The linear quadratic optimal control problem for linear descriptor systems with variable coefficients

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Abstract

We study linear quadratic optimal control problems for linear variable coefficient descriptor systems. The generalization from the case of standard control problems leads to several difficulties. We will discuss a behavioural approach that solves some of these difficulties. Furthermore an analysis of general rectangular systems is given and generalized Euler-Lagrange equations and Riccati differential algebraic equations are discussed.

Keywords: Descriptor systems, differential algebraic equations, optimal control, strangeness index, Riccati differential algebraic equations, Euler-Lagrange equations, linear feedback.

AMS(MOS) subject classification: 93C50, 34H05, 34A40, 93B10, 93B11, 93B40

1 Introduction

In this paper we study the linear-quadratic optimal control problem to minimize

$$\frac{1}{2}\int_{t_0}^{t_f}[y(t)^T K_1(t)y(t) + u(t)^T K_2(t)u(t)]\,dt \tag{1}$$

subject to the linear variable coefficient descriptor control system

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t),\ x(t_0) = x_1^0 \tag{2}$$

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with output

$$y(t) = C(t)x(t) + D(t)u(t). \tag{3}$$

If we denote by $C^r([t_0, t_f], \mathcal{R}^{n,\ell})$ the set of r-times continuously differentiable functions from the interval $[t_0, t_f]$ to the vector space $\mathcal{R}^{n,\ell}$ of real $n \times \ell$ matrices then we assume that

$$\begin{array}{l} E(t), A(t) \in C([t_0, t_f], \mathcal{R}^{n,\ell}), \\ B(t) \in C([t_0, t_f], \mathcal{R}^{n,m}), \\ C(t) \in C([t_0, t_f], \mathcal{R}^{p,l}), \\ K_1(t) \in C([t_0, t_f], \mathcal{R}^{p,p}),\ K_1(t) = K_1(t)^T, \\ K_2(t) \in C([t_0, t_f], \mathcal{R}^{m,m}),\ K_2(t) = K_2(t)^T, \\ x(t) \in C^1([t_0, t_f], \mathcal{R}^{\ell}) \text{ is the state of the system,} \\ u(t) \in C([t_0, t_f], \mathcal{R}^{m}) \text{ is the control or input of the system,} \\ y(t) \in C([t_0, t_f], \mathcal{R}^{p}) \text{ is the output of the system.} \end{array} \tag{4}$$

Optimal control problems of this form arise in mechanical multibody systems [16, 15, 35] or electrical circuits [14] or as linearizations of general nonlinear systems along trajectories [9]. Note that in contrast to common believe, the correct local linearization of nonlinear implicit control problem leads to a linear system with variable coefficients and not to a time invariant system [9, 23].

The theory for the optimal control problem is closely related to the theory of the corresponding differential-algebraic equations, which has undergone several changes in the last 10 years. See [13, 7, 8, 4, 20, 31, 32, 21, 24, 22]. Several concepts to generalize the theory of linear differential-algebraic equations to descriptor systems have been discussed in the literature [27, 10, 11, 28, 6]. Our analysis will be based on the concepts introduced in [20, 21, 22]. For a comparison of the different concepts and their advantages and disadvantages, see [6].

In this paper we will discuss several topics. The first part is concerned with the analysis of the control system (2)–(3). We will show that we may consider the system as an underdetermined differential-algebraic equation if we do not distinguish between the states and the controls, i.e., if we take a behavioural approach [36]. For this approach we can employ the theory as well as the numerical procedures developed in [20, 21, 22] to identify dependent and free variables, which we can interpret as states or inputs, respectively. This method can be used in practice to determine from an arbitrary model a standard model with states, inputs and outputs and leads to a reduced order and much simpler system. We will discuss this subject in Section 2. If, as is the case in some applications, the controls and states are predetermined, i.e., have to be distinguished, then a more complicated approach can be taken to obtain a reduced order system. We will not discuss this approach here, see [6, 33, 34]. Both approaches lead to reduced order control systems of strangeness index 0 [20] or differentiation index 1 [4], for which the linear quadratic optimal control problem (1)–(3) is well studied, [19, 28, 30]. One can also apply the usual variational approach directly to the problem (1)–(3) and obtains as necessary and sufficient condition a system of generalized Euler-Lagrange equations. This topic is discussed in Section 3. The system of Euler-Lagrange equations is again a linear time varying differential-algebraic system,

but with two-point boundary conditions. We could apply the canonical form results of [20] to this system but this would mean that we destroy the generalized Hamiltonian structure of this system. We therefore simplify the system using equivalence transformations which respect the structure of the system. From the resulting form we can then obtain new existence and uniqueness results for the generalized Euler-Lagrange equations. We finally analyze, when the solution of the generalized Euler-Lagrange equations can be obtained via the solution of a differential-algebraic Riccati equation and show that this is in general not possible.

2 Reduced Order Systems

In this section we describe an approach to transform the control problem (1)–(3) to a simpler, square, reduced order system, which can be analyzed by standard theory. This can be done in different ways.

One approach that avoids mixing states and inputs is discussed in [6, 33, 34]. It is essentially obvious how the mainly theoretical techniques discussed there can be applied to the optimal control case. For this reason we refrain here from discussing this approach in detail and rather discuss a new approach that is based on the general existence and uniqueness theory for linear differential algebraic systems [20, 21, 22] and can be interpreted as a behavioural approach [36].

We introduce a new vector $z = \begin{bmatrix} x \\ u \end{bmatrix}$, which combines states and inputs, and we rewrite problem (1)–(3) as

$$\begin{aligned} \tilde{E}(t)\dot{z}(t) &:= \begin{bmatrix} E(t) & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} \\ &= \begin{bmatrix} A(t) & B(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} =: \tilde{A}(t)z(t) \end{aligned} \tag{5}$$

with initial condition

$$\tilde{z}(t_0) = x_1^0, \tag{6}$$

output

$$y(t) = \tilde{C}(t)z(t) := \begin{bmatrix} C(t) & D(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \tag{7}$$

and with the new cost functional

$$\int_{t_0}^{t_f} z(t)^T \tilde{K}(t) z(t) dt, \tag{8}$$

where

$$\tilde{K}(t) := \begin{bmatrix} C(t)^T K_1(t) C(t) & C(t)^T K_1(t) D(t) \\ D(t)^T K_1(t) C(t) & D(t)^T K_1(t) D(t) + K_2(t) \end{bmatrix}. \tag{9}$$

Note that in (5) the derivative of the original input $u(t)$ appears only formally.

We then apply the canonical form of [20, 22] to the homogeneous descriptor system

$$\tilde{E}(t)\dot{z}(t) = \tilde{A}(t)z(t). \tag{10}$$

There, a new index was introduced (the strangeness-index in the following abbreviated as s-index) which generalizes the differentiation index [4] to systems with nonunique solutions. For the exact relationship between these indices see [24]. The important result we need here for our homogeneous, rectangular system is the following [20]:

Theorem 1 *Let the s-index μ be well-defined for the pair of matrix functions $(\tilde{E}, \tilde{A})$ as in (10). Then the solution sets of system (10) and the differential-algebraic system*

$$\begin{array}{llll} a) & \dot{z}_1(t) & = \tilde{A}_{13}(t)z_3(t) & (d_\mu \text{ equations}) \\ b) & 0 & = z_2(t) & (a_\mu \text{ equations}) \\ c) & 0 & = 0 & (u_\mu^l \text{ equations}) \end{array} \tag{11}$$

are in one-to-one correspondence.

Proof. The proof follows directly from Theorem 5.1 in [20]. □

Remark 1 The assumptions of Theorem 1 are hidden in the statement that the s-index μ is well-defined. In [24] it is shown that this is the case if the ranks of certain submatrices of the system matrices are constant in the interval under consideration. These constant rank assumptions can be significantly relaxed, essentially what is needed is that the number of equations in parts a), b), c) of (11) is constant, see [24], while the index $\mu(t)$ may vary. Otherwise one can apply Theorem 1 piecewise to intervals, where the required quantities are constant.

If the system satisfies the assumptions of Theorem 1, then part c) displays the redundancy in the system. These equations can simply be removed.

Part b) is an algebraic constraint and fixes part of the mixed state/control vector to be zero in the basis obtained via the equivalence transformation. It also displays the restrictions that hold for the initial conditions.

Part a) represents a standard control problem for an ordinary differential equation, i.e.,

$$\dot{\tilde{x}} = \tilde{B}(t)\tilde{u}(t)$$

with control variables $\tilde{u}(t) = z_3(t)$, $\tilde{A} \equiv 0$, $\tilde{B}(t) = \tilde{A}_{13}(t)$ and $\tilde{x}(t) = z_1(t)$.

Applying the equivalence transformations leading to (11) also to (7) and (8), we obtain

$$y = \begin{bmatrix} \tilde{C}_1(t) & \tilde{C}_3(t) & \tilde{C}_3(t) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \tilde{C}_1(t)\tilde{x}(t) + \tilde{C}_3(t)\tilde{u}(t) \tag{12}$$

and the cost functional transforms analogously as

$$\int_{t_0}^{t_f} \begin{bmatrix} \tilde{x}(t) \\ \tilde{u}(t) \end{bmatrix}^T \begin{bmatrix} R_{11}(t) & R_{12}(t) \\ R_{21}(t) & R_{22}(t) \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{u}(t) \end{bmatrix} dt. \tag{13}$$

In this way we have obtained a standard optimal control problem for which the classical theory applies, see [1, 18] for the case of a semidefinite cost functional and [2] for the indefinite case.

Note also that the whole analysis of the properties of the system can actually be carried out numerically with the algorithm introduced in [22] that is implemented in the code GELDA [25].

From the mathematical point of view the presented procedure essentially solves the problem of analyzing the linear quadratic control problem. But from the point of view of applications it may not be satisfactory, since some of the components of $u(t)$ in the system may be fixed to be inputs, and cannot be mixed up with the states as done previously. This is certainly a disadvantage of this approach, but it can be avoided by using a seperate treatment of states and inputs as discussed in [6, 33, 34]. Another problem with the behavioural approach is that, at least formally, higher order derivatives of the original input function $u(t)$ occur in the canonical form. This means that the usual set of admissable input functions has to be restricted. It is currently under investigation what the exact differentiability conditions for $u(t)$ are.

Since both approaches have certain disadvantages, we are lead to the question whether we can directly apply the standard calculus of variations to (1)–(3) and then apply the analysis of the differential-algebraic system to the Euler-Lagrange equations. This topic will be discussed in the next section.

3 Generalized Euler-Lagrange equations

In the case of constant coefficient systems with nonsingular E it is well-known [1, 28] that a necessary condition for the optimal control is that a costate function $\lambda(t)$ exists, so that x, λ, u satisfy the two-point boundary value problem of Euler-Lagrange equations

$$\begin{aligned} \begin{bmatrix} E & 0 & 0 \\ 0 & -E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \\ \dot{u} \end{bmatrix} &= \begin{bmatrix} A & 0 & B \\ Q & A^T & S \\ S^T & B^T & R \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ u \end{bmatrix} \\ x(t_0) &= x^0, \; E^T\lambda(t_f) = 0, \end{aligned} \tag{14}$$

where $R := K_2 + D^T K_1 D$, $S := D^T K_1 C$, $Q := C^T K_1 C$.

If the matrix $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is positive semidefinite then (14) is also a sufficient condition. Furthermore if R is positive definite, then the optimal control is a linear feedback

$$u(t) = -R^{-1}(B^T X(t) E + S)x(t), \tag{15}$$

where $X(t)$ is the solution of the Riccati differential equation

$$\begin{aligned} -E^T\dot{X}(t)E &= E^TX(t)(A-BR^{-1}S^T)+(A-BR^{-1}S^T)^TX(t)E \\ &- SR^{-1}S^T - E^TX(t)BR^{-1}B^TE+Q,\ E^TX(t_f)E=0. \end{aligned} \tag{16}$$

The same results also hold in the variable coefficient case if $E(t)$ is invertible.

So it is natural to ask whether these results can be generalized to the case of singular $E(t)$. For constant coefficient systems of s-index zero this has been done in [3, 12, 27, 28] but, as we will show now, this result also holds in general in the variable coefficient case.

Theorem 2 *Suppose that the optimal control problem (1)–(3) has a solution, then there exists a costate function $\lambda(t)$ such that*

$$\begin{aligned} \begin{bmatrix} E(t) & 0 & 0 \\ 0 & -E(t)^T & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \\ \dot{u}(t) \end{bmatrix} \\ = \begin{bmatrix} A(t) & 0 & B(t) \\ Q(t) & A(t)^T+\dot{E}(t)^T & S(t) \\ S(t)^T & B(t)^T & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \\ u(t) \end{bmatrix} \\ x(t_0) = x^0,\ E(t_f)^T\lambda(t_f)=0, \end{aligned} \tag{17}$$

where $R(t) := K_2(t) + D(t)^TK_1(t)D(t)$, $S(t) := D(t)^TK_1(t)C(t)$ *and* $Q(t) := C(t)^TK_1(t)C(t)$.

Proof. The proof is analogous to the constant coefficient case, see [28]. In the following we omit the dependence on t in the matrices. Introduce a Hamiltonian

$$H(x,\lambda,u)=\frac{1}{2}\int_{t_0}^{t_f}[y^TK_1y+u^TK_2u+2\lambda^T(E\dot{x}-Ax-Bu)]dt. \tag{18}$$

and variations

$$x+\epsilon\delta x,\ u+\epsilon\delta u,\ \lambda+\epsilon\delta\lambda,$$

with $\delta x\in C^1([t_0,t_f],\mathcal{R}^\ell)$, $\delta\lambda\in C^1([t_0,t_f],\mathcal{R}^n)$, where δu is smooth enough so that

$$E\frac{d(x+\epsilon\delta x)}{dt}=A(x+\epsilon\delta x)+B(u+\epsilon\delta u) \tag{19}$$

has a classical solution for all $\epsilon\in\mathcal{R}$ small enough in modulus, see [20, 24]. Furthermore we require that $\delta x(t_0)=0$. To prove that we can couple the differential equation via a Lagrange multiplier to the cost functional we can apply the methods of Section 2 and consider the underlying system of s-index 0. For this system the standard variational approach applies after solving the algebraic equations, see [28, 6]. We have

$$\begin{aligned} & H(x+\epsilon\delta x,\lambda+\epsilon\delta\lambda,u+\epsilon\delta u)=H(x,\lambda,u) \\ + & \epsilon\frac{1}{2}\int_{t_0}^{t_f}\left(\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} C^TK_1C & C^TK_1D \\ D^TK_1C & D^TK_1D+K_2 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix}\right. \\ + & \delta\lambda^T(E\dot{x}-Ax-Bu) \\ + & \left.\lambda^T(E\dot{\delta x}-A\delta x-B\delta u)\right)dt+\mathcal{O}(\epsilon^2). \end{aligned}$$

Using partial integration we obtain

$$\int_{t_0}^{t_f} \lambda^T E\delta\dot{x} = \lambda^T E\delta x|_{t_0}^{t_f} - \int_{t_0}^{t_f} (\dot{\lambda}^T E + \lambda^T \dot{E})\delta x \; dt. \tag{20}$$

and hence

$$\begin{aligned} & H(x+\epsilon\delta x, \lambda+\epsilon\delta\lambda, u+\epsilon\delta u) = H(x,\lambda,u) + \epsilon\lambda^T E\delta x|_{t_0}^{t_f} \\ - \; & \epsilon\int_{t_0}^{t_f} [x^T C^T K_1 C + u^T D^T K_1 C - \lambda^T A - \dot{\lambda}^T E - \lambda^T \dot{E})\delta x \\ + \; & (x^T C^T K_1 D - \lambda^T B + u^T (D^T K_1 D + K_2))\; \delta u \\ + \; & (E\dot{x} - Ax - Bu)^T \delta\lambda]\; dt + \mathcal{O}(\epsilon^2). \end{aligned}$$

Using the freedom in the variation we obtain (17) as a necessary condition. Here we have to make use of the fact that the set of $k-$times differentiable functions is dense in the set of piecewise continuous functions, so we still obtain that the third equation is necessary despite the fact that higher smoothness conditions are required for $u(t)$, see [17] . □

We also have the corresponding sufficiency result:

Theorem 3 *Suppose that* $\begin{bmatrix} x_*(t) \\ \lambda(t) \\ u_*(t) \end{bmatrix}$ *satisfies the two point boundary value problem (17) and suppose further that the matrix valued function*

$$\begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix}$$

is positive semidefinite for all $t \in [t_0, t_f]$ *then*

$$\int_{t_0}^{t_f} \begin{bmatrix} x \\ u \end{bmatrix}^T \tilde{K}(t) \begin{bmatrix} x \\ u \end{bmatrix} \geq \int_{t_0}^{t_f} \begin{bmatrix} x_* \\ u_* \end{bmatrix}^T \tilde{K}(t) \begin{bmatrix} x_* \\ u_* \end{bmatrix} \tag{21}$$

for all $x(t)$, $u(t)$ *satisfying (2), where* $\tilde{K}(t)$ *is as in (9).*

Proof. The proof is identical with the proof for the constant coefficient case in [28]. □

Having obtained necessary and sufficient conditions for the optimal control we have to analyze the two-point boundary value problem (14) with respect to existence and uniqueness of solutions. Since this is a homogeneous linear problem, we can use the canonical form of [20] for the analysis, but this would mean that the special structure of the two system matrices is not respected in the analysis. Moreover again this would mean that inputs, costates and states are mixed and the different differentiability properties are not reflected. If, however, as is the case in most applications, the matrix $R = K_2(t) + D(t)^T K_1(t) D(t)$ is positive definite in $[t_0, t_f]$ (actually invertible is enough) then we can reduce (14) by solving for $u(t)$ and obtain

$$u(t) = R(t)^{-1}[S(t)^T x(t) + B(t)^T \lambda(t)], \tag{22}$$

where $\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$ satisfies

$$\begin{aligned} & \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} \qquad (23) \\ = & \begin{bmatrix} A - BR^{-1}S^T & -BR^{-1}B^T \\ -Q + SR^{-1}S^T & -(A - BR^{-1}S^T)^T - \dot{E}^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \\ =: & \begin{bmatrix} F & G \\ H & -F^T - \dot{E}^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}, \qquad (24) \\ x(t_0) = & x^0,\ E^T(t_f)\lambda(t_f) = 0. \end{aligned}$$

In this system, we do not have extra differentiability restrictions and hence we can directly apply [20, 22]. But this would destroy the structure, which is that of a generalized Hamiltonian system, see, e.g., [28]. In order to keep this structure we can use generalizations of different types of symplectic equivalence transformations.

The first type represents changes of bases in x and λ:

$$\begin{aligned} \tilde{E} &= T_1 E T_2,\ \tilde{F} = T_1 F T_2 - T_1 E \dot{T}_2,\ \tilde{H} = T_2^T H T_2, \qquad (25) \\ \tilde{G} &= T_1 G T_1^T,\ \tilde{x} = T_2^{-1} x,\ \tilde{\lambda} = T_1^{-1}\lambda, \end{aligned}$$

for nonsingular $T_1 \in C([t_0, t_f], \mathcal{R}^{n,n})$, $T_2 \in C([t_0, t_f], \mathcal{R}^{\ell,\ell})$. Note that if $T_1 = T_2^{-1}$, $E = I$ and $\ell = n$ then actually $\begin{bmatrix} T_1 & 0 \\ 0 & T_2^T \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_1^{-T} \end{bmatrix}$ is a symplectic matrix, see [28].

The second type of transformations is of the form

$$\begin{aligned} \tilde{E} &= E,\ \tilde{F} = F + VH,\ \tilde{H} = H \qquad (26) \\ \tilde{G} &= G - (F + VH)V^T - VF^T - V\dot{E}^T,\ \tilde{x} = x + V^T\lambda,\ \tilde{\lambda} = \lambda, \end{aligned}$$

for $V \in C([t_0, t_f], \mathcal{R}^{n,\ell})$ satisfying $VE^T = 0$. Again, if $\ell = n$ and $V = V^T$ then $\begin{bmatrix} I_n & V \\ 0 & I_n \end{bmatrix}$ is a symplectic matrix.

The third type of transformations is analogous to (26).

$$\begin{aligned} \tilde{E} &= E,\ \tilde{F} = F - GV^T,\ \tilde{H} = H + F^T V^T + VF - VGV^T - \dot{E}^T V^T \\ \tilde{G} &= G,\ \tilde{x} = x,\ \tilde{\lambda} = \lambda - V^T x, \qquad (27) \end{aligned}$$

for $V \in C([t_0, t_f], \mathcal{R}^{\ell,n})$ satisfying $VE = 0$. Again, if $\ell = n$ and $V = V^T$ then $\begin{bmatrix} I_n & 0 \\ V & I_n \end{bmatrix}$ is a symplectic matrix.

Using these transformations, we can now analyze system (23). If we assume that $\operatorname{rank} E(t)$ is constant then there exist nonsingular T_1, T_2 as in (25) such that

$$\begin{aligned} \tilde{E} &= T_1 E T_2 = \begin{bmatrix} I_{s_1} & 0 \\ 0 & 0 \end{bmatrix},\ \tilde{F} = T_1 F T_2 - T_1 E \dot{T}_2 = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} - \tilde{E}, \\ \tilde{H} &= T_2^T H T_2 = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},\ \tilde{G} = T_1 G T_1^T = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \qquad (28) \end{aligned}$$

In this new coordinate system the boundary value problem takes the form

$$\begin{bmatrix} I_{t_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{t_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & G_{11} & G_{12} \\ F_{21} & F_{22} & G_{21} & G_{22} \\ H_{11} & H_{12} & -F_{11}^T & -F_{21}^T \\ H_{21} & H_{22} & -F_{12}^T & -F_{22}^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$x_1(t_0) = x_1^0,\ x_2(t_0) = x_2^0 \quad , \quad \lambda_1(t_f) = 0. \tag{29}$$

We now have the obvious characterization when the system has s-index 0 (differentiation index 1).

Proposition 4 *Assume that in the homogenous system (23) $E(t)$ has constant rank in $[t_0, t_f]$. Then the s-index is 0 if and only if the matrix valued function*

$$\mathcal{H}_{22} := \begin{bmatrix} F_{22} & G_{22} \\ H_{22} & -F_{22}^T \end{bmatrix} \in C([t_0, t_f], \mathcal{R}^{n+\ell-2s_1, n+\ell-2s_1}) \tag{30}$$

is nonsingular for all $t \in [t_0, t_f]$.

Proof. The proof follows directly from the canonical form for such systems as derived in [20, 22]. □

If $\mathcal{H}_{22}$ is nonsingular, then we can solve for $\begin{bmatrix} x_2 \\ \lambda_2 \end{bmatrix}$ and obtain

$$\begin{bmatrix} x_2 \\ \lambda_2 \end{bmatrix} = - \begin{bmatrix} F_{22} & G_{22} \\ H_{22} & -F_{22}^T \end{bmatrix}^{-1} \begin{bmatrix} F_{21} & G_{21} \\ H_{21} & -F_{12}^T \end{bmatrix} \begin{bmatrix} x_1 \\ \lambda_1 \end{bmatrix} \tag{31}$$

Note that $\mathcal{H}_{22}$ is a *Hamiltonian matrix*, i.e., $\mathcal{H}_{22} J_{n-s_1, \ell-s_1}$ is symmetric, where $J_{n-s_1,\ell-s_1} := \begin{bmatrix} 0 & I_{\ell-s_1} \\ -I_{n-s_1} & 0 \end{bmatrix}$. Then it is clear that the inverse is also Hamiltonian satisfying that $\mathcal{H}_{22}^{-1} J_{n-s_1,\ell-s_1}^T$ is symmetric, and hence it is immediate that the reduced order system obtained from inserting (31) has the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{\lambda}_1 \end{bmatrix} = \tilde{\mathcal{H}} \begin{bmatrix} x_1 \\ \lambda_1 \end{bmatrix},\ x_1(t_0) = x_1^0,\ \lambda_1(t_f) = 0, \tag{32}$$

where

$$\tilde{\mathcal{H}} := \begin{bmatrix} F_{11} & G_{11} \\ H_{11} & -F_{11}^T \end{bmatrix} - \begin{bmatrix} F_{12} & G_{12} \\ H_{12} & -F_{21}^T \end{bmatrix} \begin{bmatrix} F_{22} & G_{22} \\ H_{22} & -F_{22}^T \end{bmatrix}^{-1} \begin{bmatrix} F_{21} & G_{21} \\ H_{21} & -F_{12}^T \end{bmatrix}$$

is again Hamiltonian, i.e. $\tilde{\mathcal{H}} J_{s_1,s_1}$ is symmetric. This is now a standard Hamiltonian system as it arises from a control problem for an ordinary differential equation. The analysis for this problem is well understood, [18]. Note that this result follows regardless whether the original system was rectangular or square. A difficulty may occur if the original initial

conditions were not consistent, since then (31) represents a linear relationship between $x_2(t_0), x_1(t_0)$ which are given and $\lambda_1(t_0), \lambda_2(t_0)$, which are determined via (32).

Before we study the case that $\mathcal{H}_{22}$ is singular, let us analyze what happens if $E(t)$ does not have constant rank. We consider two cases, that $E(t)$ changes rank at a single point or in a whole interval.

Suppose that $E(t)$ is analytic, which implies that the rank is constant except for isolated points t_i where it is smaller. Then we can carry out a transformation as in (28) and (leaving off the dependence on t) we obtain a system

$$\begin{bmatrix} \Sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & G_{11} & G_{12} \\ F_{21} & F_{22} & G_{21} & G_{22} \\ H_{11} & H_{12} & -F_{11}^T - \dot{\Sigma} & -F_{21}^T \\ H_{21} & H_{22} & -F_{12}^T & -F_{22}^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$
$$x_1(t_0) = x_1^0 \quad , \quad \lambda_1(t_f) = 0, \tag{33}$$

where $\Sigma(t)$ is a diagonal matrix which changes rank at t_i. Such a smooth transformation can for example be obtained via the smooth singular value decomposition [5, 37, 29]. Suppose that the matrix in (30) is nonsingular and we solve for the algebraic part as in (31). Then we obtain the boundary value problem

$$\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \lambda_1 \end{bmatrix} = \tilde{\mathcal{H}} \begin{bmatrix} x_1 \\ \lambda_1 \end{bmatrix}, \; x_1(t_0) = x_1^0, \; \lambda_1(t_f) = 0. \tag{34}$$

If a smooth solution to (34) exists, then necessarily

$$\tilde{\mathcal{H}}(t_i) \begin{bmatrix} x_1(t_i) \\ \lambda_1(t_i) \end{bmatrix} = 0.$$

This is a condition that can be easily checked.

If $E(t)$ changes rank on complete intervals, then we can consider the problem piecewise in intervals where the rank of $E(t)$ is constant and paste the solution together by choosing appropriate initial and termimal conditions in the subintervals, provided that the s-index is 0 in each of the subintervals. It is clear that if the rank increases in intermediate intervals, then we do not have enough boundary conditions and thus the system does not have a unique solution, but if the rank drops, then the extra algebraic equation will provide interior boundary conditions. We demonstrate this via an example.

Example 1 Let $[t_0, t_f] = [0, 3]$, and suppose that

$$E(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & t < 1 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & 1 \le t \le 2 \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & t > 2 \end{cases},$$

$F(t) = \begin{bmatrix} 1+t^2 & 0 \\ 0 & 1+t^2 \end{bmatrix}$, $G(t) = 4I_2$, $H(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $x_1(0) = 1$ and $\lambda_2(3) = 0$.

In the interval $[0, 1)$ we have

$$\begin{bmatrix} x_2(t) \\ \lambda_2(t) \end{bmatrix} = 0, x_1(0) = 1, \begin{bmatrix} \dot{x_1}(t) \\ \lambda_1(t) \end{bmatrix} = \begin{bmatrix} 1+t^2 & 4 \\ 1 & -(1+t^2) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \lambda_1(t) \end{bmatrix}.$$

In $[1, 2]$ both x and λ are zero and in $(2, 3]$

$$\begin{bmatrix} x_1(t) \\ \lambda_1(t) \end{bmatrix} = 0, \lambda_2(3) = 0, \begin{bmatrix} \dot{x_2}(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} 1+t^2 & 4 \\ 0 & -(1+t^2) \end{bmatrix} \begin{bmatrix} x_2(t) \\ \lambda_2(t) \end{bmatrix}.$$

Using the trivial solution in $[1, 2]$ we obtain that the other two subsystems are overdetermined, i.e. we have extra boundary conditions $\lambda_1(1) = 0$ and $x_2(2) = 0$.

The other important case that we need to consider is that of nonzero s-index, i.e., the Hamiltonian submatrix

$$\mathcal{H}_{22} = \begin{bmatrix} F_{22} & G_{22} \\ H_{22} & -F_{22}^T \end{bmatrix}$$

is singular. In order to be able to reduce the system further in a structured way we need to assume that there exist $V(t) \in C([t_0, t_f], \mathcal{R}^{\ell - s_1, n - s_1})$ such that

$$H_{22} + VF_{22} + F_{22}^T V^T - VG_{22}V^T = 0, \text{ for all } t \in [t_0, t_f] \tag{35}$$

This is an algebraic Riccati equation in each t and the solution can be obtained via the separation of invariant subspaces. Several conditions for the existence of an appropriate solution $V(t)$ are known, see [26]. If a solution to (35) exists then

$$\begin{bmatrix} I_{n-s_1} & 0 \\ V & I_{\ell - s_1} \end{bmatrix} \begin{bmatrix} F_{22} & G_{22} \\ H_{22} & -F_{22}^T \end{bmatrix} \begin{bmatrix} I_{\ell - s_1} & 0 \\ -V^T & I_{n-s_1} \end{bmatrix} \tag{36}$$

$$= \begin{bmatrix} F_{22} - G_{22}V^T & G_{22} \\ 0 & -F_{22}^T + VG_{22} \end{bmatrix}. \tag{37}$$

If $F_{22} - G_{22}V^T$ has constant rank, then let T_1, T_2 be nonsingular matrices of appropriate dimensions such that $T_1(F_{22} - G_{22}V^T)T_2 = I_{s_2}$. Applying a transformation of type (25) we obtain a transformed matrix

$$\begin{bmatrix} I_{s_2} & & \hat{G}_{11} & \hat{G}_{12} \\ & 0 & \hat{G}_{21} & \hat{G}_{22} \\ & & -I_{s_2} & \\ & & & 0 \end{bmatrix}. \tag{38}$$

Let

$$
\begin{aligned}
S_1 &:= \begin{bmatrix} I_{s_2} & & & \\ & I_{n-s_1-s_2} & \hat{G}_{21} & 0 \\ & & I_{s_2} & \\ & & & I_{\ell-s_1-s_2} \end{bmatrix}, \\
S_2 &:= \begin{bmatrix} I_{s_2} & & & -\hat{G}_{12} \\ & I_{n-s_1-s_2} & & 0 \\ & & I_{s_2} & \\ & & & I_{\ell-s_1} \end{bmatrix}, \\
S_3 &:= \begin{bmatrix} I_{s_2} & & \hat{G}_{11}/2 & 0 \\ & I_{n-s_1-s_2} & 0 & 0 \\ & & I_{s_2} & \\ & & & I_{\ell-s_1-s_2} \end{bmatrix}.
\end{aligned}
$$

Transforming with $S_2 S_1$ from the left and $S_2 S_3^{-1}$ from the right, we obtain

$$
\begin{bmatrix} I_{s_2} & & 0 & 0 \\ & 0 & 0 & \hat{G}_{22} \\ & & -I_{s_2} & \\ & & & 0 \end{bmatrix}. \tag{39}
$$

If $\hat{G}_{22}$ has constant rank, then using another transformation of the form (25) with $T_2 = I$ and T_1 such that $\hat{G}_{22} = T_1 \begin{bmatrix} D_{s_3} & 0 \\ 0 & 0 \end{bmatrix} T_1^T$ with D_{s_3} nonsingular, (for example via a spectral factorization), we obtain the required form.

Embedding the transformations into transformations for the complete system, we obtain that (29) is equivalent to

$$
\begin{bmatrix} I_{s_1} & & & & & & & \\ & 0 & & & & & & \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & I_{s_1} & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \\ \dot{\lambda}_4 \end{bmatrix}
$$

$$= \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & G_{11} & G_{12} & G_{13} & G_{14} \\ F_{21} & I_{s_2} & 0 & 0 & G_{21} & 0 & 0 & 0 \\ F_{31} & 0 & 0 & 0 & G_{31} & 0 & D_{s_3} & 0 \\ F_{41} & 0 & 0 & 0 & G_{41} & 0 & 0 & 0 \\ H_{11} & H_{12} & H_{13} & H_{14} & -F_{11}^T & -F_{21}^T & -F_{31}^T & -F_{41}^T \\ H_{21} & 0 & 0 & 0 & -F_{12}^T & -I_{s_2} & 0 & 0 \\ H_{31} & 0 & 0 & 0 & -F_{13}^T & 0 & 0 & 0 \\ H_{41} & 0 & 0 & 0 & -F_{14}^T & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}.$$

Consider the subsystem

$$\begin{bmatrix} F_{41} & G_{41} \\ H_{31} & -F_{13}^T \\ H_{41} & -F_{14}^T \end{bmatrix} \begin{bmatrix} x_1 \\ \lambda_1 \end{bmatrix} = 0. \tag{40}$$

This system yields an explicit relationship between x_1 and λ_1 and thus consistency conditions for the boundary values. If (40) does not have full row rank then the boundary value problem is not uniquely solvable, since then

$$\begin{bmatrix} -F_{41}^T & H_{13} & H_{14} \\ G_{14} & F_{13} & F_{14} \end{bmatrix} \begin{bmatrix} \lambda_4 \\ x_3 \\ x_4 \end{bmatrix} = w$$

does not determine $\begin{bmatrix} \lambda_4 \\ x_3 \\ x_4 \end{bmatrix}$ uniquely. If (40) has full row rank but not full column rank, then we may transform further with a transformation of type (25) and obtain

$$\begin{aligned} \begin{bmatrix} F_{41} \\ H_{31} \\ H_{41} \end{bmatrix} T_1 T_1^{-1} x_1 &= \begin{bmatrix} \tilde{F}_{41} & 0 \\ \tilde{H}_{31} & 0 \\ \tilde{H}_{41} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \\ \begin{bmatrix} G_{41} \\ -F_{13}^T \\ -F_{14}^T \end{bmatrix} T_2 T_2^{-1} \lambda_1 &= \begin{bmatrix} \tilde{G}_{41} & 0 \\ -\tilde{F}_{13}^T & 0 \\ -\tilde{F}_{14}^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{bmatrix} \end{aligned} \tag{41}$$

with block columns of sizes $\tilde{s}_1, s_1 - \tilde{s}_1$ in the first and $\tilde{p}_1, s_1 - \tilde{p}_1$ in the second matrix. If $\tilde{s}_1 \neq \tilde{p}_1$, then the Hamiltonian structure cannot be kept further, since solving for $\tilde{x}_1$, $\tilde{\lambda}_1$ will lead to a non-Hamiltonian restsystem. In this case we can carry on with the analysis as in [20], since it makes no sense to try keeping the structure. If $\tilde{p}_1 = \tilde{s}_1$, then we can embed the transformation (41) into one for the complete system and solve for $\tilde{x}_1$, $\tilde{\lambda}_1$, x_3, x_4, λ_3 and λ_4 and insert the solution. The resulting system then again has Hamiltonian structure and takes the form

$$\begin{bmatrix} I_{\tilde{s}_1} & & & \\ & 0 & & \\ & & I_{\tilde{s}_1} & \\ & & & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}_2 \\ \dot{x}_2 \\ \dot{\tilde{\lambda}}_2 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_{22} & F_{23} & \tilde{G}_{22} & G_{23} \\ \tilde{F}_{32} & I_{s_2} & \tilde{G}_{32} & 0 \\ \tilde{H}_{22} & H_{23} & -\tilde{F}_{22}^T & -F_{23}^T \\ \tilde{H}_{32} & 0 & -\tilde{F}_{23}^T & -I_{s_2} \end{bmatrix} \begin{bmatrix} \tilde{x}_2 \\ x_2 \\ \tilde{\lambda}_2 \\ \lambda_2 \end{bmatrix}. \tag{42}$$

This system now has s-index 0, and we could solve for x_2, λ_2, to obtain again a standard Hamiltonian problem as it arises from a control problem for ordinary differential equations.

Note that here again it is necessary that the solution for the dependent variables is well-conditioned if we want to apply this approach numerically and we see from the analysis that for a reasonable numerical approach that also preserves the Hamiltonian structure, we would need that the Euler-Lagrange equations have s-index 0.

Another possibility would be to try the common Riccati trick to transform the linear two-point boundary value problem into a nonlinear initial value problem, i.e., to solve the Euler-Lagrange equations via the Riccati differential-algebraic equation

$$\begin{aligned} -\frac{d}{dt}(E^T X E) &= E^T X(A - BR^{-1}S^T) + (A - BR^{-1}S^T)^T XE - \\ SR^{-1}S^T &- E^T XBR^{-1}B^T XE + Q,\ E^T(t_f)X(t_f)E(t_f) = 0. \end{aligned} \tag{43}$$

Unfortuntely this approach can only be used in very special cases, since for $E(t)$ singular the solutions of (43) and (17) are not related via (15) as in the case of non-singular $E(t)$. We illustrate the difficulties with (43) via an example.

Example 2 Consider the control problem

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,\ x_1(0) = 1,\ y = x \tag{44}$$

in $[0, 1]$ and the cost functional given by

$$K_1 = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix},\ K_2 = 1. \tag{45}$$

The Euler-Lagrange equations (17) then are

$$\begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 1 \\ -\alpha & 0 & 0 & -a_{21} & 0 \\ 0 & -\beta & -a_{12} & -a_{22} & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \\ u \end{bmatrix},$$

$$x_1(0) = 1,\ \lambda(1) = 0. \tag{46}$$

The associated Riccati differential-algebraic equation takes the form

$$\begin{aligned} \begin{bmatrix} -\dot{x}_{11} & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} a_{21}x_{21} & 0 \\ a_{12}x_{11} + a_{22}x_{21} & 0 \end{bmatrix} + \begin{bmatrix} a_{21}x_{21} & a_{12}x_{11} + a_{22}x_{21} \\ 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} - \begin{bmatrix} x_{12}^2 & 0 \\ 0 & 0 \end{bmatrix},\ x_{11}(1) = 0. \end{aligned} \tag{47}$$

We consider different cases:

Case I: $a_{12} = a_{21} = 0$, $a_{22}^2 + \beta \neq 0$, $\beta \neq 0$.

In this case (2) and (17) have strangeness index 0 (differentiation index 1). (17) has the unique solution

$$\begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \alpha(1-t) \\ 0 \\ 0 \end{bmatrix}$$

but (47) is not solvable, since $\beta \neq 0$.

If $\beta = 0$, then (43) is solvable but the solution is not unique since x_{22} is free. The nonuniqueness of the solution has alrady been observed in [19, 28]. Note that the optimal control given by (15) is never-the-less unique.

Case II: $a_{12}, a_{21} \neq 0$, $a_{22} = 0$, $\beta \neq 0$.

In this case (2) has s-index 1 (differentiation index 2) but (17) has s-index 0 and has a unique solution

$$\begin{bmatrix} x_1 \\ a_{12}/(\lambda_1 \beta) \\ \lambda_1 \\ a_{21} x_1 \\ -a_{21} x_1 \end{bmatrix},$$

where x_1, λ_1 are uniquely determined from the two point boundary value problem

$$\begin{bmatrix} \dot{x}_1 \\ \dot{\lambda}_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_{12}a_{21}/\beta \\ \alpha + a_{21}a_{12} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \lambda_1 \end{bmatrix}, \; x_1(0) = 1, \; \lambda_1(1) = 0. \tag{48}$$

Again (47) is not solvable, since $\beta \neq 0$.

Case III: $a_{12}, a_{21} \neq 0$, $a_{22} = 0$, $\beta = 0$. Then (2) has s-index 1 (differentiation index 2), and (17) has s-index 1 (differentiation index is not defined). x_2 can be freely chosen, so the solution of (17) is not unique, but solutions exist. From (43) we obtain $x_{11} = 0$, x_{22} is free and x_{12} is a solution of the quadratic equation

$$-x_{12}^2 + \alpha + 2a_{21}x_{21} = 0,$$

which may or may not have a real solution.

In this section we have seen that we can analyze directly the set of Euler-Lagrange equations via special equivalence transformations and obtain (under certain rank assumptions) that the system can be reduced to a smaller system which represents again a standard control problem for an ordinary differential equation. Only for this reduced problem the Riccati differential-algebraic equation (43) can be used directly. Otherwise we need restrictions in the cost functional and also we need that the Euler-Lagrange equation (17) has s-index 0. But even in this case the solution to (43) is unique only if $E(t)$ is nonsingular, while the solution to the set of Euler-Lagrange equation is unique. This shows that one has to be exremely careful, when transferring the standard theory for the control of ordinary differential equations to the case of differential-algebraic systems. .

4 Conclusion

We have analyzed linear quadratic optimal control problems with variable coefficients. Under suitable assumptions any such problem is equivalent to the analogous problem for a descriptor system with s-index 0. Several approaches can be taken, all having certain disadvantages. The most promising approach is to use directly the set of Euler-Lagrange equations and apply a modified version of the theory for the non structured case. The construction of this equivalent problem is, however, currently not possible via a numerically procedure. To achieve such a transformation is still an open problem and currently under investigation.

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References

[1] M. Athans and P.L. Falb. *Optimal Control.* Mac Graw Hill, New York, 1966.

[2] T. Basar and G.J. Olsder. *Dynamic Noncooperative Game Theory.* Academic Press, New York, 1982.

[3] D.J. Bender and A.J. Laub. The linear-quadratic optimal regulator for descriptor systems. *IEEE Transactions on Automatic Control*, AC-32:672–688, 1987.

[4] K. Brenan, S. Campbell, and L. Petzold. *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations.* Elsevier, North Holland, New York, NY, 1989.

[5] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols. Numerical computation of an analytic singular value decomposition of a matrix valued function. *Numerische Mathematik*, 60:1–40, 1991.

[6] R. Byers, P. Kunkel, and V. Mehrmann. Regularization of linear descriptor systems with variable coefficients. *SIAM J. Control and Optimization*, 35:117–133, 1997.

[7] S. L. Campbell. A general form for solvable linear time varying singular systems of differential equations. *SIAM Journal Mathematical Analysis*, 18:1101–1114, 1987.

[8] S. L. Campbell. The numerical solution of higher index linear time varying singular systems of differential equations. *SIAM Journal Scientific and Statistical Computation*, 6:334–348, 1988.

[9] S.L. Campbell. Linearization of DAEs along trajectories. *Zeitschrift für angewandte Mathematik und Physik*, 46:70–84, 1995.

[10] S.L. Campbell, N.K. Nichols, and W.J. Terrell. Duality, observability, and controllability for linear time-varying descriptor systems. *Circuits, Systems and Signal Processing*, 10:455–470, 1991.

[11] S.L. Campbell and W.J. Terrell. Observability for linear time-varying descriptor systems. *SIAM Journal Matrix Analysis and Applications*, 12:484–496, 1991.

[12] D. Cobb. Descriptor variable systems and optimal state regulation. *IEEE Transactions on Automatic Control*, AC-28:601–611, 1983.

[13] E. Griepentrog and R. März. *Differential-Algebraic Equations and Their Numerical Treatment.* Teubner Verlag, Leipzig, 1986.

[14] M. Günther and P. Rentrop. Multirate row-methods and latency of electric circuits. Technical Report TUM-M9208, Mathematisches Institut, TU München, München, FRG, 1992.

[15] S. Hagel. Systematische Modellbildung mechanischer Systeme auf der Basis von Teilkomponenten. In P.C. Müller et al, editor, *Notes of the Workshop 'Identifizierung-, Analyse und Entwurfsmethoden für mechanische Mehrkörpersysteme in Deskriptorform' Paderborn 1994*, pages 67–72, 1994.

[16] H. Hahn and R. Wehage. Dynamic simulation of terrain vehicles. In W. Schiehlen, editor, *Multibody Systems Handbook*, pages 491–503. Springer Verlag Berlin, 1990.

[17] M. R. Hestenes. *Calculus of Variations and Optimal Control Theory.* John Wiley & Sons, New York, 1966.

[18] H.W. Knobloch and H. Kwakernaak. *Lineare Kontrolltheorie.* Springer Verlag, Berlin, 1985.

[19] P. Kunkel and V. Mehrmann. Numerical solution of differential-algebraic Riccati equations. *Linear Algebra and its Applications*, 137/138:39–66, 1990.

[20] P. Kunkel and V. Mehrmann. Canonical forms for linear differential-algebraic equations with variable coefficients. *Journal of Computational and Applied Mathematics*, 56:225–251, 1994.

[21] P. Kunkel and V. Mehrmann. A new look at pencils of matrix valued functions. *Linear Algebra and its Applications*, 212/213:215–248, 1994.

[22] P. Kunkel and V. Mehrmann. A new class of discretization methods for the solution of linear differential-algebraic equations. *SIAM Journal on Numerical Analysis*, 33:1941–1961, 1996.

[23] P. Kunkel and V. Mehrmann. Regular solutions of nonlinear differential-algebraic equations and their numerical determination. Technical Report 96-11, Fak. f. Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, FRG, 1996.

[24] P. Kunkel and V. Mehrmann. Local and global invariants of linear differential-algebraic equation and their relation. *Electronic Transactions on Numerical Analysis*, 1997. to appear.

[25] P. Kunkel, V. Mehrmann, W. Rath, and J. Weickert. A new software package for the solution of linear differential algebraic equations. *SIAM J. Scientific Computations*, 18:115–138, 1997.

[26] P. Lancaster and L. Rodman. *Algebraic Riccati Equations*. Oxford University Press, Oxford, 1995.

[27] V. Mehrmann. Existence, uniqueness and stability of solutions to singular linear quadratic optimal control problems. *Linear Algebra and its Applications*, 121:291–331, 1989.

[28] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem*. Springer-Verlag, Berlin, 1991.

[29] V. Mehrmann and W. Rath. Numerical methods for the computation of analytic singular value decompositions. *Electronic Transactions on Numerical Analysis*, 1:72–88, 1993.

[30] P.C. Müller. Linear-quadratic optimal control of mechanical descriptor systems. In R. Mennicken U. Helmke and J. Sauer, editors, *Proceedings of the MTNS 94*, pages 361–366. Akademie Verlag, Berlin, 1994.

[31] P.J. Rabier and W.C. Rheinboldt. A geometric treatment of implicit differential–algebraic equations. *Journal on Differential Equations*, 109:110–146, 1995.

[32] P.J. Rabier and W.C. Rheinboldt. Classical and generalized solutions of time-dependent linear differential algebraic equations. *Linear Algebra and its Applications*, 1996. To appear.

[33] W. Rath. *Feedback design and regularization for linear descriptor systems with variable coefficients*. Dissertationsschrift, Fak. f. Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, FRG, 1996.

[34] W. Rath. Dervative and proportional state feedback for linear descriptor systems with variable coefficients. *Linear Algebra and its Applications*, 1997. To appear.

[35] B. Simeon, F. Grupp, C. Führer, and P. Rentrop. A nonlinear truck model and its treatment as a multibody system. Technical Report TUM-M9204, Mathematisches Institut, TU München, München, FRG, 1992.

[36] J.C. Willems. Paradigms and puzzles in the theory of dynamical systems. *IEEE Transactions on Automatic Control*, AC-36:259–294, 1991.

[37] K. Wright. Differential equations for the analytic singular value decomposition of a matrix. *Numerische Mathematik*, 3:283–295, 1992.