A new look at pencils of matrix valued functions

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ABSTRACT

Matrix pencils depending on a parameter and their canonical forms under equivalence are discussed.

The study of matrix pencils or generalized eigenvalue problems is often motivated by applications from linear differential-algebraic equations (DAEs). Based on the Weierstraß-Kronecker canonical form of the underlying matrix pencil one gets existence and uniqueness results for linear constant coefficient DAEs.

In order to study the solution behaviour of linear DAEs with variable coefficients one has to look at new types of equivalence transformations. This then leads to new canonical forms and new invariances for pencils of matrix valued functions. We give a survey of recent results for square pencils and extend these results to nonsquare pencils. Furthermore we partially extend the results for canonical forms of Hermitian pencils and give new canonical forms there, too.

Based on these results we obtain new existence and uniqueness theorems for differential algebraic systems, which generalize the classical results of Weierstraß and Kronecker.

LINEAR ALGEBRA AND ITS APPLICATIONS xxx:1–xx (1992)

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655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

1. Introduction

In this paper we study matrix pencils

$$\alpha E(t) - \beta A(t) \tag{1}$$

where $E, A \in C([t_0, t_1], \mathcal{C}^{n,l})$. Here $C^m([t_0, t_1], \mathcal{C}^{n,l})$ denotes the set of m-times continuously differentiable functions from the interval $[t_0, t_1]$ to the complex vector space $\mathcal{C}^{n,l}$. We always talk synonymously about the matrix pencil $\alpha E(t) - \beta A(t)$ and the pair of matrices $(E(t), A(t))$.

The study of such matrix pencils is mainly motivated by the analysis of initial value problems for linear differential-algebraic equations with variable coefficients

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \ t \in [t_0, t_1] \tag{2}$$

with $f \in C([t_0, t_1], \mathcal{C}^n)$ together with an initial condition

$$x(t_0) = x_0 \in \mathcal{C}^n. \tag{3}$$

The first study of matrix pencils is usually dated back to the last century and the work of Sylvester [47], Weierstraß [54] and others who studied Hermitian pencils. For a historical overview see papers of Uhlig [50, 49, 51]. A general canonical form under congruence for Hermitian pencils was given by Thompson [48]. It should be noted that this case still receives quite a lot of attention, in particular due to its importance in the study of mechanical systems, e.g. [27, 45, 59, 58] and optimal control problems, e.g. [18, 9, 42].

For nonhermitian regular pencils strict equivalence and a canonical from were first established by Weierstraß [55] and Kronecker [36] who extended the results of Weierstraß to the case of singular pencils. These results are well documented in textbooks, e.g. [23], and they are well studied also from the point of view of perturbation theory, e.g. [46] and numerical analysis, e.g. [28, 1].

We will review the theory for constant pencils in Section 2 and for the corresponding linear differential-algebraic equation in Section 5.

For the case of parameter dependent pencils the situation is completely different. Such pencils arise mainly in the analysis of linear differential-algebraic equations (DAEs) with variable coefficients. The interest in this field was revived by a paper of Gear [25] and this is now an area of very active research. Such DAEs are studied for example as differential equations on manifolds, e.g. [44, 30, 29], or from the numerical point of view [31, 5, 32], as well as in application areas like control theory, where they are called descriptor or singular systems, e.g. [19, 42].

Many attempts have been made to study the global solution behaviour of DAEs by looking locally at the linearization, (i.e. the matrix pencil) at a fixed time. But it was observed very quickly, e.g. [11, 26], that the study of the constant matrix pencil $\alpha E(\hat{t}) - \beta A(\hat{t})$ at a fixed point $\hat{t}$ is not enough to characterize the local solution behaviour of the DAE, not even for the linear time varying case. We will discuss this point by studying a new local linearization which was first introduced in [41].

For the general nonlinear case it is often better to use linearizations along trajectories, which leads to linear time varying systems [16]. Thus it is important to study this problem separately.

The approaches that discuss existence and uniqueness are rarely algebraic, but often based on differential geometric or analytic tools, e.g. [30]. Nonetheless algebraic approaches have been taken but usually in the context of the construction of numerical methods, [26, 12, 17, 13, 14, 2, 34, 15].

All these approaches do not include equations which have nonunique solutions and most of them lack an appropriate discussion of the local behaviour. The first approach that covers also the case of nonunique solutions and gives a treatment of general linear variable coefficient DAEs via the study of square pencils of matrix valued functions is in [41]. This approach in used for the construction of numerical methods in [40].

We will extend these results here to the general case of nonsquare pencils in Section 3, where the local solution behaviour is discussed and in Section 4, where we study the global solution behaviour with an algebraic approach.

To do this we have to study new types of equivalence transformations for matrix pencils, and their canonical forms, which generalize the classical Weierstraß-Kronecker canonical form. Based on these results we present a general existence and uniqueness theory for DAEs of the form (2) in Section 5.

For the computation of the invariants in the Weierstraß-Kronecker canonical form numerically stable methods have been introduced in recent years and are now available as public domain software, e.g. [52, 53, 20, 21, 1]. We will briefly discuss this approach in Section 6. In order to solve linear differential algebraic equations these methods can be modified to obtain solution methods for the constant coefficient case, see [6]. For the variable coefficient case new discretization methods were given in [40].

In this paper we mainly discuss the general case of nonsquare and nonhermitian pencils. But in many applications from mechanics or control theory the matrices E, A are symmetric or Hermitian [27, 45, 59, 58, 18, 9, 42]. We will give a new local canonical form for the Hermitian case. It is still an open problem to extend these results to the variable coeffcient case, since in general variable congruence transformations destroy the symmetry.

2. Constant pencils

In this section, we survey the case of a constant pencil

$$\alpha E - \beta A \tag{4}$$

as it for example arises in the analysis of linear differential-algebraic equations with constant coefficients

$$E\dot{x} = Ax + f(t), \quad t \in [t_0, t_1], \tag{5}$$

where $E, A \in \mathcal{C}^{n,l}$ and $f \in C([t_0, t_1], \mathcal{C}^n)$.

The standard way to treat (5) is to look at all regular transformations which take (5) into an equation of the same form. This leads to the definition of so-called (strong) equivalence.

DEFINITION 1. Two pairs of matrices (E_i, A_i), $i = 1, 2$, with $E_i, A_i \in \mathcal{C}^{n,l}$ are called *(strongly) equivalent* if there are nonsingular matrices $P \in \mathcal{C}^{n,n}$, $Q \in \mathcal{C}^{l,l}$ with

$$(E_2, A_2) = P(E_1, A_1) \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}. \tag{6}$$

Clearly, this defines an equivalence relation. The canonical form connected with this equivalence is the well-known Weierstraß-Kronecker canonical form (see [24, 52] for details).

THEOREM 2. *[52] Let $E, A \in \mathcal{C}^{n,l}$. Then, there exist nonsingular $P \in \mathcal{C}^{n,n}, Q \in \mathcal{C}^{l,l}$ such that*

$$\begin{aligned} &P(\alpha E - \beta A)Q \\ &\quad = \operatorname{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, M_{\eta_1}, \ldots, M_{\eta_q}, F_{\rho_1}, \ldots, F_{\rho_v}, S_{\sigma_1}, \ldots, S_{\sigma_w}), \end{aligned} \tag{7}$$

where

$$
\begin{array}{ll}
(a) & L_{\epsilon_j} \text{ is an } \epsilon_j \times (\epsilon_j+1) - \text{bidiagonal block, } \epsilon_j \in \mathcal{N}_0 \\
 & \alpha \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \beta \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \\
(b) & M_{\eta_j} \text{ is an } (\eta_j+1) \times \eta_j - \text{bidiagonal block, } \eta_j \in \mathcal{N}_0 \\
 & \alpha \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \beta \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} \\
(c) & F_{\rho_j}(\lambda_j) = \alpha I - \beta J_{\rho_j}(\lambda_j), \text{ where } J_{\rho_j}(\lambda_j) \text{ is a } \rho_j \times \rho_j - \text{Jordan block,} \\
(d) & S_{\sigma_j} = \alpha N_{\sigma_j} - \beta I, \text{ where } N_{\sigma_j} \text{ is a } \sigma_j \times \sigma_j - \text{nilpotent Jordan block.}
\end{array} \tag{8}
$$

All quantities on the right hand side of (7) are invariants for the pair (E, A), i.e. up to order each canonical form of (E, A) consists of the same blocks.

DEFINITION 3. A matrix pencil $\alpha E - \beta A,\ E, A \in \mathcal{C}^{n,l}$, is called *regular* if $n = l$ and if the characteristic polynomial

$$P(\alpha, \beta) = \det(\alpha E - \beta A) \tag{9}$$

does not vanish identically, otherwise *singular*. The quantity

$$k = \begin{cases} 0 & \text{for } w = 0 \\ \max\{\sigma_j \mid j = 1, \ldots, w\} & \text{for } w > 0 \end{cases} \tag{10}$$

with w as in (7) is called the *index* of $\alpha E - \beta A$ and is denoted by $k = \text{ind}(E, A)$.

Regularity of the pencil means that no blocks of the types L_ϵ or M_η occur in the Weierstraß-Kronecker canonical form and guarantees that there exists a unique solution of (5) for each sufficiently smooth f and consistent initial condition (3).

Note that the numerical computation of this canonical form is an ill-conditioned problem, since arbitrarily small perturbations can drastically change the structure. Also the transformation matrices P, Q can be very large in norm, which may lead to large round-off errors.

In the case that E, A are Hermitian matrices we use congruence transformations to preserve the Hermitian structure, and also the inertia structure of the system.

DEFINITION 4. Two pairs of matrices (E_i, A_i), $i = 1, 2$, with $E_i, A_i \in \mathcal{C}^{n,n}$ Hermitian are called *(strongly) congruent* if there exists a nonsingular matrix $P \in \mathcal{C}^{n,n}$, such that

$$(E_2, A_2) = P^*(E_1, A_1) \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}. \tag{11}$$

Clearly, this also defines an equivalence relation.

THEOREM 5. *[48] Let $E, A \in \mathcal{C}^{n,n}$ be Hermitian. Then, there exists a nonsingular matrix $P \in \mathcal{C}^{n,n}$ such that*

$$\begin{aligned} &P^*(\alpha E - \beta A)P \\ &\quad = \operatorname{diag}(\Delta_{\epsilon_1}, \ldots, \Delta_{\epsilon_p}, \Theta_{\eta_1}, \ldots, \Theta_{\eta_q}, \Lambda_{\rho_1}, \ldots, \Lambda_{\rho_v}, \Psi_{\sigma_1}, \ldots, \Psi_{\sigma_w}), \end{aligned} \tag{12}$$

where

(a) Δ_{ϵ_j} *is an* $(2\epsilon_j - 1) \times (2\epsilon_j - 1) -$ *block,* $\epsilon_j \in \mathcal{N}$

$$\alpha \begin{bmatrix} 0 & L_1 \\ L_1^T & 0 \end{bmatrix} - \beta \begin{bmatrix} 0 & L_2 \\ L_2^T & 0 \end{bmatrix}, \text{ with}$$

$$L_1 = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 1 \end{bmatrix}, \; L_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

(b) Θ_{η_j} *is an* $(\eta_j + 1) \times \eta_j -$ *bidiagonal block,* $\eta_j \in \mathcal{N}_0$

$$\alpha Z - \beta \hat{J}(\lambda_j) = \alpha \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} - \beta \begin{bmatrix} & & \lambda_j \\ & \iddots & 1 \\ \lambda_j & 1 & \end{bmatrix}$$

corresponding to a Jordan block for a real eigenvalue λ_j

(c) Λ_{ρ_j} *is a* $2\rho_j \times 2\rho_j -$ *Jordan block,* $\rho_j \in \mathcal{N}$,

$$\alpha \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix} - \beta \begin{bmatrix} 0 & \hat{J}(\lambda_j) \\ \hat{J}(\bar{\lambda}_j) & 0 \end{bmatrix} \text{ with } Z, \hat{J}(\lambda_j) \text{ as in (b)},$$

corresponding to a pair of complex conjugate eigenvalues ,

(d) Ψ_{σ_j} *is a* $\sigma_j \times \sigma_j -$ *nilpotent block,* $\sigma_j \in \mathcal{N}$

$$\alpha \begin{bmatrix} & & & 0 \\ & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{bmatrix} - \beta \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}$$

(13)

Note again, that the numerical computation of this canonical form is an illconditioned problem.

We will see in the following two sections how one can generalize these canonical forms.

3. Local canonical form

Turning back to the case of matrix valued functions one could try to generalize the concepts of the previous section in the following manner. Instead of the indices $\epsilon_j, \eta_j, \rho_j, \sigma_j, k$ and the eigenvalues λ_j in (7), (12), and k in (10) we could consider functions $\epsilon_j, \eta_j, \rho_j, \sigma_j, k: [t_0, t_1] \to \{0, \ldots, n\}$, by computing pointwise the canonical forms of the pair $(E(t), A(t))$.

This approach is not suitable for the analysis of existence and uniqueness of solutions for (3) as is shown in [30, 33, 41].

EXAMPLE 6. A short computation shows that

$$\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x, \ t \in [-1, 1] \tag{14}$$

is uniformly regular and has uniform index 2 but that

$$x(t) = c(t) \begin{bmatrix} t \\ 1 \end{bmatrix} \tag{15}$$

is a solution for all $c \in C^1([-1, 1], \mathcal{C})$. Especially there are more than one solution for consistent initial conditions.

EXAMPLE 7. The equation

$$\begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix} x + f(x), \ t \in [-1, 1], \tag{16}$$

with $f \in C^2([-1, 1], \mathcal{C}^2)$ is not uniformly regular because the pencil $(E(t), A(t))$ is singular for all $t \in [-1, 1]$. Nevertheless, it has the unique solution

$$\begin{aligned} x_1(t) &= f_1(t) + t\ f_2(t) - t\ f_1(t), \quad x = (x_1, x_2)^T \\ x_2(t) &= f_2(t) - \dot{f}_1(t), \quad f = (f_1, f_2)^T \end{aligned}$$

for each consistant initial condition.

The reason for this *strange* behaviour is that for (2) we need to include non-constant transformations. Setting $x(t) = Q(t)y(t)$ and pre-multiplying (2) by $P(t)$, the equation (2) transforms to

$$P(t)E(t)Q(t)\dot{y}(t) = (P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t))y(t) + P(t)f(t). \quad (17)$$

Therefore, one is led to the following definition:

DEFINITION 8. Two pairs of matrix functions $(E_i(t), A_i(t)),\ E_i, A_i \in C([t_0, t_1], \mathcal{C}^{n,l}), i = 1, 2$ are called *equivalent* if there are $P \in C([t_0, t_1], \mathcal{C}^{n,n})$ and $Q \in C^1([t_0, t_1], \mathcal{C}^{l,l})$ with $P(t), Q(t)$ nonsingular for all $t \in [t_0, t_1]$ such that

$$(E_2(t), A_2(t)) = P(t)(E_1(t), A_1(t)) \begin{bmatrix} Q(t) & -\dot{Q}(t) \\ 0 & Q(t) \end{bmatrix}. \quad (18)$$

Standard rules for differentiation show that this is indeed an equivalence relation. For an analysis of DAEs this approach will turn out to be useful, but for a numerical solution the occurence of $\dot{Q}(t)$ creates difficulties. For the numerical solution it is usually important to consider local quantities which are numerically computable and which give information on the global behaviour of the solution in the neighborhood of a fixed point $t \in [t_0, t_1]$.

Taking into account that at a fixed point $t \in [t_0, t_1]$ we can choose $Q(t)$ and $\dot{Q}(t)$ independently (see [26]), we modify (6) in the following way:

DEFINITION 9. Two pairs of matrices $(E_i, A_i),\ E_i, A_i \in \mathcal{C}^{n,l}, \quad i = 1, 2$ are called *equivalent* if there are matrices $P \in \mathcal{C}^{n,n}$, $Q, B \in \mathcal{C}^{l,l}$ with P, Q nonsingular such that

$$(E_2, A_2) = P(E_1, A_1) \begin{bmatrix} Q & -B \\ 0 & Q \end{bmatrix}. \quad (19)$$

We study the local equivalence, first because it is the basis for the global equivalence and second because it is numerically computable.

Since we get (6) back as special case for $B = 0$, we obtain a simpler canonical form compared with the Weierstraß-Kronecker canonical form. With the notion that a matrix is basis of a vector space if this is valid for the set of its column vectors, we get the following canonical form for the equivalence relation defined in Definition 9. The result generalizes the corresponding result for square pencils given in [41].

THEOREM 10. *Let* $E, A \in \mathcal{C}^{n,l}$ *and*

$$\begin{array}{ll} (a) & T \text{ basis of } \operatorname{kernel} E \\ (b) & Z \text{ basis of } \operatorname{corange} E = \operatorname{kernel} E^* \\ (c) & T' \text{ basis of } \operatorname{cokernel} E = \operatorname{range} E^* \\ (d) & V \text{ basis of } \operatorname{corange}(Z^*AT). \end{array} \tag{20}$$

Then the quantities (with the convention $\operatorname{rank} \emptyset = 0$*)*

$$\begin{array}{lll} (a) & r = \operatorname{rank} E & \text{(rank)} \\ (b) & a = \operatorname{rank}(Z^*AT) & \text{(algebraic part)} \\ (c) & s = \operatorname{rank}(V^*Z^*AT') & \text{(strangeness)} \\ (d) & d = r - s & \text{(differential part)} \\ (e) & u^{\mathrm{l}} = n - r - a - s & \text{(left undetermined part)} \\ (f) & u^{\mathrm{r}} = l - d - a - s & \text{(right undetermined part)} \end{array} \tag{21}$$

are invariant under (19) and (E, A) *is equivalent to the canonical form*

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{matrix} s \\ d \\ a \\ s \\ u^{\mathrm{l}} \end{matrix}, \tag{22}$$

where the last block column in both matrices has width u^{r}*.*

Proof. The proof is a small modification of the proof for the square pencil case given in [41] and is therefore omitted here. □

REMARK 11. A few comments should be spent on the use of the word strangeness in (20c). The two blocks of size s (if occuring) in the local canonical form (22) are responsible for the unusual *strange* behaviour of the system, when it is considered pointwise as demonstrated in Examples 6, 7. Contributing to these blocks are not only higher index blocks but also singular blocks in the Weierstraß-Kronecker canonical form. To see this, note that the equivalence relation (6) is included in (19). Thus, we can first transform to Kronecker canonical form and then treat the single blocks separately. Denoting the i-th canonical basis vector of length n by $e_i^{(n)}$ and a nilpotent Jordan block of size ν by N_ν, we obtain for the different types of blocks:

(a) Kronecker block L_ϵ

$$(E,A) = \left(\begin{bmatrix} 0 & I_\epsilon \end{bmatrix}, \begin{bmatrix} e_1^{(\epsilon)} & N_\epsilon \end{bmatrix} \right)$$
$$T = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ Z = \emptyset,\ T' = \begin{bmatrix} 0 \\ I_\epsilon \end{bmatrix},\ V = \emptyset$$
$$Z^*AT = \emptyset,\ V^*Z^*AT' = \emptyset$$
$$r = \epsilon,\ a = 0,\ s = 0,\ d = \epsilon,\ u^{\mathrm{l}} = 0,\ u^{\mathrm{r}} = 1$$

(b) Kronecker block M_η

$$(E,A) = \left(\begin{bmatrix} I_\eta \\ 0 \end{bmatrix}, \begin{bmatrix} N_\eta \\ e_\eta^{(\eta)T} \end{bmatrix} \right)$$
$$T = \emptyset,\ T' = I_\eta,\ Z = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ V = \begin{bmatrix} 1 \end{bmatrix},$$
$$Z^*AT = \emptyset,\ V^*Z^*AT' = \begin{bmatrix} e_\eta^{(\eta)T} \end{bmatrix}$$
$$r = \eta,\ a = 0,\ s = \begin{cases} 0 & \text{for } \eta = 0 \\ 1 & \text{for } \eta \neq 0 \end{cases},\ d = \begin{cases} 0 & \text{for } \eta = 0 \\ \eta - 1 & \text{for } \eta \neq 0 \end{cases},$$
$$u^{\mathrm{l}} = \begin{cases} 1 & \text{for } \eta = 0 \\ 0 & \text{for } \eta \neq 0 \end{cases},\ u^{\mathrm{r}} = 0$$

(c) Kronecker block F_ρ

$$(E,A) = (I_\rho, J_\rho)$$
$$T = \emptyset,\ Z = \emptyset,\ T' = I_\rho,\ V = \emptyset$$
$$r = \rho,\ a = 0,\ s = 0,\ d = \rho,\ u^{\mathrm{l}} = 0,\ u^{\mathrm{r}} = 0$$

(d) Kronecker block S_σ

$$(E,A) = (N_\sigma, I_\sigma)$$
$$T = e_1^{(\sigma)},\ Z = e_\sigma^{(\sigma)},\ T' = (e_2^{(\sigma)}, \ldots, e_\sigma^{(\sigma)}),$$
$$V = \begin{cases} \emptyset & \text{for } \sigma = 1 \\ \begin{bmatrix} 1 \end{bmatrix} & \text{for } \sigma \neq 1 \end{cases}, Z^*AT = \begin{cases} \begin{bmatrix} 1 \end{bmatrix} & \text{for } \sigma = 1 \\ \emptyset & \text{for } \sigma \neq 1 \end{cases}$$
$$V^*Z^*AT' = \begin{cases} \emptyset & \text{for } \sigma = 1 \\ \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} & \text{for } \sigma \neq 1 \end{cases}$$
$$r = \sigma - 1,\ a = \begin{cases} 1 & \text{for } \sigma = 1 \\ 0 & \text{for } \sigma \neq 1 \end{cases},\ s = \begin{cases} 0 & \text{for } \sigma = 1 \\ 1 & \text{for } \sigma \neq 1 \end{cases}$$
$$d = \begin{cases} \sigma - 1 & \text{for } \sigma = 1 \\ \sigma - 2 & \text{for } \sigma \neq 1 \end{cases},\ u^{\mathrm{l}} = 0,\ u^{\mathrm{r}} = \begin{cases} 0 & \text{for } \sigma = 1 \\ 1 & \text{for } \sigma \neq 1 \end{cases}.$$

It is obvious that singular blocks of type M_η, $\eta \neq 0$ and higher index blocks contribute to a non-vanishing strangeness s. If, however, as is done in most other research, it is excluded by assumption that singular blocks occur, then only higher index blocks contribute to s.

We also wish to have a local canonical form for a pair of symmetric or Hermitian matrices E, A. There are several reasons why it is important to have such a form. Usually the symmetry of the pencils reflects a physical property of the system. If we would operate on such a pencil with equivalence transformations which destroy the structure, these physical properties are obscured, but what is worse, if we use numerical methods for the solution of such problems, then one might even get physically meaningless results. For an instructive example see [8]. The local Hermitian equivalence may also be the basis for a global Hermitian equivalence, which is still an open problem.

To study the local equivalence, we modify the definition of congruence.

DEFINITION 12. Two pairs of Hermitian matrices (E_i, A_i), $E_i, A_i \in \mathcal{C}^{n,n}$, $i = 1, 2$ are called *congruent* if there are matrices $P, B \in \mathcal{C}^{n,n}$, with P nonsingular such that

$$(E_2, A_2) = P^*(E_1, A_1) \begin{bmatrix} P & -B \\ 0 & P \end{bmatrix}. \tag{23}$$

and E_2, A_2 are again Hermitian.

Observe that not any matrix B in (23) will lead again to an Hermitian pencil. Examples of possible matrices B are matrices such that $E_1 B$ vanishes or matrices WP, where W is a real polynomial in E_1. A general characterization of the possible matrices B depends strongly on the structure of E_1.

In the following we will develop a canonical form under congruence (23). In order to do this, we need the following Lemma which is closely related to the hyperbolic singular value decomposition recently introduced in [4] and the HR-decomposition of [7].

LEMMA 13. *Let*

$$D = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \in \mathcal{C}^{p+q,p+q}, \; A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \in \mathcal{C}^{p+q,v} \tag{24}$$

be partitioned analogously. Then there exist nonsingular matrices $P \in \mathcal{C}^{p+q,p+q}$ *and* $Q \in \mathcal{C}^{v,v}$ *such that*

$$PDP^* = \begin{bmatrix} I_{v_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{v_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{v_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{w_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{v_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_{w_3} \end{bmatrix},$$
$$PAQ = \begin{bmatrix} I_{v_1} & 0 & 0 & 0 \\ I_{v_1} & 0 & 0 & 0 \\ 0 & I_{v_2} & 0 & 0 \\ 0 & 0 & I_{w_2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{25}$$

Proof. We give a constructive proof:
Perform a singular value decomposition (SVD), e.g. [28],

$$A_1 = U_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^*$$

with Σ_1 $p_0 \times p_0$ diagonal, nonsingular and set

$$P_1 := \begin{bmatrix} U_1^* & 0 \\ 0 & I_q \end{bmatrix}, \; Q_1 := V_1 \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

Then we partition

$$P_1 D P_1^* =: \begin{bmatrix} I_{p_0} & 0 & 0 \\ 0 & I_{p-p_0} & 0 \\ 0 & 0 & -I_q \end{bmatrix}, \; P_1 A Q_1 =: \begin{bmatrix} I_{p_0} & 0 \\ 0 & 0 \\ A_3 & A_4 \end{bmatrix}. \tag{26}$$

Perform a singular value decomposition

$$A_4 = U_4 \begin{bmatrix} \Sigma_4 & 0 \\ 0 & 0 \end{bmatrix} V_4^*$$

with Σ_4 $q_0 \times q_0$ diagonal, nonsingular and set

$$P_2 := \begin{bmatrix} I_{p_0} & 0 & 0 \\ 0 & I_{p-p_0} & 0 \\ 0 & 0 & U_4^* \end{bmatrix}, \; Q_2 := \begin{bmatrix} I_{p_0} & 0 \\ 0 & V_4 \end{bmatrix}, \; P := P_2 P_1, \; Q := Q_1 Q_2.$$

Then partition

$$PDP^* =: \begin{bmatrix} I_{p_0} & 0 & 0 & 0 \\ 0 & I_{p-p_0} & 0 & 0 \\ 0 & 0 & -I_{q_0} & 0 \\ 0 & 0 & 0 & -I_{q-q_0} \end{bmatrix}, \; PAQ =: \begin{bmatrix} I_{p_0} & 0 & 0 \\ 0 & 0 & 0 \\ A_{31} & \Sigma_4 & 0 \\ A_{41} & 0 & 0 \end{bmatrix}. \tag{27}$$

Set

$$Q_3 := \begin{bmatrix} I_{p_0} & 0 & 0 \\ -\Sigma_4^{-1} A_{31} & \Sigma_4^{-1} & 0 \\ 0 & 0 & I_{v-p_0-q_0} \end{bmatrix}, \; Q := QQ_3$$

and

$$PAQ =: \begin{bmatrix} I_{p_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_{q_0} & 0 \\ A_{41} & 0 & 0 \end{bmatrix}. \tag{28}$$

Perform a singular value decomposition

$$A_{41} = U_{41} \begin{bmatrix} I_{p_1} & 0 & 0 \\ 0 & \Sigma_{41} & 0 \\ 0 & 0 & 0 \end{bmatrix} V_{41}^*$$

with Σ_{41} diagonal, nonsingular having no singular values 1 and set

$$P_3 := \begin{bmatrix} V_{41}^* & 0 & 0 & 0 \\ 0 & I_{p-p_0} & 0 & 0 \\ 0 & 0 & I_{q_0} & 0 \\ 0 & 0 & 0 & U_{41}^* \end{bmatrix}, \; Q_4 := \begin{bmatrix} V_{41} & 0 & 0 \\ 0 & I_{q_0} & 0 \\ 0 & 0 & I_{v-p_0-q_0} \end{bmatrix},$$

$P := P_3 P, Q := Q Q_4$. Then partition

$$PDP^* =: \begin{bmatrix} I_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{p_0-p_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{p-p_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{q_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{p_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_{p_0-p_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_{q-q_0-p_0} \end{bmatrix},$$

$$PAQ =: \begin{bmatrix} I_{p_1} & 0 & 0 & 0 & 0 \\ 0 & I_{p_0-p_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 I_{q_0} & 0 \\ I_{p_1} & 0 & 0 & 0 & 0 \\ 0 & \Sigma_{41} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

As a next step we construct a hyperbolic matrix to eliminate Σ_{41}. Set $W = |(I_{p_0-p_1} - \Sigma_{41}^2)|$, where the absolute value is taken elementwise. Now W is nonsingular, since no diagonal element of Σ_{41} is equal to 1. Set

$$P_4 := \begin{bmatrix} I_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & W^{-1/2} & 0 & 0 & 0 & -W^{-1/2}\Sigma_{41} & 0 \\ 0 & 0 & I_{p-p_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{q_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{p_1} & 0 & 0 \\ 0 & -W^{-1/2}\Sigma_{41} & 0 & 0 & 0 & W^{-1/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_{q-q_0-p_0} \end{bmatrix},$$

$$Q_5 =: \begin{bmatrix} I_{p_1} & 0 & 0 & 0 & 0 \\ 0 & (I - \Sigma_{41}^2)^{-1} W^{1/2} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I_{q_0} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$P := P_4 P$, $Q := Q Q_5$ and partition

$$PDP^* =: \begin{bmatrix} I_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{p-p_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{q_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{p_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_{q-q_0-p_0} \end{bmatrix},$$

$$PAQ =: \begin{bmatrix} I_{p_1} & 0 & 0 & 0 & 0 \\ 0 & I_{p_0-p_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{q_0} & 0 \\ I_{p_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{30}$$

where D_1, D_2 are diagonal matrices with elements 1 or -1 on the diagonal. The final form follows by an appropriate block permutation, with the following blocksizes: $v_1 = p_1$, v_2 is the number of positive diagonal elements of D_1, w_2 is equal to q_0 plus the number of negative diagonal elements of D_1, v_3 is equal to $p_0 - p_1$ plus the number of positive diagonal elements of D_2 and $w_3 = n - 2v_1 - v_2 - w_2 - v_3$. □

Observe that the transformation is constructed from unitary and hyperbolic transformations and inversions of diagonal matrices. Thus at each step a bound for the error can be computed. The algorithm given in the proof could be easily modified for numerical computation.

We now have the following canonical form for the equivalence relation (23).

THEOREM 14. *Let $E, A \in \mathcal{C}^{n,n}$ be Hermitian and consider matrices*

$$\begin{array}{ll} (a) & T \text{ basis of } \operatorname{kernel} E \\ (b) & T' \text{ basis of } \operatorname{cokernel} E \\ (c) & V \text{ basis of } \operatorname{corange}(T^*AT) \\ (d) & V' \text{ basis of } \operatorname{range}(T^*AT). \end{array} \tag{31}$$

The following quantities are invariant under congruence: (with the notation

of the previous Lemma):

(a)	s^{p}	(number of pos. eigenv. of $V^*(T')^*ET'V$)
(b)	s^{n}	(number of neg. eigenv. of $V^*(T')^*ET'V$)
(c)	s^{d}	(number of strange eigenvalue pairs)
(d)	$s = s^{\mathrm{p}} + s^{\mathrm{n}}$	(strangeness)
(e)	$d = r - s = d^{\mathrm{p}} + d^{\mathrm{n}}$	(differential part)
(f)	$r = \mathrm{rank}(E)$	(rank)
(g)	a^{p}	(number of pos. eigenvalues of T^*AT)
(h)	a^{n}	(number of neg. eigenvalues of T^*AT)
(i)	$a = a^{\mathrm{p}} + a^{\mathrm{n}} = \mathrm{rank}(T^*AT)$	(algebraic part)
(j)	$u = n - r - a - s$	(undetermined part)

(32)

Furthermore (E, A) is congruent to the canonical form

$$\left(\begin{bmatrix}
I_{s^{\mathrm{d}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -I_{s^{\mathrm{d}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{v_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -I_{w_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{d^{\mathrm{p}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I_{d^{\mathrm{n}}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s^{\mathrm{d}}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s^{\mathrm{d}}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{v_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{w_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{a^{\mathrm{p}}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_{a^{\mathrm{n}}} & 0 & 0 & 0 & 0 \\
I_{s^{\mathrm{d}}} & I_{s^{\mathrm{d}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{v_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{w_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right) \tag{33}$$

where $v_2 = s^{\mathrm{p}} - s^{\mathrm{d}}$, $w_2 = s^{\mathrm{n}} - s^{\mathrm{d}}$ and the last column has width u.

Proof. The proof is constructive by the following sequence of transformations:

Let $E = QD^{1/2}SD^{1/2}Q^*$ be a spectral factorization of E, where

$$S = \begin{bmatrix} I_{i^{\mathrm{p}}} & 0 & 0 \\ 0 & -I_{i^{\mathrm{n}}} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

Q is unitary , D is positive diagonal, $(i^{\mathrm{p}}, i^{\mathrm{n}}, n-r)$ is the inertia of E and $r = i^{\mathrm{p}} + i^{\mathrm{n}}$. Set $P = D^{-1/2}Q^*$ and partition

$$A := PAP^* = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{35}$$

and $E := PEP^* = S$ analogously.

Perform a spectral factorization of A_{33},

$$A_{33} = Q_3 D_3^{1/2} \tilde{S} D_3^{1/2} Q_3^*, \tag{36}$$

where

$$\tilde{S} = \begin{bmatrix} I_{a^{\mathrm{p}}} & 0 & 0 \\ 0 & -I_{a^{\mathrm{n}}} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

Q_3 is unitary , D_3 is positive diagonal, $(a^{\mathrm{p}}, a^{\mathrm{n}}, n-r-a)$ is the inertia of A_{33} and $a = a^{\mathrm{p}} + a^{\mathrm{n}}$. Set

$$P_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & D_3^{-1/2} Q_3^* \end{bmatrix},$$

and partition

$$A := P_1 A P_1^* = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & I_{a^{\mathrm{p}}} & 0 & 0 \\ A_{41} & A_{42} & 0 & -I_{a^{\mathrm{n}}} & 0 \\ A_{51} & A_{52} & 0 & 0 & 0 \end{bmatrix} \tag{38}$$

and $E := P_1 E P_1^*$ analogously. We then eliminate A_{31}, A_{32}, A_{23}, A_{13}, A_{41}, A_{42}, A_{24}, A_{14} by a congruence transformation with

$$P_2 = \begin{bmatrix} I & 0 & -A_{13} & A_{14} & 0 \\ 0 & I & -A_{23} & A_{24} & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

and obtain with changed matrices A_{11}, A_{12}, A_{21}, A_{22}

$$A := P_2 A P_2^* = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{15} \\ A_{21} & A_{22} & 0 & 0 & A_{25} \\ 0 & 0 & I_{a^{\mathrm{p}}} & 0 & 0 \\ 0 & 0 & 0 & -I_{a^{\mathrm{n}}} & 0 \\ A_{51} & A_{52} & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

and $E := P_2 E P_2^*$. We then eliminate the upper left corner of A, by choosing

$$B = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 \\ -A_{21} & -A_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A := A - EB = \begin{bmatrix} 0 & 0 & 0 & 0 & A_{15} \\ 0 & 0 & 0 & 0 & A_{25} \\ 0 & 0 & I_{a^{\mathrm{p}}} & 0 & 0 \\ 0 & 0 & 0 & -I_{a^{\mathrm{n}}} & 0 \\ A_{51} & A_{52} & 0 & 0 & 0 \end{bmatrix}. \tag{40}$$

We then apply Lemma 13 to the two matrices

$$D = \begin{bmatrix} I_{i^{\mathrm{p}}} & 0 \\ 0 & -I_{i^{\mathrm{n}}} \end{bmatrix}, \; A = \begin{bmatrix} A_{15} \\ A_{25} \end{bmatrix}$$

and yield the required canonical form with $s^{\mathrm{p}} = v_1 + v_2$, $s^{\mathrm{n}} = v_1 + w_2$, $s^{\mathrm{d}} = v_1$, $d^{\mathrm{p}} = v_3$, and $d^{\mathrm{n}} = w_3$. □

REMARK 15. Again it is clear that congruence includes strong congruence and hence we could first transform to the canonical form for Hermitian pencils under congruence and then reduce this form further.

4. Global canonical form

We now apply the local canonical forms (22),(33) for each fixed value $t \in [t_0, t_1]$. Then we obtain integer valued functions $r, a, s, u^{\mathrm{l}}, u^{\mathrm{r}}\colon [t_0, t_1] \to \mathcal{N}_0$, $d^{\mathrm{p}}, d^{\mathrm{n}}, a^{\mathrm{p}}, a^{\mathrm{n}}, s^{\mathrm{p}}, s^{\mathrm{n}}, s^{\mathrm{d}}, u\colon [t_0, t_1] \to \mathcal{N}_0$, respectively.

If we do not pose any further restrictions, then it is possible that these quantities change their values with t. In order to demonstrate some of the possibilities consider the following simple examples:

EXAMPLE 16. Consider the scalar equation

$$t\dot{x} = f. \tag{41}$$

Here $a(t) \equiv 0$, $s(t) \equiv 0$ but $r(t)$ has a jump at the origin from 1 to 0. A necessary condition for solvability is $f(0) = 0$.

For the algebraic equation

$$tx = f \tag{42}$$

we have $r(t) \equiv 0$, $s(t) \equiv 0$ but $a(t)$ has a jump at the origin from 1 to 0 and we have the same necessary condition for existence of solutions. As a third example consider:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} x + f. \tag{43}$$

Here $r(t) \equiv 0, a(t) \equiv 0$ but $s(t)$ has a jump at the origin. We obtain $f_2(t) = -tx_1(t)$, where x_1 is continuously differentiable. A necessary condition for the existence of solutions is that f_2 is continuously differentiable and $f_2(0) = 0$ and the change in the strangeness seems to be responsible for the higher smoothness requirement on f.

Currently it is not completely understood how to characterize these conditions at interior points. For this reason, we exclude such phenomena by assuming

$$r(t) \equiv r,\ a(t) \equiv a,\ s(t) \equiv s,\ u^{\mathrm{l}}(t) \equiv u^{\mathrm{l}},\ u^{\mathrm{r}}(t) \equiv u^{\mathrm{r}} \tag{44}$$

and

$$\begin{aligned} &d^{\mathrm{p}}(t) \equiv d^{\mathrm{p}},\ d^{\mathrm{n}}(t) \equiv d^{\mathrm{n}},\ a^{\mathrm{p}}(t) \equiv a^{\mathrm{p}},\ a^{\mathrm{n}}(t) \equiv a^{\mathrm{n}}, \\ &s^{\mathrm{p}}(t) \equiv s^{\mathrm{p}},\ s^{\mathrm{n}}(t) \equiv s^{\mathrm{n}},\ s^{\mathrm{d}}(t) \equiv s^{\mathrm{d}},\ u(t) \equiv u, \end{aligned} \tag{45}$$

respectively, throughout the rest of this paper.

We should point out that there are other assumptions, which allow to partially drop assumptions (44) or (45) by assuming higher differentiability of the inhomogeniety and uniqueness of solutions, see [13, 5]. It is currently under investigation how to relax conditions (44) or (45) without extra assumptions.

Applying equivalence (18) to the pair $(E(t), A(t))$, we obtain the following canonical form:

THEOREM 17. *Let E, A in (2) be sufficiently smooth and let (44) hold. Then, $(E(t), A(t))$ is equivalent to a pair of matrix functions of the form*

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) \\ 0 & 0 & 0 & A_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \begin{matrix} s \\ d \\ a \\ s \\ u^{\mathrm{l}} \end{matrix}, \tag{46}$$

where the last block column in both matrices has width u^{r}.

Proof. The proof is again similar to the proof for the square case given in [41] and is therefore omitted. □

We do not know a corresponding result for the Hermitian case, mainly since we do not know an appropriate definition of congruence in the variable coefficient case.

By considering Examples 6, 7 we observe that the reduction to the form (46) is not sufficient to explain the different solution behaviour, since in both cases we obtain $(r, a, s) = (1, 0, 1)$. The consequence is that we have to allow further transformations.

Writing down the system of differential-algebraic equations that corresponds to (46), we get

$$\begin{array}{ll} (a) & \dot{x}_1(t) = A_{12}(t)x_2(t) + A_{14}(t)x_4(t) + f_1(t) \\ (b) & \dot{x}_2(t) = A_{24}(t)x_4(t) + f_2(t) \\ (c) & 0 = x_3(t) + f_3(t) \\ (d) & 0 = x_1(t) + f_4(t) \\ (e) & 0 = f_5(t). \end{array} \tag{47}$$

Here, we can insert equation (47d) in (47a), which then becomes an algebraic equation. This corresponds to passing from (46) to

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) \\ 0 & 0 & 0 & A_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right), \tag{48}$$

for which we again compute characteristic values $r, a, s, d, u^{\rm l}, u^{\rm r}$.

The above procedure therefore leads to an inductive definition of a sequence of pairs of matrix functions $(E_i(t), A_i(t))$, $i \in \mathcal{N}_0$, where $(E_0(t), A_0(t)) = (E(t), A(t))$ and $(E_{i+1}(t), A_{i+1}(t))$ is derived from $(E_i(t), A_i(t))$ by bringing it into the form (46) and passing then to (48). Here we must assume (44) for every occuring pair of matrices. Connected with this sequence, we then have sequences $r_i, a_i, s_i, d_i, u_i^{\rm l}, u_i^{\rm r}$, $i \in \mathcal{N}_0$ of nonnegative integers, which are characteristic for the given pair $(E(t), A(t))$, that is, that they do not depend on the specific way they are obtained. Furthermore the sequence stops after finitely many (say m) steps with $s_i = 0$. The quantity m is called the *strangeness index* of the pencil $(E(t), A(t))$.

Both results are proved for the square case in [41] and the proofs there are simple to modify.

Note that for square systems, for which $u^{\rm r} = u^{\rm l} = 0$ and for which the same smoothness and rank assumptions hold, the strangeness index is closely related to the differentiation index, e.g. [5]. The two indices are equal if $m = 0, a_0 = 0$ and the differentiation index is $m+1$ otherwise. But, since we also allow nonuniqueness of solutions for which the differentiation index is not defined, we termed a new expression to distinguish the different indices.

With the finite sequences $r_i, a_i, s_i, d_i, u_i^{\rm l}, u_i^{\rm r}$ $i \in \{0, \ldots, m\}$, we now obtain an appropriate generalization of the Kronecker canonical form in the case of variable pencils $(E(t), A(t))$.

THEOREM 18. *Let the strangeness index m be well-defined for the pair $(E(t), A(t))$ of smooth matrix functions. Let $r_i, a_i, s_i, d_i, u_i^{\rm l}, u_i^{\rm r}$, $i \in \{0, \ldots, m\}$ be the related characteristic values as above. Define*

$$\begin{array}{lll}
(a) & b_0 = a_0, & b_i = \operatorname{rank}\left(\left[\begin{array}{c} A_{14}^{(i-1)}(t) \end{array}\right]\right), \\
(b) & c_0 = a_0 + s_0, & c_i = \operatorname{rank}\left(\left[\begin{array}{cc} A_{12}^{(i-1)}(t) & A_{14}^{(i-1)}(t) \end{array}\right]\right), \\
(c) & w_0 = u_0^{\rm l}, & w_i = u_i^{\rm l} - u_{i-1}^{\rm l}, \qquad i = 1, \ldots, m.
\end{array} \tag{49}$$

We then have

$$\begin{array}{lll}
(a) & c_i = b_i + s_i, & i = 0, \ldots, m \\
(b) & w_i = s_{i-1} - c_i, & i = 1, \ldots, m
\end{array} \tag{50}$$

and the pair $(E(t), A(t))$ is equivalent to a pair of matrix functions of the

form (without arguments)

$$\left(\left[\begin{array}{cccccc} I & 0 & 0 & * & \dots & * \\ 0 & 0 & 0 & F_m & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & F_1 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & G_m & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & G_1 \\ 0 & 0 & 0 & & & 0 \end{array}\right], \left[\begin{array}{cccccc} * & * & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & I & & & \\ \vdots & \vdots & & \ddots & & \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & & & & I \end{array}\right]\right) \begin{array}{c} d_m \\ w_m \\ \vdots \\ \vdots \\ w_0 \\ c_m \\ \vdots \\ \vdots \\ c_0 \end{array}, \tag{51}$$

where

$$\operatorname{rank}\left(\left[\begin{array}{c} F_i \\ G_i \end{array}\right]\right) = c_i + w_i = s_{i-1} \le c_{i-1}. \tag{52}$$

and the second block column in both matrices has width u_m^{r}.

Proof. The proof is a simple modification of the proof for the square case in [41]. □

REMARK 19. Again it is natural to look at a constant pair (E, A) in Kronecker canonical form (51) and compute the strangeness index of the different blocks of the Kronecker canonical form separately. Let again N_ν denote a nilpotent Jordan block of size ν.

(a) Kronecker block L_ϵ

$$(E, A) = \left(\left[\begin{array}{cc} 0 & I_\epsilon \end{array}\right], \left[\begin{array}{cc} e_1^{(\epsilon)} & N_\epsilon \end{array}\right]\right) \sim \left(\left[\begin{array}{cc} I_\epsilon & 0 \end{array}\right], \left[\begin{array}{cc} N_\epsilon & e_1^{(\epsilon)} \end{array}\right]\right)$$

$$m = 0, \quad d_m = \epsilon, \ c_m = 0, \ w_m = 0, u_m^{\mathrm{l}} = 0, \ u_m^{\mathrm{r}} = 1$$

(b) Kronecker block M_η

$$(E, A) = \left(\left[\begin{array}{c} I_\eta \\ 0 \end{array}\right], \left[\begin{array}{c} N_\eta \\ e_\eta^{(\eta)T} \end{array}\right]\right) \sim \left(\left[\begin{array}{c} e_1^{(\eta)T} \\ N_\eta^T \end{array}\right], \left[\begin{array}{c} 0 \\ I_\eta \end{array}\right]\right)$$

$$\begin{array}{ll} m = \eta, & c_0 = \ldots = c_{m-1} = 1, \ c_m = 0 \\ & w_0 = \ldots = w_{m-1} = 0, \ w_m = 1 \\ & s_0 = \ldots = s_{m-1} = 1, \ s_m = 0 \\ & b_0 = \ldots = b_{m-1} = 0, \ b_m = 0 \\ & d_m = 0, \ a_m = \sum_{i=0}^{m} c_i = \eta \\ & u_m^{\mathrm{l}} = \sum_{i=0}^{m} w_i = 1, \ u_m^{\mathrm{r}} = l - d_m - a_m = 0 \end{array}$$

(c) Jordan block F_ρ

$$(E, A) = (I_\rho, J_\rho)$$

$$m = 0,\ d_m = \rho,\ a_m = 0,\ u_m^{\mathrm{l}} = 0,\ u_m^{\mathrm{r}} = 0$$

(d) Nilpotent block S_σ

$$(E, A) = (N_\sigma, I_\sigma)$$

$$\begin{aligned} m = \sigma - 1, \quad & c_0 = \ldots = c_{m-1} = 1,\ c_m = 1 \\ & w_0 = \ldots = w_{m-1} = 0,\ w_m = 0 \\ & s_0 = \ldots = s_{m-1} = 1,\ s_m = 0 \\ & b_0 = \ldots = b_{m-1} = 0,\ b_m = 1 \\ & d_m = 0,\ a_m = \textstyle\sum_{i=0}^{m} c_i = \sigma,\ u_m^{\mathrm{l}} = \sum_{i=0}^{m} w_i = 0,\ u_m^{\mathrm{r}} = 0. \end{aligned}$$

REMARK 20. It is now clear from the analysis of the sequences r_i, a_i, s_i, d_i, u_i what is the difference between the two examples. In Example 6, we obtain $(r_1, a_1, s_1, d_1, u_1) = (0, 1, 0, 0, 1)$, while in Example 7, we obtain $(r_1, a_1, s_1, d_1, u_1) = (0, 2, 0, 0, 0)$.

REMARK 21. As already mentioned it is an **open problem** to generalize Theorem 17 and consequently also Theorem 18 to Hermitian pencils, since we do not know in general how to get an appropriate time varying congruence transformation, which keeps the symmetry at each step.

There are examples however, where such transformations exist, and in this case it is often important to use them and not to destroy the symmetry.

If for example the transformations from the right operate in the right nullspace of E, then we keep the symmetry. Consider the following example

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 2 & 1+t^2 & 1 \\ 1+t^2 & 2t^2-1 & 0 \\ 1 & 0 & -1 \end{bmatrix} x + f, \tag{53}$$

then a congruence transformation with

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & t^2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

yields the symmetric system

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dot{y} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} y + g, \tag{54}$$

which obviously has a pencil of index 3. The local canonical form (33) at time $t \neq 0$ has indices $s^{\mathrm{p}} = s^{\mathrm{n}} = s^{\mathrm{d}} = 1$ and hence displays via a symmetric transformation that the system has a nonvanishing strangeness index.

Another example, where we can obtain a global Hermitian canonical form is when $(E(t), A(t))$ are commuting pairs. Ignoring the symmetry we end up with an Hermitian canonical form

$$(E(t), A(t)) \sim \left(\begin{bmatrix} I_d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & 0 \end{bmatrix} \right), \tag{55}$$

where each diagonal block may be missing. In any case we have $m = 0$.

5. Existence and uniqueness of solutions of DAE's

We can now apply the results obtained in the previous sections to characterize the solutions of linear DAEs. To do this we recall the following definition from [41].

DEFINITION 22. A function $x\colon [t_0, t_1] \to \mathcal{C}^n$ is called *solution* of (2) if $x \in C^1([t_0, t_1], \mathcal{C}^n)$ and x satisfies (2) pointwise.
It is called *solution of the initial value problem* (2), (3) if x is solution of (2) and x satisfies (3).
An initial condition (3) is called *consistent* if the corresponding initial value problem is solvable, i.e. has at least one solution.

Note that this definition differs slightly form the definition used in some of the literature, e.g. [5].

Using the results of Section 4, we can transform (2) to an equivalent differential-algebraic equation of a very special structure. Equivalence here means that there is a one-to-one correspondence of their solutions.

Note that, since the constant coefficient case is a special case of the variable coefficient case, we obtain the standard results on existence and uniqueness of solutions and consistency of initial conditions as given for example in [23] as a corollary of the following theorem.

THEOREM 23. *Let the strangeness index m be well-defined for the pair $(E(t), A(t))$ in (2) and $f \in C^m([t_0, t_1], \mathcal{C}^n)$. Then, (2) is equivalent to a differential-algebraic equation of the form*

$$\begin{array}{ll} (a) & \dot{x}_1(t) = A_{13}(t)x_3(t) + f_1(t) \\ (b) & 0 = x_2(t) + f_2(t) \\ (c) & 0 = f_3(t), \end{array} \tag{56}$$

where the inhomogeneity is determined by $f^{(0)}, \ldots, f^{(m)}$. In particular, $d_m, a_m, u_m^{\mathrm{r}}$ are the number of differential, algebraic and undetermined components of the unknown x in (a), (b) respectively, while u_m^{l} is the number of conditions in (c).

Proof. Inductively transforming $(E(t), A(t))$ to the form (46) and then passing to (48) until $s_m = 0$ yields a pair of matrix functions of the form

$$\left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \right),$$

with block sizes $d_m, a_m, u_m^{\mathrm{l}}$ for the rows and $d_m, a_m, u_m^{\mathrm{r}}$ for the columns. All steps in the procedure are reversible, and in each step the inhomogeneity is differentiated once. □

Then we have the following results characterizing existence and uniqueness of solutions and consistency of initial conditions.

COROLLARY 24. *Under the assumptions of Theorem 23 the following statements hold if in addition $f \in C^{m+1}([t_0, t_1], \mathcal{C}^n)$.*

Equation (2) is solvable if and only if the u_{l_m} functional consistency conditions

$$f_3(t) \equiv 0 \tag{57}$$

are satisfied.

An initial condition (3) is consistent if and only if in addition the a_m conditions

$$x_2(t_0) = -f_2(t_0) \tag{58}$$

hold.

The initial value problem (2), (3) is uniquely solvable if again in addition we have

$$u_m^{\mathrm{r}} = 0. \tag{59}$$

Otherwise, we can choose $x_3 \in C^1([t_0, t_1], \mathcal{C}^{u_m^{\mathrm{r}}})$ arbitrarily.

Proof. Observing that we need the higher differentiability of f to guarantee that x_2 is differentiable, the results are direct conclusions from Theorem 23. □

For the case of constant coefficients we can use the indices computed in Remark 19 to characterize existence and uniqueness. This gives a different proof for the following well-known result:

COROLLARY 25. *Consider the linear DAE with constant coefficients (5) in Weierstraß-Kronecker canonical form (7).*

There exists a solution of (5) for all $f \in C^{m+1}([t_0, t_1], \mathcal{C}^n)$ and for all consistent initial conditions (3) if and only if no blocks of type (8b) occur in the canonical form (8). A solution of (5) is unique if and only if no blocks of types (8a) occur in the canonical form (8).

Proof. The proof follows from Theorem 23 and Remark 19. We have a solution for all sufficiently smooth f and all consistent initial conditions if and only if $u_m^{\mathrm{l}} = 0$, which is true if and only if no blocks of type (8b) occur in the canonical form.

If a solution exists, it is unique if and only $u_m^{\mathrm{r}} = 0$, which is true if and only if no blocks of type (8a) occur in the canonical form. □

Note that this result differs from other versions of existence and uniqueness results for linear constant coefficient systems, e.g. [10, 5] due to the slightly different definition of solvability.

REMARK 26. Here we have discussed so far the case of general pencils. As already mentioned above, the case of Hermitian pencils is also very important in applications. We do not know of a general result in the variable coefficient case, that gives a canonical form under transformations that keep the symmetry like congruence transformations do in the constant coefficient case. Observe that the transformation (18) destroys symmetry of E and A, even if $P = Q^*$, due to the derivative introduced in an unsymmetric way.

Summarizing the results of this section, we have shown that three quantities are sufficient to discuss the solution behaviour of a differential–algebraic equation whose coefficients satisfy some indispensable rank and smoothness assumptions. These are the strangeness index m and the final numbers d_m and a_m of differential and algebraic components. Dependent on these quantities we have the parameters $u_m^{\mathrm{l}}, u_m^{\mathrm{r}}$, from which we directly see the solvability and number of free components. The quantity m is closely related to other well-known indices for differential-algebraic equations, see [41].

6. Numerical methods for general pencils

As we have already mentioned before, the numerical computation of the Weierstraß-Kronecker canonical form of a constant pencil pencil is an ill-conditioned problem, since the transformation matrices are in general not norm bounded and arbitrarily small perturbations may change the block structure. In finite precision arithmetic, we cannot expect accurate sizes of the indices if we do not use unitary transformations. The only hope we have to compute the invariants accurately is to restrict the transformation matrices to be unitary and therefore to reduce to a less condensed form. Such a form is the generalized Schur decomposition of an arbitrary pencil. This decomposition has been studied widely in recent years and good reliable software is now available in public domain software, see [20, 21, 1] and the references therein. Similar approaches are for example due to Van Dooren, [52, 53, 3], Wilkinson [56, 57], Kublanovskaya [37] and Kågström [35]. See [20] for a comparison of the different approaches.

Here we briefly discuss the GUPTRI form given in [20, 21].

THEOREM 27. *Let $E, A \in \mathcal{C}^{n,l}$ then there exist unitary matrices $P \in \mathcal{C}^{n,n}, Q \in \mathcal{C}^{l,l}$ such that*

$$(PEQ, PAQ) = \left(\begin{bmatrix} E^{\mathrm{r}} & * & * & * & * \\ 0 & E^{\mathrm{z}} & * & * & * \\ 0 & 0 & E^{\mathrm{f}} & * & * \\ 0 & 0 & 0 & E^{\mathrm{i}} & * \\ 0 & 0 & 0 & 0 & E^{\mathrm{l}} \end{bmatrix}, \begin{bmatrix} A^{\mathrm{r}} & * & * & * & * \\ 0 & A^{\mathrm{z}} & * & * & * \\ 0 & 0 & A^{\mathrm{f}} & * & * \\ 0 & 0 & 0 & A^{\mathrm{i}} & * \\ 0 & 0 & 0 & 0 & A^{\mathrm{l}} \end{bmatrix} \right), \tag{60}$$

where the diagonal blocks describe the Kronecker structure of the pair (E, A) in the following way:

$(E^{\mathrm{r}}, A^{\mathrm{r}})$	contains all the blocks of type $8a$
$(E^{\mathrm{z}}, A^{\mathrm{z}})$	contains all Jordan blocks to the eigenvalue 0
$(E^{\mathrm{f}}, A^{\mathrm{f}})$	contains all Jordan blocks to finite nonzero eigenvalues
$(E^{\mathrm{i}}, A^{\mathrm{i}})$	contains all Jordan blocks to infinite eigenvalues
$(E^{\mathrm{l}}, A^{\mathrm{l}})$	contains all the blocks of type $8b$

(61)

The Jordan structure to the zero and infinite eigenvalues is explicitly exposed in this form, while the Jordan structure for the other finite eigenvalues is not exposed.

From this form we can directly decide on existence, uniqueness and index of the corresponding constant coefficient DAE. More work has to be done

though to characterize consistent initial values. For the solution of linear constant coefficient DAEs, this has been discussed in [56, 6]. In the latter a reduction procedure based on singular value decompositions is given, that successively determines consistency conditions or nonunique components of the solution and finally uses a method for ordinary differential equations to solve the differential part.

For the variable coefficient case many different approaches have been taken, see for example [12, 43, 22, 14, 2, 15]. All these approaches deal partially with higher index DAEs but none is able to treat DAEs with free components as they for example occur in the numerical solution of optimal control problems for descriptor systems [38, 39, 42]. The most general approach that can also deal with such cases was given in [40] and is based on reduced forms for nonconstant matrix pencils that are obtained by numerically stable methods and that similarly to the global canonical form discussed in Section 4 expose the solution behaviour of the original DAE. They are computed from the DAE (5) and its derivatives and are not any more related to the matrix pencil with variable coefficients. We therefore refrain from presenting them here.

7. Conclusion

We have presented equivalence relations and corresponding canonical forms for matrix pencils that depend on a parameter, as they arise for example in differential algebraic equations. They can be viewed as direct generalizations of the Weierstraß-Kronecker canonical form of constant matrix pencils.

Based on these canonical forms existence, uniqueness for differential-algebraic equations and consistency of initial conditions can be characterized. For numerical computation these forms are not suitable but they can be modified in such a way that the important invariants can be computed in a numerically stable way and that also differential algebraic equations can be solved.

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