the Floer homological representation based on the Legendre dual Hamiltonian action functional.

THEOREM 4.4. *Assume that L is non-degenerate and satisfies (L_1), (L_2) above and that H is the Legendre transform of L. Let J be a sufficiently small generic perturbation of J_{LC} for some Riemannian metric g_o on Q, and let g be a generic Riemannian structure on the Hilbert manifolds Λ^1Q and Ω^1 close to $\langle\cdot,\cdot\rangle_{1,2}$ from g_o. Then, there exists a chain complex isomorphism for both loop space cases Λ and Ω,*

$$\Theta\colon\big(CM_*(\mathcal{E}_L),\partial(L,g)\big)\;\xrightarrow{\cong}\;\big(CF_*(H),\partial(H,J)\big)$$

of the form

$$\Theta(q)\;=\sum_{x\in\mathcal{P}(H),\,\mu(x)=i(q)}d^+(q,x)\cdot x\qquad\textit{f.a. }\ q\in\mathcal{P}(L)$$

with $d^+(q,x)=0$ whenever $\mathcal{E}_L(q)\le\mathcal{A}_H(x)$ except for $d^+(q,x)=\pm1$ if $\pi(x)=q$. In particular, Θ is also an isomorphism between the restricted chain complexes $CM^a_*(\mathcal{E}_L)$ and $CF^a_*(H)$ for $a\in\mathbb{R}$.

We remark that the conditions (H_1) and (H_2) are automatically satisfied for $H=\mathfrak{L}_L(L)$ if L satisfies (L_1) and (L_2). In the following we sketch the proof of 4.4 which is given in full terms in [AS04]. For simplicity, we drop the notational distinction between ΛQ and ΩQ which are considered completely analogously.

Proof. Recall first that the generators of $CM_*(\mathcal{E}_L)$ and $CF_*(H)$ allow a canonical identification by

$$\mathcal{P}(L)=\operatorname{Crit}\mathcal{E}_L\ni q\ \leftrightarrow\ x\in\operatorname{Crit}\mathcal{A}_H=\mathcal{P}(H),$$

via $\pi(x)=q$ where $\pi\colon T^*Q\to Q$ is the cotangent bundle projection, and where $\mathcal{E}_L(q)=\mathcal{A}_H(x)$ for those critical points, in view of the Legendre duality. Choose a numbering $(q_n)_{n\in\mathbb{N}}$ of $\operatorname{Crit}\mathcal{E}_L$ such that $\big(\mathcal{E}_L(q_n)\big)$ is monotone non-decreasing, and let (x_n) be the corresponding sequence in $\operatorname{Crit}\mathcal{A}_H$. The crucial issue is the observation that the operator Θ is represented in terms of this infinite basis by a matrix $D=\big(d^+(q_i,x_j)\big)_{i,j\in\mathbb{N}}$ which has the properties

1. D is lower-triangular, i.e. $d^+(q_i,x_j)=0$ for $i<j$,
2. $d^+_{ii}=\pm1$ for all $i\in\mathbb{N}$,
3. $D\,\triangle_{\mathcal{E}}=\triangle_{\mathcal{A}}\,D$,

where $\triangle_{\mathcal{E}}$ and $\triangle_{\mathcal{A}}$ are the matrices representing the chain boundary operators $\partial_{\mathcal{E}}\colon CM_*\to CM_{*-1}$ and $\partial_{\mathcal{A}}\colon CF_*\to CF_{*-1}$. It is an immediate consequence from (1) and (2) that D is invertible and with (3) we obtain that $\Theta\colon\big(CM_*,\partial_{\mathcal{E}}\big)\xrightarrow{\cong}\big(CF_*,\partial_{\mathcal{A}}\big)$ is a chain complex isomorphism. It remains to give the construction of $d^+(q,x)$ for $q\in\mathcal{P}(L)$, $x\in\mathcal{P}(H)$ such that (1), (2) and (3) are satisfied.

Let $q\in\operatorname{Crit}\mathcal{E}_L$ and $x\in\operatorname{Crit}\mathcal{A}_H$ and define

$$\begin{aligned}\mathcal{M}^{\mathcal{E}\to\mathcal{A}}_{q;x}\ =\ \big\{\,&(\gamma,u)\in W^u(q)\times C^\infty([0,\infty)\times\mathbb{T},T^*Q)\ |\\ &u_s-J(u_t-X_H(t,u))=0,\\ &u(+\infty)=x,\ \pi(u(0,t))=\gamma(0)(t)\,\text{f.a. }\ t\in\mathbb{T}\,\big\}\,.\end{aligned}\tag{24}$$

Here, we use the loop $\gamma(0)$ in the finite-dimensional unstable manifold as a 1-parameter family of Lagrangian subspaces $(T^*_{\gamma(0)(t)}Q)_{t\in\mathbb{T}}$ providing a t-dependent family of boundary conditions for the pseudoholomorphic half-cylinder u where the boundary condition identifies the loop parameter for $u(0)$ with the family parameter. This is again an admissible nonlinear Fredholm problem, for which we can prove

PROPOSITION 4.5. (a) *For generic J and g as in Theorem 4.4, the mixed solution space $\mathcal{M}^{\mathcal{E}\to\mathcal{A}}_{q;x}$ is a smooth manifold of dimension $i(q)-\mu(x)$.*

(b) *$\mathcal{M}^{\mathcal{E}\to\mathcal{A}}_{q;x}$ is compact in dimension 0 and compact, up to simply splitting of either Morse trajectories on the negative end or Floer trajectories on the positive end, in dimension 1.*

(c) *We have $\mathcal{M}^{\mathcal{E}\to\mathcal{A}}_{q;x}\neq\emptyset$ only if $\mathcal{E}_L(q)\ge\mathcal{A}_H(x)$ with $\mathcal{E}_L(q)=\mathcal{A}_H(x)$ if and only if $\pi(x)=q$ and $\mathcal{M}^{\mathcal{E}\to\mathcal{A}}_{q;x}=\big\{(q,x)\big\}$, i.e. $\gamma(0)=q$ and $u(0)=x$ in (24) above.*

Defining now $d^+(q,x)=\#_{\mathsf{alg}}\mathcal{M}^{\mathcal{E}\to\mathcal{A}}_{q;x}$ by counting with signs, which are determined by a concept of coherent orientations described completely in [AS04], we obtain the properties (1)-(3) as claimed. Concerning this proposition, the essential new argument is based on Main Lemma 4.1. Namely, for any $(\gamma,u)\in\mathcal{M}^{\mathcal{E}\to\mathcal{A}}_{q;x}$ we have $\mathcal{E}_L(\gamma(0))\ge\mathcal{A}_H(u(0))$ and thus $\mathcal{E}_L(q) \geq \mathcal{A}_H(x)$ with equality if and only if γ and u are constant with respect to the flow parameter. This implies an a priori energy bound required for the compactness statements in (b). Moreover, we see that $d^+(q_i, x_j) = 0$ for $i < j$ with respect to the above ordering of the generators.

Note that the transversality property of (a) in case of the constant solution $(q, x) \in \mathcal{M}^{\mathcal{E}\to\mathcal{A}}_{q;x}$ cannot be achieved by perturbing J or g. In fact, we can prove that for $\pi(x) = q$ we have automatic transversality which follows from the second variation estimate

$$d^2\mathcal{A}_H(x)[\zeta, \zeta] \;\leq\; d^2\mathcal{E}_L(q)\big[D\pi(x)\zeta, D\pi(x)\zeta\big], \quad \text{for all } \zeta \in T_x\Lambda(T^*Q)\,.$$

This estimate is again a consequence of Main Lemma 4.1. For details, see [AS04]. Thus, the core argument is based on Legendre duality in an essential way.

Another important remark is that there is no analogous construction of the inverse chain homomorphism

$$\Theta^{-1} \colon CF_* \to CM_*$$

via a correspondingly reversed mixed solution space. If we consider the problem

$$\begin{aligned}\mathcal{M}^{\mathcal{A}\to\mathcal{E}}_{x;q} \;=\; \big\{\, &(u, \gamma) \in C^\infty((-\infty, 0] \times \mathbb{T}, T^*Q) \times W^s(q) \;| \\ &u_s - J(u_t - X_H(t, u)) = 0,\; u(-\infty) = x,\; \pi(u(0)) = \gamma(0) \,\big\},\end{aligned}$$

there is no possibility to obtain an analogous statement as in Proposition 4.5. Both, the Fredholm property and the compactness property will fail in general, because $\dim W^s(q) = \infty$ and $\mathcal{E}_L(\gamma(0) \geq \mathcal{A}_H(u(0))$ which allows no a priori energy bound. □

5. Products in Morse-Homology

In this last section, we sketch the Morse-homological definitions of the product and intersection operations $\circ$, $\cap$ and $\#$ from Section 2.1. These are needed in order to prove that the explicit isomorphism Θ intertwines the product structures on both sides as claimed in Theorem 2.5.

The finite-dimensional model of the homomorphism $\cap$ in Morse homology is given by the following construction. Let M^m, N^n be smooth, finite-dimensional manifolds, for simplicity closed. Let $i \colon M \hookrightarrow N$ be a smooth embedding and $f^M \colon M \to \mathbb{R}$, $f^N \colon N \to \mathbb{R}$ be Morse functions such that

$$\operatorname{Crit} f^N \cap i(M) = \emptyset\,. \tag{25}$$

Then we obtain induced homomorphisms of Morse homology,

$$\begin{aligned} i_* &\colon HM_*(M, f^M) \to HM_*(N, f^N), \\ i_! &\colon HM_*(N, f^N) \to HM_{*-(n-m)}(M, f^M), \end{aligned}$$

where $i_!$ equals $\cap$ from Section 2.1. The construction goes as follows on chain level. By assumption (25), which is generically fulfilled, we can find generic Riemannian metrics g^M, g^N on M and N such that we have the transverse intersections

$$\begin{aligned} i(M) \pitchfork W^s(x; f^N, g^N) \;&=\; \big\{\, \gamma \colon [0, \infty) \to N \,\big|\, \dot\gamma + \nabla_{g^N} f^N(\gamma) = 0, \\ &\qquad \gamma(0) \in i(M),\; \gamma(\infty) = x \,\big\}, \\ i(M) \pitchfork W^u(x; f^N, g^N) \;&=\; \big\{\, \gamma \colon (-\infty, 0] \to N \,\big|\, \dot\gamma + \nabla_{g^N} f^N(\gamma) = 0, \\ &\qquad \gamma(0) \in i(M),\; \gamma(-\infty) = x \,\big\}, \end{aligned} \tag{26}$$

for all $x \in \operatorname{Crit} f^N$, and

$$
\begin{aligned}
\mathcal{M}^{i_*}_{x,y} &= W^u(x; f^M, g^M) \pitchfork_i W^s(y; f^N, g^N)\\
&= \big\{ (\gamma^M, \gamma^N) \in C^\infty((-\infty,0], M) \times C^\infty([0,\infty), N) \,\big|\\
&\qquad \dot\gamma^K + \nabla_{g^K} f^K(\gamma^K) = 0,\ K \in \{M,N\},\\
&\qquad \gamma^M(-\infty) = x,\ \gamma^N(+\infty) = y,\ i(\gamma^M(0)) = \gamma^N(0) \,\big\},\\
\mathcal{M}^{i_!}_{y,x} &= W^u(y; f^N, g^N) \pitchfork_i W^s(x; f^M, g^M)\\
&= \big\{ (\gamma^N, \gamma^M) \in C^\infty((-\infty,0], N) \times C^\infty([0,\infty), M) \,\big|\\
&\qquad \dot\gamma^K + \nabla_{g^K} f^K(\gamma^K) = 0,\ K \in \{M,N\},\\
&\qquad \gamma^N(-\infty) = x,\ \gamma^M(+\infty) = y,\ i(\gamma^M(0)) = \gamma^N(0) \,\big\}\\
&= \big(W^u(y; f^N, g^N) \pitchfork_N i(M)\big) \pitchfork_M W^s(x; f^M, g^M),
\end{aligned} \tag{27}
$$

with

$$
\dim \mathcal{M}^{i_*}_{x,y} = i_{f^M}(x) - i_{f^N}(y), \quad \dim \mathcal{M}^{i_!}_{y,x} = i_{f^N}(y) - i_{f^M}(x) - (n-m),
$$

for $x \in \operatorname{Crit} f^M$ and $y \in \operatorname{Crit} f^N$. Here, $\pitchfork_M$, $\pitchfork_N$ indicate the ambient manifold of the transversely intersecting submanifolds. Then, counting the solution of (27) in dimension 0 defines i_* and $i_!$ on chain-level.

Let us use this concept now in the infinite-dimensional Morse-homological framework of the codimension-n embedding

$$
i\colon \Omega_{q_o} Q \hookrightarrow \Lambda Q
$$

of the Hilbert manifolds, where the Morse function is given by the Lagrangian action functional for some non-degenerate Lagrangian L satisfying (L_1) and (L_2).

The transversality condition (25) is now given by the assumption that for all $c \in \mathcal{P}^\Lambda(L)$ we have $c(0) \neq q_o$, i.e.

$$
\mathcal{P}^\Lambda(L) \cap i(\Omega_{q_o} Q) = \emptyset, \tag{28}
$$

which is fulfilled generically for (L, q_o). Choosing generic Riemannian metrics g^Λ and g^Ω on ΛQ and ΩQ we construct

$$
i_*\colon HM_*(\mathcal{E}^\Omega_L) \to HM_*(\mathcal{E}^\Lambda_L)
$$

on chain level by

$$
i_*(a) \;=\; \sum_{b \in \mathcal{P}^\Lambda(L),\, i(b) = i(a)} \#_{\mathsf{alg}}\big(W^u(a; \mathcal{E}^\Omega, g^\Omega) \pitchfork W^s(b; \mathcal{E}^\Lambda, g^\Lambda)\big) \cdot b
$$

for all $a \in \operatorname{Crit} \mathcal{E}^\Omega_L$, where the intersection means counting $\gamma\colon \mathbb{R} \to \Lambda Q$ such that $\dot\gamma(s) = -\nabla_{g^\Omega} \mathcal{E}^\Omega(\gamma)$ and $\gamma(s)(0) = q_o$ for all $s \leq 0$ and $\dot\gamma(s) = -\nabla_{g^\Lambda} \mathcal{E}^\Lambda(\gamma)$ for all $s \geq 0$. Similarly, we define

$$
i_!\colon HM_*(\mathcal{E}^\Lambda_L) \to HM_{*-n}(\mathcal{E}^\Omega_L)
$$

on chain level by

$$
i_!(a) \;=\; \sum_{b \in \mathcal{P}^\Omega(L),\, i(b) = i(a) - n} \#_{\mathsf{alg}}\Big(\big[W^u(a; \mathcal{E}^\Lambda, g^\Lambda) \pitchfork \operatorname{ev}_0^{-1}(q_o)\big] \pitchfork W^s(b; \mathcal{E}^\Omega, g^\Omega)\Big) \cdot b
$$

for $a \in \mathcal{P}^\Lambda$, where the double intersection means counting $\gamma^-\colon (-\infty, 0] \to \Lambda Q$ and $\gamma^+\colon [0,\infty) \to \Omega Q$ such that $\dot\gamma^\pm = -\nabla_{g^{\Lambda \text{ resp.} \Omega}} \mathcal{E}^{\Lambda \text{ resp.} \Omega}(\gamma^\pm)$, $\gamma^-(-\infty) = a$, $\gamma^+(\infty) = b$,

$\mathrm{ev}_0\big(\gamma^-(0)\big) = q_o$ and $\gamma^-(0) = \gamma^+(0)$. One can show that, assuming (28), all intersections are transverse for generic g^Λ, g^Ω.

Next we consider the codimension-n embedding,

$$\#\colon \Omega Q \times \Omega Q \hookrightarrow \Omega_2 Q \subset C^0\big([0,2],Q\big),$$
$$\#(c,d)(t) = \begin{cases} c(t), & 0 \le t \le 1, \\ d(t), & 1 \le t \le 2, \end{cases}$$

which for convenience we view into the based loop space of period 2. The generic assumption on (L,q_o) for transversality is now that

$$c(1) \neq q_o \qquad \text{for all } c \in \mathcal{P}_2^\Omega(L)\,. \tag{29}$$

Then, for generic Riemannian metrics g_1, g_2 on ΩQ on g_3 on $\Omega_2 Q$ we have

$$\#\big(W^u(a;\mathcal{E}^\Omega, g_1) \times W^u(b;\mathcal{E}^\Omega, g_2)\big) \pitchfork W^s(c;\mathcal{E}^{\Omega_2}, g_3)$$

for all $a,b \in \mathcal{P}_1^\Omega(L)$, $c \in \mathcal{P}_2^\Omega(L)$. Counting in dimension 0 with signs thus defines on chain level an operator which on homology gives

$$\#_*\colon HM_k(\mathcal{E}_L^\Omega) \otimes HM_l(\mathcal{E}_L^\Omega) \to HM_{k+l}(\mathcal{E}_L^{\Omega_2})\,.$$

Let us now turn to the codimension-n submanifold

$$X = \big\{\, (c,d) \in \Lambda Q \times \Lambda Q \,\big|\, c(0) = d(0) \,\big\} \overset{\#}{\hookrightarrow} \Lambda_2 Q = W^{1,2}(\mathbb{R}/2\mathbb{Z}, Q)\,.$$

In order to construct

$$i_!\colon HM_k(\mathcal{E}^\Lambda_{L_1}) \otimes HM_l(\mathcal{E}^\Lambda_{L_2}) \to HM_{k+l-n}(X;\mathcal{E}^\Lambda_{L_1\#L_2})$$

and

$$\#_*\colon HM_{k+l-n}(X;\mathcal{E}^\Lambda_{L_1\#L_2}) \to HM_{k+l-n}(\mathcal{E}^{\Lambda_2}_{L_1\#L_2})$$

we assume

$$\begin{aligned} a(0) &\neq b(0) && \text{for all } (a,b) \in \mathcal{P}^\Lambda(L_1) \times \mathcal{P}^\Lambda(L_2), \quad \text{and} \\ c &\notin X && \text{for all } c \in \mathcal{P}_2^\Lambda(L_1\#L_2), \end{aligned} \tag{30}$$

where $L_1(1,\cdot) = L_2(0,\cdot)$, $L = L_1\#L_2\colon \mathbb{R}/2\mathbb{Z} \times TQ \to \mathbb{R}$, with $L_{|[0,1]} = L_1$ and $L_{|[1,2]} = L_2$. By assumption (30) we can find generic Riemannian metrics g_1, g_2 on ΛQ and g_3 on $\Lambda_2 Q$ such that we have transverse intersections

$$\begin{aligned} &Y = \big(W^u(a;\mathcal{E}^\Lambda_{L_1}, g_1) \times W^u(b;\mathcal{E}^\Lambda_{L_2}, g_2)\big) \pitchfork X, \quad \text{and} \\ &Z(a,b;c) = \#(Y) \pitchfork W^s(c;\mathcal{E}^{\Lambda_2}_{L_1\#L_2}, g_3) \end{aligned}$$

for all $(a,b) \in \mathcal{P}^\Lambda(L_1) \times \mathcal{P}^\Lambda(L_2)$ and $c \in \mathcal{P}^{\Lambda_2}(L_1\#L_2)$. We define

$$\begin{aligned} &\mathsf{o}\colon CM_k(\mathcal{E}^\Lambda_{L_1}) \otimes CM_l(\mathcal{E}^\Lambda_{L_2}) \to CM_{k+l-n}(\mathcal{E}^{\Lambda_2}_{L_1\#L_2}), \\ &a\mathsf{o}b = \sum_{i(c)=i(a)+i(b)-n} \#_{\mathsf{alg}} Z(a,b;c)\cdot c, \end{aligned} \tag{31}$$

which on homology gives $\mathsf{o} = \#\circ i_!$, the Morse-homological definition of the loop product.

5.1. **Ring Isomorphism between Morse homology and Floer homology.** Finally, we sketch very briefly the proof of the ring homomorphism property of

$$\Theta_*\colon \big(HM_*(\mathcal{E}_L^\Omega), \#_*\big) \xrightarrow{\cong} \big(HF_*(\mathcal{A}_H^\Omega), m^\Omega\big)$$

in the based loop space case. Full details, also for the free loop space case, will be contained in [AS05].

We show that on chain level, the compositions of Θ and $\#$ and m^Ω and Θ, respectively, are chain homotopy equivalent,

$$\Theta\circ\# \simeq m^\Omega\circ(\Theta\otimes\Theta)\colon CM_*^\Omega(\mathcal{E}_L)\otimes CM_*^\Omega(\mathcal{E}_L)\to CF_*^{\Omega_2}(\mathcal{A}_H)\,. \tag{32}$$

Consider a generic triple of Riemannian metrics g_1, g_2 on ΩQ and g_3 on $\Omega_2 Q$ as before, and J generic on T^*Q for $\mathcal{A}_H$ on $\Lambda_2(T^*Q)$. Then, by gluing, the composition $(\Theta\circ\#_*)(a,b)$ is given by counting

$$\begin{aligned}\mathcal{M}^I_{a,b;x}(R) = \Big\{ (\gamma_1,\gamma_2,\gamma_3,u)\,\Big|\,&\gamma_i\colon(-\infty,0]\to\Omega Q,\ i=1,2,\\ &\gamma_3\colon[0,R]\to\Omega_2 Q,\ \dot\gamma_i+\nabla_{g_i}\mathcal{E}(\gamma_i)=0,\ i=1,2,3,\\ &\gamma_1(-\infty)=a,\ \gamma_2(-\infty)=b,\ \gamma_3(0)=\gamma_1(0)\#\gamma_2(0),\\ &u\colon[R,\infty)\times[0,2]\to T^*Q,\ u_s-J(u)(u_t-X_H(t,u))=0,\\ &u(s,0),\ u(s,2)\in T^*_{q_o}Q,\ R\le s<\infty,\\ &u(+\infty)=x,\ u(R,t)\in T^*_{\gamma_3(R)(t)}Q,\ 0\le t\le 2\Big\}\end{aligned}$$

for $\mu^\Omega(x)=i(a)+i(b)$ and for $R\gg 1$ sufficiently large depending on a,b. For the other

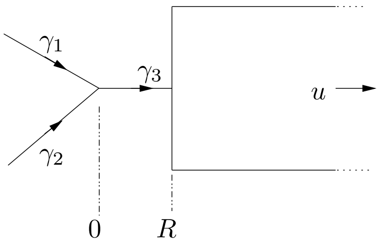

FIGURE 6. The domain for $\mathcal{M}^I_{a,b;x}(R)$

composition $m^\Omega\circ(\Theta\otimes\Theta)(a,b)$ we have to count for $r\gg 1$

$$\begin{aligned}\mathcal{M}^{II}_{a,b;x}(r) = \Big\{ (\gamma_1,\gamma_2,u)\,\Big|\,&\gamma_i\colon(-\infty,0]\to\Omega Q,\ i=1,2,\\ &\dot\gamma_i+\nabla_{g_i}\mathcal{E}(\gamma_i)=0,\ i=1,2,\ \gamma_1(-\infty)=a,\ \gamma_2(-\infty)=b,\\ &u\colon\Sigma_r\to T^*Q,\ u_s-J(u)(u_t-X_H(t,u))=0,\\ &u(s,0),\ u(s,2)\in T^*_{q_o}Q,\ \text{for }\ -r\le s<\infty,\\ &u(s,1\pm)\in T^*_{q_o}Q,\ \text{for }\ -r\le s\le 0,\\ &u(+\infty)=x,\ u(-r,t)\in\begin{cases}T^*_{\gamma_1(0)(t)}Q, & 0\le t\le 1-,\\ T^*_{\gamma_2(0)(t-1)}Q, & 1+\le t\le 2,\end{cases}\Big\}\end{aligned}$$

where $x \in \mathcal{P}_2^\Omega(H)$ with $\mu^\Omega(x) = i(a) + i(b)$, and Σ_r is the domain

$$\Sigma_r \;=\; [-r,\infty)\times[0,1]\,\dot\cup\,[-r,\infty)\times[1,2]/\sim,$$
$$\text{with }\;(s,1-)\sim(s,1+)\text{ for all }\;s\geq 0\,.$$

One can now show that both solution spaces $\mathcal{M}^I_{a,b;x}(R)$ and $\mathcal{M}^{II}_{a,b;x}(r)$ are compactly, up

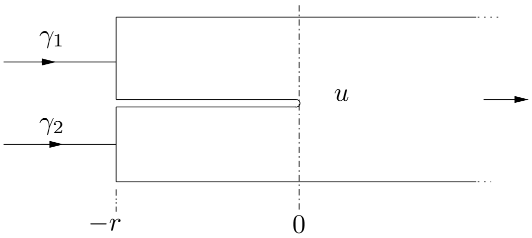

FIGURE 7. The domain for $\mathcal{M}^{II}_{a,b;x}(r)$

to simple splitting, cobordant to

$$\begin{aligned}\mathcal{M}^0_{a,b;x} = \Big\{ (\gamma_1,\gamma_2,u)\,\Big|\, &\gamma_i\colon(-\infty,0]\to\Omega Q,\; i=1,2,\\ &\dot\gamma_i+\nabla_{g_i}\mathcal{E}(\gamma_i)=0,\; i=1,2,\;\gamma_1(-\infty)=a,\;\gamma_2(-\infty)=b,\\ &u\colon[0,\infty)\times[0,2]\to T^*Q,\; u_s-J(u)(u_t-X_H(t,u))=0,\\ &u(s,0),\;u(s,2)\in T^*_{q_o}Q,\;\text{ for }\;0\leq s<\infty,\\ &u(+\infty)=x,\;u(0,t)\in\begin{cases}T^*_{\gamma_1(0)(t)}Q, & 0\leq t\leq 1-,\\ T^*_{\gamma_2(0)(t-1)}Q, & 1+\leq t\leq 2,\end{cases}\Big\}\,.\end{aligned}$$

This shows that

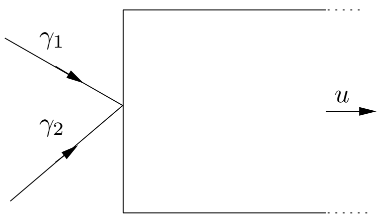

FIGURE 8. The intermediate domain for $\mathcal{M}^0_{a,b;x}$

$$\Theta\circ\#-m^\Omega\circ(\Theta\otimes\Theta)\;=\;\partial^\Omega(J,H)\circ S\pm S\circ\big(\operatorname{id}\otimes\partial^\Omega(\mathcal{E}^\Omega,g_2)\pm\partial^\Omega(\mathcal{E}^\Omega,g_1)\otimes\operatorname{id}\big)$$

for some chain homotopy operator

$$S\colon CM_*(\mathcal{E}^\Omega_L)\otimes CM_*(\mathcal{E}^\Omega_L)\to CF_{*+1}(\mathcal{A}^{\Omega_2}_H),$$

thus proving (32).

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