The second closed geodesic on the complex projective plane *

Hans-Bert Rademacher

Abstract

We show the existence of at least two geometrically distinct closed geodesics on a complex projective plane with a bumpy and non-reversible Finsler metric

2000 MSC classification: 53C22; 53C60; 58E10

Keywords: closed geodesic, energy functional, bumpy Finsler metric, Morse inequalities, equivariant Morse theory

1 Introduction

On a compact and simply-connected manifold M with a non-reversible Finsler metric there always exists a closed geodesic. There are non-reversible Finsler metrics on compact rank one symmetric spaces carrying only finitely many (geometrically distinct) closed geodesics, the geometry of this so-called *Katok examples* is explained in [Zi]. To prove the existence of several geometrically distinct closed geodesics one can consider bumpy metrics. For a *bumpy metric* all closed geodesics are non-degenerate, i.e. there are no non-trivial and periodic Jacobi fields along a closed geodesic. In this case the energy functional on the free loop space is a Morse function with non-degenerate critical S^1-orbits. In [Ra1, ch.4] the author has shown that on the 2-sphere with a bumpy metric there are at least two closed geodesics. Recently Bangert and Long proved in [BL] that for *every* non-reversible Finsler metric on S^2 there are two geometrically distinct closed geodesics.

*Front. Math. China 3 (2008) 253-258

Independently Duan and Long [DL] and the author [Ra3] showed that on an n-dimensional sphere with a bumpy non-reversible Finsler metric there are at least two geometrically distinct closed geodesic for all $n > 2$. A recent survey on existence results for closed geodesics on Finsler manifolds is [Lo]. In this short note we show that one can obtain a similar result for manifolds of the rational homotopy type of the complex projective plane $\mathbb{C}P^2$.

Theorem. *Let M be a compact and simply-connected manifold of the rational homotopy type of the complex projective plane $\mathbb{C}P^2$ carrying a bumpy and non-reversible Finsler metric. Then there are at least* two *geometrically distinct closed geodesics.*

It is likely that two is not the optimal number, the Katok examples on $\mathbb{C}P^2$ carry six closed geodesics. Existence results for closed geodesics for metrics with positive flag curvature are given in [Ra3].

2 The Proof

We assume that the manifold M satisfies the assumptions of the Theorem and we assume that there is only a single closed geodesic c. Hence there is a prime closed geodesic $c : S^1 \to M$ such that any other closed geodesic is geometrically equivalent to c. In other words any closed geodesic $\tilde{c}$ is up to the choice of a starting point of the form $\tilde{c} = c^m$ for some $m \geq 1$, here $c^m(t) = c(mt)$ is the m-th iterate of the prime closed geodesic c. Let

$$v_i := \#\{m \geq 1; \mathrm{ind}(c^m) = i \equiv \mathrm{ind}(c) \pmod 2\}\,. \tag{1}$$

If $\gamma = \gamma_c \in \{\pm 1/2, \pm 1\}$ is defined by $\gamma > 0$ if and only if $\mathrm{ind}(c)$ is even and $|\gamma| = 1/2$ if and only if $\mathrm{ind}(c^2) - \mathrm{ind}(c)$ is odd then we can also express the number v_i of homologically visible critical points of index i as follows:

$$v_i = \#\{m \geq 1; \mathrm{ind}(c^m) = i \text{ and } m\gamma \in \mathbb{Z}\}.$$

Then the Morse inequalities are

$$v_i = b_i + q_i + q_{i-1} \tag{2}$$

for a sequence q_i of non-negative integers, cf. [Ra1, (2.3)]. Here $b_i = b_i\left(\Lambda\mathbb{C}P^2/S^1, \Lambda^0\mathbb{C}P^2; \mathbb{Q}\right)$ are the rational Betti numbers of the quotient $\Lambda\mathbb{C}P^2/S^1$ of the free loop space $\Lambda\mathbb{C}P^2$ of the complex projective plane $\mathbb{C}P^2$ with respect to the canonical S^1 action. These Betti numbers are determined in [Ra1, (2.6)]:

$$b_i = \begin{cases} 1 & ; & i & = & 1 & & \\ 2 & ; & i & = & 3 & & \\ 3 & ; & i & = & 2k+5 & ; & k \geq 0 \\ 0 & ; & i & = & 2k & ; & k \geq 0 \end{cases} \tag{3}$$

Therefore the topological invariant $B(\mathbb{C}P^2) = B(2,2)$ introduced in [Ra1, (2.1)] satisfies:

$$B(2,2) = \lim_{k \to \infty} \sum_{i=0}^{k} (-1)^k b_k = -\frac{3}{2}\,. \tag{4}$$

Equation 2 implies that $v_i > 0$ only if i is odd. Therefore the sequence q_i vanishes identically, i.e. for all $i \geq 0$:

$$v_i = b_i\,. \tag{5}$$

The average index

$$\alpha = \alpha_c = \lim_{m \to \infty} \frac{\mathrm{ind}(c^m)}{m}$$

satisfies

$$\alpha = \frac{2}{3}\,|\gamma| \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}\,, \tag{6}$$

cf. [Ra1, Thm. 31] and Equation 4. The sequence $\mathrm{ind}(c^m)$ can be expressed by Bott's formula in terms of a function $I = I_c : S^1 = \{z \in \mathbb{C}; |z| = 1\} \to \mathbb{Z}^{\geq 0}$ as follows:

$$\mathrm{ind}(c^m) = \sum_{z^m = 1} I(z)\,. \tag{7}$$

with the following properties:

(a) $I(z) = I(\overline{z})$.

(b) The function I is locally constant with the possible exception at points z belonging to the set $\mathrm{Spec}(P_c)$ of eigenvalues of the linearized Poincaré mapping.

(c) Let F be a bumpy metric and let $0 = t_0 < t_1 < t_2 < \ldots < t_l < t_{l+1} = 1/2$ be the Poincaré exponents. Hence the set of eigenvalues z with $|z| = 1, \mathrm{Im}(z) > 0$ is given by $\left\{\exp\left(2\pi\sqrt{-1}\,t_1\right)\ldots, \exp\left(2\pi\sqrt{-1}\,t_l\right)\right\}$ for some $l \in \{0,1,2,3\}$ and the numbers t_j are irrational.

(d) With the help of the function I_c we get the following expression for the average index. Let $I_1 = I_c(0) = I_c(\exp(2\pi it)); t \in [0, t_1)$ and for $j \in \{1, 2, \ldots, l-1\} : I_j := I_c(\exp(2\pi it)); t \in (t_{j-1}, t_j)$ and $I_{l+1} := I_c(-1) = I_c(\exp(2\pi it)); t \in (t_l, 1/2]$. Hence

$$\left\{I_c(\exp(z)) \,|\, z \in S^1 - \mathrm{Spec}(P_c)\right\} = \{I_1, I_2, \ldots, I_{l+1}\}.$$

Then Bott's formula Equation 7 implies

$$\alpha_c = \int_0^1 I_c(\exp(2\pi it))\, dt = 2\, I_1 t_1 + 2 \sum_{j=1}^{l-1} I_j\,(t_j - t_{j-1}) + I_l\,(1 - 2t_l) \tag{8}$$

(e) The *total splitting number* $S = S_c$ of a closed geodesic on an 4-dimensional manifold satisfies

$$S = \sum_{j=1}^{l} |I_j - I_{j+1}| \leq 3\,. \tag{9}$$

We define the function $e : [0, 1/2] \to \mathbb{C}, e(a) = \exp\left(2\pi\sqrt{-1}\,a\right)$. Now we determine the values $I(e(p/q))$ for $q = 1, 2, \ldots$ until we obtain with the help of Bott's formula 7 a contradiction to Equation 5 derived from the Morse inequalities. We divide the proof into the following steps:

Claim 1. *$I(e(0)) = I(0) = \mathrm{ind}(c) = 1\,;\ \mathrm{ind}(c^m) \geq 2$ for all $m \geq 2$.*

Proof. Since $ind(c^m) \geq \mathrm{ind}(c)$ for all $m \geq 1$ we conclude from Equation 5 and Equation 3 that

$$I(1) = I(e(0)) = I_1 = 1. \tag{10}$$

Since $v_1 = b_1 = 1$ we conclude that $\mathrm{ind}(c^m) \geq 2$ for all $m \geq 2$. □

Claim 2. *$I(e(1/2)) = I(-1) = 2\,;\ \mathrm{ind}(c^2) = 3\,;\ \gamma = -1\,;\ \alpha = 2/3\,;\ t_2 - t_1 > 1/6\,;\ t_l > 1/3$*

Proof. Since $b_1 = v_1 = 1$ it also follows that $\mathrm{ind}(c^m) > 1$ for all $m > 1$. Since $\alpha \in (0, 1)$ by Equation 6 we conclude from Equation 9 and Equation 8 that $I(e(1/2)) \leq 2$ which implies: $\mathrm{ind}(c^2) \in \{2, 3\}$. If $\gamma = -1/2$ then $\mathrm{ind}(c^2)$ is even, i.e. $\mathrm{ind}(c^2) = 2$ resp. $I(e(1/2)) = 1$. Since $S \leq 3$ it follows in this case that $\max\{I_1, \ldots, I_l\} = 1$, i.e.

$$\mathrm{ind}(c^3) = 1 + 2I(e(1/3)) = 3. \tag{11}$$

On the other hand Equation 6 implies $\alpha = 1/3$, therefore $l = 2$ and $I_1 = 1, I_2 = 0, I_3 = 1$. Equation 8 implies $t_2 - t_1 = 1/3$ from which we conclude that $1/3 \in (t_1, t_2)$ i.e. $I(e(1/3)) = 0$ contradicting Equation 11. Therefore $\gamma = -1, I_0 = 1, I_1 = 0, l \in \{2, 3\}$ and

$$I(e(1/2)) = I_l = 2\,;\ \mathrm{ind}(c^2) = 3\,. \tag{12}$$

Since $\alpha = 2/3$ we obtain $t_2 - t_1 > 1/6$ and $t_l > 1/3$. □

Claim 3. *$I(e(1/3)) = 1\,;\ \mathrm{ind}(c^3) = 3\,;\ \mathrm{ind}(c^m) \geq 5$ for all $m \geq 4\,.$*

Proof. Since $\mathrm{ind}(c^3) = 1 + 2I(e(1/3)) \geq 3$ and $t_l > 1/3$ by Claim 2 we obtain $I(e(1/3)) = 1$, which implies $\mathrm{ind}(c^3) = 3$. Since $\mathrm{ind}(c^2) = \mathrm{ind}(c^3) = 3$ and $v_3 = b_3 = 2$ we conclude $\mathrm{ind}(c^m) \geq 5$ for all $m \geq 4$. □

Claim 4. *$I(e(1/4)) = 1\,;\ \mathrm{ind}(c^4) = 5\,;\ t_2 < 1/4\,;\ l = 3\,,\ t_3 > 5/12$ and $\mathrm{ind}(c^m) \geq 5$ for all $m \geq 4$.*

Proof. By Claim 3: $\mathrm{ind}(c^4) = 3 + 2I(e(1/4)) \geq 5$, i.e. $I(e(1/4)) \geq 1$. On the other hand $t_l > 1/3$ by Claim 2 hence $I(e(1/4)) = 1$. Since $t_2 - t_1 > 1/6$ and $I(e(t)) = 0$ for $t \in (t_1, t_2)$ it follows from $I(e(1/4)) = I(e(1/3)) = 1, I(e(1/2)) = 2$ that $t_2 < 1/4$ and hence $l = 3$. But then $2/3 = \alpha > 1/2 + 2(1/2 - t_l) = 3/2 - 2t_3$ which implies $t_3 > 5/12$. □

Claim 5. *$I(e(1/5)) = I(e(2/5)) = 1\,;\ \mathrm{ind}(c^5) = 5\,;\ t_2 < 1/5\,;\ t_1 < 1/30$*

Proof. $\mathrm{ind}(c^5) = 1 + 2I(e(1/5)) + 2I(e(2/5)) \geq 5$ by Claim 4. Since $t_3 > 5/12 > 2/5$ by Claim 4 we conclude $I(e(2/5)) = 1$ and hence $I(e(1/5)) = 1$. Since $t_2 - t_1 > 1/6\,;\ t_2 < 1/4$ we conclude $t_2 < 1/5$. On the other hand $t_2 < 1/5$ and $t_2 - t_1 > 1/6$ implies $t_1 < 1/30$. □

Claim 6. *$I(e(1/6)) = 0\,;\ \mathrm{ind}(c^6) = 5$ and $\mathrm{ind}(c^m) \geq 7$ for all $m \geq 7$*

Proof. Since $t_2 - t_1 > 1/6$ and $I(e(1/5)) = I(e(1/4)) = 1$ it follows that $t_2 > 1/6$ i.e. $I(e(1/6)) = 0$. Then $\mathrm{ind}(c^6) = 1 + 2I(e(1/6)) + 2I(e(1/3)) + I(e(1/2)) = 5$. Since $\mathrm{ind}(c^4) = \mathrm{ind}(c^5) = \mathrm{ind}(c^6) = 5$ and $b_5 = 3$ it follows that $\mathrm{ind}(c^m) \geq 7$ for all $m \geq 7$. □

Claim 7. *$I(e(1/7)) = 0\,;\ I(e(2/7)) = 1\,;\ I(e(3/7)) = 2$ resp. $t_3 < 3/7$.*

Proof. Since $t_1 < 1/30$ and $t_2 > 1/6$ we obtain $I(e(1/7)) = 0$. Since $t_2 < 2/7 < t_3$ by Claim 4 we conclude $I(e(2/7)) = 1$. Since $\mathrm{ind}(c^7) = 1 + 2I(e(1/7)) + 2I(e(2/7)) + 2I(e(3/7)) = 3 + 2I(e(3/7)) \geq 7$ by Claim 6 we get $I(e(3/7)) = 2$ resp. $t_3 < 3/7$. $\square$

Now we obtain the final contradiction:

Claim 8. $\alpha > 26/35$

Proof. Since $t_2 < 1/5$ (Claim 5) and $t_3 < 3/7$ (Claim 7) we conclude $\alpha > 3/5 + 1/7 = 26/35$. $\square$

But by Claim 2: $\alpha = 2/3$. This contradiction finishes the proof of Theorem 1.

References

[BL] V.Bangert & Y.Long: *The existence of two closed geodesics on every Finsler 2-sphere.* `arXiv:0709.1243`

[DL] H.Duan & Y.Long: *Multiple closed geodesics on bumpy Finsler n-spheres.* J.Diff.Eq. **233** (2007) 221–240

[Lo] Y.Long: *Multiplicity and stability of closed geodesics on Finsler 2-spheres.* J.Eur.Math.Soc. **8** (2006) 341–353

[Ra1] H.B.Rademacher: *On the average indices of closed geodesics.* J.Differential Geom. **29** (1989) 65–83

[Ra2] ______: *Existence of closed geodesics on positively curved Finsler manifolds.* Ergod.Th.& Dyn.Syst. **27** (2007) 957–969

[Ra3] ______: *The second closed geodesic on Finsler spheres of dimension $n > 2$.* (to appear in: Trans. Amer. Math. Soc.) `arXiv:math/0608160`

[Zi] W.Ziller: *Geometry of the Katok examples.* Ergod.Th.& Dyn.Syst. **3** (1982) 135–157

Universität Leipzig, Mathematisches Institut
D-04081 Leipzig, Germany
rademacher@math.uni-leipzig.de
www.math.uni-leipzig.de/~rademacher