REGULARIZATION OF LINEAR DESCRIPTOR SYSTEMS WITH VARIABLE COEFFICENTS

RALPH BYERS *, PETER KUNKEL † AND VOLKER MEHRMANN ‡

Abstract. We study linear descriptor control systems with rectangular variable coefficient matrices. We introduce condensed forms for such systems under equivalence transformations and use these forms to detect, whether the system can be transformed to a uniquely solvable closed loop system via state or derivative feedback. We show that under some mild assumptions every such system consists of an underlying square subsystem that behaves essentially like a standard state space system, plus some solution components that are constrained to be zero.

Keywords: Descriptor systems, differential algebraic equations, condensed forms, smooth singular value decomposition, strangeness index, regularization, linear feedback

AMS(MOS) subject classification: 93C50, 34H05, 34A40, 93B10, 93B11, 93B40

1. Introduction. In this paper we study linear variable coefficient descriptor systems

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{1}$$

in the interval $[t_0, t_1] \subset \mathcal{R}$ together with an initial condition

$$x(t_0) = x_0. \tag{2}$$

If we denote by $C^r([t_0,t_1],\mathcal{C}^{n,\ell})$ the set of r-times continuously differentiable functions from the interval $[t_0,t_1]$ to the vector space $\mathcal{C}^{n,\ell}$ of complex $n \times \ell$ matrices then we assume that

$$\begin{array}{ll} E(t), A(t) \in C([t_0,t_1],\mathcal{C}^{n,\ell}), & \\ B(t) \in C([t_0,t_1],\mathcal{C}^{n,m}), & \\ x(t) \in C([t_0,t_1],\mathcal{C}^{\ell}) & \text{is the state of the system} \\ u(t) \in C([t_0,t_1],\mathcal{C}^{m}) & \text{is the control of the system.} \end{array} \tag{3}$$

Descriptor systems of the form (1) are used in modeling control problems for mechanical multibody systems [32, 30, 31] or electrical circuits [16]. They are also obtained as linearizations of general nonlinear systems along trajectories [6].

In order to study the properties of such systems one has to have first a good understanding of the behaviour of the corresponding differential algebraic equations. The main difficulty in doing this is that different generalizations of the concepts of solvability, index, etc from linear constant coefficient systems to variable coefficient systems are

* Department of Mathematics, University of Kansas, Lawrence, Kansas 66045, USA. Partial support received from National Science Foundation grant INT-8922444. Some support was also received from University of Kansas General Research Allocation 3514-20-0038.

† Fachbereich Mathematik, Carl von Ossietzky Universität, Postfach 2503, D-26111 Oldenburg Fed. Rep. Germany. Supported by DFG research grant Me 790/5-1 Differentiell algebraische Gleichungen.

‡ Fakultät für Mathematik, Technische Universität Chemnitz-Zwickau, D-09107 Chemnitz, Fed. Rep. Germany. Supported by DFG research grant Me 790/5-1 Differentiell algebraische Gleichungen.

possible and have been discussed in the literature, see [1, 18]. Based on these different concepts one can then obtain also different results for linear systems, although only few results have been achieved so far in this direction [7, 8]. In particular, the latter results use the solvability concepts for differential algebraic equations as described in [1, 9, 10, 6].

Now in recent papers, Kunkel and Mehrmann have discussed a more general solvability concept and have developped new canonical forms [18, 20], new existence and uniqueness theorems [18, 21] and new numerical methods [22, 23] for linear variable coefficient differential-algebraic equations. Extensions to this approach have recently been given by Rabier and Rheinboldt [27].

Several other concepts to study the index of differential-algebraic equations and their generalization to descriptor systems have also been discussed in the literature. The approach in [7, 8] is based on the solvability concept in [9]. Another approach based on generic solvability is discussed in [14]. Since it is our ultimate goal to develop numerical methods that allow the computation of the invariants in finite precision arithmetic, we do not discuss the generic approach here, which is suitable for computer algebra computation.

Instead we will briefly discuss the two different solvability concepts of [9] and [18] in Section 2 and give some extensions of solvability results discussed in [18, 20, 27].

We then show in Section 3 that analogous methods can be used to study the properties of linear descriptor systems with variable coefficients. We obtain condensed forms for linear systems from which properties of the system can be read off.

In Section 4 we then show that under some mild assumptions every rectangular variable coefficient descriptor system has an underlying square system which in principle behaves like a standard linear state space system, together with some purely algebraic equations and some solution components which are constrained to be zero.

In Section 5 we will then study the question whether the solvability properties of descriptor systems can be improved via different types of feedback, i.e. whether appropriate linear time-varying feedbacks can be chosen, so that the closed loop system is uniquely solvable for all consistent initial values. This topic has been discussed for linear, constant coefficient descriptor systems in [5]. There, it has been shown (in the square case $n = \ell$) that uncontrollable higher index modes are constrained to be zero. Thus, regularizable descriptor systems consist of a subsystem that can be made index one via feedback together with some zero components of the state. In this paper we come to essentially the same conclusions for time varying descriptor systems despite the fact that the transformations become more complicated in this case.

2. Existence and uniqueness of solutions of linear time varying DAEs. We begin our analysis of the descriptor system (1), (2) with the following definition from [18, 20] on the solvability of linear variable coefficient DAEs of the form

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \ t \in [t_0, t_1] \subset \mathcal{R} \tag{4}$$

with initial condition (2), E, A as in (3) and $f \in C([t_0, t_1], \mathcal{C}^n)$.

DEFINITION 2.1. A function $x : [t_0, t_1] \to \mathcal{C}^\ell$ is called *solution* of (4) if $x \in C^1([t_0, t_1], \mathcal{C}^\ell)$ and x satisfies (4) pointwise.
It is called *solution of the initial value problem* (4), (2) if x is solution of (4) and x satisfies (2).
An initial condition (2) is called *consistent* if the corresponding initial value problem is solvable, i.e. has at least one solution.

The definition of solvability is still a subject of discussion in the literature. Often it is required that a solution exists for all sufficiently differentiable inhomogeneities [1, Page 22]. Alternatively, only a well behaved manifold of solutions is required [28]. Also, unique dependence on initial conditions is sometimes incorporated in the definition of solvability [1, 7].

The difficulty is illustrated by the following examples.

EXAMPLE 1. Consider the DAE

$$\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(t), \ t \in [-1, 1].$$

We obtain that

$$x(t) = c(t) \begin{bmatrix} t \\ 1 \end{bmatrix}$$

is a solution for all $c \in C^1([-1, 1], \mathcal{C})$. So clearly a solution exists. Nonetheless this DAE is not solvable in the sense of [1].

EXAMPLE 2. The linear DAE

$$\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

has the solution $x_1(t) = -t\dot{f}_2(t) - f_1(t)$ and $x_2(t) = -f_2(t)$. Although the matrix E changes rank, no singularity appears in the solution. This DAE does not satisfy the hypothesis of the solvability theorem in [18], but as shown in [27] the requirement of constant rank for E can be relaxed in this case and an extension of the solvability theorem under weaker assumptions is true. This system also satisfies the hypothesis of solvability theorem in [1, Page 30].

EXAMPLE 3. A special case of (4) is the purely algebraic equation $0 = A(t)x(t) + f(t)$. If $A(t)$ is nonsingular, then there is a unique solution $x(t) = -A(t)^{-1}f(t)$ regardless of the smoothness of $f(t)$. However, this DAE is not solvable in the sense of [1] unless $f(t)$ is differentiable, because the equivalent ODE is $A(t)\dot{x}(t) = -\dot{A}(t)x(t) + \dot{f}(t)$. Although this system is also not solvable in the sense of [18], the normal form given in [18] exists and directly suggests the introduction of a weaker solution concept under which the problem is then solvable, see also [15]. If $A(t)$ drops rank for some t, then one has to apply the extension of the theory in [18, 20] given in [27] to show solvability. The weakness in [1], which is used in the context of control problems in [7, 8] is that it requires differentiability of all components of the inhomogeneity which is usually not the case in the applications from control. Another weakness of this concept is that

it does not apply to rectangular systems. The concept introduced in [18] with the extensions given in [20, 27] is more general, more suitable to control problems, allows rectangular systems and even distributional solutions. This is the reason why we use [18] in preference to [1]. In this paper we will, however, discuss only classical solutions in the sense of Definition 2.1.

REMARK 1. A simple but useful trick that removes part but not all of the discussed difficulties with the solvability concepts is the following. If we add the term $\dot{E}(t)x(t)$ on both sides of (4) we obtain

$$\frac{d}{dt}(E(t)x(t)) = (A(t) + \dot{E}(t))x(t) + f(t). \tag{5}$$

In this form we would have to require sufficient smoothness of $x(t)$ only in the range of $E(t)$. This would allow weaker differentiability assumptions for x on the cost of smoothness assumptions for E. Such an approach would be more suitable for index one problems in particular, since exactly the differentiability that is needed is displayed. But as is shown in [18] for higher index problems it is still not possible to identify the exact differentiability conditions without going to a canonical form. In order to avoid confusion with the existing literature we therefore use the solvability condition introduced in Definition 2.1.

Another modification of the solvability concept that has been used frequently with constant coefficient systems is to restrict the initial conditions to the range of E by requiring

$$E(t_0)x(t_0) = E(t_0)x_0. \tag{6}$$

This would be in line with (5). We will determine in general what the exact consistency conditions for the initial values are in the following. Since these not only include modifications like (6), but also others we will use the more general condition in Definition 2.1.

The standard variable coefficient transformations that can be applied to linear DAEs are changes of bases, i.e., $x(t) = Q(t)y(t)$ and pre–multiplication of (1) by $P(t)$. Under these transformations (4) transforms to

$$P(t)E(t)Q(t)\dot{y}(t) = (P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t))y(t) + P(t)f(t). \tag{7}$$

DEFINITION 2.2. Two pairs of matrix functions $(E_i(t), A_i(t))$, $E_i, A_i \in C([t_0, t_1], \mathcal{C}^{n,\ell}), i = 1, 2$ are called *equivalent* if there exist $P \in C([t_0, t_1], \mathcal{C}^{n,n})$ and $Q \in C^1([t_0, t_1], \mathcal{C}^{\ell,\ell})$ with $P(t), Q(t)$ nonsingular for all $t \in [t_0, t_1]$ such that

$$(E_2(t), A_2(t)) = P(t)(E_1(t), A_1(t)) \begin{bmatrix} Q(t) & -\dot{Q}(t) \\ 0 & Q(t) \end{bmatrix}. \tag{8}$$

Based on suitable equivalence transformations we will now extend the solvability theorems of [18, 20, 27].

To simplify the notation in the condensed forms, we denote in the following by $\Sigma_j(t)$ a square diagonal matrix valued function of dimension $j \times j$, which is invertible for

all but finitely many $t \in [t_0, t_1]$. We also denote blocks of a matrix which are not specifically needed but which are not necessarily identically zero by $*$ and zero blocks of all dimensions by 0.

We construct the condensed form via smooth unitary equivalence transformation. This displays the invariants of the system but does not produce a canonical form but rather a condensed staircase form in the sense of Van Dooren [33] from which invariants can be read off. To do the transformations we use smooth singular value decompositions as they were introduced in [2] and for which several numerical methods are available [2, 25, 34]. To apply these transformations, we need derivatives of the right transformations. These can be obtained numerically from the original matrix function $E(t)$ and its derivatives using the method described in [17]. Note, however, that the following theorem and the construction procedure in the proof cannot be applied as a practical numerical algorithm. Nevertheless it gives an indication how the computation of the invariants can be carried out locally. For a discussion of this topic for DAEs see [18], the extension to descriptor systems is currently investigated. One step of the construction given in the proof of the following theorem can be carried out locally. If more steps are needed, then local computation is not applicable in the given form. For the use of determining the required information on the global invariants via local computation see [22].

THEOREM 2.3. *Given analytic matrix valued functions $E(t)$, $A(t)$ as in (3) there exist unitary analytic matrix valued functions $P(t)$, $Q(t)$, as in (8) such that the matrices $P(t)E(t)Q(t)$, $P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t)$ have the following form*

$$\begin{array}{c} \\ d_\mu \\ a_\mu \\ u_\mu^l \end{array} \begin{bmatrix} E_{11}(t) & E_{12}(t) & 0 \\ 0 & E_{22}(t) & 0 \\ 0 & E_{32}(t) & 0 \end{bmatrix}, \begin{bmatrix} A_{11}(t) & A_{12}(t) & A_{13}(t) \\ 0 & A_{22}(t) & 0 \\ 0 & A_{32}(t) & 0 \end{bmatrix} \tag{9}$$

where $E_{11}(t)$ is diagonal and nonsingular except for isolated points, $E_{22}(t)$, $E_{32}(t)$ and $A_{32}(t)$ are block upper triangular with zero diagonal blocks and $A_{22}(t)$ is block upper triangular with diagonal block which are nonsingular except for isolated points. The block columns have sizes d_μ, a_μ, u_μ^r. (Note that 0×0 matrices are diagonal and invertible, e.g. [11].)

Proof. The proof is constructive using a sequence of analytic singular value decompositions (ASVDs), see [2]. In the following we drop the dependence on t in the formulas. Consider the following recursive procedure:

Begin: Let $E_0 = E$, $A_0 = A$ and set $j = 0$, $n_j = n$, $\ell_j = \ell$.

1) Let P_1, Q_1 be unitary matrices of appropriate dimensions that produce an ASVD of E_0 such that

$$E_1 := P_1^* E_0 Q_1 = \begin{bmatrix} \Sigma_{r_j} & 0 \\ 0 & 0 \end{bmatrix}.$$

Set

$$A_1 := P_1^* A_0 Q_1 - P_1^* E_0 \dot{Q}_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

If $r_j = n_j$, i.e. E_1 has full row rank except for isolated points, then we **STOP** the process here.

2) Let $\tilde{P}_2, Q_2$ be unitary matrices that produce a permuted ASVD of $\begin{bmatrix} A_{21} & A_{22} \end{bmatrix}$:

$$\tilde{P}_2^* \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} Q_2 = \begin{bmatrix} 0 & \Sigma_{t_j} \\ 0 & 0 \end{bmatrix}.$$

Set

$$P_2 := \begin{bmatrix} I_{r_j} & 0 \\ 0 & \tilde{P}_2 \end{bmatrix},$$

and set

$$E_2 := P_2^* E_1 Q_2 = \left[\begin{array}{c|c} \tilde{E}_{11} & \tilde{E}_{12} \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right], \; A_2 := P_2^* A_1 Q_2 - P_2^* E_1 \dot{Q}_2 = \left[\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline 0 & \Sigma_{t_j} \\ 0 & 0 \end{array}\right],$$

The row dimensions are now $r_j, t_j, p_j = n_j - r_j - t_j$ and the column dimensions are $\ell_j - t_j, t_j$.

We then set $n_{j+1} := r_j$, $\ell_{j+1} := \ell_j - t_j$, $E_0 := \tilde{E}_{11}$, $A_0 := \tilde{A}_{11}$ and $j = j+1$ and repeat the process from Step 1) by applying transformations always to the complete system via an appropriate embedding of the transformation matrices.

end

It is clear that the procedure is finite, since n_j decreases in each step. At the end of this recursion we have the form

$$P(t)E(t)Q(t) = \left[\begin{array}{c|c|c|c|c|c} \Sigma_{n_\mu}(t) & 0 & * & * & \dots & * \\ \hline 0 & 0 & 0 & * & \dots & * \\ 0 & 0 & 0 & * & \dots & * \\ \hline & & & 0 & \ddots & \vdots \\ & & & 0 & & \vdots \\ \hline & & & & \ddots & \vdots \\ \hline & & & & & 0 \\ & & & & & 0 \end{array}\right] \begin{array}{l} n_\mu \\ \hline t_{\mu-1} \\ p_{\mu-1} \\ \hline t_{\mu-2} \\ p_{\mu-2} \\ \hline \vdots \\ \hline t_0 \\ p_0 \end{array}, \tag{10}$$

$$P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t) = \left[\begin{array}{c|c|c|c|c|c} \hat{A}_{11}(t) & \hat{A}_{12}(t) & * & * & \dots & * \\ \hline 0 & 0 & \Sigma_{t_{\mu-1}}(t) & * & \dots & * \\ 0 & 0 & 0 & * & \dots & * \\ \hline & & & \Sigma_{t_{\mu-2}}(t) & \dots & * \\ & & & 0 & \dots & * \\ \hline & & & & \ddots & \vdots \\ \hline & & & & & \Sigma_{t_0}(t) \\ & & & & & 0 \end{array}\right] \begin{array}{l} n_\mu \\ \hline t_{\mu-1} \\ p_{\mu-1} \\ \hline t_{\mu-2} \\ p_{\mu-2} \\ \hline \vdots \\ \hline t_0 \\ p_0 \end{array}. \tag{11}$$

We then set $E_{11}(t) = \Sigma_{n_\mu}(t)$ and permute the second block column to the end and the block rows in the order $1, 2, 4, 6, \ldots, 3, 5, 7, \ldots$ to obtain the final form with $d_\mu = n_\mu$, $a_\mu = t_0 + \ldots + t_\mu$, $u_\mu^l = n - a_\mu - d_\mu$ and $u_\mu^r = \ell - a_\mu - d_\mu$. □

REMARK 2. Note that the analyticity assumption on the coefficient matrices can be relaxed to the condition that all smooth singular value decompositions exist and that rank changes in the factored matrices occur only at isolated points. This property is, however, hard to quantify, since already infinite differentiability of the coefficient matrices may not be enough to guarantee the existence of such a decomposition with once differentiable unitary factors, see [2]. The construction given in the proof of Theorem 2.3 is similar to the construction given in [18] but it needs fewer assumptions, in particular no constant rank assumptions are needed.

REMARK 3. The block sizes $t_{\mu-1}, \ldots, t_0$, $p_{\mu-1}, \ldots, p_0$ can be combined to determine the invariants of the equivalence transformation but the form (10) is not a canonical form but rather a condensed form, since not all the invariants are displayed. However, the quantities $d_\mu, a_\mu, u_\mu^\ell, u_\mu^r$ are invariants, see [18], and they determine existence and uniqueness as is shown in the following corollary. The condensed form can be viewed as an analogue to the staircase form of Van Dooren [33] compared with the Kronecker canocical form for constant matrix pencils. The reason why such condensed forms are useful is that they allow to compute the relevant information via unitary transformations, which can be implemented in a numerically stable way. The quantity μ is called the *strangeness index* of the DAE and it is a generalization of the differentiation index, e.g. [1, 18, 27, 21]. A variant of the solvability theorem of [20] is then:

COROLLARY 2.4. *Let $(E(t), A(t))$ be analytic matrix valued functions as in (3) and $f \in C^\mu([t_0, t_1], \mathcal{C}^n)$. Then, (4) is equivalent to a differential–algebraic equation of the form*

$$
\begin{array}{ll}
(a) & \Sigma_{d_\mu}(t)\dot{x}_1(t) + E_{12}(t)\dot{x}_2(t) = \\
 & \qquad A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + A_{13}(t)x_3(t) + f_1(t) \\
(b) & E_{22}(t)\dot{x}_2(t) = A_{22}(t)x_2(t) + f_2(t) \\
(c) & E_{32}(t)\dot{x}_2(t) = A_{32}(t)x_2(t) + f_3(t),
\end{array}
\tag{12}
$$

where the inhomogeneity is determined by $f^{(0)}$, ..., $f^{(\mu)}$ and $E_{22}(t)$, $E_{32}(t)$, $A_{22}(t)$, $A_{32}(t)$ are as in (9). In particular, d_μ, a_μ, u_μ^r are the number of differential, algebraic and undetermined components of the unknown x in (a), (b) respectively, while u_μ^l is the number of conditions in (c). In particular, if in addition $f \in C^{\mu+1}([t_0, t_1], \mathcal{C}^n)$, then we have:

Equation (4) is solvable if and only the following properties hold:

i) At isolated points where the diagonal matrix $\Sigma_{d_\mu}(t)$ or the diagonal blocks of $A_{22}(t)$ are singular, the components in $x_3(t)$ (if they occur) can be chosen so that the solution can be completed in a continuously differentiable way.

ii) The conditions (12(b)) are satisfied at the initial point.

iii) The u_μ^l functional consistency conditions (12(c)) are satisfied for $x_2(t)$ which is fixed by (12(b)). (Observe that (12(b)) fixes $x_2(t)$ by recursive insertion and differentiation except for points where the diagonal blocks of $A_{22}(t)$ are singular, using the nilpotency structure of $E_{22}(t)$.)

An initial condition (2) is consistent if and only if the a_μ conditions

$$E_{22}(t_0)\dot{x}_2(t_0) = A_{22}(t_0)x_2(t_0) + f_2(t_0) \tag{13}$$

yield an $x_2(t_0)$ which coincides with solution of (12(b)) at t_0.

The initial value problem (1), (2) is uniquely solvable if we also have

$$u_\mu^r = 0. \tag{14}$$

Otherwise, we can choose $x_3(t) \in C^1([t_0, t_1], \mathcal{C}^{u_\mu^r})$ arbitrarily.

Proof. The proof follows directly from Theorem 2.3. Considering the second block equation, we obtain from the form of $E_{22}(t), A_{22}(t)$, that we can recursively solve for the solution components. The diagonal blocks of $A_{22}(t)$ are diagonal matrices which are invertible except possibly at isolated points. In these points we have to be able to complete the solution in a smooth way, since these components then have to be differentiated to continue the solution process. There are μ differentiations necessary to completely solve for the second block. Inserting $\dot{x}_2, x_2$ and chosing x_3 we can solve equation (12(a)) except at points where the matrix $\Sigma_{d_\mu}(t)$ is singular. The same argument as above applies in these points. The remaining assertions are straightforward. □

In this section we have given a generalization of the condensed form and the solvability results of [18, 20, 27]. We do not need to apply constant rank assumptions, but we need assumptions that guarantee the existence of all ASVDs according to Remark 2. Here we could generalize the construction to weak solvability, which would allow us to drop some further smoothness assumptions, see [27]. In the next section we perform the corresponding construction for linear systems.

3. Condensed forms for linear descriptor systems. In this section we discuss the set of equivalence transformations that we will apply to variable coefficient descriptor systems and canonical forms under these transformations. Using these forms we then obtain information about the system properties. For constant coefficient systems such forms have been studied for general transformations in [24], and for unitary transformations in [3, 4, 5]. The results that we give here generalize the results for the unitary case even for constant coefficient systems.

Observe that we cannot apply directly the solvability result of Section 2, since *usually we cannot assume* that the input functions $u(t)$ are sufficiently differentiable. In principle we can apply differentiation of components only in the uncontrollable subspace, i.e. the part of the system operating in the left nullspace of $B(t)$. Note that this is a major difference to the approach in [7, 8], where it is assumed that the input functions are sufficiently smooth.

We use the following global equivalence transformations for the triple of matrix valued functions $(E(t), A(t), B(t))$.

DEFINITION 3.1. Two triples of matrix functions $(E_i(t), A_i(t), B_i(t))$, $E_i, A_i \in C([t_0, t_1], \mathcal{C}^{n,\ell})$, $B_i \in C([t_0, t_1], \mathcal{C}^{n,m}), i = 1, 2$ are called *equivalent* if there exist

$P \in C([t_0,t_1],\mathcal{C}^{n,n})$, $Q \in C^1([t_0,t_1],\mathcal{C}^{\ell,\ell})$ and $W \in C([t_0,t_1],\mathcal{C}^{m,m})$ with $P(t), Q(t), W(t)$ nonsingular for all $t \in [t_0,t_1]$ such that

$$(15) \quad (E_2(t),A_2(t),B_2(t)) = P(t)(E_1(t),A_1(t),B_1(t)) \begin{bmatrix} Q(t) & -\dot{Q}(t) & 0 \\ 0 & Q(t) & 0 \\ 0 & 0 & W(t) \end{bmatrix}.$$

It is easily checked that the above transformations describe equivalence transformations.

We obtain the following condensed form:

THEOREM 3.2. *Given analytic matrix valued functions $E(t)$, $A(t)$, $B(t)$ as in (3) there exist unitary analytic matrix valued functions $P(t)$, $Q(t)$, $W(t)$ as in (15) such that the three matrices $P(t)E(t)Q(t)$, $P(t)A(t)Q(t)-P(t)E(t)\dot{Q}(t)$, $P(t)B(t)W(t)$ have the following form*

$$(16) \quad \begin{array}{c} d_\nu \\ v_\nu \\ \hline s_\nu \\ \hline \tilde{u}^l_\nu \\ \hline u^l_\nu \end{array} \left[\begin{array}{cc|c|c} \Sigma_{d_\nu}(t) & 0 & E_{13}(t) & 0 \\ 0 & 0 & E_{23}(t) & 0 \\ \hline 0 & 0 & E_{33}(t) & 0 \\ \hline 0 & 0 & E_{43}(t) & 0 \\ \hline 0 & 0 & E_{53}(t) & 0 \end{array}\right], \quad \left[\begin{array}{cc|c|c} A_{11}(t) & A_{12}(t) & A_{13}(t) & A_{14}(t) \\ A_{21}(t) & A_{22}(t) & A_{23}(t) & A_{24}(t) \\ \hline 0 & 0 & A_{33}(t) & 0 \\ \hline A_{41}(t) & A_{42}(t) & A_{43}(t) & A_{44}(t) \\ \hline 0 & 0 & A_{53}(t) & 0 \end{array}\right], \quad \left[\begin{array}{ccc} B_{11}(t) & B_{12}(t) & B_{13}(t) \\ \Sigma_{v_\nu}(t) & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & \Sigma_{\tilde{u}^l_\nu}(t) & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

with $E_{33}(t)$ block upper triangular with zero diagonal blocks, and $A_{33}(t)$ block upper triangular with diagonal blocks which are diagonal matrices that are nonsingular except for isolated points. The block columns in E, A have sizes d_ν, v_ν, s_ν, u^r_ν.

Proof. The proof is constructive using again a sequence of analytic singular value decompositions (ASVDs). We again drop the dependence on t in the formulas. Consider the following recursive procedure:

Begin:

Let $E_0 = E$, $A_0 = A$, $B_0 = B$ and set $j = 0$, $n_j = n$, $\ell_j = \ell$.

1) Let P_1, Q_1 be unitary matrices of appropriate dimensions that produce an ASVD of E_0 such that

$$E_1 := P_1^* E_0 Q_1 = \begin{bmatrix} \Sigma_{d_j} & 0 \\ 0 & 0 \end{bmatrix}.$$

Set

$$A_1 := P_1^* A_0 Q_1 - P_1^* E_0 \dot{Q}_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B_1 := P_1 B_0 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}.$$

2) Let $\tilde{P}_2, W_2$ be unitary matrices of appropriate dimensions that produce an ASVD of B_{21}. Set

$$\tilde{P}_2^* B_{21} W_2 = \begin{bmatrix} \Sigma_{c_j} & 0 \\ 0 & 0 \end{bmatrix}, \; P_2 := \begin{bmatrix} I_{d_j} & 0 \\ 0 & \tilde{P}_2 \end{bmatrix},$$

and

$$E_2 := P_2^* E_1 = \begin{bmatrix} \Sigma_{d_j} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; A_2 := P_2^* A_1 = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \\ \tilde{A}_{31} & \tilde{A}_{32} \end{bmatrix},$$
$$B_2 := P_2^* B_1 W_2 = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \Sigma_{c_j} & 0 \\ 0 & 0 \end{bmatrix}.$$

If B_{21} had full row rank except at isolated points, then we **STOP** the process here.

3) Let $\tilde{P}_3, Q_3$ be unitary matrices that produce a permuted ASVD of $\begin{bmatrix} \tilde{A}_{31} & \tilde{A}_{32} \end{bmatrix}$. Set

$$\tilde{P}_3^* \begin{bmatrix} \tilde{A}_{31} & \tilde{A}_{32} \end{bmatrix} Q_3 = \begin{bmatrix} 0 & \Sigma_{k_j} \\ 0 & 0 \end{bmatrix}, \; P_3 := \begin{bmatrix} I_{d_j} & 0 & 0 \\ 0 & I_{c_j} & 0 \\ 0 & 0 & \tilde{P}_3 \end{bmatrix}$$

and

$$E_3 := P_3^* E_2 Q_3 = \left[\begin{array}{c|c} \hat{E}_{11} & \hat{E}_{12} \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right], \; A_3 := P_3^* A_2 Q_3 - P_3^* E \dot{Q}_3 = \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \\ \hline 0 & \Sigma_{k_j} \\ 0 & 0 \end{array}\right], \tag{17}$$
$$B_3 := P_3^* B_2 = \left[\begin{array}{c|c} \hat{B}_{11} & \hat{B}_{12} \\ \Sigma_{c_j} & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right].$$

with block rows of sizes $d_j, c_j, k_j, q_j = n_j - d_j - c_j - k_j$ and block columns of sizes $\ell_j - k_j, k_j$ in E_3, A_3, and $c_j, m - c_j$ in B_3.

Now we set $n_{j+1} = d_j + c_j$, $\ell_{j+1} = \ell_j - k_j$ and we set E_0, A_0 to be the $n_{j+1} \times \ell_{j+1}$ upper left submatrices of E_3, A_3 and B_0 to be the upper $n_{j+1} \times m$ submatrix of B_3. Set $j = j + 1$ and repeat the process from Step 1) by applying transformations always to the complete system via an appropriate embedding of the transformation matrices.
end

It is clear that the procedure is finite, since n_j decreases in each step. At the end of

this recursion we have the following forms for the transformed E, A, B:

$$\left[\begin{array}{cc|c|c|c|c} \Sigma_{d_\nu}(t) & 0 & * & * & \dots & * \\ 0 & 0 & * & * & \dots & * \\ \hline & & 0 & * & \dots & * \\ & & 0 & * & \dots & * \\ \hline & & & \ddots & \vdots & \\ \hline & & & & 0 & * \\ & & & & 0 & * \\ \hline & & & & & 0 \\ & & & & & 0 \end{array}\right] \begin{array}{l} d_\nu \\ c_\nu \\ \hline k_{\nu-1} \\ q_{\nu-1} \\ \hline \vdots \\ \hline k_1 \\ q_1 \\ \hline k_0 \\ q_0 \end{array},$$

$$\left[\begin{array}{cc|c|c|c} A_{11}(t) & A_{12}(t) & * & * & \dots & * \\ A_{21}(t) & A_{22}(t) & * & * & \dots & * \\ \hline & & \Sigma_{k_{\nu-1}}(t) & * & \ddots & \vdots \\ & & 0 & * & & \vdots \\ \hline & & & \ddots & \ddots & \vdots \\ \hline & & & & \Sigma_{k_1}(t) & * \\ & & & & 0 & * \\ \hline & & & & & \Sigma_{k_0}(t) \\ & & & & & 0 \end{array}\right] \begin{array}{l} d_\nu \\ c_\nu \\ \hline k_{\nu-1} \\ q_{\nu-1} \\ \hline \vdots \\ \hline k_1 \\ q_1 \\ \hline k_0 \\ q_0 \end{array},$$

$$\left[\begin{array}{cc} B_{11}(t) & B_{12}(t) \\ \Sigma_{c_\nu}(t) & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline \vdots & \vdots \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \begin{array}{l} d_\nu \\ c_\nu \\ \hline k_{\nu-1} \\ q_{\nu-1} \\ \hline \vdots \\ \hline k_1 \\ q_1 \\ \hline k_0 \\ q_0 \end{array}.$$

The columns of E, A have sizes $d_\nu, \ell - k_{\nu-1} - \ldots - k_0, k_{\nu-1}, \ldots, k_0$.

We now split the second block row and column further, so that we obtain a square diagonal block of size $v_\nu = \min(c_\nu, \ell - k_{\nu-1} - \ldots - k_0)$ in the (2,2) position. Set $u_\nu^r := \ell - k_{\nu-1} - \ldots - k_0 - v_\nu$, and $u_\nu^l := c_\nu - v_\nu$. The final form is then obtained by a block permutation which moves the new third block column to the end and permutes

the block rows in the order $1, 2, 3, 5, \ldots, 4, 6, \ldots$. It is as follows:

$$P(t)E(t)Q(t) = \left[\begin{array}{cc|cccc|c} \Sigma_{d_\nu} & 0 & * & * & \ldots & * & 0 \\ 0 & 0 & 0 & * & \ldots & * & 0 \\ \hline & & 0 & * & \ldots & * & 0 \\ & & & 0 & \ddots & \vdots & 0 \\ & & & & \ddots & * & 0 \\ & & & & & 0 & 0 \\ \hline & & * & * & \ldots & * & 0 \\ \hline & & 0 & * & \ldots & * & 0 \\ & & & 0 & \ddots & \vdots & 0 \\ & & & & \ddots & * & 0 \\ & & & & & 0 & 0 \end{array}\right] \begin{array}{l} d_\nu \\ v_\nu \\ \hline k_{\nu-1} \\ k_{\nu-2} \\ \vdots \\ k_0 \\ \hline \tilde{u}_\nu^l \\ \hline q_{\nu-1} \\ \vdots \\ q_1 \\ q_0 \end{array},$$

$$P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t) =$$

$$\left[\begin{array}{cc|ccccc|c} A_{11}(t) & A_{12}(t) & * & * & \ldots & * & * & * \\ A_{21}(t) & A_{22}(t) & * & * & \ldots & * & * & 0 \\ \hline & & \Sigma_{k_{\nu-1}}(t) & * & \ddots & * & 0 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & \Sigma_{k_1}(t) & * & 0 \\ & & & & & \Sigma_{k_0}(t) & 0 \\ \hline * & * & * & \ldots & \ldots & * & * \\ \hline & & 0 & * & \ldots & * & 0 \\ & & & 0 & \ddots & \vdots & \vdots \\ & & & & 0 & * & 0 \\ & & & & & 0 & 0 \end{array}\right] \begin{array}{l} d_\nu \\ v_\nu \\ \hline k_{\nu-1} \\ k_{\nu-2} \\ \vdots \\ k_0 \\ \hline \tilde{u}_\nu^l \\ \hline q_{\nu-1} \\ \vdots \\ q_1 \\ q_0 \end{array},$$

$$P(t)B(t)W(t) = \left[\begin{array}{ccc} * & * & * \\ \hline \Sigma_{v_\nu}(t) & 0 & 0 \\ \hline 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_{\tilde{u}_\nu^l} & 0 \\ \hline 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} d_\nu \\ v_\nu \\ \hline k_{\nu-1} \\ k_{\nu-2} \\ \vdots \\ k_0 \\ \hline \tilde{u}_\nu^l \\ \hline q_{\nu-1} \\ \vdots \\ q_1 \\ q_0 \end{array},$$

where the widths of the block columns in the transformed E, A are d_ν, v_ν, $k_{\nu-1}$, ..., k_0, u_ν^r and v_ν, $\tilde{u}_\nu^l$, $m - v_\nu - \tilde{u}_\nu^l$ in the transformed B. We then combine all the k_j-blocks and q_j-blocks to blocks of sizes $s_\nu = \sum_{j=0}^{\nu} k_j$, $u_\nu^r = \sum_{j=0}^{\nu} q_j$, respectively. □

Note again that the analyticity assumption on the coefficient matrices can be relaxed to the condition that all smooth singular value decompositions exist and that rank changes in the factored matrices occur only at isolated points, see Remark 2. If only one step of the procedure given in the proof of Theorem 3.2 needs to be performed, i.e. $\nu = 0$, then the invariant quantities can be obtain via local rank computations.

REMARK 4. As for the condensed of Theorem 2.3 the condensed form of Theorem 3.2 is not a canonical form but it displays the relevant information. The corresponding canonical form, using non-unitary transformations has subsequently been developed by Rath [29].

In this section we have determined invariants of the system under global equivalence transformations. We will apply these results in the next section to analyze system properties.

4. The square subsystem. We will now analyze the system (1) after transformation to the condensed form (16).

$$
\begin{array}{c}
\begin{matrix} d_\nu \\ v_\nu \\ \hline s_\nu \\ \hline \tilde{u}_\nu^l \\ \hline u_\nu^l \end{matrix}
\left[\begin{array}{cc|c|c}
\Sigma_{d_\nu}(t) & 0 & E_{13}(t) & 0 \\
0 & 0 & E_{23}(t) & 0 \\ \hline
0 & 0 & E_{33}(t) & 0 \\ \hline
0 & 0 & E_{43}(t) & 0 \\ \hline
0 & 0 & E_{54}(t) & 0
\end{array}\right]
\left[\begin{array}{c} \dot{x}_1 \\ x_2 \\ \hline x_3 \\ \hline x_4 \end{array}\right] = \\
\left[\begin{array}{cc|c|c}
A_{11}(t) & A_{12}(t) & A_{13}(t) & A_{14}(t) \\
A_{21}(t) & A_{22}(t) & A_{23}(t) & A_{24}(t) \\ \hline
0 & 0 & A_{33}(t) & 0 \\ \hline
A_{41}(t) & A_{42}(t) & A_{43}(t) & A_{44}(t) \\ \hline
0 & 0 & A_{53}(t) & 0
\end{array}\right]
\left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \\ \hline x_4 \end{array}\right] +
\left[\begin{array}{ccc}
B_{11}(t) & B_{12}(t) & B_{13}(t) \\ \hline
\Sigma_{v_\nu}(t) & 0 & 0 \\ \hline
0 & 0 & 0 \\ \hline
0 & \Sigma_{\tilde{u}_\nu^l}(t) & 0 \\ \hline
0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}\right],
\end{array}
\tag{18}
$$

with columns of sizes d_ν, v_ν, s_ν, u_r^ν in E, A and columns of sizes v_ν, $\tilde{u}_\nu^l$, $m - u_\nu - \tilde{u}_\nu^l$ in B. We immediately make the following observations:

1. From the third block equation we obtain by recursive substitution that $x_3(t) = 0$ almost everywhere and since we want a smooth solution, then we obtain $x_3(t) \equiv 0$.
2. Since $x_3(t) \equiv 0$, the equations given by the last block row are fulfilled trivially. So we might leave these equations off altogether.

Thus we may consider the subsystem

$$
\begin{array}{c}
\begin{matrix} d_\nu \\ v_\nu \\ \hline \tilde{u}_\nu^l \end{matrix}
\left[\begin{array}{cc|c}
\Sigma_{d_\nu}(t) & 0 & 0 \\
0 & 0 & 0 \\ \hline
0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c} \dot{x}_1 \\ x_2 \\ \hline x_4 \end{array}\right] = \\
\left[\begin{array}{cc|c}
A_{11}(t) & A_{12}(t) & A_{14}(t) \\
A_{21}(t) & A_{22}(t) & A_{24}(t) \\ \hline
A_{41}(t) & A_{42}(t) & A_{44}(t)
\end{array}\right]
\left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_4 \end{array}\right] +
\left[\begin{array}{ccc}
B_{11}(t) & B_{12}(t) & B_{13}(t) \\ \hline
\Sigma_{v_\nu}(t) & 0 & 0 \\ \hline
0 & \Sigma_{\tilde{u}_\nu^l}(t) & 0
\end{array}\right]
\left[\begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}\right],
\end{array}
\tag{19}
$$

with columns of sizes d_ν, v_ν, u_r^ν in E, A and columns of sizes v_ν, $\tilde{u}_\nu^l$, $m - u_\nu - \tilde{u}_\nu^l$ in B.

If in this reduced form $u_r^\nu = \tilde{u}_{\nu^l}$ then this pencil is square and if we would compute the condensed form for this subsystem via Theorem 3.2, we would obtain $\nu = 0$, $s_0 = 0$, $\tilde{u}_\nu^l = u_\nu^l$.

If $u_r^\nu \neq \tilde{u}_{\nu^l}$ then we have many possibilities to reconsider this system. Solution component $x_4(t)$ can be chosen arbitrarily and hence could be viewed as an **extra input** to the system. Compare this with the results concerning generalized inverses of differential-algebraic operators in [19]. Also if $\Sigma_{\tilde{u}_\nu^l}$ is pointwise nonsingular, we could use a nonunitary equivalence transformation to eliminate the block $B_{12}(t)$ in B. If we do this and choose some x_4, u_1, u_3, then the first two equations form a subsystem independent from the rest and if its solution is computed, then u_2 is fixed. Thus we could interpret u_2 as an **extra state variable** rather than a control variable. Combining these ideas we could replace (18) by the system

$$
\begin{array}{l}
\begin{matrix} d_\nu \\ v_\nu \\ s_\nu \\ \tilde{u}_\nu^l \end{matrix}
\begin{bmatrix} \Sigma_{d_\nu} & 0 & E_{13} & 0 \\ 0 & 0 & E_{23} & 0 \\ 0 & 0 & E_{33} & 0 \\ 0 & 0 & E_{43} & 0 \end{bmatrix}
\begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \\ u_2 \end{bmatrix} = \\
\begin{bmatrix} A_{11} & A_{12} & A_{13} & B_{12} \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & 0 & A_{33} & 0 \\ A_{41} & A_{42} & A_{43} & \Sigma_{\tilde{u}_\nu^l} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ u_2 \end{bmatrix} +
\begin{bmatrix} B_{11} & A_{14} & B_{13} \\ \Sigma_{v_\nu} & A_{24} & 0 \\ 0 & 0 & 0 \\ 0 & A_{44} & 0 \end{bmatrix}
\begin{bmatrix} u_1 \\ x_4 \\ u_3 \end{bmatrix},
\end{array}
\tag{20}
$$

where we have left out the dependence on t for convenience. This is now a square system.

But it is clear that we can extract a square subsystem in many different ways by reinterpreting states as inputs or vice versa. A suitable choice will certainly depend on the application. In any case we wish to have a unique solution for suitably chosen inputs, thus we cannot allow the pencil to have more columns then rows. On the other hand if there are more rows than columns, then the set of controls that will lead to continuous solutions is restricted by these extra algebraic equations.

The previous observations suggests that the design of a practical control problem should be done in such a way, that components which are identically zero should be left off already in the model and the reinterpretation of components as states or controls should be done beforehand.

Thus we will assume in the following that the system has been reordered so that it has square matrices $E(t), A(t)$ with $\nu = 0$, $s_0 = 0$, $\tilde{u}_\nu^l = u_\nu^l$ if we transform it to the condensed form of Theorem 3.2. We call such a subsystem *an underlying square subsystem.*

Observe that although all the transformations that we used are unitary transformations, which can in principle be carried out in a numerically stable way, the rank decisions are still an ill-conditioned problem and small perturbations can change the picture completely. See the remarks for constant coefficient systems in [5], which hold here, too.

5. Regularization by Feedback. For constant coefficient systems regularizability, i.e. the question whether there exist proportional and/or derivative feedbacks such that the closed loop system has a regular pencil, i.e. is solvable for all consistent initial vectors, has been studied by several authors, see for example [13, 26, 3, 5]. We now generalize these results to the variable coefficient case. We introduce the following concepts.

DEFINITION 5.1.

(a) The descriptor system (1) is called *regularizable by proportional feedback* if there exists a (proportional state) feedback $u(t) = F(t)x(t) + w(t)$ such that the closed loop system

$$E(t)\dot{x}(t) = (A(t) + B(t)F(t))x(t) + B(t)w(t),\ x(t_0) = x_0 \tag{21}$$

is uniquely solvable for every consistent initial vector x_0 and any given control $w(t)$.

(b) The descriptor system (1) is called *regularizable by derivative feedback* if there exists a (derivative) feedback $u(t) = G(t)\dot{x}(t) + w(t)$ such that the closed loop system

$$(E(t) + B(t)G(t))\dot{x}(t) = A(t)x(t) + B(t)w(t),\ x(t_0) = x_0 \tag{22}$$

is uniquely solvable for all consistent initial vectors x_0 and any given control $w(t)$.

It is clear from the discussion in the previous section that we need $\tilde{u}_\nu^l = u_\nu^r$ in order to obtain regularizability, otherwise we cannot expect a unique solution or we have to reinterpret certain variables. If $\tilde{u}_\nu^l > u_\nu^r$ then we cannot apply arbitrary controls and if $u_\nu^r > \tilde{u}_\nu^l$ then the solution will not be unique.

As the following theorem shows this gives a necessary and sufficient condition if the matrices Σ_{d_ν}, Σ_{v_ν} occurring in the condensed form are invertible everywhere in the given interval, i.e. no rank drops occur, not even at isolated points.

THEOREM 5.2. *Consider system (1) in the condensed form (16) and assume that the diagonal matrices $\Sigma_{d_\nu}(t)$, $\Sigma_{v_\nu}(t)$, $\Sigma_{\tilde{u}_\nu^l}(t)$ are pointwise nonsingular in the whole interval $[t_0, t_1]$.*

System (1) can be regularized by proportional state feedback if and only if $\tilde{u}_\nu^l = u_\nu^r$.

System (1) can be regularized by derivative feedback if and only if $\tilde{u}_\nu^l = u_\nu^r$.

Proof. We have already observed that $\tilde{u}_\nu^l = u_\nu^r$ is a necessary condition. In order to show that this is also sufficient observe that in this case the system can be permuted (by exchanging the last two block rows and columns) to the form

$$\begin{array}{c} d_\nu \\ v_\nu \\ \tilde{u}_\nu^l = u_\nu^r \\ s_\nu \end{array} \begin{bmatrix} \Sigma_{d_\nu} & 0 & 0 & E_{13} \\ 0 & 0 & 0 & E_{23} \\ 0 & 0 & 0 & E_{43} \\ 0 & 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} =$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{14} & A_{13} \\ A_{21} & A_{22} & A_{24} & A_{23} \\ A_{41} & A_{42} & A_{44} & A_{43} \\ 0 & 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ \Sigma_{v_\nu} & 0 & 0 \\ 0 & \Sigma_{\tilde{u}_\nu^l} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Since Σ_{d_ν} and $\Sigma_{\tilde{u}^l_\nu}$ are nonsingular in the whole interval, we can choose the proportional feedback

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} F_{11}(t) & F_{12}(t) & 0 & 0 \\ F_{21}(t) & F_{22}(t) & F_{23}(t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_4(t) \\ x_3(t) \end{bmatrix} + w(t)$$

such that $\Sigma_{v_\nu}(t) \begin{bmatrix} F_{11}(t) & F_{12}(t) \end{bmatrix} = \begin{bmatrix} -A_{21}(t) & I - A_{22}(t) \end{bmatrix}$ and $\Sigma_{\tilde{u}^l_\nu}(t) \begin{bmatrix} F_{21}(t) & F_{22}(t) & F_{23}(t) \end{bmatrix} = \begin{bmatrix} -A_{41}(t) & -A_{42}(t) & I - A_{44}(t) \end{bmatrix}$.

This choice gives a closed loop system

$$\begin{array}{c} d_\nu \\ v_\nu \\ \tilde{u}^l_\nu = u^r_\nu \\ s_\nu \end{array} \begin{bmatrix} \Sigma_{d_\nu} & 0 & 0 & E_{13} \\ 0 & 0 & 0 & E_{23} \\ 0 & 0 & 0 & E_{43} \\ 0 & 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} =$$
$$\begin{bmatrix} A_{11} & A_{12} & A_{14} & A_{13} \\ 0 & I & A_{24} & A_{23} \\ 0 & 0 & I & A_{43} \\ 0 & 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ \Sigma_{v_\nu} & 0 & 0 \\ 0 & \Sigma_{\tilde{u}^l_\nu} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Recall that the solutions components x_3 are constrained to be zero. If we remove these equations, then the remaining system has strangeness index $\mu = 0$, i.e. is uniquely solvable for all consistent initial values.

Similarly in the case of derivative feedback we choose the derivative feedback

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} 0 & F_{12}(t) & 0 & 0 \\ 0 & 0 & F_{13}(t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \\ x_4(t) \\ x_3(t) \end{bmatrix} + w(t)$$

such that $\Sigma_{v_\nu} F_{12} = -I$ and $\Sigma_{\tilde{u}^l_\nu} F_{23} = -I$

This choice gives a closed loop system

$$\begin{array}{c} d_\nu \\ v_\nu \\ \tilde{u}^l_\nu = u^r_\nu \\ s_\nu \end{array} \begin{bmatrix} \Sigma_{d_\nu} & 0 & 0 & E_{13} \\ 0 & I & 0 & E_{23} \\ 0 & 0 & I & E_{43} \\ 0 & 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} =$$
$$\begin{bmatrix} A_{11} & A_{12} & A_{14} & A_{13} \\ A_{21} & A_{22} & A_{24} & A_{23} \\ A_{43} & A_{42} & A_{44} & A_{43} \\ 0 & 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ \Sigma_{v_\nu} & 0 & 0 \\ 0 & \Sigma_{\tilde{u}^l_\nu} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

which is as required. □

It is clear that weaker assumptions can be considered in Theorem 5.2, by allowing rank jumps in the matrices Σ_{d_ν}, Σ_{v_ν} at isolated points and using a weak solvability concept. This topic is currently under investigation.

Note further that there is still quite a lot of freedom in the choice of the feedback and the freedom may be used to improve the robustness of the system as it was done for constant coefficient systems in [3, 4, 12]. Unfortunately so far it is not really clear what robustness means for variable coefficient systems of the type considered.

6. Conclusion. We have shown that under some smoothness assumptions every linear time varying descriptor system can be transformed to a condensed form which displays free state components that can be interpreted as inputs, fixed controls, that can be interpreted as states, (which together form a regularizable subsystem), some solution components which are constrained to be zero coming from higher index components that are unchanged by feedback, plus some removable equations which hold trivially. In principle this structure can be obtained via a sequence of smooth singular value decompositions for which numerical methods are available. From a practical point of view, however, the uncontrollable higher index part and the other removable parts are very sensitive to perturbations, which may change the whole system structure. In view of this, modeling or linearization which leads to such components should be avoided.

Acknowledgement. We thank Werner Rath for helpful comments.

REFERENCES

[1] K. Brenan, S. Campbell, and L. Petzold, *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations*, Elsevier, North Holland, New York, NY, 1989.

[2] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols, *Numerical computation of an analytic singular value decomposition of a matrix valued function*, Numerische Mathematik, 60 (1991), pp. 1–40.

[3] A. Bunse-Gerstner, V. Mehrmann, and N. Nichols, *Regularization of descriptor systems by derivative and proportional state feedback*, SIAM Journal Matrix Analysis and its Applications, 13 (1992), pp. 46–67.

[4] ——, *Regularization of descriptor systems by output feedback*, IEEE Transactions on Automatic Control, 39 (1994), pp. 1742–1748.

[5] R. Byers, T. Geerts, and V. Mehrmann, *Descriptor systems without controllability at infinity*, tech. report, Fakultät für Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, Germany, 1994. Submitted for publication.

[6] S. Campbell, *Linearization of DAEs along trajectories*, Zeitschrift für angewandte Mathematik und Physik, 46 (1995), pp. 70–84.

[7] S. Campbell, N. Nichols, and W. Terrell, *Duality, observability, and controllability for linear time-varying descriptor systems*, Circuits, Systems and Signal Processing, 10 (1991), pp. 455–470.

[8] S. Campbell and W. Terrell, *Observability for linear time-varying descriptor systems*, SIAM Journal Matrix Analysis and Applications, 12 (1991), pp. 484–496.

[9] S. L. Campbell, *A general form for solvable linear time varying singular systems of differential equations*, SIAM Journal Mathematical Analysis, 18 (1987), pp. 1101–1114.

[10] ——, *The numerical solution of higher index linear time varying singular systems of differential equations*, SIAM Journal Scientific and Statistical Computation, 6 (1988), pp. 334–348.

[11] C. De Boor, *An empty exercise*, SIGNUM Newsletter, 25 (1990), pp. 2–6.

[12] L. Elsner, C. He, and V. Mehrmann, *Completion of a matrix so that the inverse has minimum norm. Application to the regularization of descriptor control problems*, in Linear Algebra for Control Theory, P. Van Dooren and B. Wyman, eds., Springer Verlag, New York, 1993, pp. 75–86.

[13] L. Fletcher, J. Kautsky, and N. Nichols, *Eigenstructure assignment in descriptor systems*, IEEE Trans. Automat. Control, AC-31 (1986), pp. 1138–1141.

[14] M. Fliess, J. Levine, and P. Rouchon, *Index of a general linear time-varying implicit system*, in European Control Conference 1991, Hermes, Paris, 1991, pp. 769–772.

[15] E. Griepentrog and R. März, *Differential-Algebraic Equations and Their Numerical Treatment*, Teubner Verlag, Leipzig, 1986.

[16] M. Günther and P. Rentrop, *Multirate row-methods and latency of electric circuits*, Tech. Report TUM-M9208, Mathematisches Institut, Technische Universität München, Bibliothek der Fakultät für Mathematik und Informatik, Arcisstr. 21, Postfach 202420, D-8000 München 2, 1992.

[17] P. Kunkel and V. Mehrmann, *Smooth factorizations of matrix valued functions and their derivatives*, Numerische Mathematik, 60 (1991), pp. 115–132.

[18] ——, *Canonical forms for linear differential-algebraic equations with variable coefficients*, Journal of Computational and Applied Mathematics, 56 (1994), pp. 225–251.

[19] ——, *Generalized inverses of differential-algebraic operators*, tech. report, Fakultät für Mathematik, TU Chemnitz-Zwickau, D-09107 Chemnitz, FRG, 1994. to appear in SIAM J. Matrix Analysis.

[20] ——, *A new look at pencils of matrix valued functions*, Linear Algebra and its Applications, 212/213 (1994), pp. 215–248.

[21] ——, *Local and global invariants of linear differential-algebraic equations and their relation*, Fakultät für Mathematik SPC 95-25, Technical University Chemnitz, D-09107 Chemnitz, 1995.

[22] ——, *A new class of discretization methods for the solution of linear differential-algebraic equations*, SIAM Journal on Numerical Analysis, to appear (1995).

[23] P. Kunkel, V. Mehrmann, W. Rath, and J. Weickert, *'GELDA: A software package for the solution of general linear differential algebraic equations*, Fakultät für Mathematik SPC 95-8, Technical University Chemnitz, D-09107 Chemnitz, 1995.

[24] J. J. Loiseau, K. Özçaldiran, M. Malabre, and N. Karcanias, *Feedback canonical forms of singular systems*, Kybernetika, 27 (1991), pp. 289–305.

[25] V. Mehrmann and W. Rath, *Numerical methods for the computation of analytic singular value decompositions*, Electronic Transactions on Numerical Analysis, 1 (1993), pp. 72–88.

[26] K. Özcaldiran and F. Lewis, *On regularizability of singular systems*, IEEE Trans. Automat. Control, AC-35 (1990), pp. 1156–1160.

[27] P. Rabier and W. Rheinboldt, *Classical and generalized solutions of time-dependent linear differential algebraic equations*, Linear Algebra and its Applications, to appear (1995).

[28] ——, *A geometric treatment of implicit differential–algebraic equations*, Journal on Differential Equations, to appear (1995).

[29] W. Rath, *Canonical forms for linear descriptor systems with variable coefficients*, Fakultät für Mathematik SPC 95-16, Technical University Chemnitz, D-09107 Chemnitz, 1995.

[30] B. Simeon, C. Führer, and P. Rentrop, *Differential-algebraic equations in vehicle system dynamics*, Surveys for Mathematics in Industry, 1 (1991), pp. 1–37.

[31] B. Simeon, F. Grupp, C. Führer, and P. Rentrop, *A nonlinear truck model and its treatment as a multibody system*, Tech. Report TUM-M9204, Mathematisches Institut, TU München, München, FRG, 1992.

[32] T. Schmidt and M. Hou, *Rollringgetriebe*. Internal Report, Sicherheitstechnische Regelungs- und Meßtechnik, Bergische Universität, GH Wuppertal, Wuppertal, FRG, 1992.

[33] P. Van Dooren, *The computation of Kronecker's canonical form of a singular pencil*, Linear Algebra and its Applications, 27 (1979), pp. 103–141.

[34] K. Wright, *Differential equations for the analytic singular value decomposition of a matrix*, Numerische Mathematik, 3 (1992), pp. 283–295.