Canonical forms for linear differential–algebraic equations with variable coefficients

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Abstract

We give a new set of local characterizing quantities for the treatment of linear differential–algebraic equations with variable coefficients. This leads to new global invariances under which we can find a generalization of the global (or differentiation) index and get a new existence and uniqueness theorem.

Keywords. Linear differential-algebraic equations, characteristic values, invariants, index, canonical forms, generalized Kronecker canonical form, existence and uniqueness.

1 Introduction

In this paper, we study linear differential–algebraic equations with variable coefficients

$$E(t)\dot{x}(t) = A(t)x(t) + f(t),\ t \in [\underline{t}, \bar{t}] \tag{1}$$

with $E, A \in C([\underline{t}, \bar{t}], \mathbb{C}^{n,n}),\ f \in C([\underline{t}, \bar{t}], \mathbb{C}^n)$ together with an initial condition

$$x(t_0) = x_0,\ t_0 \in [\underline{t}, \bar{t}],\ x_0 \in \mathbb{C}^n. \tag{2}$$

Here $C^\ell([\underline{t}, \bar{t}], \mathbb{C}^n)$ denotes the set of ℓ-times continuously differentiable functions from the interval $[\underline{t}, \bar{t}]$ to the n-dimensional complex vector space $\mathbb{C}^n$.

Definition 1 A function $x : [\underline{t}, \bar{t}] \to \mathbb{C}^n$ is called *solution* of (1) if $x \in C^1([\underline{t}, \bar{t}], \mathbb{C}^n)$ and x satisfies (1) pointwise.
It is called *solution of the initial value problem* (1), (2) if x is solution of (1) and x satisfies (2).
An initial condition (2) is called *consistent* if the corresponding initial value problem is solvable, i.e. has at least one solution.

We are interested in the following questions:

- Under which conditions has (1) a solution?
- How many solutions do exist in this case?
- What is the set of consistent initial values?
- Under which conditions are there unique solutions?

In the case of constant coefficients $E(t) \equiv E, A(t) \equiv A$, the above problem is well–understood by the Kronecker canonical form (see, e.g. [7]) in terms of regularity and index of the underlying matrix pencil $\lambda E - A$. But dealing with the general case (1), concepts based on generalizations of these now local quantities, like uniform regularity or uniform index turned out to be unsatisfactory or even meaningless (see [6, 10] but also [12, 14]) especially for so–called higher index problems. The only known invariant quantity for (1) seems to be the global index or differentiation index (see, e.g. [8]) which, however, is not a straight–forward generalization of the index of a matrix pencil.

The main reason for this observation is that the Kronecker canonical form is based on constant equivalence transformations whereas for (1) we must allow for time–dependent transformations.

The aim of the present paper is to develop a system of local quantities (that is, which can be derived from local information) which also bear information on the global behaviour of (1).

For this, we first give the necessary notation and a brief sketch of the results for the constant coefficient case in Section 2. In Section 3 we develop a set

of local characterizing quantities and a corresponding canonical form especially adapted to the global transformation behaviour of (1). Then we show how these quantities can be used to give a deeper insight into the given problem class (1) (Section 4) and how they give a new existence and uniqueness theorem (Section 5). At last, we treat the question how the present approach is related to other known invariances in Section 6 and give some conclusions in Section 7.

2 The case of constant coefficients

In this section, we first treat the simpler case of a differential–algebraic equation with constant coefficients

$$E\dot{x} = Ax + f(t), \quad t \in [\underline{t}, \bar{t}], \tag{3}$$

where $E, A \in \mathbb{C}^{n,n}$ and $f \in C([\underline{t}, \bar{t}], \mathbb{C}^n)$.

The main purpose here is to introduce some notation and to quote the most important results for later reference. Therefore, we do not include any proofs in this part.

The standard way to treat (3) is to look at all regular transformations which take (3) into an equation of the same form. This leads to the definition of so–called (strong) equivalence. Two pairs of matrices (E_i, A_i), $i = 1, 2$, of the above form are called *(strongly) equivalent* if there are nonsingular matrices $P, Q \in \mathbb{C}^{n,n}$ with

$$(E_2, A_2) = P(E_1, A_1) \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}. \tag{4}$$

Clearly, this defines an equivalence relation which can be found in connection with the examination of linear matrix functions $\lambda E - A$, also called matrix pencils. Of course, all these formulations are equivalent. A canonical form connected with this equivalence is the Kronecker canonical form (see [7] for details).

Theorem 2 *Let $E, A \in \mathbb{C}^{n,n}$. Then, there exist nonsingular $P, Q \in \mathbb{C}^{n,n}$ such that*

$$\begin{aligned} &P(\lambda E - A)Q \\ &\quad = \operatorname{diag}(\boldsymbol{L}_{\epsilon_1}, \ldots, \boldsymbol{L}_{\epsilon_p}, \boldsymbol{M}_{\eta_1}, \ldots, \boldsymbol{M}_{\eta_q}, \boldsymbol{J}_{\rho_1}, \ldots, \boldsymbol{J}_{\rho_v}, \boldsymbol{N}_{\sigma_1}, \ldots, \boldsymbol{N}_{\sigma_w}), \end{aligned} \tag{5}$$

where

$$
\begin{aligned}
&(a)\quad \boldsymbol{L}_{\epsilon_j} \text{ is an } \epsilon_j \times (\epsilon_j+1) - \text{bidiagonal block, } \epsilon_j \in \mathbb{N}_0\\
&\qquad \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}\\
&(b)\quad \boldsymbol{M}_{\eta_j} \text{ is an } (\eta_j+1) \times \eta_j - \text{bidiagonal block, } \eta_j \in \mathbb{N}_0\\
&\qquad \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}\\
&(c)\quad \boldsymbol{J}_{\rho_j} \text{ is a } \rho_j \times \rho_j - \text{Jordan block, } \rho_j \in \mathbb{N},\ \lambda_j \in \mathbb{C}\\
&\qquad \lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}\\
&(d)\quad \boldsymbol{N}_{\sigma_j} \text{ is a } \sigma_j \times \sigma_j - \text{nilpotent block, } \sigma_j \in \mathbb{N}\\
&\qquad \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}
\end{aligned}
\tag{6}
$$

Note that all quantities on the right hand side of (5) are characteristic for the pair (E, A), i.e. up to order each canonical form of (E, A) consists of the same blocks. For the treatment of (3), the following invariances play an important role.

Definition 3 A matrix pencil $\lambda E - A,\ E, A \in \mathbb{C}^{n,n}$, is called *regular* if the characteristic polynomial

$$p(\lambda) = \det(\lambda E - A) \tag{7}$$

does not vanish identically, otherwise *singular*. The quantity

$$k = \begin{cases} 0 & \text{for } w = 0 \\ \max\{\sigma_j \mid j = 1, \ldots, w\} & \text{for } w > 0 \end{cases} \tag{8}$$

with w as in (5) is called the *index* of $\lambda E - A$ and is denoted by $k = \mathrm{ind}(E, A)$.

A typical existence and uniqueness theorem for (3) then has the following form (see e.g. [1, 12]).

Theorem 4 *Let (3) be given with regular pencil $\lambda E - A$ and let $f \in C^k([\underline{t}, \bar{t}], \mathbb{C}^{n,n})$ with $k = \mathrm{ind}(E, A)$. Then, (3) is solvable and every consistent initial condition (2) fixes a unique solution.*

We will see in the following sections how one can generalize this result to the case of variable coefficients.

3 Local canonical form

Turning back to the variable coefficient case one could try to generalize the concepts of the previous section in the following manner. Instead of k in (8), we now have a function $k : [\underline{t}, \bar{t}] \to \{0, \ldots, n\}$ with $k(t) = \mathrm{ind}(E(t), A(t))$, which is commonly called *local index*. If $k(t) \equiv k$, we call (3) to be of *uniform index* k.

In addition, we can define *uniform regularity* for (1) in the sense that $(E(t), A(t))$ shall be regular for all $t \in [\underline{t}, \bar{t}]$. But these terms are not suited for statements similar to Theorem 4 as the following examples show. For more details, we refer to [12, 14].

Example 1 A short computation shows that

$$\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(t), \; t \in [-1, 1]$$

is uniformly regular and of uniform index $k = 2$ but that

$$x(t) = c(t) \begin{bmatrix} t \\ 1 \end{bmatrix}$$

is a solution for all $c \in C^1([-1, 1], \mathbb{C})$. In particular, there are more than one solution for consistent initial conditions.

Example 2 The equation

$$\begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \; t \in [\underline{t}, \bar{t}],$$

with $f_i \in C^2([\underline{t}, \bar{t}], \mathbb{C}^2), i = 1, 2$ is not uniformly regular because the pencil $(E(t), A(t))$ is singular for all $t \in [\underline{t}, \bar{t}]$. Nevertheless, it has the unique solution

$$\begin{aligned} x_1(t) &= f_1(t) + t f_2(t) - t \dot{f}_1(t), & x &= (x_1, x_2)^T \\ x_2(t) &= f_2(t) - \dot{f}_1(t), & f &= (f_1, f_2)^T \end{aligned}$$

for each consistent initial condition.

The intrinsic reason for these phenomena is that the above notion is based on the equivalence relation (4), while it seems appropriate for (1) to include non–constant transformations. Setting $x(t) = Q(t)y(t)$ and pre–multiplying (1) by $P(t)$, the equation (1) transforms to

$$P(t)E(t)Q(t)\dot{y}(t) = (P(t)A(t)Q(t) - P(t)E(t)\dot{Q}(t))y(t) + P(t)f(t). \quad (9)$$

Therefore, one is led to the following definition:

Definition 5 Two pairs of matrix functions $(E_i(t), A_i(t))$, $E_i, A_i \in C([\underline{t}, \bar{t}], \mathbb{C}^{n,n}), i = 1, 2$ are called *equivalent* if there are $P \in C([\underline{t}, \bar{t}], \mathbb{C}^{n,n})$ and $Q \in C^1([\underline{t}, \bar{t}], \mathbb{C}^{n,n})$ with $P(t), Q(t)$ nonsingular for all $t \in [\underline{t}, \bar{t}]$ such that

$$(E_2(t), A_2(t)) = P(t)(E_1(t), A_1(t)) \begin{bmatrix} Q(t) & -\dot{Q}(t) \\ 0 & Q(t) \end{bmatrix}. \quad (10)$$

Standard rules for differentiation show that this is indeed an equivalence relation.

Remark 1 Examples 1 and 2 are obtained by non–constant transformations applied to differential–algebraic equations with constant coefficients, where the underlying matrix pencil is singular in the first case and regular with index $k = 2$ in the second case.

It is obvious that the above problems occur because (4) is not a proper local version of (10). But one would like to have local quantities, i.e. characteristic values of $(E(t), A(t))$ for fixed $t \in [\underline{t}, \bar{t}]$, at hand which also give information on the global problem.

Taking into account that at a fixed point $t \in [\underline{t}, \bar{t}]$ we can choose $Q(t)$ and $\dot{Q}(t)$ independently (see [10]), we modify (4) in the following way:

Definition 6 Two pairs of matrices (E_i, A_i), $E_i, A_i \in \mathbb{C}^{n,n}$, $i = 1, 2$ are called *equivalent* if there are matrices $P, Q, B \in \mathbb{C}^{n,n}$ with P, Q nonsingular such that

$$(E_2, A_2) = P(E_1, A_1) \begin{bmatrix} Q & -B \\ 0 & Q \end{bmatrix}. \quad (11)$$

Again, it is clear that this is an equivalence relation. Note, however, that (11) cannot be applied to the differential–algebraic equation (1) because it would transform x and $\dot{x}$ independently. Therefore, we cannot expect existence and uniqueness results on the basis of (11). Nevertheless, it will be helpful for a better understanding of (10).

Since we get (4) back as special case for $B = 0$, we can expect a simpler set of characterizing quantities and a simpler canonical form compared with the Kronecker canonical form. With the notion that a matrix is basis of a vector space if this is valid for the set of its column vectors, we get the following result.

Theorem 7 *Let $E, A \in \mathbb{C}^{n,n}$ and*

$$\begin{array}{ll} (a) & T \text{ basis of kernel } E \\ (b) & Z \text{ basis of corange } E = \text{kernel } E^* \\ (c) & T' \text{ basis of cokernel } E = \text{range } E^* \\ (d) & V \text{ basis of corange}(Z^*AT). \end{array} \tag{12}$$

Then, the quantities (with the convention $\operatorname{rank} \emptyset = 0$*)*

$$\begin{array}{lll} (a) & r = \operatorname{rank} E & \text{(rank)} \\ (b) & a = \operatorname{rank}(Z^*AT) & \text{(algebraic part)} \\ (c) & s = \operatorname{rank}(V^*Z^*AT') & \text{(strangeness)} \\ (d) & d = r - s & \text{(differential part)} \\ (e) & u = n - r - a - s & \text{(undetermined part)} \end{array} \tag{13}$$

are invariant under (11) and (E, A) is equivalent to the canonical form

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{matrix} s \\ d \\ a \\ s \\ u \end{matrix} . \tag{14}$$

Proof. Let (E_i, A_i), $i = 1, 2$, be equivalent. Since

$$\operatorname{rank}(E_2) = \operatorname{rank}(PE_1Q) = \operatorname{rank}(E_1),$$

r is invariant. For a and s, we must first show that they are well–defined with respect to the choice of the bases. Each change of bases can be represented by

$$\tilde{T} = TM_T, \ \tilde{Z} = ZM_Z, \ \tilde{T}' = T'M_{T'}, \ \tilde{V} = M_Z^{-1}VM_V$$

with nonsingular matrices $M_T, M_Z, M_{T'}, M_V$ and the well–defineteness follows from

$$\operatorname{rank}(\tilde{Z}^*A\tilde{T}) = \operatorname{rank}(M_Z^*Z^*ATM_T) = \operatorname{rank}(Z^*AT)$$

and

$$\begin{aligned} \operatorname{rank}(\tilde{V}^*\tilde{Z}^*A\tilde{T}') &= \operatorname{rank}(M_V^*V^*M_Z^{-*}M_Z^*Z^*AT'M_{T'}) \\ &= \operatorname{rank}(V^*Z^*AT'). \end{aligned}$$

Let now bases T_2, Z_2, T_2', V_2 be given for (E_2, A_2), i.e.

$$\begin{array}{llll} \operatorname{rank}(E_2T_2) = 0, & T_2^*T_2 & \text{nonsingular}, & \operatorname{rank}(T_2^*T_2) = n - r \\ \operatorname{rank}(Z_2^*E_2) = 0, & Z_2^*Z_2 & \text{nonsingular}, & \operatorname{rank}(Z_2^*Z_2) = n - r \\ \operatorname{rank}(E_2T_2') = r, & T_2'^*T_2' & \text{nonsingular}, & \operatorname{rank}(T_2'^*T_2') = r \\ \operatorname{rank}(V_2^*Z_2^*A_2T_2) = 0, & V_2^*V_2 & \text{nonsingular}, & \operatorname{rank}(V_2^*V_2) = \hat{a}_2, \end{array}$$

with $\hat{a}_2 = \dim(\mathrm{corange}(Z_2^* A_2 T_2))$. Using (11) and setting

$$T_1 = QT_2,\ Z_1^* = Z_2^* P,\ T_1' = QT_2',\ V_1^* = V_2^*$$

we obtain the same relations for (E_1, A_1) and the above T_1, Z_1, T_1', V_1. Thus, T_1, Z_1, T_1' form bases according to (12). Since

$$\begin{aligned}\hat{a}_2 &= \dim(\mathrm{corange}(Z_2^* A_2 T_2)) \\ &= \dim(\mathrm{corange}(Z_2^* P A_1 Q T_2 - Z_2^* P E_1 B T_2)) \\ &= \dim(\mathrm{corange}(Z_1^* A_1 T_1)) = \hat{a}_1\end{aligned}$$

where we used $Z_1^* E_1 = 0$, this also holds for V_1. Moreover, with the same technique, the invariance of a and s and therefore also of d and u follows immediately.

For the derivation of the canonical form (14), we take a basis Z' of range E and a basis V' of range (Z^*AT). Note that the block matrices $(T', T), (Z', Z), (V', V)$ are then nonsingular so that we obtain the following sequence of equivalent $(\sim)$ matrix pairs :

$$\begin{aligned}(E, A) &\sim \left(\begin{bmatrix} Z'^* E T' & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} Z'^* A T' & Z'^* A T \\ Z^* A T' & Z^* A T \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} Z'^* E T' & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ Z^* A T' & Z^* A T \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ V'^* Z^* A T' & I_a & 0 \\ V^* Z^* A T' & 0 & 0 \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_a & 0 \\ V^* Z^* A T' & 0 & 0 \end{bmatrix} \right),\end{aligned}$$

which at last leads to (14) by a similar third transformation step. □

Remark 2 Since the equivalence relation (4) is included in (11), we can first transform to Kronecker canonical form and then treat the single blocks separately. Note that since (11) only applies for quadratic matrices, we must treat the bidiagonal blocks in pairs, which is possible, since in (5) for quadratic matrices we have $p = q$. In the following, we denote the i–th canonical basis vector of length n by $e_i^{(n)}$.

(a) Kronecker pair $\boldsymbol{L}_\epsilon \oplus \boldsymbol{M}_\eta$

$$(E, A) = \left(\begin{bmatrix} 0 & I_\epsilon & 0 \\ 0 & 0 & I_\eta \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} e_1^{(\epsilon)} & N_\epsilon & 0 \\ 0 & 0 & N_\eta \\ 0 & 0 & e_\eta^{(\eta)T} \end{bmatrix} \right)$$

$$T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ T' = \begin{bmatrix} 0 & 0 \\ I_\epsilon & 0 \\ 0 & I_\eta \end{bmatrix},\ V = \begin{bmatrix} 1 \end{bmatrix}$$

$$Z^*AT = \begin{bmatrix} 0 \end{bmatrix},\ V^*Z^*AT' = \begin{bmatrix} 0 & e_\eta^{(\eta)T} \end{bmatrix}$$

$$r = \epsilon + \eta,\ a = 0,\ s = \begin{cases} 0 & \text{for } \eta = 0 \\ 1 & \text{for } \eta \neq 0 \end{cases}$$

$$d = \begin{cases} \epsilon & \text{for } \eta = 0 \\ \epsilon + \eta - 1 & \text{for } \eta \neq 0 \end{cases},\quad u = \begin{cases} 1 & \text{for } \eta = 0 \\ 0 & \text{for } \eta \neq 0 \end{cases}$$

(b) Jordan block $\boldsymbol{J}_\rho$

$$(E, A) = (I_\rho, J_\rho)$$

$$T = \emptyset,\ Z = \emptyset,\ T' = I_\rho,\ V = \emptyset$$

$$r = \rho,\ a = 0,\ s = 0,\ d = \rho,\ u = 0$$

(c) Nilpotent block $\boldsymbol{N}_\sigma$

$$(E, A) = (N_\sigma, I_\sigma)$$

$$T = e_1^{(\sigma)},\ Z = e_\sigma^{(\sigma)},\ T' = (e_2^{(\sigma)}, \ldots, e_\sigma^{(\sigma)}),\ V = \begin{cases} \emptyset & \text{for } \sigma = 1 \\ \begin{bmatrix} 1 \end{bmatrix} & \text{for } \sigma \neq 1 \end{cases}$$

$$Z^*AT = \begin{cases} \begin{bmatrix} 1 \end{bmatrix} & \text{for } \sigma = 1 \\ \emptyset & \text{for } \sigma \neq 1 \end{cases}$$

$$V^*Z^*AT' = \begin{cases} \emptyset & \text{for } \sigma = 1 \\ \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} & \text{for } \sigma \neq 1 \end{cases}$$

$$r = \sigma - 1,\ a = \begin{cases} 1 & \text{for } \sigma = 1 \\ 0 & \text{for } \sigma \neq 1 \end{cases},\ s = \begin{cases} 0 & \text{for } \sigma = 1 \\ 1 & \text{for } \sigma \neq 1 \end{cases}$$

$$d = \begin{cases} \sigma - 1 & \text{for } \sigma = 1 \\ \sigma - 2 & \text{for } \sigma \neq 1 \end{cases},\ u = 0$$

Note that so–called higher index problems, i.e. problems with $k \geq 2$, are indicated by a non–vanishing strangeness s.

4 Global canonical form

Applying now the results for the local canonical form (14) to equation (1) yields functions $r, a, s : [\underline{t}, \bar{t}] \to \{0, \ldots, n\}$. Without any further restrictions, we are in particular faced with problems of the following kind.

Example 3 Let $n = 1$ and $t\dot{x}_1 = f_1(t)$. Then $a(t) \equiv 0$, $s(t) \equiv 0$ but $r(t)$ has a jump at the origin from 1 to 0. Necessary for solvability is $f_1(0) = 0$. Now let $n = 1$ and $0 = tx_1 + f_1(t)$. In this case $r(t) \equiv 0$, $s(t) \equiv 0$ but $a(t)$ has a jump at the origin from 1 to 0. Necessary for solvability is again $f_1(0) = 0$. Finally let $n = 2$ and $\dot{x}_1 = f_1(t)$, $0 = tx_1 + f_2(t)$. Here $r(t) \equiv 1$, $a(t) \equiv 0$

but $s(t)$ has a jump at the origin from 1 to 0. It follows that f_2 must satisfy $f_2(t) = -tx_1(t)$ where x_1 is continuously differentiable. So we have as necessary condition for solvability that f_2 is continuously differentiable and $f_2(0) = 0$ where the requirement for more smoothness of the inhomogeneity seems to be induced by the nonzero strangeness.

To exclude phenomena like the above interior point conditions, we assume

$$r(t) \equiv r,\ a(t) \equiv a,\ s(t) \equiv s \tag{15}$$

throughout the rest of this paper. Note that for analytical E, A this is violated only at a finite number of points (see e.g. [2] or [6]). Points violating these conditions must be treated separately whether in the context of the subsequent discussion or by different techniques. But this would be beyond the scope of this paper.

Because of (15), we make now use of the following property (see e.g. [16, 18]).

Lemma 8 *Let $E \in C^\ell(\underline{I}, \bar{I}],\ \mathbb{C}^{n,n})$, $\ell \in \mathbb{N}_0$ and $\operatorname{rank} E(t) = r$ for all $t \in \underline{I}, \bar{I}]$. Then there exist $U, V \in C^\ell(\underline{I}, \bar{I}], \mathbb{C}^{n,n})$ with $U(t), V(t)$ nonsingular (unitary) for every $t \in \underline{I}, \bar{I}]$ such that*

$$U(t)^* E(t) V(t) = \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix},\ t \in \underline{I}, \bar{I}], \tag{16}$$

where $\Sigma \in C^\ell(\underline{I}, \bar{I}], \mathbb{C}^{r,r})$.

In particular, this means that $\Sigma(t)$ is nonsingular for all $t \in \underline{I}, \bar{I}]$ and we have $\Sigma^{-1} \in C^\ell(\underline{I}, \bar{I}], \mathbb{C}^{r,r})$ for the function Σ^{-1} defined by $\Sigma^{-1}(t) = \Sigma(t)^{-1},\ t \in \underline{I}, \bar{I}]$.

Using now the equivalence relation (10) on the pair $(E(t), A(t))$, the first question is, whether under the assumption (15) there is an equivalent pair which according to (14) reflects this property. From the restriction of B to $\dot{Q}$ one expects a more complex answer to this question.

Theorem 9 *Let E, A in (1) be sufficiently smooth and let (15) hold. Then, $(E(t), A(t))$ is equivalent to a pair of matrix functions of the form*

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) & A_{15}(t) \\ 0 & 0 & 0 & A_{24}(t) & A_{25}(t) \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{matrix} s \\ d \\ a \\ s \\ u \end{matrix}. \tag{17}$$

Proof. Using Lemma 8, we have (omitting the argument t and using the word "new" on top of the equivalence operator to mark that we have changed the

notation according to the new block structure of the matrices)

$$
\begin{aligned}
(E,A) &\sim \left(\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \overset{\text{new}}{\sim} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \\
&\sim \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12}\,V_1 \\ U_1^* A_{21} & U_1^* A_{22}\,V_1 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \dot{V}_1 \end{bmatrix}\right) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & I & 0 \\ A_{31} & 0 & 0 \end{bmatrix}\right) \\
&\sim \left(\begin{bmatrix} V_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11}V_2 & A_{12} & A_{13} \\ A_{21}V_2 & I & 0 \\ U_2^* A_{31} V_2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{V}_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} V_{11} & V_{12} & 0 & 0 & 0 \\ V_{21} & V_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} \\ 0 & A_{32} & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\sim \left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} \\ 0 & A_{32} & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & -A_{32} & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}\right) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & A_{22} & A_{23} & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\sim \left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & A_{22} & 0 & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\overset{\text{new}}{\sim} \left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & A_{22}Q_2 - \dot{Q}_2 & 0 & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)
\end{aligned}
$$

$$\overset{\text{new}}{\sim}\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & 0 & 0 & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right).$$

In the last step, Q_2 was chosen to be the solution of the initial value problem

$$\dot{Q}_2 = A_{22}Q_2,\ Q(t_0) = I,$$

which is nonsingular at every point $t \in [\underline{t}, \bar{t}]$. □

Recalling Remarks 1 and 2, which say that for the Examples 1 and 2 (consisting of a Kronecker pair $\boldsymbol{L}_0 \oplus \boldsymbol{M}_1$ and of a nilpotent block $\boldsymbol{N}_2$ respectively), we have as triple $(r, a, s) = (1, 0, 1)$ in both cases, it follows that (17) is not sufficient for the discussion of (1) with respect to the questions posed in the beginning. In particular, we must expect that the matrix functions $A_{ij}(t)$ in (17) contain important information concerning these questions.

Writing down the system of differential–algebraic equations that belongs to (17), we get

$$\begin{array}{ll} (a) & \dot{x}_1(t) = A_{12}(t)x_2(t) + A_{14}(t)x_4(t) + A_{15}(t)x_5(t) + f_1(t) \\ (b) & \dot{x}_2(t) = A_{24}(t)x_4(t) + A_{25}(t)x_5(t) + f_2(t) \\ (c) & 0 = x_3(t) + f_3(t) \\ (d) & 0 = x_1(t) + f_4(t) \\ (e) & 0 = f_5(t). \end{array} \tag{18}$$

Here, we can in principle differentiate equation (18d) and insert it in (18a), which then becomes an algebraic equation. This corresponds to passing from (17) to

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) & A_{15}(t) \\ 0 & 0 & 0 & A_{24}(t) & A_{25}(t) \\ 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) \begin{matrix} s \\ d \\ a \\ s \\ u \end{matrix}, \tag{19}$$

for which we again compute characteristic values r, a, s, d, u.

The above procedure therefore leads to an inductive definition of a sequence of pairs of matrix functions $(E_i(t), A_i(t))$, $i \in \mathbb{N}_0$, where $(E_0(t), A_0(t)) = (E(t), A(t))$ and $(E_{i+1}(t), A_{i+1}(t))$ is derived from $(E_i(t), A_i(t))$ by bringing it into the form (17) and passing then to (19). Here we must assume (15) for every occuring pair of matrices. Connected with this sequence, we then have sequences r_i, a_i, s_i, d_i, u_i, $i \in \mathbb{N}_0$ of nonnegative integers.

The next theorem shows that these sequences are characteristic for the given pair $(E(t), A(t))$, that is, that they do not depend on the specific way they are obtained.

Theorem 10 *Let $(E(t), A(t)), (\tilde{E}(t), \tilde{A}(t))$ be equivalent and of form (17). Then the modified pairs $(E_{\rm mod}(t), A_{\rm mod}(t)), (\tilde{E}_{\rm mod}(t), \tilde{A}_{\rm mod}(t))$ obtained by passing to (19) are also equivalent.*

Proof. By assumption, there are smooth, pointwise nonsingular matrix functions P, Q such that (omitting arguments)

$$P\tilde{E} = EQ, \ P\tilde{A} = AQ - E\dot{Q}.$$

From the first relation, we deduce

$$\begin{bmatrix} P_{11} & P_{12} & 0 & 0 & 0 \\ P_{21} & P_{22} & 0 & 0 & 0 \\ P_{31} & P_{32} & 0 & 0 & 0 \\ P_{41} & P_{42} & 0 & 0 & 0 \\ P_{51} & P_{52} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

if we partition P, Q according to (17).

With this, we obtain for the last three block rows of the second relation

$$\begin{bmatrix} P_{34} & 0 & P_{33} & 0 & 0 \\ P_{44} & 0 & P_{43} & 0 & 0 \\ P_{54} & 0 & P_{53} & 0 & 0 \end{bmatrix} = \begin{bmatrix} Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} \\ Q_{11} & Q_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In terms of the matrix Q, we therefore have

$$P = \begin{bmatrix} Q_{11} & 0 & P_{13} & P_{14} & P_{15} \\ Q_{21} & Q_{22} & P_{23} & P_{24} & P_{25} \\ 0 & 0 & Q_{33} & Q_{31} & Q_{35} \\ 0 & 0 & 0 & Q_{11} & P_{45} \\ 0 & 0 & 0 & 0 & P_{55} \end{bmatrix}, \ Q = \begin{bmatrix} Q_{11} & 0 & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ Q_{31} & 0 & Q_{33} & 0 & 0 \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} \end{bmatrix}$$

and $Q_{11}, Q_{22}, Q_{33}, P_{55}, \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{54} & Q_{55} \end{bmatrix}$ must be nonsingular. From the first two block rows of the second relation, we now get

$$\begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \tilde{A}_{12} & \tilde{A}_{14} & \tilde{A}_{15} \\ 0 & \tilde{A}_{24} & \tilde{A}_{25} \end{bmatrix} = \begin{bmatrix} A_{12} & A_{14} & A_{15} \\ 0 & A_{24} & A_{25} \end{bmatrix} \begin{bmatrix} Q_{22} & 0 & 0 \\ Q_{42} & Q_{44} & Q_{45} \\ Q_{52} & Q_{54} & Q_{55} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ \dot{Q}_{22} & 0 & 0 \end{bmatrix}.$$

Thus it follows that $(\tilde{E}_{\rm mod}, \tilde{A}_{\rm mod})$ is equivalent to

$$\left(\begin{bmatrix} Q_{11} & 0 & & & \\ Q_{21} & Q_{22} & & & \\ & & I & & \\ & & & I & \\ & & & & I \end{bmatrix}\begin{bmatrix} 0 & & & & \\ & I & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix},\right.$$
$$\left.\begin{bmatrix} Q_{11} & 0 & & & & \\ Q_{21} & Q_{22} & & & & \\ & & I & & & \\ & & & I & & \\ & & & & I & \end{bmatrix}\begin{bmatrix} 0 & \tilde{A}_{12} & 0 & \tilde{A}_{14} & \tilde{A}_{15} \\ 0 & 0 & 0 & \tilde{A}_{24} & \tilde{A}_{25} \\ 0 & 0 & 0 & I & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$
$$\sim \left(\begin{bmatrix} 0 & & & & \\ & Q_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & 0 & 0 & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & Q_{42} & 0 & Q_{44} & Q_{45} \\ 0 & Q_{52} & 0 & Q_{54} & Q_{55} \end{bmatrix}\right.$$
$$\left. - \begin{bmatrix} 0 & & & & \\ & \dot{Q}_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}\right)$$
$$\sim \left(\begin{bmatrix} 0 & & & & \\ & Q_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & Q_{22}^{-1} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & * & 0 & * & * \\ 0 & * & 0 & * & * \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & 0 & 0 & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right. ,$$
$$- \begin{bmatrix} 0 & & & & \\ & \dot{Q}_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & Q_{22}^{-1} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & * & 0 & * & * \\ 0 & * & 0 & * & * \end{bmatrix}$$
$$\left. - \begin{bmatrix} 0 & & & & \\ & Q_{22} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \dot{Q}_{22}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & * \\ 0 & * & 0 & * & * \end{bmatrix}\right)$$
$$\sim \left(\begin{bmatrix} 0 & & & & \\ & I & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} \\ 0 & X & 0 & A_{24} & A_{25} \\ 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

with

$$X = -(\dot{Q}_{22}Q_{22}^{-1} + Q_{22}\overbrace{Q_{22}^{-1}}^{\cdot}) = -(\overbrace{Q_{22}Q_{22}^{-1}}^{\cdot}) = -\dot{I} = 0.$$

□

Having now shown that under the above assumptions the sequences $r_i, a_i, s_i, d_i, u_i,\ i \in \mathbb{N}_0$, are well-defined, we can state some basic properties of these quantities:

Lemma 11 *Let E, A in (1) be sufficiently smooth and such that the sequences $(E_i(t), A_i(t)),\ i \in \mathbb{N}_0$, and $r_i, a_i, s_i, d_i, u_i,\ i \in \mathbb{N}_0$, are well–defined by the above process. Let furthermore*

$$\begin{array}{l}(E_i(t), A_i(t)) \\ \sim \left(\begin{bmatrix} I_{s_i} & 0 & 0 & 0 & 0 \\ 0 & I_{d_i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}^{(i)}(t) & 0 & A_{14}^{(i)}(t) & A_{15}^{(i)}(t) \\ 0 & 0 & 0 & A_{24}^{(i)}(t) & A_{25}^{(i)}(t) \\ 0 & 0 & I_{a_i} & 0 & 0 \\ I_{s_i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{array}{l} s_i \\ d_i \\ a_i \\ s_i \\ u_i \end{array}.\end{array} \tag{20}$$

Then, we have (for all $t \in [\underline{t}, \bar{t}]$)

$$\begin{array}{ll} (a) & r_{i+1} = r_i - s_i \\ (b) & a_{i+1} = a_i + s_i + \operatorname{rank}\left(\begin{bmatrix} A_{14}^{(i)}(t) & A_{15}^{(i)}(t) \end{bmatrix} \right) \\ (c) & s_{i+1} = \operatorname{rank}(V_i(t)^* A_{12}^{(i)}(t)) \\ (d) & d_{i+1} = r_{i+1} - s_{i+1} = d_i - \operatorname{rank}(V_i(t)^* A_{12}^{(i)}(t)) \\ (e) & u_{i+1} = n - r_{i+1} - a_{i+1} - s_{i+1} \\ & \quad = u_i + (s_i - \operatorname{rank}\left(\begin{bmatrix} A_{12}^{(i)}(t) & A_{14}^{(i)}(t) & A_{15}^{(i)}(t) \end{bmatrix} \right), \end{array} \tag{21}$$

with $V_i(t) = \operatorname{corange}\left(\begin{bmatrix} A_{14}^{(i)}(t) & A_{15}^{(i)}(t) \end{bmatrix} \right)$.

There exists a number $m \in \mathbb{N}_0$ (strangeness index) defined by

$$m = \min\{i \in \mathbb{N}_0 \mid s_i = 0\} \tag{22}$$

and the above sequences have the properties

$$\begin{array}{llllllll} (a) & r_i > r_{i+1} & \text{for} & i < m, & r_i = r_m & \text{for} & i \geq m \\ (b) & a_i < a_{i+1} & \text{for} & i < m, & a_i = a_m & \text{for} & i \geq m \\ (c) & s_i \geq s_{i+1} & \text{for} & i < m, & s_i = 0 & \text{for} & i \geq m \\ (d) & d_i \geq d_{i+1} & \text{for} & i < m, & d_i = d_m & \text{for} & i \geq m \\ (e) & u_i \leq u_{i+1} & \text{for} & i < m, & u_i = u_m & \text{for} & i \geq m. \end{array} \tag{23}$$

Proof. Replacing I_{s_i} by 0 in the first block matrix of (20) yields (21) by direct application of (13). Since $A_{12}^{(i)}$ is a (s_i, d_i)–matrix, we have $s_i \geq s_{i+1}$ and s_i is

a monotonically decreasing sequence. Since r_i is nonnegative, s_i must become zero after a finite number of steps. Thus, m is well–defined by (22). Finally, (23) is a direct consequence of (21) and (22). □

The hope is now that the finite sequences $r_i, a_i, s_i,\ i \in \{0, \ldots, m\}$, (recall that d_i, u_i are not independent of these) are sufficient to describe the possible phenomena for (1) and to yield an appropriate generalization of the Kronecker canonical form in the case of variable coefficients. One can expect that such a canonical form, if it reflects all the above quantities, plays an important role in analyzing linear differential–algebraic equations with variable coefficients.

Theorem 12 *Let m from (22) be well–defined for the pair $(E(t), A(t))$ of smooth matrix functions. Let $r_i, a_i, s_i, d_i, u_i,\ i \in \{0, \ldots, m\}$ be the related characteristic values as above. Furthermore define (in the notation of Lemma 11)*

$$\begin{aligned}
&(a) \quad b_0 = a_0, && b_i = \operatorname{rank}\left(\begin{bmatrix} A_{14}^{(i-1)}(t) & A_{15}^{(i-1)}(t) \end{bmatrix}\right),\\
&(b) \quad c_0 = a_0 + s_0, && c_i = \operatorname{rank}\left(\begin{bmatrix} A_{12}^{(i-1)}(t) & A_{14}^{(i-1)}(t) & A_{15}^{(i-1)}(t) \end{bmatrix}\right),\\
&(c) \quad w_0 = u_0, && w_i = u_i - u_{i-1}, \qquad i = 1, \ldots, m.
\end{aligned} \tag{24}$$

We then have

$$\begin{aligned}
&(a) \quad c_i = b_i + s_i, && i = 0, \ldots, m\\
&(b) \quad w_i = s_{i-1} - c_i, && i = 1, \ldots, m
\end{aligned} \tag{25}$$

and the pair $(E(t), A(t))$ is equivalent to a pair of matrix functions of the form (without arguments)

$$\left(\begin{bmatrix}
I & 0 & \ldots & 0 & 0 & * & \ldots & * \\
0 & 0 & \ldots & 0 & 0 & F_m & & * \\
\vdots & \vdots & & \vdots & & \ddots & \ddots & \\
\vdots & \vdots & & \vdots & & & \ddots & F_1 \\
0 & 0 & \ldots & 0 & & & & 0 \\
0 & 0 & \ldots & 0 & 0 & G_m & & * \\
\vdots & \vdots & & \vdots & & \ddots & \ddots & \\
\vdots & \vdots & & \vdots & & & \ddots & G_1 \\
0 & 0 & \ldots & 0 & & & & 0
\end{bmatrix},
\begin{bmatrix}
* & * & \ldots & * & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & & & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 0 & I & & & \\
\vdots & \vdots & & \vdots & & \ddots & & \\
\vdots & \vdots & & \vdots & & & \ddots & \\
0 & 0 & \ldots & 0 & & & & I
\end{bmatrix}\right)
\begin{matrix}
d_m \\ w_m \\ \vdots \\ \vdots \\ w_0 \\ c_m \\ \vdots \\ \vdots \\ c_0
\end{matrix}, \tag{26}$$

where

$$\operatorname{rank}\left(\begin{bmatrix} F_i \\ G_i \end{bmatrix}\right) = c_i + w_i = s_{i-1} \le c_{i-1}. \tag{27}$$

Proof. From

$$\begin{aligned}
&\operatorname{rank}\left(\begin{bmatrix} A_{12}^{(i-1)} & A_{14}^{(i-1)} & A_{15}^{(i-1)} \end{bmatrix}\right)\\
&\quad = \operatorname{rank}\left(\begin{bmatrix} A_{14}^{(i-1)} & A_{15}^{(i-1)} \end{bmatrix}\right) + \operatorname{rank}\left(\begin{bmatrix} V_{i-1}^* A_{12}^{(i-1)} \end{bmatrix}\right)
\end{aligned}$$

we at once obtain (25a), while (25b) follows from (21e).

Starting from (17) in the permutated form (if we do not perform the last transformation in the proof of Theorem 9)

$$\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}\right) \begin{matrix} d_0 \\ s_0 \\ w_0 \\ s_0 \\ b_0 \end{matrix}$$

we obtain the following equivalent pairs of matrix functions in the i–th step (omitting superscripts $^{(i)}$ and denoting by $\begin{bmatrix} A_{ij} & \dots & A_{ij} \end{bmatrix}$ a block entry A_{ij} which extends over several block columns)

$$\left(\begin{bmatrix}
I & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & \dots & \dots & \dots & 0 & I & 0 & * & \dots & \dots & * \\
 & & 0 & & & & & & & F_i & & * & \\
 & & & \ddots & & & & & & 0 & \ddots & & \\
 & & & & \ddots & & & & & & \ddots & \ddots & \\
 & & & & & \ddots & & & & & & \ddots & F_1 \\
 & & & & & & 0 & & & & & & 0 \\
 & & & & & & & 0 & 0 & W_{11} & * & \dots & * \\
 & & & & & & & 0 & 0 & W_{21} & * & \dots & * \\
 & & & & & & & & & 0 & G_{i-1} & & * \\
 & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & \ddots & G_1 \\
 & & & & & & & & & & & & 0
\end{bmatrix}, \right.$$

$$\left.\begin{bmatrix}
A_{11} & A_{12} & A_{13} & \dots & \dots & \dots & A_{13} & & & & & & \\
A_{21} & A_{22} & A_{23} & \dots & \dots & \dots & A_{23} & & & & & & \\
 & & 0 & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & \\
 & & & & \ddots & & & & & & & & \\
 & & & & & \ddots & & & & & & & \\
 & & & & & & 0 & & & & & & \\
 & & & & & & & I & 0 & & & & \\
 & & & & & & & 0 & I & & & & \\
 & & & & & & & & & I & & & \\
 & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & I
\end{bmatrix}\right) \begin{matrix} d_i \\ s_i \\ w_i \\ \vdots \\ \vdots \\ \vdots \\ w_0 \\ s_i \\ b_i \\ c_{i-1} \\ \vdots \\ \vdots \\ c_0 \end{matrix}$$

$$\overset{\text{new}}{\sim}\left(\begin{bmatrix}
I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & U_{11} & U_{12} & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & U_{21} & U_{22} & 0 & * & \dots & \dots & * \\
 & & & 0 & & & & & & & & F_i & & * & \\
 & & & & \ddots & & & & & & & 0 & \ddots & & \\
 & & & & & \ddots & & & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & & & & & & & \ddots & F_1 \\
 & & & & & & & 0 & & & & & & & 0 \\
 & & & & & & & & 0 & & & W_{11} & * & \dots & * \\
 & & & & & & & & & \ddots & & W_{21} & * & \dots & * \\
 & & & & & & & & & & 0 & W_{31} & * & \dots & * \\
 & & & & & & & & & & & 0 & G_{i-1} & & * \\
 & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & \ddots & G_1 \\
 & & & & & & & & & & & & & & 0
\end{bmatrix},$$

$$\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \dots & \dots & \dots & A_{14} & & & & & & & \\
A_{21} & I & 0 & 0 & \dots & \dots & \dots & 0 & & & & & & & \\
A_{31} & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & & & \\
 & & & 0 & & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & & \\
 & & & & & & \ddots & & & & & & & & \\
 & & & & & & & 0 & & & & & & & \\
 & & & & & & & & I & 0 & 0 & & & & \\
 & & & & & & & & 0 & I & 0 & & & & \\
 & & & & & & & & 0 & 0 & I & & & & \\
 & & & & & & & & & & & I & & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & & I
\end{bmatrix}\begin{matrix}
d_i \\ b_{i+1} \\ s_i - b_{i+1} \\ w_i \\ \vdots \\ \vdots \\ \vdots \\ w_0 \\ b_{i+1} \\ s_i - b_{i+1} \\ b_i \\ c_{i-1} \\ \vdots \\ \vdots \\ c_0
\end{matrix}$$

$$\overset{\text{new}}{\sim}\left(\begin{bmatrix}
I & 0 & 0 & 0 & 0 & & & & & 0 & * & \dots & \dots & * \\
0 & I & 0 & 0 & 0 & & & & & 0 & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & & & & & U_{11} & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & & & & & U_{21} & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & & & & & U_{31} & * & \dots & \dots & * \\
 & & & & & 0 & & & & & F_i & & * & \\
 & & & & & & \ddots & & & & 0 & \ddots & & \\
 & & & & & & & \ddots & & & & \ddots & \ddots & \\
 & & & & & & & & \ddots & & & & \ddots & F_1 \\
 & & & & & & & & & 0 & & & & 0 \\
 & & & & & & & & & & 0 & G_i & * & \dots & * \\
 & & & & & & & & & & & 0 & G_{i-1} & * & \\
 & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & \ddots & G_1 \\
 & & & & & & & & & & & & & & 0
\end{bmatrix},\right.$$

$$\left.\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & \dots & \dots & \dots & A_{16} & & & & & \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & \dots & \dots & \dots & A_{26} & & & & & \\
A_{31} & A_{32} & I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
0 & I & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
 & & & & & 0 & & & & & & & & & \\
 & & & & & & \ddots & & & & & & & & \\
 & & & & & & & \ddots & & & & & & & \\
 & & & & & & & & \ddots & & & & & & \\
 & & & & & & & & & 0 & & & & & \\
 & & & & & & & & & & I & & & & \\
 & & & & & & & & & & & I & & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & & I
\end{bmatrix}\right)\begin{matrix}
d_{i+1} \\ s_{i+1} \\ b_{i+1} \\ s_{i+1} \\ w_{i+1} \\ w_i \\ \vdots \\ \vdots \\ \vdots \\ w_0 \\ c_i \\ c_{i-1} \\ \vdots \\ \vdots \\ c_0
\end{matrix}$$

$$\overset{\text{new}}{\sim}\left(\begin{bmatrix}
I & 0 & 0 & 0 & 0 & & & & & * & * & \dots & \dots & * \\
0 & I & 0 & 0 & 0 & & & & & * & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & & & & & W_{21} & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & & & & & W_{11} & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & 0 & & & & & F_{i+1} & * & \dots & \dots & * \\
 & & & & & 0 & & & & & F_i & & * & \\
 & & & & & & \ddots & & & & 0 & \ddots & & \\
 & & & & & & & \ddots & & & & \ddots & \ddots & \\
 & & & & & & & & \ddots & & & & \ddots & F_1 \\
 & & & & & & & & & 0 & & & & 0 \\
 & & & & & & & & & & 0 & G_i & * & \dots & * \\
 & & & & & & & & & & & 0 & G_{i-1} & * & \\
 & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & \ddots & G_1 \\
 & & & & & & & & & & & & & & 0
\end{bmatrix}\right.$$

$$\left.\begin{bmatrix}
A_{11} & 0 & 0 & A_{12} & A_{13} & A_{14} & \dots & \dots & \dots & A_{14} & & & & & \\
A_{21} & 0 & 0 & A_{22} & A_{23} & A_{24} & \dots & \dots & \dots & A_{24} & & & & & \\
0 & 0 & I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
0 & I & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & \\
 & & & & & 0 & & & & & & & & & \\
 & & & & & & \ddots & & & & & & & & \\
 & & & & & & & \ddots & & & & & & & \\
 & & & & & & & & \ddots & & & & & & \\
 & & & & & & & & & 0 & & & & & \\
 & & & & & & & & & & I & & & & \\
 & & & & & & & & & & & I & & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & & I
\end{bmatrix}\right)\begin{matrix}
d_{i+1} \\ s_{i+1} \\ b_{i+1} \\ s_{i+1} \\ w_{i+1} \\ w_i \\ \vdots \\ \vdots \\ \vdots \\ w_0 \\ c_i \\ c_{i-1} \\ \vdots \\ \vdots \\ c_0
\end{matrix}$$

$$\sim \left(\begin{bmatrix}
I & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & * & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & I & 0 & * & * & \dots & \dots & * \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & F_{i+1} & * & \dots & \dots & * \\
 & & & 0 & & & & & & & 0 & F_i & & * & \\
 & & & & \ddots & & & & & & & 0 & \ddots & & \\
 & & & & & \ddots & & & & & & & \ddots & \ddots & \\
 & & & & & & \ddots & & & & & & & \ddots & F_1 \\
 & & & & & & & 0 & & & & & & & 0 \\
 & & & & & & & & 0 & & W_{11} & * & \dots & \dots & * \\
 & & & & & & & & & 0 & W_{21} & * & \dots & \dots & * \\
 & & & & & & & & & & 0 & G_i & * & \dots & * \\
 & & & & & & & & & & & 0 & G_{i-1} & * & \\
 & & & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & & & \ddots & G_1 \\
 & & & & & & & & & & & & & & 0
\end{bmatrix}, \right.$$

$$\left. \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \dots & \dots & \dots & A_{14} & & & & & & & \\
A_{21} & A_{22} & A_{23} & A_{24} & \dots & \dots & \dots & A_{24} & & & & & & & \\
0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & & & & & & & \\
 & & & 0 & & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & & \\
 & & & & & \ddots & & & & & & & & & \\
 & & & & & & \ddots & & & & & & & & \\
 & & & & & & & 0 & & & & & & & \\
 & & & & & & & & I & & & & & & \\
 & & & & & & & & & I & & & & & \\
 & & & & & & & & & & I & & & & \\
 & & & & & & & & & & & I & & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & & I
\end{bmatrix} \right) \begin{matrix}
d_{i+1} \\ s_{i+1} \\ w_{i+1} \\ w_i \\ \vdots \\ \vdots \\ \vdots \\ w_0 \\ s_{i+1} \\ b_{i+1} \\ c_i \\ c_{i-1} \\ \vdots \\ \vdots \\ c_0
\end{matrix}$$

Thus, (26) follows by induction and (27) holds, since $\begin{bmatrix} F_{i+1} \\ G_{i+1} \end{bmatrix}$ is obtained by nonsingular transformations applied to $\begin{bmatrix} U & 0 \end{bmatrix}$, with nonsingular U, where U is the transformation used above in the first step. □

Example 4 Concluding this section, we determine the canonical form (26) of the various blocks of the Kronecker canonical form (see Remark 2).

(a) Kronecker pair $\boldsymbol{L}_\epsilon \oplus \boldsymbol{M}_\eta$

$$(E,A) = \left(\begin{bmatrix} 0 & I_\epsilon & 0 \\ 0 & 0 & I_\eta \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} e_1^{(\epsilon)} & N_\epsilon & 0 \\ 0 & 0 & N_\eta \\ 0 & 0 & e_\eta^{(\eta)^T} \end{bmatrix} \right)$$
$$\sim \left(\begin{bmatrix} I_\epsilon & 0 & 0 \\ 0 & 0 & e_\eta^{(\eta)^T} \\ 0 & 0 & N_\eta^T \end{bmatrix}, \begin{bmatrix} N_\epsilon & e_1^{(\epsilon)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_\eta \end{bmatrix} \right)$$

$$\begin{aligned} m = \eta, \quad & c_0 = \ldots = c_{m-1} = 1,\ c_m = 0 \\ & w_0 = \ldots = w_{m-1} = 0,\ w_m = 1 \\ & s_0 = \ldots = s_{m-1} = 1,\ s_m = 0 \\ & b_0 = \ldots = b_{m-1} = 0,\ b_m = 0 \\ & d_m = \epsilon,\ a_m = \textstyle\sum_{i=0}^m c_i = \eta,\ u_m = \sum_{i=0}^m w_i = 1 \end{aligned}$$

(b) Jordan block $\boldsymbol{J}_\rho$

$$(E,A) = (I_\rho, J_\rho)$$

$$m = 0,\ d_m = \rho,\ a_m = 0,\ u_m = 0$$

(c) Nilpotent block $\boldsymbol{N}_\sigma$

$$(E,A) = (N_\sigma, I_\sigma)$$

$$\begin{aligned} m = \sigma - 1, \quad & c_0 = \ldots = c_{m-1} = 1,\ c_m = 1 \\ & w_0 = \ldots = w_{m-1} = 0,\ w_m = 0 \\ & s_0 = \ldots = s_{m-1} = 1,\ s_m = 0 \\ & b_0 = \ldots = b_{m-1} = 0,\ b_m = 1 \\ & d_m = 0,\ a_m = \textstyle\sum_{i=0}^m c_i = \sigma,\ u_m = \sum_{i=0}^m w_i = 0 \end{aligned}$$

5 Existence and uniqueness of solutions

On the background of the suite of results obtained in the previous sections, we now turn to the questions posed in the beginning of this paper. Note that especially the notion of solvability may differ from those sometimes used in the literature (see, e.g. [1, 4, 5, 6] or [14] for an overview). The notion used here is a straightforward generalization of the one known from ordinary differential equations. Using the results of Section 4, we can transform (1) to an equivalent differential–algebraic equation of a very special structure. Equivalence here means that there is a one–to–one correspondence of their solutions.

Theorem 13 *Let m from (22) be well–defined for the pair $(E(t), A(t))$ in (1) and $f \in C^m([\underline{t}, \bar{t}], \mathbb{C}^n)$. Then, (1) is equivalent to a differential–algebraic equation of the form*

$$\begin{array}{ll} (a) & \dot{x}_1(t) = A_{13}(t)x_3(t) + f_1(t) \\ (b) & 0 = x_2(t) + f_2(t) \\ (c) & 0 = f_3(t), \end{array} \tag{28}$$

where the inhomogeneity is determined by $f^{(0)}, \ldots, f^{(m)}$. In particular, d_m, a_m, u_m are the number of differential, algebraic and undetermined components of the unknown x in (a), (b), (c) respectively.

Proof. Inductively transforming $(E(t), A(t))$ to the form (17) and then passing to (19) until $s_i = 0$ yields a pair of matrix functions of the form

$$\left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{array}{l} d_m \\ a_m \\ u_m \end{array},$$

where all steps are reversible, and in each step the inhomogeneity is differentiated once. □

Now, we can state the answers to the raised questions.

Corollary 14 *Under the assumptions of Theorem 13 the following statements hold if in addition $f \in C^{m+1}([\underline{t}, \bar{t}], \mathbb{C}^n)$.*

The equation (1) is solvable if and only if the u_m functional consistency conditions

$$f_3(t) \equiv 0 \tag{29}$$

are satisfied.

An initial condition (2) is consistent if and only if in addition the a_m conditions

$$x_2(t_0) = -f_2(t_0) \tag{30}$$

hold.

The initial value problem (1), (2) is uniquely solvable if again in addition we have

$$u_m = 0. \tag{31}$$

Otherwise, we can choose $x_3 \in C^1([\underline{t}, \bar{t}], \mathbb{C}^{u_m})$ arbitrarily.

Proof. Observing that we need the higher differentiability of f to guarantee that x_2 is differentiable, the results are direct conclusions from Theorem 13. □

We remark here that the special form (28) of a differential–algebraic equation may suggest a weaker form of solvability because there seems to be no need in requiring x_2 to be differentiable. To circumvent differentiation in the general form (1) when x_2 is only continuous, we find in [17] the definition of a so–called modified matrix pencil. Realizing that under the assumption of constant rank

the Moore–Penrose pseudo–inverse $E(t)^+$ of $E(t)$ is as smooth as $E(t)$ and as the projector $\Pi(t) = E(t)^+E(t)$ (see e.g. [16]), we can replace (1) by

$$\begin{aligned} E(t)\overbrace{(\Pi(t)x(t))}^{\cdot} &= E(t)\dot{\Pi}(t)x(t) + E(t)\Pi(t)\dot{x}(t) \\ &= E(t)\dot{\Pi}(t)x(t) + E(t)\dot{x}(t) \\ &= (A(t) + E(t)\dot{\Pi}(t))x(t) + f(t). \end{aligned} \tag{32}$$

For this equation, it is now sufficient to require $x \in C([\underline{t}, \bar{t}], \mathbb{C}^n)$ with $\Pi x \in C^1([\underline{t}, \bar{t}], \mathbb{C}^n)$, and Corollary 14 becomes valid for $f \in C^m([\underline{t}, \bar{t}], \mathbb{C}^n)$ with arbitrary $x_3 \in C([\underline{t}, \bar{t}], \mathbb{C}^n)$.

It is now time to illustrate the obtained results by applying them to the motivating Examples 1 and 2 of Section 3. In the following we use the equivalence operator with the abbreviation "dif" to indicate a step from (17) to (19).

Example 5 Treating the problem of Example 1, we get

$$\begin{aligned} &\left(\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) \\ &\sim \left(\begin{bmatrix} 0 & 1 \\ 1 & -t \end{bmatrix}\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -t \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) \\ &\sim \left(\begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & t \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ &\sim \left(\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\right) \overset{\text{dif}}{\sim} \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \end{aligned}$$

and therefore $m = 1$ with

$$\begin{aligned} &r_0 = 1,\ a_0 = 0,\ s_0 = 1,\ d_0 = 0,\ u_0 = 0, \\ &r_1 = 0,\ a_1 = 1,\ s_1 = 0,\ d_1 = 0,\ u_1 = 1. \end{aligned}$$

Thus, the given problem consists of one algebraic equation with one undetermined component. Following Corollary 14, the problem is solvable, since $f(t) = 0$, but not uniquely solvable, since $u_m \neq 0$. The general solution $y(t)$ of the transformed equation is given by

$$y(t) = \begin{bmatrix} 0 \\ y_2(t) \end{bmatrix}.$$

Transforming back then yields

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y_2(t) \end{bmatrix} = y_2(t)\begin{bmatrix} t \\ 1 \end{bmatrix}.$$

Example 6 Treating the problem of Example 2, we get (including the inhomogeneity)

$$\begin{aligned}
&\left(\begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix}, \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}\right) \\
&\sim \left(\begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right. \\
&\qquad\qquad \left. - \begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}\right) \\
&\sim \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}\right) \\
&\overset{\text{dif}}{\sim} \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} f_1(t) \\ f_2(t) - \dot{f}_1(t) \end{bmatrix}\right)
\end{aligned}$$

and therefore $m = 1$ with

$$\begin{aligned}
&r_0 = 1,\ a_0 = 0,\ s_0 = 1,\ d_0 = 0,\ u_0 = 0, \\
&r_1 = 0,\ a_1 = 2,\ s_1 = 0,\ d_1 = 0,\ u_1 = 0.
\end{aligned}$$

Thus, the given problem consists of two algebraic equations. Following Corollary 14, the problem is uniquely solvable for each consistent initial condition. Transforming back the solution

$$y(t) = \begin{bmatrix} f_1(t) \\ f_2(t) - \dot{f}_1(t) \end{bmatrix}$$

of the transformed equation, we obtain

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} f_1(t) \\ f_2(t) - \dot{f}_1(t) \end{bmatrix} = \begin{bmatrix} f_1(t) + tf_2(t) - t\dot{f}_1(t) \\ f_2(t) - \dot{f}_1(t) \end{bmatrix}.$$

Summarizing the results of the previous sections, we have shown that three quantities are sufficient to discuss the solution behaviour of a differential–algebraic equation whose coefficients satisfy some indispensable rank and smoothness assumptions. These are the strangeness index m and the final numbers d_m and a_m of differential and algebraic components.

6 Relation to other global invariances

In trying to generalize the concept of (local) index (8) for linear differential–algebraic equation with variable coefficients or even for general nonlinear differential–algebraic equations, one can find definitions for a whole suite of (global) indices in the literature as for example the global index ([10]), the differentiation index ([4, 5, 8]), the tractability index ([12]), the geometric index ([11]) and last but not least the perturbation index ([13]). In fact, with

the exception of the last one, they are all equal for linear differential–algebraic equations modulo some differences in the necessary technical assumptions. For more details, see [3, 9, 11, 15]. In the following, we show that the strangeness index m from (22) directly leads to a generalization of the differentiation index k which we define according to [4, 5] as follows:

Definition 15 Let E, A in (1) be sufficiently smooth and let

$$E_\ell = (\frac{d}{dt})^\ell E,\ A_\ell = (\frac{d}{dt})^\ell A,\ x_\ell = (\frac{d}{dt})^\ell x,\ f_\ell = (\frac{d}{dt})^\ell f \tag{33}$$

for $\ell = 0, 1, \ldots$. Formal differentiation of (1) then leads to linear equations of the form

$$M_\ell(t) z_\ell = N_\ell(t) x_0 + g_\ell(t) \tag{34}$$

for any ℓ, where

$$\begin{array}{lll} (a) & (M_\ell)_{ij} = \left(\begin{array}{c} i \\ j \end{array} \right) E_{i-j} - \left(\begin{array}{c} i \\ j+1 \end{array} \right) A_{i-j-1}, & \\ (b) & (N_\ell)_i = A_i, & i, j = 0, \ldots, \ell \\ (c) & (z_\ell)_i = x_{i+1}, & \\ (d) & (g_\ell)_i = f_i. & \end{array} \tag{35}$$

The *differentiation index* k of (1) is now defined to be the smallest value $\ell \in \mathbb{N}_0$ for which $M_\ell(t)$ is *smoothly 1-full*, i.e. for which there is a continuous matrix function R with $R(t)$ nonsingular and

$$R(t) M_\ell(t) = \left[\begin{array}{cc} I_n & 0 \\ 0 & H(t) \end{array} \right] \tag{36}$$

for all $t \in [\underline{t}, \bar{t}]$.

Note that by definition k gives the number of differentiations we must apply to (1) to obtain an ordinary differential equation (the so–called underlying ordinary differential equation), which is equivalent to (1). We can now prove the following relation between k and m.

Theorem 16 *Let k and m be well–defined. Then, $u_m = 0$ and*

$$k = \left\{ \begin{array}{ll} 0 & \text{for } m = 0,\ a_0 = 0 \\ m+1 & \text{otherwise} \end{array} \right. . \tag{37}$$

Proof. If $u_m \neq 0$, i.e. if there is an undetermined part, there is no unique solution even for consistent initial conditions and (1) cannot be equivalent by differentiation to an ordinary differential equation by Corollary 14. Thus, for $u_m \neq 0$ the differentiation index k is not well–defined. Let now $u_m = 0$. Then,

$u_i = 0$ for $i = 0, \ldots, m$ by Lemma 11. Since k is invariant under the transformation (9), see [1], we can make use of the canonical form (26) with $w_i = 0$ for $i = 0, \ldots, m$. For $m = 0$ and $a_0 = 0$, we obtain

$$(E, A) \sim (I, 0)$$

by (26). Therefore, we have $M_0 = I$ and hence $k = 0$.

For $m = 0$ and $a_0 \neq 0$, we obtain

$$(E, A) \sim (\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}).$$

Therefore M_0 is rank deficient, but

$$M_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \end{bmatrix}$$

is smoothly 1-full and so $k = 1$. For the general case $m \neq 0$, we must have a closer look at the matrix M_ℓ of (34a). First, we observe that $(M_\ell)_{ij} = 0$ for $i < j$, i. e. M_ℓ is block lower triangular. Then, without loss of generality, we can replace G_i by $-G_i$ and perform some row scaling to obtain

$$(M_\ell)_{i,i} = \begin{bmatrix} I & 0 & & & & * & \\ 0 & 0 & G_m & & & & \\ & & 0 & G_{m-1} & & * & \\ & & & \ddots & \ddots & * & \\ & & & & \ddots & G_1 & \\ & & & & & 0 & \end{bmatrix} \quad \text{for } i = 0, \ldots, \ell$$

and

$$(M_\ell)_{i,i-1} = \begin{bmatrix} 0 & 0 & & & & \\ 0 & I & & * & & \\ & & I & & & \\ & & & \ddots & * & \\ & & & & I & \end{bmatrix}, \qquad (M_\ell)_{i,j-2} = \begin{bmatrix} 0 & 0 & & & & \\ 0 & 0 & & * & & \\ & & 0 & & & \\ & & & \ddots & * & \\ & & & & 0 & \end{bmatrix}$$

$$\text{for } i = 1, \ldots, \ell, \qquad \text{for } i, j = 2, \ldots, \ell,\ i > j,$$

as relevant structure of the nonzero entries of M_ℓ. Recall from Theorem 12 that G_i is a (c_i, c_{i-1})–matrix of full row rank c_i with $c_i \neq 0$ for $i = 1, \ldots, m$. The claim now is that we must choose $\ell = m + 1$ to gain a smoothly 1–full M_ℓ.

As first step, we can use the identities in the subdiagonal block to eliminate the entries in the same column and in the same block or in the blocks below.

Therefore, one can restrict the considerations to a block bidiagonal matrix $\bar{M}_\ell$ where the diagonal is as in M_ℓ, i. e. $(\bar{M}_\ell)_{ii} = (M_\ell)_{ii}$ for $i = 0, \ldots, \ell$, and the relevant structure of the subdiagonal is given by

$$(\bar{M}_\ell)_{i,i-1} = \begin{bmatrix} 0 & 0 & & & \\ 0 & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{bmatrix} \text{ for } i = 1, \ldots, \ell.$$

Since $r_0 = \operatorname{rank} E$ and

$$r_0 = n - s_0 - b_0 = n - (b_m + \ldots + b_0) = n - c_0$$

or

$$r_0 = d_0 + s_0 = d_1 + s_1 + c_1 = d_m + (c_m + \ldots + c_1),$$

the matrix E is not of full rank. Therefore, we must require at least $\ell = 1$. In this case, we can use the $(m+2, m+2)$–identity matrix of the $(1,0)$–block to eliminate all other entries in this column and then interchange the pivot row with the corresponding row in the first block row. The $(0,0)$–block now has rank $n - c_1$. But at this point there is nothing left to do because the remaining rows in the $(1,1)$–block have full row rank. Therefore, we must require at least $\ell = 2$. But then we can eliminate as above with the help of the $(m+2, m+2)$–identity matrix in the $(2,1)$–block. By this, the $(m+1, m+1)$–identity of the $(1,0)$–block becomes free for interchange into the $(0,0)$–block which then has rank $n - c_2$.

Proceeding inductively in this way, we arrive at a $(0,0)$–block of rank n only by the choice $\ell = m + 1$ because $c_i \neq 0$, $i = 1, \ldots, m$. The same elimination procedure also shows that in this case M_ℓ is indeed smoothly 1–full and so $k = m + 1$. □

The reason that k and m differ by 1 as soon as $a_m \neq 0$ lies in the fact that we do not replace the algebraic equations by their differentiated expressions when we pass from (17) to (19). If we would do so, we would loose the information on the number of equations for which we cannot freely impose initial conditions. Instead of the statement that those problems (1) are hard for which $k \geq 2$, we here have the condition $m \neq 0$. This condition, however, can easily be checked locally by pointwise determination of the strangeness of $(E(t), A(t))$. Under the assumption of constant characteristic values, the cost of this in a numerical algorithm for solving (1) are three rank decisions in the beginning of the computations, which are the computation of the values r, a, s as in Theorem 7 and which can be obtained in a numerically stable way via three singular value decompositions. Because of (13e) in the form $u + s = n - r - a$, we can decide on $u + s \neq 0$ with the first two rank decisions. So, already at this stage, we can check for the differentiation index k to be greater than one or even undefined, like in the well-known standard test for higher index problems.

7 Conclusions

Starting with a new concept of local equivalence, we have developed a system of invariant quantities and a canonical form characterizing linear differential-algebraic equations with variable coefficients with respect to solvability, uniqueness of solutions and consistency of initial values. While the global canonical form as generalization of the Kronecker canonical form may be a powerful tool in the analysis of such equations, the numerical accessibility of local characterizing quantities which give essential information on the global solution behaviour as given in this paper is of great importance in the development of reliable numerical methods.

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