Generalized Killing spinors with imaginary Killing function and conformal Killing fields*

Hans–Bert Rademacher

1 Statement of results

We consider complete Riemannian spin manifolds (M, g) with complex spinor bundle S. S carries a hermitian product $\langle ., . \rangle$ which we assume to be complex conjugate linear in the first argument and complex linear in the second argument. The Clifford bundle $Cl(M)$ of M acts on S by Clifford multiplication which we denote by $X \cdot \psi$ for a vector field X and a spinor ψ . Clifford multiplication by a tangent vector X is skew symmetric with respect to $\langle ., . \rangle$. The Levi–Civita connection ∇ on M induces the spinor connection on S which we also denote by ∇, cf. [1]. A non–trivial spinor ψ is called *generalized Killing spinor* with *Killing function* λ if

$$\nabla_X \psi = \lambda \, X \cdot \psi \tag{1}$$

for all vector fields X and for a complex–valued function λ on M . In particular ψ is a *twistor spinor* , i.e. $\nabla_X \psi + \frac{1}{n} X \cdot D\psi = 0$ for all X , where D is the Dirac operator. If λ is constant then ψ is a *Killing spinor* with *Killing number* λ and (M^n, g) is an Einstein manifold of scalar curvature $r = 4n(n-1)\lambda^2$, see [8] or [5]. Hence three cases occur: If $\lambda = 0$ then ψ is parallel and M is Ricci flat. If $\lambda^2 > 0$ then λ is real, M is compact and $\lambda^2 n^2 = rn/(4n-4)$ is the smallest eigenvalue of the square D^2 of the Dirac operator D by results of T.Friedrich [9] and O.Hijazi [12]. If $\lambda^2 < 0$ then λ is an imaginary number. H.Baum classified in [2], [3] and [4] these manifolds, see Corollary 1. From results of O.Hijazi [12, cor.3.6] and A.Lichnerowicz [16, thm.1] it follows that a generalized Killing spinor is either a Killing spinor with real Killing number or λ is an imaginary function.

From now on we consider the second case, i.e. we assume $\lambda = ib$ for a not everywhere vanishing real function b.The function $f := \langle \psi, \psi \rangle$ is positive everywhere since equation (1) is a first order linear ordinary differential equation along a geodesic, cf. [16, prop.1]. The vector field V on M defined by $\langle V, X \rangle = i \langle \psi, X \cdot \psi \rangle$ for all X is a conformal non–isometric closed Killing field, cf. §3 .

T.Friedrich introduced in [10] the function $q_\psi := f^2 - \|V\|^2$ which is a non–negative constant. In [17, thm.] , [18, thm.4] A.Lichnerowicz proved that locally M is the warped product of an open interval and a manifold carrying a parallel spinor if $q_\psi = 0$ and that M is globally isometric to a warped product of the real line $\mathbb{R}$ with a complete manifold carrying a parallel spinor if $q_\psi = 0$ and b has no zero [17, prop.2] , [18, prop.2] .

We obtain the above quoted results of H.Baum and A.Lichnerowicz and the global structure of the manifold M from the classification of complete Riemannian manifolds

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carrying a non–isometric conformal closed Killing field. This classification is given in theorem 2, related results are due to H.W.Brinkmann [7], Y.Tashiro [19], J.P. Bourguignon [6], Y.Kerbrat [13] and W.Kühnel [14] [15]. In [15] W.Kühnel studies the complete Riemannian manifolds of constant scalar curvature carrying a non–isometric conformal closed Killing field. Corollary 2 b) contains an explicit example.

Our main result is

Theorem 1 *Let M be a complete Riemannian spin manifold with a generalized Killing spinor with an imaginary Killing function, i.e.*

$$\nabla_X \psi = ib\, X \cdot \psi$$

with a not everywhere vanishing real function b .

a) If $q_\psi = 0$, then there is a positive function h on $\mathbb{R}$ and a complete $(n-1)$–dimensional Riemannian spin manifold (M_*, g_*) carrying a parallel spinor such that the warped product $\mathbb{R} \times_h M_*$ (with metric $g = du^2 + h^2(u) g_*$) is a Riemannian covering of M . Here $f = \langle \psi, \psi \rangle$ and b are functions of $u \in \mathbb{R}$ alone, $f(u,x) = f(u) = h(u), (u,x) \in \mathbb{R} \times M_*$ and $b = f'/(2f)$.

If M is a proper quotient of $\mathbb{R} \times_f M_*$ then f is periodic with period $\omega > 0$ and there is an isometry γ of M_* such that M is isometric to $\mathbb{R} \times_f M_*/\Gamma$ where the group $\Gamma \cong \mathbb{Z}$ of isometries is generated by $(u,x) \mapsto (u+\omega, \gamma(x))$.

b) If $q_\psi > 0$, then M is isometric to the n–dimensional hyperbolic space $\mathrm{H}^n(-4b^2)$ of constant sectional curvature $-4b^2$.

Hence $q_\psi = 0$ iff the conformal closed vector field V has no zero, i.e. is inessential. A conformal Killing field is inessential, if it becomes an isometric Killing field after a conformal change of the metric.

H.Baum shows in [3, thm.1] that the n–dimensional hyperbolic space carries Killing spinors with imaginary Killing number with $q_\psi = 0$ for all n and with $q_\psi > 0$ if $n \neq 3, 5$. It follows from [3, lem.4] that the warped product $\mathbb{R} \times_f M_*$ of a manifold M_* carrying a parallel spinor with an arbitrary positive function f on $\mathbb{R}$ carries a generalized Killing spinor ψ with imaginary Killing function ib with $q_\psi = 0$, where $b = f'/(2f)$.

A.Lichnerowicz describes in [17, §5] , [18, §10] the following example : Let M_* be a compact manifold carrying a parallel spinor (e.g. a K_3–surface with the Calabi–Yau metric or a flat torus with the canonical spin structure) and let $f : S^1 \longrightarrow \mathbb{R}^+$ be a positive non–constant periodic function and $b = f'/(2f)$. Then $M = S^1 \times_f M_*$ is a compact spin manifold with a generalized Killing spinor with Killing function ib. From theorem 1 it follows that up to Riemannian quotients and twisting these are all such compact manifolds.

In [11] K.Habermann gives another characterization of hyperbolic space, she shows that a complete n–dimensional Einstein spin manifold with negative scalar curvature $r = kn(n-1)$ and a non–parallel twistor spinor ψ whose length function $f = \langle \psi, \psi \rangle$ attains a minimum is the hyperbolic space $\mathrm{H}^n(k)$.

From theorem 1 we obtain

Corollary 1 (H.Baum [2], [3]) *If (M,g) is a complete Riemannian manifold carrying a Killing spinor ψ with imaginary Killing number $ib, b \in \mathbb{R} - \{0\}$ then M is isometric to* *a)* $\mathbb{R} \times_{\exp(2bu)} M_*$ *where* M_* *is a complete* $(n-1)$*–dimensional Riemannian manifold carrying a parallel spinor, if* $q_\psi = 0$ *.*

b) Hyperbolic space $\mathrm{H}^n(-4b^2)$ *, if* $q_\psi > 0$ *.*

From the formula for the scalar curvature of a warped product we obtain

Corollary 2 *Let* M *be a complete Riemannian manifold which carries a generalized Killing spinor with imaginary Killing function* ib *,* $b \not\equiv 0$ *.*

a) There is a point of negative scalar curvature.

b) If b *is non–constant, if the scalar curvature* r *is constant and* $a := 1/2(-rn/(n-1))^{1/2}$ *then* M *is isometric to* $\mathbb{R} \times_f M_*$ *where* M_* *carries a parallel spinor and* $f(u) = (\cosh(au))^{2/n}$*, hence* $b(u) = a\tanh(au)/n$ *.*

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2 Conformal Killing fields

We denote by $L_V g$ the Lie derivative of the metric $g = \langle .,. \rangle$ in direction of the vector field V, i.e. $L_V g(X,Y) = \langle \nabla_X V, Y \rangle + \langle X, \nabla_Y V \rangle$. A vector field V is a *conformal Killing field* if the local flow consists of conformal transformations. This is equivalent to $L_V g = 2hg$ with a function h. By taking traces one obtains $h = \mathrm{div}\ V/n$. V is *homothetic* if h is a constant and it is *isometric* if $L_V g = 0$. V is *closed* if the corresponding 1–form $\omega = \langle V, . \rangle$ is closed . Hence if V is a conformal closed Killing field then for every point $p \in M$ there is a neighborhood U and a function F on U such that $V = \nabla F$ on U . The Hessian $\nabla^2 F(X,Y) := \langle \nabla_X \nabla F, Y \rangle = 1/2\ L_{\nabla F} g(X,Y)$ then satisfies

$$\nabla^2 F = \frac{\Delta F}{n} g \tag{2}$$

where Δ is the Laplacian. H.W.Brinkmann showed in [7, §3] that nearby a regular point of F the metric has a warped product structure. Y.Tashiro classifies in [19, lem.2.2] the complete Riemannian manifolds with a non–constant function F on M satisfying equation (2), i.e. ∇F is a conformal Killing field, cf. also W.Kühnel [14, thm.22] . Using this result we show

Theorem 2 *Let* (M^n, g) *be a complete Riemannian manifold with a non–isometric conformal closed Killing field* V *and let* N *be the number of zeros of* V*. Then* $N \leq 2$ *and:*

a) If $N = 2$*, then* M *is conformally diffeomorphic to the standard sphere* S^n*.*

b) If $N = 1$*, then* M *is conformally diffeomorphic to euclidean space* $\mathbb{R}^n$*.*

c) If $N = 0$*: Then there is a complete* $(n-1)$*–dimensional Riemannian manifold* (M_*, g_*) *and a function* $h : \mathbb{R} \to \mathbb{R}^+$ *such that the warped product* $\mathbb{R} \times_h M_*$ *is a Riemannian covering of* M *and the lift of* V *is* $h\frac{\partial}{\partial u}$ *.*

If M *is a proper quotient of* $\mathbb{R} \times_h M_*$ *then* h *is periodic with period* $\omega > 0$ *and there is an isometry* γ *of* M_* *such that* $M = \mathbb{R} \times_h M_*/\Gamma$ *, where the group* $\Gamma \cong \mathbb{Z}$ *of isometries is generated by* $(u,x) \mapsto (u+\omega, \gamma(x))$ *.*

Proof . Let $\bar{M}$ be the universal Riemannian covering of M with projection $\pi : \bar{M} \to M$, denote by $\bar{V}$ the lift of V onto $\bar{M}$ and by $\bar{N}$ the number of zeros of $\bar{V}$. If G is the group of deck transformations of $\bar{M}$ such that $M = \bar{M}/G$, then $\bar{N} = \mathrm{ord}(G)N$. Since $\bar{V}$ is closed there is a non–constant function F on $\bar{M}$ with $\bar{V} = \nabla F$ and $\nabla^2 F = (\Delta F/n)g$. From Tashiro's classification [19, lem.2.2] resp. [14, thm.21] it follows that $\bar{N} \leq 2$ and that the following cases occur:

a) $\bar{N} = 2$, then $\bar{M}$ is conformally diffeomorphic to S^n and since $\Delta F = \mathrm{div}\ \bar{V}$ has different signs in the critical points of F the vector field $\bar{V}$ does not project onto a proper quotient of S^n .

b) $\bar{N} = 1$, then $\bar{M}$ is conformally diffeomorphic to $\mathbb{R}^n$ and since $N = 1$ we have $M = \bar{M}$.

c) $\bar{N} = 0$, then $\bar{M}$ is isometric to $\mathbb{R} \times_{F'} \bar{M}_*$ for a complete $(n-1)$–dimensional Riemannian manifold $\bar{M}_*$ where F is a function of u alone, i.e. $F(u,x) = F(u)$ and $\bar{V} = F' \frac{\partial}{\partial u}$. Hence F' is also a function on M. $\bar{M}_*$ is a connected component of the submanifold $F^{-1}(F(p))$ for a point $p \in \bar{M}$ with $F'(p) \neq 0$. Since V is not–isometric we can assume in addition that $F''(p) = \mathrm{div}\ \bar{V}(p)/n \neq 0$. Then $M_* := \pi(\bar{M}_*)$ is a connected component of the submanifold $F'^{-1}(F'(\pi(p))$ on M, $\mathbb{R} \times_h M_*$ is a Riemannian covering of M and the projection $\pi_1 : \mathbb{R} \times_h M_* \to M$ can be identified with the normal exponential map of the submanifold M_* in M where $h = F'$.

If M is a proper quotient then h is periodic since $\bar{V} = h\frac{\partial}{\partial u}$ is the lift of V. Let $\omega > 0$ be the greatest number such that the restriction $\pi_1|(-\omega/2, \omega/2) \times_h M_*$ is injective, then $\gamma : M_* \to M_*$ is defined by $\gamma(x) = \pi_1(-\omega, x)$.

Remark 1 a) The cases a) and b) of theorem 2 are proved by Y.Kerbrat [13]. J.P.Bourguignon proves these cases in [6] under the additional assumption that the vector field is complete. Theorem 2 can also be found in W.Kühnel [15] .

b) If in case c) of theorem 2 M is a proper quotient then : Either the isometry γ is of finite order, then $S^1 \times_h M_*$ is a Riemannian covering and all geodesics normal to M_* are closed or otherwise no normal geodesic closes.

3 Generalized Killing spinors with imaginary Killing function

(3.1) We assume that ψ is a generalized Killing spinor with imaginary Killing function ib. From the definitions $f := \langle\psi,\psi\rangle$ and $\langle V, X\rangle := i\langle\psi, X\cdot\psi\rangle$ it follows immediately that

$$\nabla f = 2bV \quad , \quad \nabla_X V = 2bfX \quad . \tag{3}$$

Hence V is closed and since $L_V g = 4bfg$ we have that V is a non–isometric conformal Killing field. It follows from equation (3) that $q_\psi := f^2 - \|V\|^2$ is constant. Let

$$Q(X) := \|X\cdot\psi - i\psi\|^2 = \|X\|^2 f + 2\langle V, X\rangle + f \ ,$$

then X is a minimum of Q if $X = -V/f$ and $q_\psi = fQ(-V/f) \geq 0$. Hence q_ψ is a non–negative constant and if $q_\psi = 0$ then $(-V/f)\cdot\psi = i\psi$, cf. [10].

(3.2) Let $(e_1,\ldots,e_n)$ be a local orthonormal frame, then $D\psi = \sum_{i=1}^n e_i\cdot\nabla_{e_i}\psi$ is the *Dirac operator*, $\nabla^*\nabla\psi = -\sum_{i=1}^n \nabla_{e_i}\nabla_{e_i}\psi + \nabla_{\nabla_{e_i}e_i}\psi$ is the *connection Laplacian* . We obtain

$$D\psi = -ibn\psi \ , \ D^2\psi = -b^2n^2\psi - in\nabla b\cdot\psi$$

and

$$\nabla^*\nabla\psi = -b^2n\psi - i\nabla b\cdot\psi \ .$$

Then one obtains from Lichnerowicz's formula $D^2 = \nabla^*\nabla + \frac{1}{4}r$ where r is the scalar curvature that $\nabla b\cdot\psi = i\left(b^2n + r/(4n-4)\right)\psi$. Hence

$$|\langle\nabla b, V\rangle| = |i\langle\nabla b\cdot\psi,\psi\rangle| = \|\nabla b\| f \leq \|\nabla b\|\|V\|.$$

If b is non–constant then it follows that $q_\psi \leq 0$ i.e. $q_\psi = 0$.

Now we prove theorem 1 stated in the first section

Proof of Theorem 1.

a) If $q_\psi = 0$ then $\|V\| = f$ has no zero. By theorem 2c) M has the warped product $\mathbb{R}\times_f M_*$ of $\mathbb{R}$ with a complete $(n-1)$–dimensional manifold M_* as a Riemannian covering. Here $f(u,x) = f(u)$ is a function of $u\in\mathbb{R}$ alone and $V = f\frac{\partial}{\partial u}$ is the lift of V. Since $\nabla f = f'\frac{\partial}{\partial u} = 2bV = 2bf\frac{\partial}{\partial u}$ by equation (3) also b is a function of u alone and $b = f'/(2f)$.

Since

$$g = du^2 + f^2(u)g_* = f^2(u)(dv^2 + g_*)$$

with $\frac{dv}{du} = \frac{1}{f^2(u)}$ we compare the conformally equivalent metrics g and $\overline{g} = dv^2 + g_*$, i.e. $\overline{g}$ is the product metric on $\mathbb{R}\times M_*$. Let $f = \exp(-h)$. $\overline{g}$ induces on $\mathbb{R}\times M_*$ a spinor bundle $\overline{S}$, such that $S \longrightarrow \overline{S}, \phi\mapsto\overline{\phi} = \exp(h/2)\phi$ is an isometry. Let $\psi_1 := \exp(h/2)\psi$, then it follows from the formula [1, 3.2.4] :

$$\overline{\nabla}_X\overline{\psi_1} = \overline{\nabla_X\psi_1} - \frac{1}{2}\overline{X\cdot\nabla h\cdot\psi_1} - \frac{1}{2}X(h)\overline{\psi_1}$$

that $\overline{\psi_1}$ is a parallel spinor of $\mathbb{R}\times M_*$, cf. [17, §4]. This implies that M_* carries a parallel spinor , cf. [3, lem.4] .

b) If $q_\psi > 0$ then b is a non–zero constant by (3.2), i.e. ψ is a Killing spinor with imaginary Killing number b and (M,g) is an Einstein manifold with negative scalar curvature $r = -4n(n-1)b^2$. From equation (3) it follows that $\nabla^2 f = 4b^2 fg$, hence $\nabla f = 2bV$ is a non–homothetic conformal Killing field. If f has no critical point then by theorem 2 resp. the classification by Y.Tashiro [19, lem.2.2] we have that M is isometric to $\mathbb{R} \times_{f'} M_*$ where f is a function of $u \in \mathbb{R}$ alone. f satisfies $f'' = 4b^2 f$ and since f and f' both have no zero $f' = 2bf$. Then $q_\psi = 0$.

Hence f has a critical point, so by theorem 2 M is conformally diffeomorphic to a simply–connected space of constant sectional curvature. Since M is Einstein with $r = -4n(n-1)b^2$ it follows that M is isometric to $\mathrm{H}^n(-4b^2)$.

Remark 2 Since in theorem 2 M_* is Ricci flat it follows from the formulae for the curvature tensor of a warped product that the scalar curvature r of M is given by

$$r = -(n-2)(n-1)\frac{f'^2}{f^2} - 2(n-1)\frac{f''}{f} \quad (4)$$

$$= -4n(n-1)b^2 - 4(n-1)b' \quad (5)$$

Proof of Corollary 2 .

a) Let $y := f^{n/2}$ then one obtains from equation (4) $y'' + rny/(4n-4) = 0$. If $r \geq 0$ then y has a zero since f is non–constant. This contradicts $f > 0$.

b) For constant r it follows from a) that $r < 0$. Let $a := 1/2(-rn/(n-1))^{1/2}$, i.e. $y'' - a^2 y = 0$. Since $y'/y = nb$ is non–constant y' has a zero . Let $y'(0) = 0$ then $y(u) = y(0)\cosh(au)$. By scaling g_* we can assume $y(0) = 1$.

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Mathematisches Institut der Universität Bonn
Wegelerstr.10, D–W–5300 Bonn 1 , Federal Republic of Germany