Conformal Transformations of Pseudo-Riemannian manifolds

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Abstract. This is a survey about conformal mappings between pseudo-Riemannian manifolds and, in particular, conformal vector fields defined on such.

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Contents

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1. Introduction

Conformal transformations and conformal vector fields are important concepts in both Riemannian and pseudo-Riemannian geometry. Liouville's theorem made clear already in the 19th century that in dimensions $n \geq 3$ conformal mappings are more rigid than in dimension 2. Conformally flat spaces have been characterized by Cotton, Finzi and Schouten in the early 20th century. In General Relativity conformal transformations are important since they preserve the causal structure up to time orientation and light-like geodesics up to parametrization. Already in the early days of Einstein's relativity theory, Kasner studied the *question whether two fields both obeying Einstein's equations of gravitation can ever have the same light rays.* Motivated by this question about light rays, Kasner [Ks21a] proved the following: *When a conformal representation of an Einstein manifold on a flat space is possible, the manifold is isometric to flat space.* In modern terminology this is the statement that a vacuum spacetime which is locally conformally flat must be flat. Brinkmann [Br'25] investigated conformal transformations between two Einstein spaces as well as conditions for a space to be conformal to an Einstein space. He solved the differential equation $\nabla^2 f = (\Delta f/n) \cdot g$ on Riemannian and pseudo-Riemannian manifolds of arbitrary dimension and – as a by-product – found those metrics which were later called *pp*-waves. For global conformal geometry, the conformal development map was introduced by Kuiper [Ku'49] in 1949. Using this concept he proved that a compact and simply connected Riemannian manifold which is locally conformally flat must be globally conformally equivalent with the standard sphere, see Section 3. Similarly, in the pseudo-Riemannian case a simply connected and locally conformally flat space admits a conformal development map into the quadric Q which is the conformal compactification of pseudo-Euclidean space.

Conformal vector fields can be considered as a natural generalization of Killing vector fields. They are also called *conformal Killing fields* or infinitesimal conformal transformations. Those which become Killing after some conformal change of the metric are considered as *inessential*. *Essential* conformal vector fields on Riemannian spaces have been studied by Obata, Lelong-Ferrand and Alekseevskii [Al'72], [La'88], their results are given in Section 4. Conformal gradient fields are essentially solutions of the differential equation $\nabla^2 f = (\Delta f/n) \cdot g$. After Brinkmann this equation has been investigated by Fialkow, Yano, Obata, Kerbrat and others, these results are presented in Section 5 and Section 7. For Riemannian manifolds the theorem of Obata-Ferrand states that a compact Riemannian manifold carrying an essential conformal vector field must be locally conformally flat and, therefore, is conformally equivalent with the standard sphere. In pseudo-Riemannian geometry any conformal vector field V induces a conservation law for lightlike geodesics since the quantity $g(V, \gamma')$ is constant along such a geodesic γ. Therefore, a classification of pseudo-Riemannian metrics admitting a conformal vector field is a challenge. In the pseudo-Riemannian case the authors started in [KR95] and [KR97b] a systematic approach to the structure of conformal gradient fields with isolated singularities including a conformal classification theorem which we present in Section 7. The ultimate pseudo-Riemannian analogue of the Obata-Ferrand theorem seems still to be missing, compare [Fs'05]. Already Brinkmann investigated the question which manifolds are conformal to an Einstein metric. In section 6 we give tensorial conditions for metrics to be conformally Einstein in particular following Listing [Li'01] as well as Gover and Nurowski [GN'06]. Conformal symmetries of four-dimensional spacetimes were investigated by Hall and others, cf. for example [Hl'04]. Among the four-dimensional spacetimes, which we discuss in section 8, the *pp*-waves play a special role. In the vacuum case they are the only ones admitting non-homothetic conformal vector fields. Plane waves occur as the so-called Penrose limit of arbitrary spacetimes. We review this construction in section 9. Introduced by Penrose [Pe'76] in 1976 the Penrose limit recently gained much attention in papers investigating background metrics for models in supergravity and string theory, cf. for example papers by Blau, Figueroa-O'Farrill et al. [BF'02] [BP'04]. One can introduce twistor spinors as solutions of a conformally covariant field equation and they come with an associated conformal vector field called the Dirac current. Twistor spinors can be seen as conformal extension of the concept of parallel and Killing spinors. We review shortly results by the authors in the Riemannian case in Section 10 before discussing results about twistor spinors and their Dirac currents in the Lorentzian setting which are mainly due to Baum and Leitner [BL'04].

2. Basic concepts

We consider a pseudo-Riemannian manifold (M, g), which is defined as a smooth manifold M (here *smooth* means of class C^∞) together with a pseudo-Riemannian metric of arbitrary signature $(k, n-k), 0 \leq k \leq n$. A conformal mapping between two pseudo-Riemannian manifolds $(M, g), (N, h)$ is a smooth mapping $F : (M, g) \to (N, h)$ with the property $F^*h = \alpha^2 g$ for a smooth positive function $\alpha : M \to \mathbb{R}^+$. In more detail this means that the equation

$$h_{F(x)}(dF_x(X), dF_x(Y)) = \alpha^2(x) g_x(X, Y)$$

holds for all tangent vectors $X, Y \in T_xM$. Particular cases are homotheties resp. dilatations, for which $\alpha = const$ is constant and isometries, for which $\alpha = 1$.

A (local) one-parameter group Φ_t of conformal mappings generates a conformal (Killing) vector field V, sometimes also called an infinitesimal conformal transformation, by $V = \frac{\partial}{\partial t}\Phi_t$. Vice versa, any conformal vector field generates a local one-parameter group of conformal mappings. In terms of derivatives of tensors this is expressed as follows:

Definition 2.1. *A vector field V is called* conformal *if and only if the* Lie derivative *$\mathcal{L}_V g$ of the metric g in direction of the vector field V satisfies the equation*

$$\mathcal{L}_V g = 2\sigma g$$

for a certain smooth function $\sigma : M \to \mathbb{R}$. The Lie derivative *of the metric in direction of a vector field V is defined as the symmetrization of the derivative ∇V as follows: For any given tangent vectors $X, Y \in T_xM$ the equation*

$$\mathcal{L}_V g_x(X,Y) = g_x\left(\nabla_X V, Y\right) + g_x\left(X, \nabla_Y V\right) = 2\sigma(x) g_x(X,Y) \tag{1}$$

holds. Here ∇ denotes the Levi-Civita connection of the pseudo-Riemannian manifold (M,g). For computing the trace let $(e_1, e_2, \ldots, e_n)$ be an orthonormal basis with $g(e_i, e_j) = \epsilon_i \delta_{ij}$, where $\epsilon_1 = \ldots = \epsilon_k = -1$ and $\epsilon_{k+1} = \ldots = \epsilon_n = 1; \delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. Then we have the divergence

$$2\,\mathrm{div} V = 2\sum_{i=1}^{n} \epsilon_i g\left(\nabla_{e_i} V, e_i\right) = \sum_{i=1}^{n} \epsilon_i \mathcal{L}_V g(e_i, e_i) = 2\,n\,\sigma$$

i.e. $\sigma = \mathrm{div} V/n$. Particular cases of conformal vector fields are homothetic vector fields *for which $\sigma = const$ and* isometric vector fields, *also called* Killing vector fields, *for which $\sigma = 0$.*

Proposition 2.2. *The image of a lightlike geodesic under any conformal mapping is again a lightlike geodesic.*

Furthermore, for any lightlike geodesic γ and any conformal vector field V the quantity $g(\gamma', V)$ is constant along γ.

Proof. The first statement follows from the following equation for the Levi-Civita connections $\nabla, \overline{\nabla}$ of two conformally equivalent metrics $g, \overline{g} = \varphi^{-2}\, g$:

$$\overline{\nabla}_X Y - \nabla_X Y = -X(\log\varphi) Y - Y(\log\varphi) X + g(X,Y)\mathrm{grad}(\log\varphi).$$

The second statement follows from

$$\gamma' g(\gamma', V) = g(\nabla_{\gamma'} V, \gamma') = \frac{1}{2}\mathcal{L}_V g(\gamma', \gamma') = \sigma g(\gamma', \gamma') = 0.$$

□

Conformal vector fields V with non-vanishing $g(V,V)$ can be made into Killing fields within the same conformal class of metrics.

Lemma 2.3. *If V is a conformal vector field on the pseudo-Riemannian manifold (M,g) for which the function $g(V,V)$ does not have a zero, then the vector field V is an isometric vector field for the conformally equivalent metric $\overline{g} = |g(V,V)|^{-1}\, g$. This is a special case of a so-called inessential conformal vector field.*

Proof. Let $\alpha = g(V,V)^{-1}$ and $\eta = \mathrm{sign}\, g(V,V) \in \{\pm 1\}$, then $V(\alpha) = -V\left(g(V,V)\right)\alpha^2 = -2g(\nabla_V V, V)\alpha^2 = -\mathcal{L}_V g(V,V)\alpha^2 = -2\sigma\alpha$. We conclude from $\mathcal{L}_V g = 2\sigma g$:

$$\mathcal{L}_V \overline{g}(X,Y) = \eta \mathcal{L}_V(\alpha g)(X,Y) = \eta\left(V(\alpha) g(X,Y) + \alpha\,\mathcal{L}_V g(X,Y)\right) = 0$$

Hence V is an isometric vector field for the metric $\overline{g}$. □

Definition 2.4. *We call a vector field V on a pseudo-Riemannian manifold* closed *if it is locally a gradient field, i.e., if locally there exists a function f such that $V = \mathrm{grad} f$ Consequently, from Equation 3 we see that a closed vector field V is conformal if and only*

$$\nabla_X V = \sigma X \tag{2}$$

for all X or, equivalently $\nabla^2 f = \sigma g$ where ∇^2 denotes the Hessian *$(0,2)$-tensor, i.e. $\nabla^2 f(X,Y) = g\left(\nabla_X \mathrm{grad} f, Y\right)$.*

In terms of 1-forms which are dual to vector fields this is nothing but the usual condition of closedness in terms of the exterior derivative: Let ω be the 1-form dual to the vector field V with respect to the metric g, i.e. $\omega(X) = g(V,X)$. Then the exterior derivative $d\omega$ equals the skew-symmetrization of the covariant derivative ∇V , i.e.

$$d\omega(X,Y) = g\left(\nabla_X V, Y\right) - g\left(X, \nabla_Y V\right)$$

From $2\, g(\nabla_X V, Y) = \mathcal{L}_V g(X,Y) + d\omega(X,Y)$ we obtain for a conformal vector field V with $\mathcal{L}_V g = 2\sigma g$ the equation

$$g(\nabla_X V, Y) = \sigma g(X,Y) + d\omega(X,Y)\,. \tag{3}$$

Accordingly, if $\overline{g} = \alpha g$ and if $\overline{\omega}$ is the one-form dual to V with respect to $\overline{g}$ then $d\overline{\omega} = d(\alpha\omega) = d\alpha \wedge \omega + \alpha d\omega$. Therefore a vector field V for which the dual one-form ω satisfies $d\omega = \eta \wedge \omega$ for some one-form η is also called conformally closed, cf. [KR97b].

Lemma 2.5. *Let V be a closed conformal vector field of the pseudo-Riemannian manifold (M,g) for which $g(V,V)$ does not vanish. Then the vector field V is a parallel vector field of the conformally equivalent metric $\overline{g} = \alpha g$, with $\alpha = |g(V,V)|^{-1}$.*

Further notions: A vector field is called complete if the flow is globally defined as a 1-parameter group of diffeomorphisms $\Phi\colon \mathbb{R} \times M \to M$. In the particular case of a gradient field $V = \mathrm{grad}\ f$ we have $\mathcal{L}_V g = 2\nabla^2 f$, hence $\mathrm{grad}\ f$ is conformal if and only if $\nabla^2 f = \sigma \cdot g$ where $n \cdot \sigma = \Delta f = \mathrm{div}\ (\mathrm{grad}\ f)$ is the Laplacian. If the symbol $(\ \ )^\circ$ denotes the traceless part of a $(0,2)$-tensor, then $\mathrm{grad}\ f$ is conformal if and only if $(\nabla^2 f)^\circ \equiv 0$. This equation $(\nabla^2 f)^\circ = 0$ allows explicit solutions in many cases, for Riemannian as well as for pseudo-Riemannian manifolds, see the discussion in Section 7 below. A vector field V is called concircular if the local flow (Φ_t) consists of concircular mappings, i.e. conformal mappings preserving geodesic circles. A transformation of the metric $g \mapsto \bar{g} = \frac{1}{\psi^2} g$ is concircular if and only if $(\nabla^2 \psi)^\circ = 0$, see [Ta'65], equivalently if $\mathrm{Ric}^\circ_{\bar{g}} = \mathrm{Ric}^\circ_g$, see [KR95a].

We introduce some notation: As usual,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Z \tag{4}$$

denotes the (Riemann) curvature $(1,3)$-tensor. Then the Ricci tensor as a symmetric $(0,2)$-tensor is defined by the equation $\mathrm{Ric}(X,Y) = \mathrm{trace}\,\{V \mapsto R(V,X)Y\}$.

The associated $(1,1)$ tensor is denoted by ric where $\mathrm{Ric}(X,Y) = g\,(\mathrm{ric}(X),Y)$. Then $S = \mathrm{trace}\,\{V \mapsto \mathrm{ric}(V)\}$ is the scalar curvature. Then the Schouten tensor P (as a $(0,2)$-tensor) is defined by

$$P = \frac{1}{n-2}\left(\frac{S}{2(n-1)}g - \mathrm{Ric}\right).$$

The Kulkarni Nomizu product $g * h$ of symmetric $(0,2)$-tensors g, h :

$$\begin{aligned} g * h(X,Y,Z,T) := & \; g(X,T)h(Y,Z) + g(Y,Z)h(X,T) \\ & - g(X,Z)h(Y,T) - g(Y,T)h(X,Z) \end{aligned}$$

is a $(0,4)$-tensor with the algebraic symmetry properties of the curvature tensor.

The Weyl tensor (also called conformal curvature tensor) is defined by the equation (cf. [Be'87, ch.1G])

$$R = g * P + W. \tag{5}$$

The Weyl tensor is the totally tracefree part of the Riemannian curvature tensor. In dimension $n \geq 4$ the Weyl tensor vanishes if and only if the manifold is conformally flat. A manifold is conformally flat, if every point has a neighborhood which is conformally equivalent to an open subset of pseudo-Euclidean space. If h is a symmetric $(0,2)$-tensor, then the *exterior derivative* dh equals the skew-symmetrization of the covariant derivative ∇h, i.e.

$$dh(X,Y,Z) := (\nabla_X h)(Y,Z) - (\nabla_Y)h(X,Z).$$

In dimension $n = 3$ W vanishes identically. To detect conformal flatness also in dimension $n = 3$ one introduces the $(1,2)$–Cotton tensor $C = dP$. Then we have in dimension 3 that C vanishes if and only if the manifold is conformally flat. Let F be a $(0,4)$-tensor with the symmetries of the curvature operator, then we define the *divergence* $\mathrm{div}_r F; 1 \leq r \leq 4$

$$\mathrm{div}_r F(X_1,X_2,X_3) := \mathrm{trace}\,\{(V,W) \mapsto \nabla_V F(X_1,\ldots,X_{r-1},W,X_{r+1},\ldots,X_3)\} \tag{6}$$

In particular we write $\mathrm{div} = \mathrm{div}_4$. The second Bianchi identitiy implies the following relations, cf. [Be'87, (16.D)]:

$$\mathrm{div}R = d\,\mathrm{Ric}\,;\; \mathrm{div}W = (n-3)\,d\,P = (n-3)\,C. \tag{7}$$

Notation: Throughout this paper we also use the notation $\langle X,Y\rangle$ instead of $g(X,Y)$ if there is no danger of confusion which metric tensor g is referred to.

3. Flat and conformally flat spaces

Conformal geometry was first studied for the flat Euclidean space and its pseudo-Euclidean analogue, compare Liouville's theorem from 1850. In the case of Euclidean space $\mathbb{E}^n$ there are the following key examples of complete conformal vector fields

1. the radial vector field $V_1(x) = x$,
2. the constant vector field $V_2(x) = x_0$.

The corresponding 1-parameter groups of conformal diffeomorphisms are

1. $\Phi_t^{(1)}(x) = e^t \cdot x$,
2. $\Phi_t^{(2)}(x) = x + t \cdot x_0$,

respectively. On the conformal compactification $S^n = \mathbb{E}^n \cup \{\infty\}$ with the standard conformal structure these two vector fields are essential meaning that they are not isometric with respect to any conformally equivalent metric. V_1 has two zeros at $0, \infty$. It is the gradient of a globally defined function on the sphere whereas V_2 is not a gradient and has only one zero at ∞. The standard metric on S^n is characterized by the existence of a conformal gradient field gradφ such that $\nabla_g^2\varphi + c^2\varphi \cdot g = 0$, see [Ta'65], [Ob'62] and Section 7 below. Any simply connected and conformally flat Riemannian manifold M of dimension n admits a conformal immersion $\delta\colon M \to S^n$, see below. Vice versa, for getting examples of conformally flat spaces one can take the preimage under δ of any open subset $A \subset S^n$ or its universal covering. This includes the example $\mathbb{R} \times H^{n-1}$ as the covering of $S^n \setminus S^{n-2}$, called a *Mercator-manifold* in [KP'94].

We denote by $\mathbb{E}_k^n$ the pseudo-Euclidean space with the metric $g = -\sum_{i\le k} dx_i^2 + \sum_{i>k} dx_i^2$. A pseudo-Riemannian manifold of the same signature is called (locally) conformally flat if it is locally conformally equivalent to $\mathbb{E}_k^n$.

Lemma 3.1. (Brinkmann [Br'23])*:*
For any conformally flat pseudo-Riemannian manifold (M_k^n, g) there exists locally an isometric immersion into $\mathbb{E}_{k+1}^{n+2}$.

Proof: Locally the metric has the form $\varphi^2(-\sum_{i\le k} dx_i^2 + \sum_{i>k} dx_i^2)$ where $x_1, \dots, x_n$ are cartesian coordinates and $\varphi \neq 0$ is a scalar function. Let $\langle x, x\rangle$ denote the pseudo-Euclidean scalar product of the point $x = (x_1, \dots, x_n)$. We define the following mapping

$$x \mapsto y = (y_0, \dots, y_{n+1}) := \Big(\tfrac{\varphi}{2}\big(\langle x, x\rangle + 1\big), \varphi x_1, \dots, \varphi x_n, \tfrac{\varphi}{2}\big(\langle x, x\rangle - 1\big)\Big).$$

Then the following conditions are easily checked:

1. $(y_0, \dots, y_{n+1}) \neq (0, \dots, 0)$,
2. y lies in the null cone $\{y \mid \langle y, y\rangle = 0\}$,
3. the induced metric of this immersion is
   $-\sum_{i\le k} dy_i^2 + \sum_{i>k} dy_i^2 = \varphi^2\big(-\sum_{i\le k} dx_i^2 + \sum_{i>k} dx_i^2\big)$.

The sphere inversion appears essentially as the mapping $y_{n+1} \mapsto -y_{n+1}$. Note that the Riemannian case $k = 0$ is included; in this case the image does not meet the hyperplane $y_0 = 0$. However, the ambient space is $\mathbb{E}^{n+2}_1$.

With respect to the pseudo-Euclidean metric, the mapping $x \mapsto y$ is conformal in any case, independent of φ. This motivates the following definition of a conformal development map into the real projective space $\mathbb{R}P^{n+1}$.

Definition 3.2. ([Ku'49], see also [AD'89]) *The* conformal development map *on a conformally flat pseudo-Riemannian manifold* (M^n_k, g) *is defined locally by* $x \mapsto y \mapsto [y_0, \dots, y_{n+1}] \in \mathbb{R}P^{n+1}$.

If M is simply connected this induces a conformal immersion $\delta\colon M \to Q^n_k \subset \mathbb{R}P^{n+1}$, the conformal development. Here Q^n_k denotes the projective quadric $\{y \mid \langle y, y\rangle = 0\}$. Q^n_k can also be regarded as the conformal compactification of $\mathbb{E}^n_k$. One observes that $\delta(\mathbb{E}^n_k) = \{[y_0, y, y_{n+1}] \in Q \mid y_{n+1} \neq y_0\}$ where the equation $\{y_{n+1} = y_0\}$ describes the 'points at infinity'. The quadric Q^n_k is diffeomorphic with $\{\langle y, y\rangle = 0\} \cap S^{n+1} \cong S^k \times S^{n-k}$ modulo the identification of antipodal pairs of points. Topologically, Q^n_k can also be regarded as a sphere bundle over $\mathbb{R}P^k$ if $k \leq n - k$ cf. [CK'82], a Euclidean model is the tensor product $S^k \otimes S^{n-k} \subset \mathbb{R}^{(k+1)(n-k+1)}$. In the special case of Minkowski 4-space $\mathbb{R}^4_1$ the Lie group $U(2)$ can be considered as the conformal compactification of $u(2)$ via the *Cayley map* $\delta\colon u(2) \to U(2)$, $\delta(x) = (1 + x)(1 - x)^{-1}$, see [BP'85].

Lemma 3.3. *The conformal transformations of the projective quadric* Q^n_k *are in* 1-1-*correspondence with those projective transformations of* $\mathbb{R}P^{n+1}$ *preserving* Q^n_k.

This lemma is essentially due to Möbius for the classical case $k = 0, n = 2$ (compare the *Möbius geometry*). For arbitrary dimensions it is stated in [Ku'49].

Theorem 3.4. (Kuiper [Ku'49]*):*

If M *is simply connected and conformally flat then* $\delta\colon M \to Q^n_k$ *is globally defined. If moreover* M *is compact then* δ *is either a diffeomorphism between* M *and* S^n *(if* $k = 0$*) or a two-fold covering (if* $2 \leq k \leq n-2$*). For* $k = 1$ *or* $k = n-1$ *the universal covering is non-compact.*

A conformally flat manifold M is called developable if the conformal development map δ is globally defined. Any simply connected conformally flat manifold is developable.

Examples The *key examples of conformal vector fields* on pseudo-Euclidean space are again the vector fields V_1 and V_2 above, extended to Q^n_k by taking limits of the flow.

The fixed points of the flow $\Phi^{(1)}_t$ are the two isolated points $[1, 0, -1]$ (the origin) and $[1, 0, 1]$ (its image under the inversion at the unit sphere) and the null cone at infinity $\{0\} \times Q^{n-2}_{k-1} \times \{0\}$.

The fixed point set of $\Phi^{(2)}_t$ depends on the type of the translation vector x_0: If $\langle x_0, x_0\rangle \neq 0$ then the only fixed point is $[1, 0, 1] = \infty$. This is a perfect analogue of the conformal flow on the standard sphere with one fixed point. If $\langle x_0, x_0\rangle = 0$

this is different. In this case let I denote the conformal inversion $I(x) = \frac{x}{|x|^2}$. Then the conjugation $I \circ \Phi_t^{(2)} \circ I$ of the 1-parameter group $\Phi_t^{(2)}$ leads to the vector field $V_3(x) = \frac{d}{dt}|_{t=0}\big(I \circ \Phi_t^{(2)} \circ I\big) = -2\langle x, x_0\rangle x + \langle x, x\rangle x_0$. This is a third type where the gradient of the conformal factor σ satisfying $\mathcal{L}_{V_3} g = 2\sigma g$ is a parallel and isotropic vector. The fixed point set is one isotropic line in the null cone. The flat metric of Minkowski 4-space in coordinates (u, v, x, y) can be written as $g = -2dudv + dx^2 + dy^2$. If we choose especially the isotropic translational vector $x_0 = \frac{1}{2}\partial_v$ then this vector field V_3 takes the form $V_3(u, v, x, y) = \big(u^2, \frac{1}{2}(x^2 + y^2), ux, uy\big)$. This is the standard type of a so-called *special conformal vector field*, see Section 8.

Corollary 3.5. *On the conformal compactification Q_k^n there exists a conformal vector field $\overline{V}_2$ with one zero, and on $Q_k^n \setminus Q_{k-1}^{n-2}$ there exists a conformal vector field $\overline{V}_1$ with two zeros. These vector fields are essential and complete. $\overline{V}_1$ is a local gradient field, $\overline{V}_2$ is not a gradient field near the zero.*

$Q_k^n \setminus Q_{k-1}^{n-2}$ is nothing but the union of $\delta(\mathbb{E}^n)$ and its image under inversion at the 'unit sphere'. This inversion transforms $\overline{V}_1$ into $-\overline{V}_1$. This space $Q_k^n \setminus Q_{k-1}^{n-2}$ is not simply connected. In fact, its fundamental group is isomorphic to the integers $\mathbb{Z}$ if $2 \le k \le n-2$, leading to a $\mathbb{Z}$-sheeted universal covering which carries a conformal vector field with infinitely many zeros.

Corollary 3.6. [KR95]

For $2 \le k \le n-2$ the universal covering of $Q_k^n \setminus Q_{k-1}^{n-2}$ defines a manifold $M(\mathbb{Z})$ together with a conformal structure such that $\delta\colon M(\mathbb{Z}) \to Q_k^n \setminus Q_{k-1}^{n-2}$ becomes a conformal covering. The conformal vector field $\overline{V}_1$ can be lifted to a vector field $V_1^{(\mathbb{Z})}$ with infinitely many zeros. These zeros are in natural bijection to $(2\mathbb{Z}) \cup (2\mathbb{Z}+1) \cong \mathbb{Z}$. Similarly, there are intermediate coverings with any even number of zeros of the vector field.

For $2 \le k \le n-2$ the universal covering of the quadric Q_k^n itself is diffeomorphic to $S^k \times S^{n-k}$. The metric can be chosen as the product of two metrics of constant curvature with opposite signs. This space carries a conformal vector field $V_2^{(2)}$ with two zeros as the lift of $\overline{V}_2$ via the conformal covering $\delta\colon S^k \times S^{n-k} \to Q_k^n$. Even if we remove one of the zeros, the vector field is still complete. The punctured $S^k \times S^{n-k}$ carries a complete conformal vector field with one zero.

In a neighborhood of a zero of a conformal gradient field the metric is conformally flat, see Section 7.

4. The Riemannian case

In the case of a Riemannian manifold any conformal vector field without zeros can be made into an isometric vector field by a conformal change, see Lemma 2.1. Such a field is called inessential, otherwise it is essential. Since much is known about the isometry groups and Killing fields, it is here more interesting to study essential conformal vector fields, that is, conformal vector fields which never become isometric under a global conformal change of the metric.

Theorem 4.1. *(Essential conformal vector fields)*

1. (Alekseevskii [Al'72], Ferrand [Fe'77],[Fe'96], Yoshimatsu [Yo'76])
Assume that (M, g) is a Riemannian manifold of dimension n admitting a complete and essential conformal vector field. Then (M, g) is conformally diffeomorphic with either the standard sphere S^n or with the Euclidean space $\mathbb{E}^n$.

2. (Obata [Ob'71], Lelong-Ferrand [LF'71], Lafontaine [La'88])
Assume that (M, g) is a compact Riemannian manifold of dimension n admitting an essential conformal vector field. Then (M, g) is conformally diffeomorphic with the standard sphere S^n.

Three key steps in the proof are the following:

1. The zeros of the vector field are isolated.
2. In a neighborhood of a zero the manifold is conformally flat.
3. The conformal development map $\delta : M \to S^n$ is injective.

Several steps in the proof were made more precise in various papers, so the result cannot really be attributed to a single person, compare [Gu'95]. The case of a complete manifold carrying a complete and closed essential conformal vector field was solved by Bourguignon [Bo'70]. No analogous result seems to be known yet in the case of a pseudo-Riemannian manifold with an indefinite metric. It is a conjecture that a compact and pseudo-Riemannian manifold carrying an essential conformal vector field is conformally flat, it is named *Lichnerowicz' conjecture* in [Fs'05].

The situation with respect to inessential conformal vector fields is totally different, even in the compact case and even under additional curvature restrictions.

Example 4.2. [Ej'81] For any n there is a compact Riemannian n-manifold of constant scalar curvature admitting a conformal vector field without zeros.

The simplest example of this kind for $n = 4$ is the product $S^1 \times S^3$ with the warped product metric $g = dt^2 + (2 + \cos t)g_1$ where g_1 is the standard metric on the unit sphere. In this case the vector field $V = \sqrt{2 + \cos t}\ \partial_t$ is conformal (and inessential), see [De'80, p.277]. There are similar examples $g = dt^2 + (f(t))^2 g_*$ in any dimension, with a periodic warping function f which can be explicitly given. It has to satisfy the ODE $n\rho f^2 + (n-2)f'^2 + 2ff'' = (n-2)\rho_*$ where ρ, ρ_* are the constant (normalized) scalar curvatures of g, g_*, respectively. These examples can be extended to the case of a pseudo-Riemannian metric, see [KR97a].

5. Conformal Transformations of Einstein Spaces

Definition 5.1. *A pseudo-Riemannian manifold of dimension $n \geq 3$ is called an* Einstein space *if the Ricci tensor is a (necessarily constant) multiple of the metric tensor. In this case the metric is called an* Einstein metric*, and in the equation*

$$Ric = \lambda g$$

the factor λ is called the Einstein constant*. Hence $\lambda = S/n$ where S is the scalar curvature.*

In general relativity the case $\lambda = 0$ is precisely the case where the Einstein field equations hold for the vacuum. 4-dimensional Einstein spacetimes with non-vanishing Einstein constants are the de Sitter space and the anti-de Sitter space. Kasner started in the early 20s [Ks21a, Ks21b] an investigation about conformal changes of Ricci flat metrics. In more generality, one can ask what happens to Einstein metrics under conformal change.

Lemma 5.2. *The following formulae hold for any conformal change $g \mapsto \overline{g} = \varphi^{-2}g$:*

$$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = \varphi^{-2}\Big((n-2)\cdot\varphi\cdot\nabla^2\varphi + \Big[\varphi\cdot\Delta\varphi - (n-1)\cdot||\nabla\varphi||^2\Big]\cdot g\Big). \tag{8}$$

Moreover, if V is a conformal vector field with $\mathcal{L}_V g = 2\sigma g$ then the formula

$$\mathcal{L}_V \mathrm{Ric} = -(n-2)\nabla^2\sigma - \Delta\sigma\cdot g \tag{9}$$

holds and the following conditions are eqivalent:

1. $\mathcal{L}_V \mathrm{Ric} = \theta g$ *for a certain function* θ
2. $\mathrm{grad}(\mathrm{div} V)$ *is conformal*
3. $(\nabla^2\sigma)^\circ = 0$

The first equation follows from the relationship between the two Levi-Civita connections $\nabla, \overline{\nabla}$ associated with g and $\overline{g}$:

$$\overline{\nabla}_X Y - \nabla_X Y = -X(\log\varphi)Y - Y(\log\varphi)X + g(X,Y)\mathrm{grad}(\log\varphi).$$

Corollary 5.3. *The Einstein property of a metric is in general not preserved under conformal changes. If g is an Einstein metric then the conformally transformed metric $\overline{g} = \varphi^{-2}g$ is Einstein if and only if*

$$(\nabla^2\varphi)^\circ = 0,$$

that is, if the Hessian of φ is a scalar multiple of the metric tensor.

This equation was already analyzed by Brinkmann [Br'24],[Br'25] in the 1920s. He was the first who proved that in the case $g(\mathrm{grad}\varphi, \mathrm{grad}\varphi) \neq 0$ the metric g is a warped product. Furthermore, he proved that in the case $g(\mathrm{grad}\varphi, \mathrm{grad}\varphi) = 0$ the metric has a specific form carrying a parallel isotropic vector field (now called a Brinkmann space) which in dimension four became later important in physics as a *pp*-wave, compare [Si'74]. For solutions of the equation $(\nabla^2\varphi)^\circ = 0$ see Section 7 below.

Corollary 5.4. *Assume that an Einstein space carries a conformal vector field V which is not homothetic or isometric. Then it carries also a conformal gradient field, namely, the gradient of* $\mathrm{div}V$*. This gradient field does not vanish identically but it can happen that it is a parallel isotropic vector field, hence isometric.*

Theorem 5.5. (Brinkmann [Br'25])
Assume that (M, g) is an Einstein space of dimension $n \geq 3$ admitting a non-constant solution f of the equation $(\nabla^2 f)^\circ = 0$. Then the following hold:

- *(a) Then around any point p with $g(\mathrm{grad}f(p), \mathrm{grad}f(p)) \neq 0$ the metric tensor is a warped product $g = \eta dt^2 + (f'(t))^2 g_*$ where $\mathrm{grad}f = f'\frac{\partial}{\partial t}$, $\eta = \pm 1$ and where the $(n-1)$-dimensional Einstein metric g_* does not depend on t. Moreover, f satisfies the ODE $f'' + \rho\eta f = 0$ where ρ denotes the normalized Einstein constant (such that $\rho = 1$ on the unit sphere in any dimension).*
- *(b) Furthermore, if $g(\mathrm{grad}f, \mathrm{grad}f) = 0$ on an open subset then $\mathrm{grad}f$ is a parallel isotropic vector field on that subset, and the metric tensor can be brought into the form $g = dudv + g_*(u)$ where $\mathrm{grad}f = \frac{\partial}{\partial u} = \mathrm{grad}v$ and where the $(n-2)$-dimensional metric $g_*(u)$ is Ricci flat for any fixed u and does not depend on v. Consequently g itself must be Ricci flat. These coordinates $u, v, x_i; i = 1, 2, \ldots, n-2$ are sometimes called* Rosen coordinates.

Corollary 5.6. *In dimension $n = 4$ any Einstein space is of constant sectional curvature if it admits either a non-trivial conformal mapping onto some other Einstein space or if it admits a non-homothetic conformal vector field V such that* $\mathrm{grad}(\mathrm{div}V)$ *is not parallel and isotropic. For a Riemannian 4-manifold the latter case cannot occur, and for a Lorentzian 4-manifold the latter case is the case of a Ricci flat pp-wave (or vacuum pp-wave).*

This result is due to Brinkmann [Br'25], compare also [GT'87]. See Section 8 below for a further discussion of the 4-dimensional Lorentzian case.

Theorem 5.7. (Yano and Nagano [YN'59])
Assume that a compact Riemannian Einstein space admits a non-homothetic conformal vector field. Then it is conformally diffeomorphic with the standard sphere.

This follows essentially from Theorem 7.7 since on a compact space the gradient of the divergence must have a critical point. Then by Theorem 3.4 the manifold is a quotient of the standard sphere. The case of a covering can be easily excluded, so only the sphere itself is possible. In a similar way one obtains the following.

Theorem 5.8. (Kanai [Ka'83])

Assume that a complete Riemannian Einstein space admits a non-homothetic conformal vector field V with a critical point of div$V$. *Then it is of constant sectional curvature.*

Without any assumption on critical points or zeros of the vector fields, there are counterexamples in form of warped products $dt^2 + (\cosh t)^2 g_*$ where g_* is Einstein with $\rho_* = -1$ but not of constant sectional curvature, a fact implicitly contained in [Br'25]. compare [BK'78]. The case of a single conformal mapping into some Einstein space which is defined on a complete Einstein space is classified in [Kü'88, Thm.27]. Here the case of a warped product $g = dt^2 + e^{2t} g_*$ comes in with a complete and Ricci flat metric g_*.

Conformal vector fields on Riemannian Einstein spaces were classified by Kanai [Ka'83]. A pseudo-Riemannian analogue is more complicated since it has to include the case of *pp*-waves and generalizations, compare [Kc'91] and [KR97a][1].

The case of pseudo-Riemannian spaces of constant scalar curvature carrying non-isometric local gradient fields can also be classified, see [KR97a, Thm.4.3]. Here we obtain generalizations of Ejiri's example at the end of Section 4, all as warped product metrics. The possible warping functions can be explicitly determined.

6. Spaces which are conformally Einstein

We call a pseudo-Riemannian manifold (M, g) conformally Einstein if every point p has an open neighborhood U such that the conformally equivalent metric $(U, \overline{g} = \lambda^2 g)$ for some function $\lambda : U \to \mathbb{R}$ is an Einstein metric. As a direct consequence of Lemma 5.2 we obtain the following:

Proposition 6.1. *An n-dimensional pseudo-Riemannian manifold (M, g) admits a conformal mapping onto an Einstein space $(M, \overline{g})$ with $\overline{g} = \varphi^{-2} g$ if and only if the factor φ satisfies the following equation*

$$\varphi \cdot \mathrm{Ric}^\circ + (n-2)\big(\nabla^2\varphi\big)^\circ = 0 \tag{10}$$

It seems that Brinkmann [Br'24] was the first who discussed this equation, which is also called *conformal Einstein equation* . Its integration is surprisingly difficult. In dimension 2 the equation is trivial. In higher dimensions the equation implies that the eigenspaces of $\nabla^2\varphi$ must coincide with the given eigenspaces of Ric. Furthermore the eigenvalues of $\nabla^2\varphi$ are determined by the eigenvalues of Ric and by φ itself.

We rewrite this equation for the function $\phi = \log\varphi$ using the following

[1]In Theorem 3.2 of this paper the proof has a gap, as kindly pointed out to the authors by Helga Baum. So possibly one case in the classification there was missing.

Definition 6.2. *For a vector field V on a pseudo-Riemannian manifold we define the* Schwarzian tensor *as the following traceless $(0,2)$ tensor:*

$$F_V(X,Y) := g(\nabla_X V, Y) + g(X,V)g(Y,V) - \frac{1}{n}\left\{\mathrm{div}V + \|V\|^2\right\} g(X,Y)\,. \tag{11}$$

Then one can rewrite the conformal Einstein equations with the help of the Schwarzian tensor as follows, see [Li'01]:

Proposition 6.3. *An n-dimensional pseudo-Riemannian manifold (M,g) admits a conformal mapping onto an Einstein space $(M,\overline{g})$ with $\overline{g} = \exp(-2\phi)g$ if and only if the factor ϕ satisfies the following equation*

$$\mathrm{Ric}^\circ = -(n-2)F_{\mathrm{grad}\phi} \tag{12}$$

It is an important question treated by many authors to characterize conformally Einstein manifolds by tensorial equations. In dimension four the Bach equation [Ba'21] is only a necessary condition, there are Bach flat spaces which are not conformally Einstein, cf. [NP'01]. Tensorial conditions in dimension 4 for certain classes of metrics are discussed for example in Szekeres [Sz'63], Kozameh, Newman and Tod [KN'85]. Extensions to arbirtrary dimensions are due to Listing [Li'01],[Li'06], Gover and Nurowski [GN'06]. Under a suitable non-degeneracy assumption for the Weyl curvature we present tensorial equations providing necessary and sufficient conditions for the metric to be conformally Einstein following [Li'01] and [GN'06].

A pseudo-Riemannian manifold (M,g) of dimension $n \geq 4$ has an harmonic Weyl tensor (or is called a *C-space*) if the divergence of the Weyl tensor vanishes, i.e. $\mathrm{div}W = 0$.

Proposition 6.4. *A pseudo-Riemannian Einstein manifold of dimension $n \geq 4$ has an harmonic Weyl tensor.*

Proof. Since $\mathrm{Ric} = (S/n)g$ and since the scalar curvature is constant the Schouten tensor P is parallel, therefore the Cotton tensor $C = dP$ as the skew-symmetrization of ∇P vanishes. We conclude from Equation 7 that $\mathrm{div}W = 0$. □

Let $\overline{g} = \varphi^{-2}g$ be a conformally equivalent metric and let $\varphi = \exp(\phi)$ and denote by $\overline{W}, \overline{\mathrm{div}W}$ the Weyl tensor resp. its divergence with respect to the metric $\overline{g}$. Then the conformal behaviour of the divergence of the Weyl tensor is given by the following equation, cf. [Li'01, Lem.1].

$$\left(\overline{\mathrm{div}}\ \overline{W}\right)(X,Y,Z) = (\mathrm{div}W)\,(X,Y,Z) + (3-n)W(X,Y,Z,\mathrm{grad}\phi) \tag{13}$$

If now $\overline{g} = \exp(-2\phi)\,g$ is an Einstein metric we conclude from Proposition 6.4 that

$$\mathrm{div}_4 W(.,.,.) + (3-n)W(.,.,.,\mathrm{grad}\phi) = 0\,. \tag{14}$$

Let $(E_1, E_2, \ldots, E_n)$ be a pseudo-orthonormal frame with $g(E_i,E_j) = \epsilon_i\delta_{ij}$ and $\epsilon_i \in \{\pm 1\}$. For a $(0,2)$-tensor h we denote by $W[h]$ the following $(0,2)$ tensor:

$$W[h](X,Y) = \sum_{i,j} \epsilon_i\epsilon_j W(E_i,X,Y,E_j)h(E_j,E_i)$$

If we take the divergence div_1 with respect to the first argument of Equation 14 we obtain:

$$\mathrm{div}_1 \mathrm{div}_4 W + (3-n) W[\nabla^2 \phi] - (n-3)^2 W[d\phi \otimes d\phi] = 0$$

Since for any function f we have $W[f\, g] = 0$ we conclude from the conformal Einstein equation 12

$$\mathrm{div}_1 \mathrm{div}_4 W + \frac{n-3}{n-2} W[\mathrm{Ric}] - (n-3)(n-4) W[d\phi \otimes d\phi] = 0 \tag{15}$$

Note that $W[\mathrm{Ric}] = W[\mathrm{Ric}^\circ] = W[(n-2)P^\circ] = .$ Equation 15 motivates the following

Definition 6.5. *For a pseudo-Riemannian manifold (M, g) of dimension $n \geq 4$ one defines the* Bach tensor

$$B = \mathrm{div}_1 \mathrm{div}_4 W + \frac{n-3}{n-2} W[\mathrm{Ric}]$$

Hence we obtain part (a) of the following theorem presenting a necessary condition for a metric to be conformally Einstein. In the 4-dimensional case this result can be found in [KN'85, Thm.2], for arbitrary dimension in [Li'01]. Following [GN'06] in part (b) a sufficient condition is formulated. The metric is called *weakly generic* in [GN'06] if $W(V, ., ., .) = 0$ holds if and only if $V = 0$, i.e. the Weyl tensor viewed as map $TM \to \bigotimes^3 TM$ is injective.

Theorem 6.6. *Let (M, g) be a pseudo-Riemannian manifold.*

(a) ($n = 4$: [KN'85, Thm.2]; $n \geq 4$: [Li'01, Rem.2])
If (M, g) is conformally Einstein such that $\overline{g} = \exp(-2\phi)\, g$ is Einstein then the Cotton tensor C and the Bach tensor B satisfy the following equations:

$$C - W(., ., ., \mathrm{grad}\phi) = 0 \tag{16}$$
$$B + (3-n)(n-4) W\,[d\phi \otimes d\phi] = 0 \tag{17}$$

(b) ($n = 4$: [KN'85, Thm.2] $n \geq 4$: [GN'06, Thm.2.2])
Let the metric be weakly generic and let for some vector field V with dual one form $V^\#$ the following equations be satisfied:

$$C - W(., ., ., V^\#) = 0 \tag{18}$$
$$B + (3-n)(n-4) W\left[V^\# \otimes V^\#\right] = 0 \tag{19}$$

then the metric is conformally Einstein. The vector field $V = \mathrm{grad}\phi$ is locally a gradient field and $\overline{g} = \exp(-2\phi)\, g$ is an Einstein metric.

If we consider the Weyl tensor as endomorphism $\mathcal{W} : \Lambda^2(T^*M) \to \Lambda^2(T^*M)$ then Listing [Li'01] calles the Weyl tensor non-degenerate if $\mathcal{W}$ has maximal rank. Then he defines the vector field

$$\mathbb{T} := \frac{1}{n-3} \sum_{i,k=1}^{n} \epsilon_i \epsilon_k \mathcal{W}^{-1} \left[\mathrm{div} W(., ., E_i)\right](E_i, E_k) E_k\, . \tag{20}$$

Using the conformal Einstein equations written in Proposition 6.3 with the help of the Schwarzian tensor one obtains the following tensorial characterization:

Theorem 6.7. ([Li'01, Thm.2], [GN'06, Prop.2.7])
A pseudo-Riemannian manifold (M, g) with non-degenerate Weyl tensor is locally conformally Einstein if and only if the the vector field $\mathbb{T}$ defined in Equation 20 satisfies:

$$\mathrm{Ric}^\circ + (n-2)F_{\mathbb{T}} = 0$$

This result can be extended to weakly generic metrics, cf. [Li'06] and it is shown in [GN'06, Prop.2.7] that the tensor field $G := \mathrm{Ric}^\circ + (n-2)F_{\mathbb{T}}$ is conformally invariant. It is used in [GN'06, Thm.2.10] to define a natural polynomial in the Riemannian curvature tensor and its covariant derivatives of conformal weight $2n(n-1)$ whose vanishing for a weakly generic metric characterizes conformally Einstein metrics. Conformally Einstein metrics can be characterized as conformal structures for which the *standard tractor bundle* admits a parallel section, cf. for example [GN'06, Sec.3]

One can also use the vector field $\mathbb{T}$ and Theorem 6.6(b) to define a generalized Bach tensor which is conformally covariant not only in dimension 4, cf. [Li'06]. For the rest of this section we consider the particular case of dimension four:
For a pseudo-Riemannian manifold of dimension 4 the definition of the Bach tensor B given in Definition 6.5 reads: $B = \mathrm{div}_1\mathrm{div}_4 W + (W[\mathrm{Ric}])/2$ and in index notation

$$B_{ij} = \sum_{k,l=1}^{4} \nabla^k \nabla^l W_{kijl} + \frac{1}{2}\sum_{k,l=1}^{4} R^{kl} W_{kijl} \,.$$

The Bach tensor in dimension 4 is a symmetric, trace free and divergence free tensor and it is conformally covariant, i.e. if $\overline{g} = f^{-2}\, g$ then $B_{\overline{g}} = f^2 B_g$. For a compact manifold it is the gradient of the functional

$$\mathcal{W}(g) = \int_M |W_g|^2 \; dV_g \tag{21}$$

We call a metric Bach-flat, if the Bach equation $B = 0$ is satisfied. Hence this equation is the Euler-Lagrange equation of the functional $\mathcal{W}$. In particular metrics which are locally conformal to an Einstein metric are Bach flat. In the Riemannian case half conformally flat metrics are also Bach flat, but they are not weakly generic.

As a consequence of Theorem 6.6 we obtain:

Corollary 6.8. [KN'85, Thm.2] *A pseudo-Riemannian and weakly generic manifold (M, g) of dimension 4 with non-degenerate Weyl tensor is conformally Einstein if and only if it is Bach-flat and conformally equivalent to a space with a harmonic Weyl tensor, i.e. if for some vector field V : $B = 0\,;\, C = W(.,.,.,V^\#)$.*

As an interesting class of conformally Einstein metrics in dimension 4 one can discuss products of surfaces, cf. [DS'00, ch. 18]. Here extremal metrics on surfaces play a particular role, we call a metric h on a surface S with Gaussian curvature

κ extremal, if $\nabla^2\kappa = \sigma h$ for some function σ, i.e. if gradκ is a conformal vector field on the surface. Then one can show that $2\sigma = \Delta\kappa$ and the so-called *first (classifying) parameter* c and the *second (classifying) parameter* p with

$$c := \Delta\kappa + \kappa^2; p := c\kappa - h(\mathrm{grad}\kappa, \mathrm{grad}\kappa) - \frac{\kappa^3}{3} \tag{22}$$

are constants. Let F be the cubic polynomial

$$F(\kappa) = c\kappa - p - \frac{\kappa^3}{3}\,, \tag{23}$$

then we conclude from Theorem 5.5 and the formula for the Gauss curvature, cf. [DS'00, Lemma 18.9]: If for some κ_1 $F(\kappa_1) \neq 0$ then we can introduce coordinates κ, θ in in a neigborhood of κ_1 such that the metric is of the following form:

$$\eta F(\kappa) d\theta^2 + \frac{1}{F(\kappa)} d\kappa^2\,, \tag{24}$$

with $\eta = \pm 1$. These metrics (in arbitrary dimension) were first introduced by Calabi in the Riemannian setting as critical Kähler metrics for certain curvature functionals, see [Be'87, ch. 11E]. As an equivalent characterization one can use that the gradient grad$s$ of the scalar curvature is a holomorphic vector field. Then one obtains the following

Proposition 6.9. *Let $(M^4, g) = (M_1^2, g_1) \times (M_2^2, g_2)$ be a product of two surfaces with a pseudo-Riemannian product metric $g = g_1 \oplus g_2$ whose scalar curvature s is nowhere vanishing. Then the pseudo-Riemannian manifold (M, g) is locally conformally Einstein if and only if both surface metrics are extremal and have the same first classifying parameter. The conformally equivalent Einstein metric $\overline{g}$ is uniquely determined up to a constant.*

Hence we can introduce coordinates $\kappa_i, \theta_i; i = 1, 2$ on (M_i, g_i) and classifying parameters c, p_1, p_2 with the corresponding cubic polynomials $F_i(\kappa_i) = c\kappa_i - p_i - \kappa_i^3/3, i = 1, 2$

$$g = \epsilon_1 F_1(\kappa_1) d\theta_1^2 + \frac{1}{F_1(\kappa_1)} d\kappa_1^2 + \epsilon_1 F_2(\kappa_2) d\theta_2^2 + \frac{1}{F_2(\kappa_2)} d\kappa_2^2 \tag{25}$$

Then the conformally equivalent Einstein metric is given by

$$\overline{g} = \frac{4}{S^2}\, g = \frac{1}{(\kappa_1 + \kappa_2)^2}\, g \tag{26}$$

and has scalar curvature $\overline{S} = 3(p_1 + p_2)$. These examples can be found in several papers, cf. for example [Wo'43]. They also occur in the context of *conformal gravity* where the field equation $B = 0$ is considered, cf. [FS'80],[DS'00].
Note that the Schwarzschild metric is a Ricci flat metric which is conformally equivalent to the product of two surface metrics one of which has constant Gaussian curvature.

7. Conformal gradient fields

This section deals with conformal gradient vector fields

$$V = \mathrm{grad} f$$

and simultaneously with vector fields which are integrable (or closed), so that they are locally gradient fields.This means that for every point $p \in M$ there is a neighborhood U and a function $f \in C^\infty(U)$ such that $V = \mathrm{grad} f$. It follows that $\mathrm{grad} f$ is conformal if and only if the *Hessian* $\nabla^2 f(X,Y) := \langle \nabla_X \mathrm{grad} f, Y\rangle$ satisfies the equation

$$\nabla^2 f = \sigma g \tag{27}$$

since $\mathcal{L}_{\mathrm{grad} f} g(X,Y) = 2\nabla^2 f(X,Y)$. The Laplacian Δf is the divergence of the gradient of f, so in the equation above the factor σ is nothing but the Laplacian, divided by the dimension:

$$\nabla^2 f = \frac{\Delta f}{n} g \tag{28}$$

From Equation 27 we obtain the following Ricci identity for the *curvature tensor* introduced in Equation 4

$$R(X,Y)\mathrm{grad} f = X(\sigma)Y - Y(\sigma)X\,. \tag{29}$$

By contraction we obtain for the Ricci tensor:

$$\mathrm{Ric}(X, \mathrm{grad} f) = (1-n)X(\sigma)\,. \tag{30}$$

It turns out that one can integrate equation 27 by reducing it to an ODE whenever the gradient of f is not isotropic. This can be done along the lines of Brinkmann's results [Br'25]. The following lemma was stated by Fialkow [Fi'39, p.471].

Lemma 7.1. *Let (M,g) be a pseudo–Riemannian manifold. Then the following conditions are equivalent:*

(1) There is a non–constant solution f of the equation $\nabla^2 f = \frac{\Delta f}{n} g$ in a neighborhood of a point $p \in M$ with $\langle \mathrm{grad} f(p), \mathrm{grad} f(p)\rangle \neq 0$.

(2) There is a neighborhood U of p, a C^∞–function $f : (-\epsilon,\epsilon) \to \mathbb{R}$ with $f'(t) \neq 0$ for all $t \in (-\epsilon,\epsilon)$ and a pseudo–Riemannian manifold (M_*, g_*) such that (U,g) is isometric to the warped product

$$\left((-\epsilon,\epsilon)\ ,\ \eta dt^2\right) \times_{f'} (M_*, g_*)) = \left((-\epsilon,\epsilon) \times M_*\ \ ,\ \ \eta dt^2 + f'(t)^2 g_*\right)$$

where $\eta := \mathrm{sign}\langle \mathrm{grad} f(p), \mathrm{grad} f(p)\rangle \in \{\pm 1\}$.

Proof. (2) $\Rightarrow$ (1): Define the function $f : (-\epsilon,\epsilon) \times M_* \to \mathbb{R}$ by $f(t,x) = f(t)$. Then $\mathrm{grad} f(t,x) = f'(t)\cdot\eta\cdot\partial_t$ and $\nabla_{\partial_t}\mathrm{grad} f = f''(t)\cdot\eta\cdot\partial_t$. Let X be a lift of a vector field on M_*, then by Equation 27 we have $\nabla_X \mathrm{grad} f = f''\cdot\eta\cdot X$.

(1) $\Rightarrow$ (2) : Let U be a neigborhood of $p \in M$ with compact closure and with $\langle \mathrm{grad} f(q), \mathrm{grad} f(q)\rangle \neq 0$ for all $q \in U$. Hence $c = f(p)$ is a regular value, let M_* be

the connected component of $f^{-1}(c)$ containing p. Then there is an $\epsilon > 0$ such that the normal exponential map $\exp^{\perp} : (-\epsilon, \epsilon) \times M_* \to M$ defines a diffeomorphism onto the image. Let $q \in U$, $g(X, \mathrm{grad} f(q)) = 0$, then it follows immediately that

$$Xg(\mathrm{grad} f, \mathrm{grad} f) = 2\frac{\Delta f}{n} g(\mathrm{grad} f, X) = 0\,. \tag{31}$$

Hence $\langle \mathrm{grad} f, \mathrm{grad} f\rangle$ is constant along the level hypersurfaces $f^{-1}(c')$ and the level hypersurfaces $f^{-1}(f(\exp(t, x_0)))$, $t \in (-\epsilon, \epsilon)$ are parallel. Therefore they coincide with the t–levels and f can be regarded as a function only of t, written as $f(t, x) = f(t)$ by slight abuse of notation and $\mathrm{grad} f(t, x) = f'(t) \cdot \eta\partial_t$ as well as

$$\nabla^2 f = 2f''\eta g = \frac{\Delta f}{n}\, g\,. \tag{32}$$

The equation $g(\partial_t, \partial_t) = \eta = \mathrm{sign}\langle \mathrm{grad} f(p), \mathrm{grad} f(p)\rangle$ follows since the curve $t \mapsto \exp(t\mathrm{grad} f(x))$ is a geodesic. Let X be a lift of a vector field on M_*, then $g(\partial_t, X) = 0$ by the Gauss Lemma. If X_1, X_2 are vectors tangential to M_* at x_0 and $X_i(t) = d\exp(t, x_0)(X_i), i = 1, 2$ then

$$\begin{aligned}
\frac{d}{dt}|_{t=s} g(X_1, X_2)(t) &= \mathcal{L}_{\partial_t} g(X_1, X_2)(s) = \frac{\eta}{f'(s)}\mathcal{L}_{\mathrm{grad} f} g(X_1, X_2)(s) = \\
\frac{2\eta}{f'(s)}\nabla^2_{X_1(s),X_2(s)} f &= 2\frac{f''(s)}{f'(s)} g(X_1, X_2)(s)\,.
\end{aligned}$$

The claim follows from the uniqueness of the solution of the ODE

$$((f')^{-2} g(X_1, X_2))'(t) = 0.$$

The metric g_* is non–degenerate since it is orthogonal to the time-like or space-like t-direction. ☐

Proposition 7.2. [Kb'76, Prop.2] [KR95]

Let V be a non–trivial closed conformal vector field on the n–dimensional pseudo–Riemannian manifold (M, g).

1. *If $V(p) = 0$, then* $\mathrm{div} V(p) = n \cdot \lambda(p) \neq 0$*, in particular all zeros of V are isolated.*

2. *Denote by $\mathcal{C} = \mathcal{C}(M, g)$ the vector space of closed conformal vector fields, then* $\dim \mathcal{C} \leq n + 1$.

If the dimension of the space of closed conformal vector fields is maximal, i.e., if $\dim C(M, g) = n + 1$, then the manifold is of constant sectional curvature.

Lemma 7.3. *Let ∂_t be the unit tangent vector in direction of the first factor of the product $I \times M_*$ and let X, Y, Z be lifts of vector fields on M_*. Here I denotes an open interval in $\mathbb{R}$. Denote by $\nabla^*, \mathrm{R}_*, \mathrm{Ric}_*, \rho_*$ the Levi–Civita covariant derivative, the Riemannian curvature tensor and the normalized scalar curvature of (M_*, g_*). (The* normalized scalar curvature *of the standard sphere with sectional curvature* 1 *is also* 1*). Then we have the following formulae for the corresponding geometric quantities $\nabla, \mathrm{R}, \mathrm{Ric}, \rho$ of the warped product metric $g = dt^2 + f^2(t)$:*

1. $\nabla_{\partial_t}\partial_t = 0$
$\nabla_{\partial_t}X = \nabla_X\partial_t = \frac{f'}{f}X$
$\nabla_X Y = -\frac{g(X,Y)}{f}\eta f'\partial_t + \nabla^*_X Y$

2. $\mathrm{R}(X,Y)Z = \mathrm{R}_*(X,Y)Z - \frac{f'^2}{f^2}\eta\{g(Y,Z)X - g(X,Z)Y\}$
$\mathrm{R}(X,Y)\partial_t = 0$
$\mathrm{R}(X,\partial_t)\partial_t = -\frac{f''}{f}X$

3. $\mathrm{Ric}(Y,Z) = \mathrm{Ric}_*(Y,Z) - \frac{\eta}{f^2}\{(n-2)f'^2 + f''f\}g(Y,Z)$
$\mathrm{Ric}(Y,\partial_t) = 0$
$\mathrm{Ric}(\partial_t,\partial_t) = -(n-1)\frac{f''}{f}$

4. $f^2\rho = \frac{n-2}{n}\rho_* - \frac{n-2}{n}f'^2\eta - \frac{2}{n}\eta f'' f$

This follows from the formulae for warped products in general, cf. [ON'83, ch.7] since

$$\nabla f = f'\eta\partial_t\,,\ \nabla^2{}_{\partial_t,\partial_t}f = g(\nabla_{\partial_t}\nabla f, \partial_t) = f''\,.$$

Note, however, that the curvature tensor in [ON'83] has the opposite sign. The formulae in the Riemannian case and the pseudo–Riemannian case coincide if we consider in the case $\eta = -1$ the warped product $\tilde g = dt^2 + f^2(t)\tilde g_*$, $\tilde g_* = -g_*$ which is anti–isometric to g (then $\tilde\rho = -\rho, \tilde\rho_* = -\rho_*, \ldots$). In particular we obtain as in the Riemannian case the

Corollary 7.4. *The warped product $(I, \eta dt^2)\times_f (M_*, g_*)$ is an Einstein metric (a metric of constant sectional curvature) if and only if g_* is an Einstein metric (a metric of constant sectional curvature) and $f'^2 + \rho\eta f^2 = \eta\rho_*$.*

Near a regular point of a function f satisfying $\nabla^2 f = \lambda g$ the metric has the structure of a warped product, cf. Lemma 7.1. Around a critical point we can use geodesic polar coordinates and obtain the following.

Proposition 7.5. [Kb'76], [KR95]
Let (M,g) be a pseudo–Riemannian manifold with a non–constant solution f of the equation $\nabla^2 f = \lambda g$ for a function λ and with a critical point $p \in M$.

1. (cf. [Ta'65], [Kü'88, lemma 18] in the Riemannian case) *Then there are functions $f_\pm$ such that the metric in geodesic polar coordinates $(r,x) \subset \mathbb{R}\times\Sigma$ in a neighborhood U of p has the form*

$$g(r,x) = \eta dr^2 + \frac{f'_\eta(r)^2}{f''_\eta(0)^2}\, g_1(x)\,;\ \eta = \langle x,x\rangle \tag{33}$$

and $f(r,x) = f_\eta(r), \lambda(r,x) = \lambda_\eta(r)$ with $\lambda_\eta(r) = \eta f''(r)$, in particular the metric is conformally flat in a neighborhood of the critical point.

2. *If all geodesics through p are defined on the whole real line $\mathbb{R}$ then the metric g is of the form above for all (r, x), as long as $f'_\eta(r)$ does not vanish.*

Definition 7.6. *We call a pseudo-Riemannian manifold with a conformal vector field C-*complete *if every point can be joined by a geodesic with some zero of the vector field. In the case of a gradient field* grad$f$ *this means that every point can be joined by a geodesic with some critical point of the function f.*

Theorem 7.7. *Let (M, g) be a pseudo–Riemannian manifold carrying a non–constant solution f of the equation $\nabla^2 f = \lambda\, g$ having critical points. We assume either that all geodesics through critical points are defined on $\mathbb{R}$ and that (M, g) is null complete or that (M, g) is C-complete.*

Then the manifold (M, g) is (locally) conformally flat. One can define neighborhoods M_j for every critical point p_j on which the metric has the form as in 33. These neighborhoods M_j cover M.

Theorem 7.7 follows from Proposition 7.5 since around each critical point the metric is conformally flat by the equation 33 in polar coordinates. On the other hand, the level (M_*, g_*) in the warped product metric according to Lemma 7.1 cannot change along the geodesic t-lines. Therefore the completeness assumption implies the assertion.

With regard to the global geometry of complete manifolds, the main results of [KR95] and [KR97b] are the following:

Theorem 7.8. [KR95]

For any signature $(k, n-k)$ with $1 \leq k \leq n-1$ there exists a smooth pseudo–Riemannian manifold of dimension n carrying a complete conformal gradient field $V =$ grad$f$ with an arbitrary prescribed number $N \geq 1$ of isolated zeros (including the case of infinitely many zeros in two different ways corresponding to $\mathbb{N}$ or $\mathbb{Z}$). These manifolds are C-complete.

Theorem 7.9. [KR95]

Let M_k^n be a geodesically complete pseudo–Riemannian manifold of signature (k, n) with $2 \leq k \leq n-2$ carrying a non–trivial conformal gradient field with at least one zero.

1. *The diffeomorphism type of M_k^n is uniquely determined by the number N of zeros. Here in the case of infinitely many zeros we have to distinguish between $\mathbb{N}$ and $\mathbb{Z}$.*

2. *Every manifold is conformally equivalent to a standard manifold $M(J)(\alpha, \beta)$ defined in* [KR95, p.468].

3. *If in addition the vector field is complete then the conformal type is uniquely determined by the number N of zeros.*

In the Lorentzian case $k = 1$ the disconnectedness of the geodesic distance spheres opens up more possibilities for the global conformal types which can be described by the *gluing graph*. In the Riemannian case part 3.) of Theorem 7.9 is given in [Bo'70].

Definition 7.10. *Let M be a Lorentzian manifold with a conformal gradient field* $\mathrm{grad} f$. *The* associated graph $G(M, f)$ *is defined as follows:*

1. *The vertices of $G(M, f)$ are the critical points of f in M.*
2. *Every vertex is contained in three edges, one space-like and two time-like ones. These correspond to the space-like cone and the two components of the time-like cone in M at that point.*
3. *Two vertices are joined by an edge if and only if there is a trajectory of the vector field from one to the other, in such a way that the trajectory passes through the corresponding space-like or time-like cones.*

Obviously each edge has a unique character $(+)$ *if it is space-like or* $(-)$ *if it is timelike. We can use this as a label for each edge. It is possible that an edge is incident with only one vertex. This is called a* free edge. *It is also possible that two vertices are joined by more than one edge.*

If M is C-complete then every point can be joined by a geodesic with some critical point of f. Consequently, every point is somehow represented by an edge in the associated graph. In the case of signature $2 \le k \le n - 2$ the analogous associated graph has to be a linear graph since the time-like cone is always connected. In the Lorentzian case $k = 1$ or $k = n - 1$ there are many possibilities and interesting properties of these graphs. They may have cycles. Furthermore, M is simply connected if and only if the associated graph is a tree.

Proposition 7.11. [Bc'98]
Let M_1, M_2 be two C-complete Lorentzian manifolds admitting conformal gradient fields $\mathrm{grad} f_1, \mathrm{grad} f_2$, *respectively, each with at least one zero. If $F : M_1 \to M_2$ is a conformal diffeomorphism transforming at least one critical point of f_1 into a critical point of f_2, then F preserves the trajectories and induces an isomorphism between the two associated graphs including the labeling.*

On the other hand, a conformal classification in general has to incorporate more than just the combinatorial structure of the associated graph with the labeling $(+)$ and $(-)$. In addition one needs a time orientation and weights on the edges. The weight is a positive real number (including ∞ for free edges) associated with an edge. Somehow the weights correspond to the lengths of the trajectories after a conformal development. Different developments lead to constant ratios of the weights. These constants have to be factorized out. The details can be found in [Bc'98].

There remains a discussion of the case of a conformal gradient field which is isotropic on an open set. Here we have the following:

Theorem 7.12. (Brinkmann [Br'25]) [Ca'06]

Assume that (M, g) *is a pseudo-Riemannian manifold of dimension* $n \geq 3$ *admitting a non-vanishing and isotropic conformal gradient field, i.e., a non-constant solution* f *of the equation 27 such that* $\mathrm{grad} f$ *is isotropic on an open subset. Then* $\mathrm{grad} f$ *is in addition parallel, and the metric tensor can be brought into the form* $g = dudv + g_*(u)$ *where* $\mathrm{grad} f = \frac{\partial}{\partial u} = \mathrm{grad} v$ *and where the* $(n-2)$*-dimensional metric* $g_*(u)$ *does not depend on* v.

Such spaces carrying a parallel isotropic vector field are often called Brinkmann spaces. The transition from a non-isotropic gradient to an isotropic one is further explained in [Ca'06]. It corresponds to passing to the limit $\alpha \to 0$ in the metric $g = -\alpha(u)du^2 + dudv + g_*(u)$.

We mention here the isotropic case in a generalized Liouville theorem which is the case of a conformal mapping preserving the Ricci tensor.

Theorem 7.13. [KR'06]

Assume that an n*-dimensional pseudo-Riemannian manifold* (M, g) *admits a conformal mapping* $F\colon M \to M$ *such that the conformal factor* φ *has an isotropic gradient* $\mathrm{grad}\varphi \neq 0$ *everywhere. Assume further that* F *preserves the Ricci tensor and the null-congruence given by the parallel and isotropic vector* $\partial_v = \mathrm{grad}\varphi$*. Then in certain coordinates* u, v, x_k $(k = 1, \ldots, n-2)$ *the metric has the form*

$$g = -2dudv + \sum_{i,j} g^{\#}_{ij}(u, x_k) dx^i dx^j$$

and, up to an isometry, F *has the form*

$$F(u, v, x_k) = \Big(-\frac{1}{cu}, cv + \zeta(u, x_k), \xi_1(u, x_k), \ldots, \xi_{n-2}(u, x_k) \Big)$$

with a constant c *and with a certain function* ζ*, where for any fixed* u, v *the transformation*

$$(x_1, \ldots, x_{n-2}) \mapsto (\xi_1, \ldots, \xi_{n-2})$$

is a homothety with respect to the metric $g^{\#}$*. The conformal factor of* F *is the function* $\varphi(u, v, x_k) = u$*, i.e.,* $F^*g = u^{-2}g$.

Conversely, Let h *be any metric on an* $(n-2)$*-dimensional space* M_* *admitting a* 1*-parameter group* Φ_u *of similarities (homothetic transformations) with* $\Phi_u^* h = u^{-2}h$*. Then on* $M = \mathbb{R}_+ \times \mathbb{R} \times M_*$ *the metric* $g = -2dudv + \Phi_u^* h$ *admits a conformal mapping* F *such that the conformal factor* u *has an isotropic gradient* $\mathrm{grad} u = \partial_v$*. In this case* F *acts on* M_* *by the similarities* Φ_u.

8. 4-dimensional Lorentzian manifolds

In General Relativity one considers 4-dimensional spacetimes of $3+1$ dimensions which can also be described as 4-dimensional Lorentzian manifolds with a metric

tensor of signature $(1,3)$, modelled after the flat Minkowski space $\mathbb{R}^4_1$ with the metric
$$g = -dt^2 + dx^2 + dy^2 + dz^2.$$
For an observer at rest time corresponds to the t-axis and space to the (x,y,z)-part. The local conformal group of this Minkowski space is 15-dimensional. It is generated by four translations, six rotations (that is, the group $O(3,1)$), one homothety $x \mapsto cx$ and four proper conformal mappings. The corresponding conformal vector fields are four infinitesimal translations, six infinitesimal rotations (i.e., the Lie algebra $o(3,1)$), the radial vector field $V(X) = X$ and the vector fields $V(X) = 2\langle X,T\rangle X - \langle X,X\rangle T$ for a fixed vector T.

It is well known [So'97] that the local conformal group of Minkowski space is isomorphic with $O(4,2)/\{\pm\}$. It is also well known that any spacetime which is not locally conformally flat has a conformal group which is at most 7-dimensional [He'91]. The case of dimension 7 is fairly interesting since in this case all Ricci flat metrics (or vacuum spacetimes) can be determined which admit a 7-dimensional conformal group which is not contained in the isometry group, see Theorem 8.3. For further related results cf. [Hl'04], [DS'99].

By Theorem 5.5 any 4-dimensional Lorentzian Einstein space (not of constant sectional curvature) is a vacuum spacetime if it admits a non-homothetic conformal vector field V. For vacuum spacetimes in turn we can formulate the following statement:

Theorem 8.1. *A vacuum spacetime admitting a non-homothetic conformal vector field is either locally flat or is locally a pp-wave (plane–fronted wave).*

Definition 8.2. *The class of pp-*waves *(or plane-fronted waves) in general is given by all Lorentzian metrics on open parts of* $\mathbb{R}^4 = \{(u,v,x,y)\}$ *which are of the form*

$$ds^2 = -2H(u,x,y)du^2 - 2dudv + dx^2 + dy^2$$

with an arbitrary function H, the potential, which does not depend on v. The subclass of plane waves *is given by all H of the form*

$$H(u,x,y) = a(u)x^2 + 2b(u)xy + c(u)y^2,$$

compare [Si'74].

Isometric, homothetic and conformal vector fields of *pp*-waves were classified in a kind of a recursive normal form in [MM'91], starting from the possible Killing fields. On the other hand, the possible isometry groups are known from the work of Ehlers-Kundt [EK'62] and Sippel-Goenner [SG'86]. Furthermore it is well known that the isometry group is of codimension at most one in the homothety group, and that in turn the homothety group is of codimension at most one in the conformal group, compare [Hl'04]. For a discussion of homothetic transformations with one-dimensional fixed point set see also [Al'85]. At the maximum dimension we have the following result:

Theorem 8.3. ([KR'04])

All vacuum spacetimes admitting a 7-dimensional conformal group (together with the vector fields themselves) can be explicitly determined in terms of elementary functions and a finite number of parameters. Moreover there is one family admitting a non-homothetic conformal vector field.

The typical candidate of a hon-homothetic conformal vector field on a *pp*-wave is the standard special conformal vector field V_3 which we already met in Section 3. The flow of V_3 is explicitly given by $\Phi_t(u,v,x,y) = \frac{1}{1-2tu}\big(u, v(1-2tu) + t(x^2 + y^2), x, y\big)$. Any fixed trajectory is a straight line. This vector field is depicted in Figure 1 and Figure 2 below where the (x,y)-plane is reduced to just the $(x,0)$-axis[2].

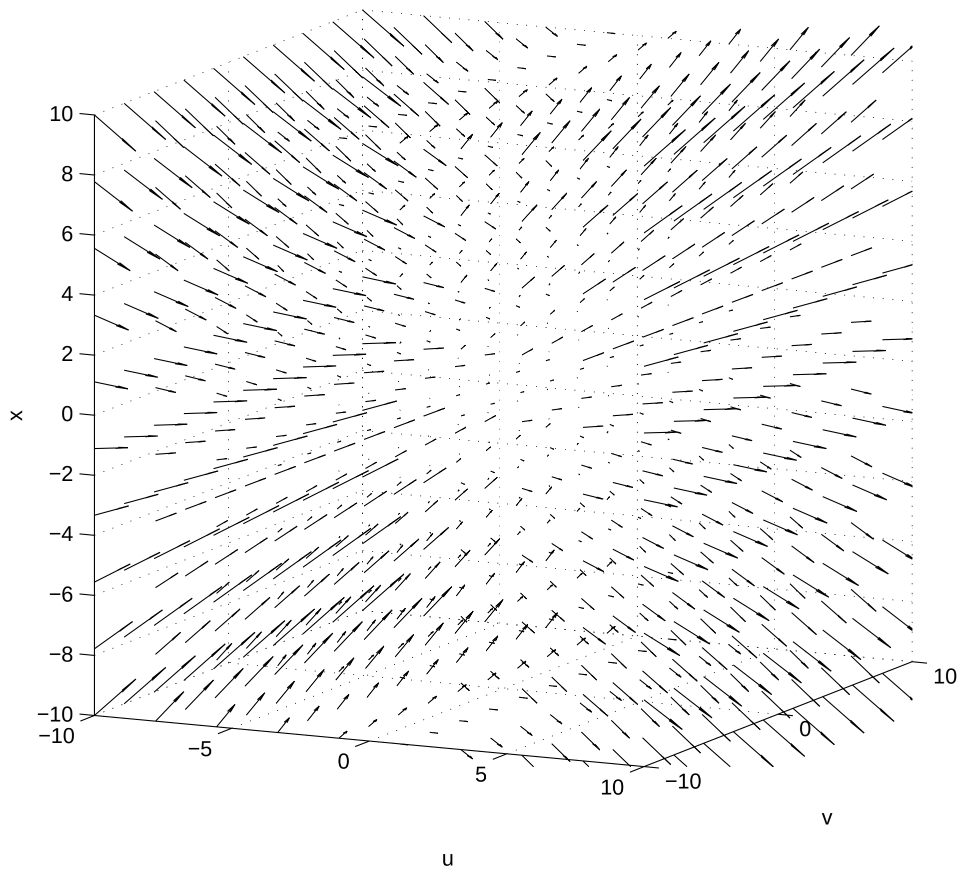

Figure 1. The standard special conformal vector field V_3 in the $(u,v,x,0)$-slice

[2] We thank Andreas App for providing these figures in Matlab

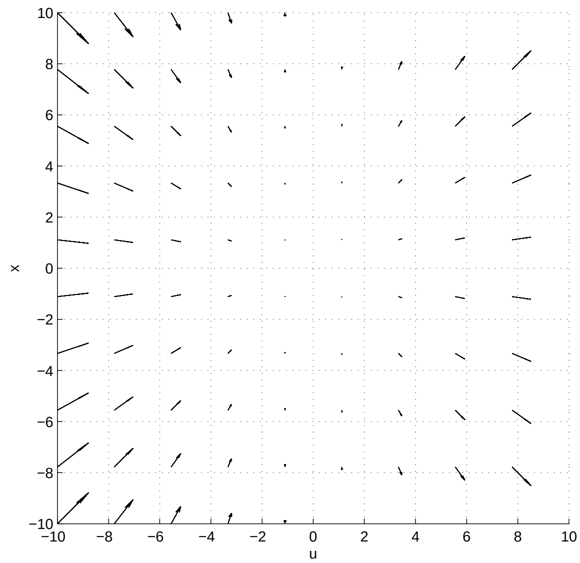

Figure 2. The projection of V_3 into the $(u, 0, x, 0)$-plane

Theorem 8.4. ([KR'04]) *Assume that a vacuum pp-wave with metric*

$$g = -2H(u, x, y)du^2 - 2dudv + dx^2 + dy^2$$

admits the standard conformal vector field $V_3 = u^2\partial_u + \frac{1}{2}(x^2+y^2)\partial_v + ux\partial_x + uy\partial_y$. *Assume further that the function H is defined in a neighborhood of $x = y = 0$ for any fixed $u_0 \neq 0$. Then in a neighborhood of u_0 H can be written as*

$$H(u, x, y) = \sum_{n \geq 0} u^{-(n+2)} P_n(x, y)$$

where P_n denotes a homogeneous polynomial of degree n in the variables x, y which is harmonic, i.e. $\Delta P_n = 0$. Vice versa, any function H of that type admits the standard conformal vector field V_3, compare Section 3.

Due to the singularity $u = 0$ this vector field does not have a zero in the spacetime (except for the flat case). A classification of conformal vector fields with zeros is still not complete. In the case of closed vector fields see Section 7. For non-closed fields there are normal forms under additional assumptions on the Petrov type of the metric, see [Hl'04], [St'06]. Typical results are the following:

Proposition 8.5. *Let (M, g) be a spacetime of constant Petrov type D, and let q be a zero of the conformal vector field V. Then the following hold:*

1. $\mathrm{div}V(q) = 0$,

2. after a conformal change of g the vector field becomes isometric, *3. in certain coordinates* $y_0, \ldots, y_3$ *we have*

$$V = \kappa y_0 \partial_{y_0} - \kappa y_1 \partial_{y_1} - \alpha y_2 \partial_{y_2} + \alpha y_3 \partial_{y_3}$$

where α, κ *are real constants.*

Proposition 8.6. *Let* (M, g) *be a spacetime of constant Petrov type III, and let* q *be a zero of the conformal vector field* V*. Then the following hold:*

1. $\mathrm{div}V(q) \neq 0$,

2. after a conformal change of g *the vector field becomes homothetic,*

3. in certain coordinates $y_0, \ldots, y_3$ *we have*

$$V = \mathrm{div}V(q)(3y_0 \partial_{y_0} - y_1 \partial_{y_1} + y_2 \partial_{y_2} + y_3 \partial_{y_3}).$$

Proposition 8.7. *Let* (M, g) *be a spacetime of Petrov type I or II, and let* q *be a zero of the conformal vector field* V*. Then* V *vanishes identically.*

It does not seem to be known what happens if the Petrov type degenerates at exactly the zero of the vector field. Possibly there are more cases to be considered. However, examples are still missing.

Any conformal vector field V on a vacuum spacetime preserves the Ricci tensor, i.e., $\mathcal{L}_V \mathrm{Ric} = 0$. This is a trivial case of a so-called *Ricci collineation.* In more generality, one can consider the case that $\mathcal{L}_V \mathrm{Ric}$ is conformal to the metric. This is a *conformal Ricci collineation* , as defined in [KR'01]. A conformal Ricci collineation preserves the eigendirections of the Ricci tensor.

Further aspects of the conformal geometry of spacetimes can be found in [HS'02].

9. The transition to the Penrose limit

In 1976 R.Penrose introduced in [Pe'76] the following construction which associates to any lightlike geodesic on a Lorentzian manifold a plane wave metric. In several recent papers about models for supergravity respectively string theory, in particular regarding the maximally supersymmetric type IIB plane wave background and its relation to $AdS_5 \times S^5$ the Penrose limit has been discussed intensively, cf. [BF'02], [BP'04] and the survey article [Sa'04].

Along a lightlike geodesic $\gamma : I \to M$ on a Lorentzian manifold of dimension n which is free of conjugate points it is possible to introduce coordinates $(U, V, Y) = (U, V, Y_1, \ldots, Y_{n-2})$ such that the metric nearby the geodesic is of the form

$$g = dV \left(2dU + a(U, V, Y)dV + 2 \sum_{i=1}^{n-2} b_i(U, V, Y)dY_i \right) + \sum_{i=1}^{n-2} g_{ij}(U, V, Y)dY_i dY_j \,. \tag{34}$$

In these coordinates the lightlike geodesic γ is of the form $\gamma(U) = (U, 0, 0)$, it is embedded in a congruence of lightlike geodesics $U \mapsto (U, V_1, Y_1)$ for any fixed (V_1, Y_1). Then we introduce a scaling of coordinates, for any positive λ let

$$(U, V, Y_1, \ldots, Y_{n-2}) = (u, \lambda^2 v, \lambda y_1, \ldots, \lambda y_{n-2}) \tag{35}$$

In the new coordinates the scaled metric $g_\lambda = \lambda^{-2} g$ is of the form

$$g_\lambda = dv\left(2du + \lambda^2 a(u, \lambda^2 v, \lambda y)dv + 2\lambda \sum_{i=1}^{n-2} b_i(u, \lambda^2 v, \lambda y)dy_i\right) + \sum_{i=1}^{n-2} g_{ij}(u, \lambda^2 v, \lambda y)dy_i dy_j\,.$$

Then Penrose [Pe'76] introduced the following construction:

Definition 9.1. *The* Penrose limit *is defined with respect to the coordinates* $(u, v, y_1, \ldots, y_{n-2})$ *introduced in Equation 35 as the metric*

$$\overline{g} := \lim_{\lambda \to 0} g_\lambda = 2dudv + \sum_{i,j=1}^{n-2} \overline{g}_{ij}(u)dy_i dy_j \tag{36}$$

with $\overline{g}_{ij}(u) = g_{ij}(u, 0, 0)$.

This is a plane wave metric in the so-called Rosen coordinates.
Now we investigate whether a conformal vector field

$$\xi(U, V, Y) = A(U, V, Y)\partial_U + B(U, V, Y)\partial_V + \sum_{i=1}^{n-2} C_i(U, V, Y)\partial_{Y_i}$$

survives under this limit construction. We express the conformal field $\lambda^2\xi$ in the coordinates (u, v, y) :

$$\lambda^2\xi(u, v, y) = B(u, \lambda^2 v, \lambda y)\partial_v + \lambda \sum_{i=1}^{n-2} C_i(u, \lambda^2 v, \lambda y)\partial_{y_i} + \lambda^2 A(u, \lambda^2 v, \lambda y)\partial_u$$

and assume that one of the coefficient functions for fixed (u, v, y): $B(u, \lambda^2 v, \lambda y)$, $\lambda \sum_{i=1}^{n-2} C_i(u, \lambda^2 v, \lambda y), \lambda^2 A(u, \lambda^2 v, \lambda y)$ has a Taylor expansion around $\lambda = 0$ with a leading term $\lambda^k f_k(u, v, y), f_k(u, v, y) \neq 0$ We also assume that $k \geq 0$ is the minimal exponent with this property. Then the limit $\overline{\xi}(u, v, y) := \lim_{\lambda \to 0} \lambda^{2-k}\xi(u, v, y)$ is a non-trivial vector field. Let $\mathcal{L}_\xi\, g = \phi g$, i.e. $\phi = \mathrm{div}\xi/n$. Since

$$\mathcal{L}_{\lambda^{2-k}\xi(u,v,y)}\left(\lambda^{-2}g\right) = \lambda^{2-k}\phi(u, \lambda^2 v, \lambda y)\left(\lambda^{-2}g\right)$$

we obtain in the limit $\lambda \to 0$ that the vector field $\overline{\xi}$ is a non-trivial conformal vector field. Therefore the property to be a conformal vector field is a *hereditary* property, the corresponding argument for Killing fields can be found for example in [BF'02, ch.4.3].

We summarize this considerations in the following

Proposition 9.2. *For a conformal and analytic vector field $\xi = \xi(U, V, Y)$ of the Lorentzian metric g in coordinates (U, V, Y) adapted to a lightlike geodesic $\gamma(U) = (U, 0, 0)$ by Equation 34 there is a non-negative integer k such that the limit $\overline{\xi}(u, v, y) := \lim_{\lambda \to 0} \left(\lambda^{2-k}\xi(u, \lambda^2 v, \lambda y)\right)$ is a conformal vector field on the Penrose limit given by Equation 36.*

The Proposition shows the importance of the conformal geometry of plane waves, resp. the description of the conformal vector fields, cf. Section 8.

One can transform the Penrose limit in Rosen coordinates given in Equation 36 into Brinkmann coordinates $x = (x_1, x_2, \ldots, x_n)$. Then the Penrose limit is of the form

$$\overline{g} = 2dx_1 dx_2 + \sum_{i,j=3}^{n} A_{ij}(x_1) x_i x_j dx_1^2 + \sum_{i=3}^{n} dx_i^2 \tag{37}$$

These is the coordinate form of a plane wave used used in the four-dimensional case in Section 8, see for example Definition 8.2. Now the wave profile $A_{ij}(x_1)$ of the plane wave coincides with the only non-vanishing curvature components

$$A_{ij} = \overline{R}_{1ij1} = \overline{R}(\partial_1, \partial_i, \partial_j, \partial_1)\,;\, i, j = 3, 4, \ldots n$$

of the plane wave metric. The coordinate transformation as well as the curvature computations are explained in detail for example in [Gü'88, ch. VIII.2].

This allows the following invariant interpretation of the Penrose limit:

Proposition 9.3. (Blau et al. [BP'04, (2.14)]):
Let $\gamma = \gamma(u)$ be a lightlike geodesic of a Lorentzian manifold (M, g). We assume that $e_1, e_2, e_3, \ldots, e_n$ is an pseudo-orthonormal frame in the tangent space $T_{\gamma(0)}M$ with $e_1 = \gamma'(0)$; $g(e_1, e_2) = g(e_i, e_i) = 1$ for all $i = 3, 4, \ldots, n$ and $g(e_i, e_j) = 0$ otherwise and let $e_1(u), e_2(u), e_3(u), \ldots e_n(u)$ be the parallel transport of $(e_1, \ldots, e_n)$ along γ. Then the Penrose limit of (M, g) associated to the lightlike geodesic is the plane wave metric

$$2dx_1 dx_2 + \sum_{i,j=3}^{n} A_{ij}(x_1) x_i x_j dx_1^2 + \sum_{i=3}^{n} dx_i^2$$

with the wave profile

$$A_{ij}(x_1) = R\,(e_1, e_i, e_j, e_1)$$

Here R is the curvature tensor of (M, g).

It follows in particular that the Penrose limit of an Einstein manifold is Ricci flat.

10. Conformal vector fields and twistor spinors

For a pseudo-Riemannian manifold (M, g) of signature $(k, n-k)$ with spin structure the tangent bundle acts on the spinor bundle Σ via the Clifford multiplication

$X \otimes \psi \in T_*M \otimes \Sigma M \mapsto X \cdot \psi \in \Sigma M$. The spinor bundle carries the spin connection ∇ and a hermitian inner product $\langle ., . \rangle$ satisfying the equations

$$\begin{aligned} \langle X \cdot \phi, \psi \rangle &= (-1)^{k+1} \langle \phi, X \cdot \psi \rangle \\ X(\langle \phi, \psi \rangle) &= \langle \nabla_X \phi, \psi \rangle + \langle \phi, \nabla_X \psi \rangle \,. \end{aligned}$$

For the details of this construction see [BF'91] and [Fr'00] in the Riemannian case and [Ba'81] in the pseudo-Riemannian case. A spinor field ψ is called parallel if $\nabla \psi = 0$, i.e. for all tangent vectors $X : \nabla_X \psi = 0$. The composition of the spin connection and the Clifford product defines the Dirac operator D. If $e_1, \ldots, e_n$ is an orthonormal frame with $g(e_i, e_j) = \epsilon_i \delta_{ij}, \epsilon_1 = \ldots = \epsilon_p = -1, \epsilon_{p+1} = \ldots = \epsilon_n = 1$, then

$$D\psi = \sum_{i=1}^{n} \epsilon_i \, e_i \cdot \nabla_{e_i} \psi \,. \tag{38}$$

Definition 10.1. *We call a spinor field ψ a* twistor spinor *if the following twistor equation is satisfied for all tangent vectors X :*

$$\nabla_X \psi + \frac{1}{n} X \cdot D\psi = 0 \,. \tag{39}$$

Twistor spinors can also be described as the kernel of a differential operator $\mathcal{D}$, called the twistor operator or Penrose operator: It is the composition of the spin connection with a projection onto the kernel of the Clifford product:

$$\mathcal{D}\psi = \sum_{i=1}^{n} \epsilon_i e_i \otimes \left(\nabla_{e_i} \psi + \frac{1}{n} e_i \cdot D\psi \right) \tag{40}$$

The Dirac operator D and the twistor operator $\mathcal{D}$ are both conformally covariant in the following sense. If $\overline{g} = \exp(4\phi)\, g$ then there is a isometry between the spinor bundles $\psi \in \Sigma_g M \to \overline{\psi} \in \Sigma_{\overline{g}} M$ such that for the Dirac operators $D = D_g$ and $\overline{D} = D_{\overline{g}}$ resp. the twistor operators $\mathcal{D} = \mathcal{D}_g$ and $\overline{\mathcal{D}} = \mathcal{D}_{\overline{g}}$ the following equations hold, cf. [BF'91, ch.(1.3),(1.4)], [Fr'89]:

$$\begin{aligned} \overline{D}\,\overline{\psi} &= e^{-(n+1)\phi} D \left(e^{(n-1)\phi} \psi \right) \\ \overline{\mathcal{D}}\,\overline{\psi} &= e^{-\phi} \mathcal{D} \left(e^{-\phi} \psi \right) \end{aligned}$$

Hence the dimension of twistor spinors is a conformal invariant, if ψ is a twistor spinor of the pseudo-Riemannian manifold (M, g) then $e^{\varphi} \overline{\psi}$ is a twistor spinor of the conformally equivalent metric $\overline{g} = \exp(4\phi)\, g$. In this sense the twistor equation $\mathcal{D}\psi = 0$ is conformally covariant. Particular twistor spinors are Killing spinors, they satisfy the equation

$$\nabla_X \psi = \lambda X \cdot \psi \tag{41}$$

for a complex number λ and for all tangent vectors X. Then one can conclude from the relation between the curvature of the spin connection and the Riemannian curvature tensor:

Proposition 10.2. *Let (M,g) be a pseudo-Riemannian manifold of signature $(k,n-k)$ with a Killing spinor ψ, i.e. $\nabla_X\psi = \lambda X\cdot\psi$ for some complex number λ.*

(a) *The scalar curvature S is constant and satisfies $S = 4n(n-1)\lambda^2$.*

(b) *The Ricci curvature as a $(1,1)$ tensor satisfies*

$$\left\{\mathrm{Ric}(X) - 4\lambda^2(n-1)X\right\}\cdot\psi = 0.$$

Therefore the complex number λ is either real or purely imaginary, then we call the Killing spinor either a real Killing spinor or an imaginary Killing spinor. If the manifold is Riemannian then the metric is an Einstein metric. In the pseudo-Riemannian case the traceless Ricci tensor of a manifold carrying a Killing spinor is lightlike. If in addition the Killing spinor is not light-like then the traceless Ricci tensor vanishes, i.e. the manifold is Einstein.
On the other hand one can show that a twistor spinor on an Einstein manifold is either parallel or the sum of two Killing spinors:

Proposition 10.3. *Let (M,g) be a pseudo-Riemannian Einstein manifold with scalar curvature S carrying a twistor spinor ψ. Then*

(a) *If $S\neq 0$ then the twistor spinor $\psi = \psi_+ + \psi_-$ is the sum of two Killing $\psi_\pm$ with $\nabla_x\psi_\pm = \pm\frac{1}{2}\sqrt{S/(n(n-1))}$.*

(b) *If $S = 0$ then either ψ or $D\psi$ is a parallel spinor.*

We are interested here in the conformal vector field which can be associated to a twistor spinor:

Definition 10.4. *For a spinor field ψ on a pseudo-Riemannian manifold (M,g) of index k we call the vector field V_ψ defined by $g(V_\psi, X) = -i^{k+1}\langle X\cdot\psi,\psi\rangle$ for all tangent vectors X the* Dirac current.

Then we obtain

Proposition 10.5. *Let (M,g) be a pseudo-Riemannian manifold of index k with spinor field ψ and Dirac current V_ψ.*

(a) *The Dirac current V_ψ of a twistor spinor ψ is a conformal vector field.*

(b) *The Dirac current V_ψ of a real (resp. imaginary) Killing spinor ψ is a Killing vector field if p is even (resp. odd).*

(c) *The Dirac current V_ψ of a parallel spinor ψ is a parallel vector field.*

Proof. We assume that ψ is a twistor spinor: We compute the Lie derivative $\mathcal{L}_V g$ for $V = V_\psi$:

$$\begin{aligned}
\mathcal{L}_V g(X,Y) &= g(\nabla_X V, Y) + g(X, \nabla_Y V)\\
&= Xg(V,Y) + Yg(X,V) - g(V,[X,Y])\\
&= -X < Y\psi,\psi > -Y < X\psi,\psi > - < [X,Y]\psi,\psi >\\
&= - < Y\nabla_X\psi,\psi > - < Y\psi,\nabla_X\psi > - < X\nabla_Y\psi,\psi >\\
&\quad - < X\psi,\nabla_Y\psi >
\end{aligned}$$

Then we conclude from the twistor equation 39:

$$\begin{aligned}
\mathcal{L}_V g(X,Y) &= -n\{< YXD\psi, \psi > + < Y\psi, XD\psi > + \\
&\quad < XYD\psi, \psi > + < X\psi, YD\psi >\} \\
&= -n\left\{< (XY+YX)D\psi, \psi > + (-1)^{k+1} < \psi, (XY+YX)D\psi >\right\} \\
&= 2n\, g(X,Y)\left\{< D\psi, \psi > + (-1)^{k+1} < \psi, D\psi >\right\}.
\end{aligned}$$

Hence V is a conformal vector field with divergence

$$\mathrm{div} V = 2b(< D\psi, \psi >) \tag{42}$$

Here $b(a)$ for a complex number a denotes the real part of a if p is odd and the imaginary part otherwise. This finishes the proof of part (a). If ψ satisfies Equation 41 then $D\psi = -n\lambda\psi$ hence by Equation 42 the Dirac current is a Killing vector field if either λ is real and p is even of λ is purely imaginary and p is odd. If ψ is parallel then $\langle \nabla_Y V, X\rangle = Y\langle \psi, X\psi\rangle - \langle V, \nabla_Y X\rangle = \langle \nabla_Y \psi, X\psi\rangle + \langle \psi, X\nabla_Y \psi\rangle = 0$ shows that ψ is parallel. □

In the Riemannian case it may very well occur that the Dirac current of a twistor spinor vanishes identically. If for example ψ is a parallel spinor on a Riemannian manifold with a Dirac current V_ψ which does not vanish identically then V_ψ is parallel and the manifold is locally a Riemannian product. A connection to the problem of presenting essential conformal vector fields (cf. section 4) twistor spinors with zeros play a particular role. On the doubled spinor bundle $E = \Sigma \oplus \Sigma$ of a pseudo-Riemannian spin manifold there is a connection ∇^E with the following property: A section (ψ, φ) of the bundle E is parallel if and only if ψ is a twistor spinor and $\varphi = D\psi$. This shows in particular that for a non-trivial twistor spinor with zero p, i.e. $\psi(p) = 0$ we have $D\psi(p) \neq 0$. Therefore one can show that $\nabla V_\psi(p) = 0$, hence also $\mathrm{div} V_\psi(p) = 0$. But since a conformal vector field W vanishes identically if for some point p the quantities $W(p) = 0; \nabla W(p) = 0$ and $\mathrm{grad}\,\mathrm{div} W(p) = 0$ vanish we conclude:

Proposition 10.6. [KR'94] *Let (M,g) be a pseudo-Riemannian spin manifold with a non-trivial twistor spinor ψ having a zero. If the Dirac current V_ψ is non-zero it is an essential conformal vector field.*

In the Riemannian case we obtain the following consequence:

Theorem 10.7. [KR'94, Thm.A] *If a Riemannian spin manifold (M,g) carries a twistor spinor with zero and with non-trivial Dirac current V_ψ then the manifold is conformally flat.*

K.Habermann showed in [Ha'94] a similar result under an additional curvature and completeness assumption. So the question was whether there are examples of twistor spinors with zeros on Riemannian manifolds which are not conformally flat. The authors show in [KR'96] for dimension $n = 4$ and for even dimensions $n \geq 4$ in [KR97c] that there are complete Riemannian spin manifolds carrying twistor spinors with zeros which are not conformally flat. In particular in this case the Dirac current vanishes identically. These examples are conformal compactifications of irreducible and asymptotically locally Euclidean manifolds carrying parallel spinors. This leads to the following

Theorem 10.8. [KR'98, Thm.1.2] *Let (M, g) be an n-dimensional Riemannian spin manifold carrying a twistor spinor ψ with non-empty zero set $Z_\psi := \{p; \psi(p) = 0\}$. Then the conformally equivalent Riemannian metric $\overline{g} = \|\psi\|^{-4} g$ on $M - Z_\psi$ is either flat or locally irreducible and Ricci flat carrying a parallel spinor. In addition corresponding to any zero point the complement $M - Z_\psi$ of the zero set Z_ψ has an end carrying an asymptotically Euclidean coordinate system of order* 3.

For a construction of compact orbifolds which are not conformally flat and which carry twistor spinors with zero we refer to [BG'04].
In contrast to the Riemannian case in the Lorentzian case the Dirac current is always non-trivial. In addition the twistor spinor as well as the Dirac current can be lightlike. For example the following result in dimension four is well known:

Theorem 10.9. (Ehlers-Kundt [EK'62])
A four-dimensional Lorentzian manifold with spin structure carrying a parallel spinor is locally isometric to a pp-metric.

In this case the Dirac current is parallel and lightlike. For results on the holonomy of pseudo-Riemannian manifolds with parallel spinors see Baum and Kath [BK'99]. A description of the local geometry of Lorentzian manifolds carrying twistor spinors without zeros up to dimension 7 is given by Baum and Leitner in [BL'04]. These geometries include Brinkmann spaces with special Kähler flag, Fefferman spaces and Lorentzian-Sasaki manifolds. For a survey on these constructions we refer to [Ba'00].
The Dirac current V_ψ is called non-twisting if the dual one-form $\omega = V_\psi^\#$ satisfies $d\omega \wedge \omega = 0$ and twisting if $d\omega \wedge \omega$ does not vanish anywhere.

Theorem 10.10. (Baum, Leitner [BL'04, Prop.4.3])
A Lorentzian manifold with spin structure carrying a twistor spinor with lightlike and non-twisting Dirac current is locally conformally equivalent to a Brinkmann space with parallel spinor.

The Dirac current also plays an important role in the classification of imaginary Killing spinors on Lorentzian manifolds presented in [Le'05]. In one of the cases the gradient of the *length function* $\langle \psi, \psi \rangle$ defines a conformal vector field, hence results discussed in Section 7 can be used. In the Riemannian case the same argument was used in [Ra'91] to classify manifolds carrying an generalized imaginary Killing spinor ψ satisfying the equation $\nabla_X \psi = ibX \cdot \psi$ for some real function b and all tangent vectors X. In [Le'04, Thm.1] it is shown that the zero set of a twistor spinor on a Lorentzian spin manfiold consists either of isolated images of lightlike geodesiscs or of isolated points. In the first case the metric is outside the zero set locally conformally equivalent to a Brinkmann space with parallel spinor. In the second case the metric is outside the zero set locally conformally equivalent ot a product metric of the form $-ds^2 + h$, where h is a Riemannian metric carrying a parallel spinor. The second case actually occurs at least in the C^1-category:

Theorem 10.11. (Leitner [Le'06])
There exists a five-dimensional manifold with a C^1-Lorentzian metric carrying a twistor spinor with an isolated zero. The metric is not conformally flat and the Dirac current is causal.

It remains open whether such examples exist with a higher order of differentiability.
On the other hand there is the following recent result:

Theorem 10.12. (Frances [Fs'06, Cor.2])
If an analytic Lorentzian manifold (M, g) admits a non-zero twistor spinor which has a zero then the manifold is conformally flat.

Frances actually proves the following statement about conformal vector fields of Lorentzian manifolds, cf. [Fs'06, Thm.1]: If a smooth Lorentzian manifold of signature $(1, n-1); n \geq 3$ carries a non-trivial conformal and causal vector field X (i.e. $\|X\| \leq 0$) then there is an open and non-empty subset on which the manifold is conformally flat.
As a generalization of conformal Killing vector fields one can consider *conformal Killing forms,* which are studied in detail by Semmelmann [Se'03]. The defining equation is also called *conformal Killing-Yano equation.* As a particular example for twistor spinors ψ_1, ψ_2 on a Riemannian spin manifold the exterior k-form

$$\omega_k(X_1, \ldots, X_k) = \langle (X_1 \wedge \ldots \wedge X_k) \cdot \psi_1, \psi_2 \rangle$$

is a conformal Killing form. The twistor equation also allows a supersymmetric interpretation, cf. Alekseevskii et al. [AC'98] and Klinker [Kl'05].

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