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Smooth factorizations of matrix valued functions and their derivatives

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Abstract. In this paper we study the numerical factorization of matrix valued functions in order to apply them in the numerical solution of differential algebraic equations with time varying coefficients. The main difficulty is to obtain smoothness of the factors and a numerically accessible form of their derivatives. We show how this can be achieved without numerical differentiation if the derivative of the given matrix valued function is known. These results are then applied in the numerical solution of differential algebraic Riccati equations. For this a numerical algorithm is given and its properties are demonstrated by a numerical example.

§ 1 Introduction.

We discuss the numerical computation of smooth transformations for time dependent matrix pencils

$$\alpha E(t) - \beta A(t), \quad E(t), A(t) : [t_0, t_f] \subset \mathbb{R} \to \mathbb{R}^{n,n}. \tag{1.1}$$

Pencils of this form arise in the study of linear differential algebraic equations

$$E(t)\dot{x}(t) := E(t)\frac{d}{dt}x(t) = A(t)x(t) + f(t) \tag{1.2}$$

and differential algebraic Riccati equations (DARE)

$$E(t)^T\dot{X}(t)E(t) = E(t)^TX(t)A(t)+A(t)^TX(t)E(t)+Q(t)-E(t)^TX(t)W(t)X(t)E(t), \tag{1.3}$$

with $E(t), A(t)$ as in (1.1), $Q(t), W(t) : [t_0, t_f] \to \mathbb{R}^{n,n}$, $Q(t) = Q(t)^T, W(t) = W(t)^T$.

In order to analyze the properties of (1.2) and (1.3) theoretically and for the construction of numerical methods one must apply time dependent transformationss to achieve the Kronecker canonical form or to separate the regular part of $E(t)$.

Typically such a transformation for (1.2) takes the form

$$\begin{aligned}
\widetilde{E}(t) &= U(t)^{-1}E(t)V(t),\\
\widetilde{A}(t) &= U(t)^{-1}(A(t)V(t) - E(t)\dot{V}(t)),\\
\widetilde{f}(t) &= U(t)^{-1}f(t),\\
\widetilde{y}(t) &= V(t)^{-1}x(t),
\end{aligned} \tag{1.4}$$

leading to the system

$$\widetilde{E}(t)\dot{y}(t) = \widetilde{A}(t)y(t) + \widetilde{f}(t), \tag{1.5}$$

while for (1.3) we obtain

$$\begin{aligned}
\widetilde{E}(t) &= U(t)^{-1}E(t)V(t),\\
\widetilde{A}(t) &= U(t)^{-1}A(t)V(t) - \frac{d}{dt}(U(t)^{-1})E(t)V(t),\\
Y(t) &= U(t)^TX(t)U(t),\\
\widetilde{Q}(t) &= V(t)^TQ(t)V(t),\\
\widetilde{W}(t) &= U(t)^{-1}W(t)U(t)^{-T},
\end{aligned} \tag{1.6}$$

leading to

$$\widetilde{E}(t)^T\dot{Y}(t)\widetilde{E}(t) = \widetilde{E}(t)^TY(t)\widetilde{A}(t)+\widetilde{A}(t)^TY(t)\widetilde{E}(t)+\widetilde{Q}(t)-\widetilde{E}(t)^TY(t)\widetilde{W}(t)Y(t)\widetilde{E}(t). \tag{1.7}$$

Differential algebraic Riccati equations like (1.3) arise in linear quadratic optimal control problems for descriptor systems, e.g. Mehrmann [M1], [M2], which have the following form:

Minimize

$$J(x(t),u(t)) = \frac{1}{2}\left[[x(t)^T M x(t)]\Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \right] \tag{1.8}$$

subject to

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x^0. \tag{1.9}$$

Here $E(t), A(t), B(t), Q(t), R(t), S(t)$ are matrix valued functions:

$$\begin{aligned} E, A, Q &: [t_0, t_f] \subset \mathbb{R} \longrightarrow \mathbb{R}^{n,n}, \\ B, S &: [t_0, t_f] \subset \mathbb{R} \longrightarrow \mathbb{R}^{m,n}, \\ R &: [t_0, t_f] \subset \mathbb{R} \longrightarrow \mathbb{R}^{m,m}, \end{aligned}$$

$\begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix}$ is symmetric positive semidefinite and $R(t)$ is positive definite for all $t \in [t_0, t_f]$. $M \in \mathbb{R}^{n,n}$ is constant, positive semidefinite and $x(t), u(t)$ are vector valued functions:

$x : [t_0, t_f] \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ is the state of the systems,

$u : [t_0, t_f] \subset \mathbb{R} \longrightarrow \mathbb{R}^M$ is the input of the system.

Under certain conditions (e.g. Kunkel/Mehrmann [KM]), the optimal solution is given by the linear feedback law

$$u(t) = -R(t)^{-1}\left(S(t)^T + B(t)^T X(t) E(t) \right) x(t), \tag{1.10}$$

where $X(t)$ is the symmetric solution of the initial value problem

$$\begin{aligned} &- E(t)^T \dot{X}(t) E(t) = A(t)^T X(t) E(t) + E(t)^T X(t) A(t) \\ &+ Q(t) - \left[B(t)^T X(t) E(t) + S(t)^T \right]^T R(t)^{-1} \left[B(t)^T X(t) E(t) + S(t)^T \right] \end{aligned} \tag{1.11}$$

with "terminal" condition

$$E(t_f)^T X(t_f) E(t_f) = M. \tag{1.12}$$

In this paper we are mainly interested in the numerical solution of problem (1.3). As shown in Kunkel/Mehrmann [KM] for the special case that $E(t)$ has constant left nullspace, one needs numerically stable transformations $U(t)$ and $V(t)$ to transform the system into a form that is treatable by standard solvers for differential algebraic equations. In the general case we additionally need the derivative $\dot{U}(t)$.

On the basis of QR -decompositions with pivoting, we show that we achieve

$$E(t) = U(t) \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} V(t)^T,$$

with $U(t), V(t)$ orthogonal, $\Sigma(t)$ nonsingular and $U(t), V(t), \Sigma(t)$ as smooth as $E(t)$ (at least locally) if we have

(1.14) rank $E(t)$ constant, independent of t.

Moreover, we give an explicit formula in terms of $\dot{E}(t)$ for $\dot{U}(t)$, thus avoiding numerical differentiation by finite differences, which would lead to undesirable discretization errors. This construction is given in Section 2.

However, as shown in Kunkel/Mehrmann [KM] we need the additional assumption that

(1.15) $\alpha E(t) - \beta A(t)$ is regular and has constant local index 1 independent of t.

We discuss this condition in Section 3.

In the case that the DARE (1.3) arises from (1.8), (1.9), i.e. we have the initial value problem (1.11), (1.12), we show that we can achieve (1.15) under some weak assumptions by introducing a preliminary feedback control, see Section 4.

Finally we describe a numerical algorithm for the solution of (1.3), generalizing the approaches of [KM] and give a numerical example in Section 5.

§ 2 The Procrustes Problem and Derivatives.

In this section we discuss a method to perform a smooth transformation

$$U(t)^T E(t) V(t) = \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.1}$$

where $\Sigma(t)$ is nonsingular and $U(t), V(t)$ are orthogonal.
Note that if we use this transformation in combination with one step solvers for differential algebraic equations, we need smoothness only locally for the current integration step. A globally smooth factorization can under stronger assumptions like analyticity of $E(t)$ be achieved by the ASVD, e.g. Bunse–Gerstner et al [BBMN].

Let us now assume that we have reached time t^* and that we want to construct a smooth decomposition of $E(t)$ of type (2.1) in a neighbourhood of t^* . Performing a QR–decomposition of $E(t)$ followed by a number of Householder reflections from the right, we can achieve the following factorization:

$$Q_1(t)^T E(t) \Pi(t) Q_2(t) = \begin{bmatrix} \widetilde{\Sigma}(t) & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.2}$$

with $Q_1(t), Q_2(t)$ orthogonal, $\Pi(t)$ a permutation matrix and $\widetilde{\Sigma}(t)$ nonsingular. We set

$$\widetilde{U}(t) := Q_1(t), \widetilde{V}(t) := \Pi(t) Q_2(t). \tag{2.3}$$

In general these factors are not smooth, due to pivoting and other stability checks employed in standard QR–decomposition procedures.

In the following we concentrate on the factor $U(t)$, because we want to follow here the transformation (1.6), in particular

$$\widetilde{A}(t) = U(t)^T A(t)V(t) - \dot{U}(t)^T E(t)V(t),$$

because of the orthogonality of $U(t), V(t)$. The treatment of $V(t)$ if needed, however, is in lines with that of $U(t)$.
To obtain smoothness of $U(t)$ without changing the form of (2.2) we treat kernel$(E(t)^T)$ and cokernel$(E(t)^T)$ separately. Let

$$\widetilde{U}(t) = (\widetilde{U}_1(t), \widetilde{U}_2(t)),\ U^* = \widetilde{U}(t^*) = (U_1^*, U_2^*) \tag{2.5}$$

be partitioned according to (2.2). We now solve the so–called orthogonal Procrustes problem (e.g. Golub/Van Loan [GV]) for cokernel$(E(t)^T)$, that is we set

$$U_1(t) = \widetilde{U}_1(t)Z(t), \tag{2.6}$$

with

$$Z(t) = \widehat{V}(t)^T \widehat{U}(t), \tag{2.7}$$

where $\widehat{U}(t), \widehat{V}(t)$ are the orthogonal factors of the singular value decomposition

$$(U_1^*)^T \widetilde{U}_1(t) = \widehat{U}(t)^T \widehat{\Sigma}(t) \widehat{V}(t). \tag{2.8}$$

Employing a result which is due to Rheinboldt [R 1], we can now show smoothness for $U_1(t)$ in a neighbourhood of t^* . We repeat the result here with proof in order to introduce the notation that we need for deriving an expression for $\dot{U}_1(t)$ in terms of $\dot{E}(t)$.

2.9 THEOREM. *(Rheinboldt [R 1]) Let $\widetilde{U}_1(t), U_1^*, Z(t)$ be as in (2.6), (2.7), (2.8). Then $U_1(t) = \widetilde{U}_1(t)Z(t)$ is as smooth as the projection*

$$P(t) = \widetilde{U}_1(t)\widetilde{U}_1(t)^T. \tag{2.10}$$

PROOF: Define the symmetric matrix $H(t) := \widehat{U}(t)^T \widehat{\Sigma}(t) \widehat{U}(t)$, using the factorization (2.8). Then, we have

$$(U_1^*)^T \widetilde{U}_1(t) = \widehat{U}(t)^T \widehat{\Sigma}(t) \widehat{V}(t) = H(t)Z(t)^T$$

and

$$((U_1^*)^T \widetilde{U}_1(t)(U_1^*)^T \widetilde{U}_1(t))^T = H(t)^2.$$

Hence,

$$H(t) := [(U_1^*)^T \widetilde{U}_1(t)\widetilde{U}_1(t)^T U_1^*]^{\frac{1}{2}} = [(U_1^*)^T P(t) U_1^*]^{\frac{1}{2}} \tag{2.11}$$

is as smooth as $P(t)$. Since $H(t^*) = I$, we have that $H(t)$ is nonsingular in a neighbourhood of t^* and

$$Z(t) = \widetilde{U}_1(t)^T U_1^* H(t)^{-1}.$$

Hence,

$$U_1(t) = \widetilde{U}_1(t)\widetilde{U}_1(t)^T U_1^* H(t)^{-1} = P(t)U_1^*[(U_1^*)^T P(t) U_1^*]^{\frac{1}{2}} \tag{2.12}$$

is as smooth as $P(t) = \widetilde{U}_1(t)\widetilde{U}_1(t)^T$. ■

In our case, $P(t)$ is the orthogonal projection on cokernel$(E(t)^T)$, i.e.

$$P(t) = E(t)E(t)^+ \tag{2.13}$$

where $E(t)^+$ denotes the Moore–Penrose inverse of $E(t)$. By assumption (1.14) it follows that $P(t)$ is as smooth as $E(t)$.

In order to obtain an explicit representation of $\dot{U}_1(t)$, we need a representation for $\dot{P}(t)$, which is given by the following Lemma:

2.14 Lemma. *Let $P(t)$ be as in (2.13). Then*

$$\dot{P}(t) = (I - P(t))\dot{E}(t)E(t)^+ + ((I - P(t))\dot{E}(t)E(t)^+)^T. \tag{2.15}$$

Proof: Differentiating the two identities

$$P(t)E(t) = E(t),\ P(t)^2 = P(t) \tag{2.16}$$

we obtain

$$\dot{P}(t)E(t) + P(t)\dot{E}(t) = \dot{E}(t),\ \dot{P}(t)P(t) + P(t)\dot{P}(t) = \dot{P}(t). \tag{2.17}$$

Inserting (2.13) and solving for $\dot{P}(t)$ yields

$$\dot{P}(t) = \dot{P}(t)E(t)E(t)^+ + E(t)E(t)^+\dot{P}(t),$$

and since $E(t)E(t)^+, P(t)$ are symmetric, we get

$$\dot{P}(t) = \dot{P}(t)E(t)E(t)^+ + (\dot{P}(t)E(t)E(t)^+)^T. \tag{2.18}$$

and therefore (2.15). ■

Now by (2.12) we have

$$U_1(t)H(t) = P(t)U_1^* \tag{2.19}$$

and by differentiation

$$\dot{U}_1(t)H(t) + U_1(t)\dot{H}(t) = \dot{P}(t)U_1^* \tag{2.20}$$

and hence

$$\dot{U}_1(t) = (\dot{P}(t)U_1^* - U_1(t)\dot{H}(t))H(t)^{-1}. \tag{2.21}$$

Thus, if we can obtain a formula for $\dot{H}(t)$ in terms of $\dot{E}(t)$, we have obtained an expression for $\dot{U}_1(t)$.

2.22 LEMMA. *Let $U_1(t), P(t), U_1^*, H(t)$ be as in (2.6), (2.13), (2.5), (2.11), then*

$$\dot{H}(t) = \widehat{U}(t)^T \widehat{R}(t)\widehat{U}(t), \tag{2.23}$$

where $\widehat{U}(t)$ is as in (2.8) and the entries of $\widehat{R}(t) = [r_{ij}(t)]$ are given by

$$r_{ij}(t) = \frac{1}{\widehat{\sigma}_i(t) + \widehat{\sigma}_j(t)}\ \gamma_{ij},\ i,j = 1,\dots,n-r, \tag{2.24}$$

where $\Gamma = [\gamma_{ij}] = \widehat{U}(t)(U_1^*)^T \dot{P}(t) U_1^* \widehat{U}(t)^T$, $r = \operatorname{rank} E(t)$ and $\widehat{\sigma}_i(t)$ are the diagonal elements of $\widehat{\Sigma}(t) = \operatorname{diag}(\widehat{\sigma}_1(t),\dots,\widehat{\sigma}_n(t))$.

PROOF: From

$$H(t)^2 = (U_1^*)^T P(t) U_1^*$$

we obtain by differentiation

$$\dot{H}(t)H(t) + H(t)\dot{H}(t) = (U_1^*)^T \dot{P}(t) U_1^*.$$

By the definition of $H(t)$ and multiplication with $\widehat{U}(t)$ from the left and $\widehat{U}(t)^T$ from the right we obtain

$$\widehat{U}(t)\dot{H}(t)\widehat{U}(t)^T\widehat{\Sigma}(t) + \widehat{\Sigma}(t)\widehat{U}(t)\dot{H}(t)\widehat{U}(t)^T = \widehat{U}(t)(U_1^*)^T\dot{P}(t)U_1^*\widehat{U}(t)^T. \tag{2.25}$$

Setting $\widehat{R}(t) := \widehat{U}(t)\dot{H}(t)\widehat{U}(t)^T$, we can compute the elements of $\widehat{R}(t) = [r_{ij}(t)]$ componetwise from (2.25) and obtain $r_{ij}(t)$ as in (2.24). ■

Combining Lemmata 2.14 and 2.22 we obtain:

2.26 THEOREM. *Let $U_1^*, U_1(t), \widehat{U}(t), H(t), P(t), \widehat{R}(t)$ be as in (2.5), (2.6), (2.8), (2.12), (2.13), (2.24), respectively. Then,*

$$\widetilde{A}(t) = U(t)^T A(t) V(t) + \begin{bmatrix} U_1(t)^T\dot{U}_1(t)\Sigma(t) & 0 \\ U_2(t)^T\dot{U}_1(t)\Sigma(t) & 0 \end{bmatrix}, \tag{2.27}$$

where

$$\begin{aligned} \dot{U}_1(t) &= [\dot{P}(t)U_1^* - U_1(t)\widehat{U}(t)^T\widehat{R}(t)\widehat{U}(t)]H(t)^{-1} \\ &= [\{(I-P(t))\dot{E}(t)E(t)^+ + ((I-P(t))\dot{E}(t)E(t)^+)^T\}U_1^* \\ &\quad - U_1(t)\widehat{U}(t)^T\widehat{R}(t)\widehat{U}(t)]H(t)^{-1}. \end{aligned} \tag{2.28}$$

PROOF: From (2.4) we obtain

$$\begin{aligned}\widetilde{A}(t) &= U(t)^T A(t)V(t) - \dot{U}(t)^T U(t) \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \\ &= U(t)^T A(t)V(t) + U(t)^T \dot{U}(t) \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \\ &= U(t)^T A(t)V(t) - \dot{U}(t)^T U(t) \begin{bmatrix} U_1(t)^T \dot{U}_1(t)\Sigma(t) & 0 \\ U_2(t)^T \dot{U}_1(t)\Sigma(t) & 0 \end{bmatrix}\end{aligned}$$

and (2.28) follows directly from Lemmata 2.14, 2.22. ■

Hence, $\widetilde{A}(t)$ is computable without numerical differentiation, when $\dot{E}(t)$ is available, in particular no discretization errors occur. Observe that we do not need $\dot{U}_2(t)$ in (2.27).

Note that the same construction is applicable if $U(t), V(t)$ are obtained from other factorizations for matrix valued functions like the analytic singular value decomposition (ASVD) of Bunse-Gerstner et al. [BBMN].

§ 3 Index and Transformations.

In this section we discuss the global and local index for a system of the form (1.2).

3.1 DEFINITION. *(Gear/Petzold [GP], Griepentrop/März [GM])*
a) The system given by (1.2) with coefficients $E(t), A(t)$ continuous, is said to be in Kronecker canonical form (KCF) with global index k if

$$E(t) = \begin{bmatrix} I & 0 \\ 0 & J(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} C(t) & 0 \\ 0 & I \end{bmatrix} \tag{3.2}$$

partitioned accordingly, where $\operatorname{kernel}(J(t)) = \operatorname{kernel}(J(t_0))$ *for all* $t \in [t_0, t_f]$, $J(t)^k \equiv 0$ *and* $J(t)^\ell \neq 0$ *for all* $\ell < k$, *for all* $t \in [t_0, t_f]$.

b) System (1.2) with coefficient matrices $E(t), A(t)$ continuous has global index k if there exist $V(t)$ continuous and $U(t)$ differentiable, $V(t), U(t)$ nonsingular on $[t_0, t_f]$ such that system (1.5) with the replacements (1.4) is in KCF with global index k .

c) For every $t \in [t_0, t_f]$ the local index of the system given by (1.3) is the index of the constant pencil $\alpha E(t) - \beta A(t)$, i.e. the size of the largest nilpotent block in the Kronecker canonical form of $\alpha E(t) - \beta A(t)$. See Gantmacher [G].

For a detailed discussion of various properties of systems like (1.2) see Brenan et al [BCP], Hansen [H] and Rheinboldt [R 2]. In the sequel the following proposition will be important:

3.3 PROPOSITION. *Let $E(t), A(t)$ be as in (1.1) then the local index is 1 if and only if the global index is 1.*

PROOF: Global index 1 and local index 1 are not changed by smooth nonlinear equivalence transformations, i.e. Gear/Petzold [GP], and hence since rank $E(t)$ is constant independent of t , we may assume that $\alpha E(t) - \beta A(t)$ is in KCF

$$\alpha \begin{bmatrix} I & 0 \\ 0 & N(t) \end{bmatrix} - \beta \begin{bmatrix} C(t) & 0 \\ 0 & I \end{bmatrix}, \tag{3.4}$$

for which local and global index 1 means that $N(t) \equiv 0$. ■

Hence, for index 1 systems, we need not to distinguish between local and global index.

We will show in the next section how we can achieve index 1 for problems of type (1.9) by a preprocessing procedure with a preliminary feedback.

We thus remain for these problems with the assumption (1.14). If this condition is not satisfied, one has to expect peculiar behaviour of the solution. Consider the following example.

3.5 Example.

Let

$$E(t) = \begin{bmatrix} t & 0 \\ -1 & t \end{bmatrix}, \; A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then rank $E(t)$ changes at $t = 0$, where the system has local index 2, while at all other t it has local index 0, the global index is not defined. For $t \neq 0$ the solution exists for all $u(t)$ continuous. But for $t = 0$ it must satisfy

$$\dot{x}_1 = x_2 = -\dot{u}(t),$$

hence requires higher differentiability for $u(t)$, which may not be given.

§ 4 Preliminary Feedback Control.

In this section we show how it is possible to adapt the procedure of Bunse–Gerstner et al. [BMN2] to preprocess a constant pencil to make it regular index 1 to the case of a time varying pencil. The main tool to do this is an analytic singular value decomposition (ASVD). The theoretical properties of the ASVD and a numerical method to achieve this decomposition is given in Bunse–Gerstner et al. [BBMN]. The adaption is as follows:

Given the system

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{4.1}$$

with $E(t), A(t), B(t)$ analytic, $E(t)$ satisfies (1.14), i.e. has constant rank r independent of t .

Perform an ASVD of $E(t)$, e.g. [BBMN],

$$E(t) = U(t) \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} V(t)^T \tag{4.2}$$

with $U(t), V(t)$ orthogonal, analytic, and $\Sigma(t)$ nonsingular diagonal. Observe that due to (1.14), $\Sigma(t)$ is nonsingular for all $t \in [t_0, t_f]$. The condition that $E(t), A(t), B(t)$ are analytic is a sufficient condition to guarantee the existence of the ASVD of $E(t)$. The procedure described in Bunse–Gerstner et al [BBMN] will produce a smooth decomposition under further conditions also in the nonanalytic case, but a general theory on the existence of smooth decompositions is not known in the nonanalytic case. Using (4.2) we set

$$\widetilde{E}(t) := \begin{matrix} \\ r\{ \\ n-r\{ \end{matrix} \begin{bmatrix} \overbrace{\Sigma(t)}^{r} & \overbrace{0}^{n-r} \\ 0 & 0 \end{bmatrix} \tag{4.3}$$

$$\widetilde{A}(t) := U(t)^T A(t)V(t) + \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \dot{V}(t)^T V(t)$$

(4.4)
$$= \begin{matrix} \\ r\{ \\ n-r\{ \end{matrix} \overbrace{\begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}}^{r \quad n-r} + \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \dot{V}(t)^T V(t),$$

$$\widetilde{B}(t) := \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} := U(t)^T B(t), \tag{4.5}$$

$$\widetilde{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := V(t)^T x(t). \tag{4.6}$$

We want to apply a preliminary feedback $u(t) = F(t)x(t) + v(t)$ such that

$$\alpha E(t) - \beta(A(t) + B(t)F(t)) \text{ is regular index 1.}$$

If we apply such a feedback to $\widetilde{E}, \widetilde{A}$, we only need $A_{22}(t), B_2(t)$, since due to the special form of $\widetilde{E}(t)$, the second block row of $\widetilde{A}(t)$ remains unchanged from $U(t)^T A(t)V(t)$.

A simplification of this first step that could also be employed is to replace the ASVD of $E(t)$ by the decomposition introduced in Section 3.

In order to guarantee that (1.15) holds, we must choose a preliminary (possibly time varying) feedback

$$u(t) = \overbrace{\begin{bmatrix} F_1(t) & F_2(t) \end{bmatrix}}^{r \quad n-r} \widetilde{x}(t) + v(t) \tag{4.7}$$

such that

$$\widehat{A}_{22}(t) = A_{22}(t) + B_2(t)F_2(t) \tag{4.8}$$

is nonsingular for all $t \in [t_0, t_f]$.

This we obtain from the following two results, which generalize well–known results for the constant coefficient case, e.g. Bunse–Gerstner et al. [BMN1].

4.9 Lemma. *Let $\alpha E(t) - \beta A(t)$ be a pencil with* rank $E(t) = r < n$ *constant independent of t. Let $\Delta(t)$ be a matrix whose columns form an orthogonal basis of* kernel $(E(t))$. *Then the following are equivalent.*

a) $\alpha E(t) - \beta A(t)$ is regular with constant local index 1.

b) rank $[E(t), A(t)\Delta(t)] = n$ *for all $t \in [t_0, t_f]$ independent of t.*

Proof: a) $\Longrightarrow$ b). We may assume that there exist $U(t), V(t)$ nonsingular on $[t_0, t_f]$ such that

$$U(t)(\alpha E(t) - \beta A(t))V(t) = \alpha \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} J(t) & 0 \\ 0 & I \end{bmatrix} \tag{4.10}$$

and $\Delta(t) = V(t)\begin{bmatrix} 0 \\ I \end{bmatrix}$, see [H] or [BMN1]. Then we have

$$\text{rank } [E(t), A(t)\Delta(t)] = \text{rank } \left[\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J(t) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \right] = \text{rank } \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = n.$$

b) $\Longrightarrow$ a). From rank $[E(t), A(t)\Delta(t)] = n$ for all $t \in [a, b]$ it follows immediately that $\alpha E(t) - \beta A(t)$ is regular, since rank $[E(t), A(t)] \geq \text{rank}[E(t), A(t)\Delta(t)] = n$, and hence rank$[E(t), A(t)] = n$. Suppose now that $\alpha E(t) - \beta A(t)$ has an index larger than 1 for some $\hat{t} \in [t_0, t_f]$. The local index is invariant under equivalence transformation, so for $\hat{t}$ we have the Kronecker canonical form

$$U(\hat{t})\Big(\alpha E(\hat{t}) - \beta A(\hat{t})\Big)V(\hat{t}) = \alpha \begin{bmatrix} I & 0 \\ 0 & N(\hat{t}) \end{bmatrix} - \beta \begin{bmatrix} J(\hat{t}) & 0 \\ 0 & I \end{bmatrix}, \tag{4.11}$$

with $U(\hat{t}), V(\hat{t})$ nonsingular and $N(\hat{t}) \neq 0$ nilpotent. Let $\widetilde{\Delta}(\hat{t})$ be a matrix whose columns form an orthogonal basis of kernel $(N(\hat{t}))$. Then $\Delta(\hat{t}) = \begin{bmatrix} 0 \\ \widetilde{\Delta}(\hat{t}) \end{bmatrix}$ is as required, and

$$\text{rank } \Big[E(\hat{t}), A(\hat{t})\Delta(\hat{t})\Big] = n = \text{rank } \left[\begin{bmatrix} I & 0 \\ 0 & N(\hat{t}) \end{bmatrix}, \begin{bmatrix} 0 \\ \widetilde{\Delta}(\hat{t}) \end{bmatrix} \right]. \tag{4.12}$$

Hence, $[N(\hat{t}), \widetilde{\Delta}(\hat{t})]$ has full rank. Let $x \neq 0$ be such that $y^T = x^T N(\hat{t})^{\ell-1} \neq 0$, where $\ell > 1$ is the index of $N(\hat{t})$. Then,

$$y^T[N(\hat{t}), \widetilde{\Delta}(\hat{t})] = [0, x^T N^{\ell-1}(\hat{t})\widetilde{\Delta}(\hat{t})] = 0,$$

which contradicts that rank $[N(\hat{t}), \widetilde{\Delta}(\hat{t})]$ is full. ■

4.13 THEOREM. *Let* $E(t) = \begin{matrix} r\{ \\ n-r\{ \end{matrix} \overbrace{\begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix}}^{r \quad n-r}$ *and* $A(t) = \begin{matrix} r\{ \\ n-r\{ \end{matrix} \overbrace{\begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}}^{r \quad n-r}$ *be as in (4.3), (4.4). Assume that* $\Sigma(t)$ *is nonsingular for all* $t \in [t_0, t_f]$. *Then the pencil* $\alpha E(t) - \beta A(t)$ *is regular and has constant local index 1, if and only if* $A_{22}(t)$ *is nonsingular for all* $t \in [a, b]$.

PROOF: The proof is analogous to that in the constant case, e.g. Bunse–Gerstner et al. [BMN 1]. If $A_{22}(t)$ is nonsingular, then $\alpha E(t) - \beta A(t)$ is equivalent to the pencil

$$\alpha \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A_{11}(t) - A_{12}(t)A_{22}^{-1}(t)A_{21}(t) & 0 \\ 0 & A_{22}(t) \end{bmatrix}$$

which is clearly regular and has local index 1.
Conversely if $\alpha E(t) - \beta A(t)$ is regular and has constant local index 1, then by Lemma 4.9

$$\text{rank } [E(t), A(t)\Delta(t)] = n \quad \text{for all } t \in [t_0, t_f],$$

where $\Delta(t)$ is as in Lemma 4.9. Hence rank $\begin{bmatrix} \Sigma(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix} = n$, which implies directly that $A_{22}(t)$ is nonsingular for all $t \in [t_0, t_f]$. ■

It follows from Theorem 4.13 that a feedback as in (4.4) has to be chosen. In the constant time case, the freedom in the choice of the feedback is used in Bunse–Gerstner et al. [BMN 2] to improve the conditioning of $\widehat{A}_{22}$ as given by (4.8). We proceed here analogously.

It is obvious that we can only achieve that $\widehat{A}_{22}(t)$ in (4.8) is nonsingular if $[A_{22}(t), B_2(t)]$ has full rank, independent of t for all $t \in [t_0, t_f]$. This is the time varying case analogue to "controllability at infinity" in the constant case, e.g. [BMN 2], [VLK].

Obviously $F_1(t)$ can be chosen arbitrarily without affecting the problem, so for simplicity we choose $F_1(t) \equiv 0$. To determine $F_2(t)$, we perform an ASVD on $B_2(t)$:

$$B_2(t) = U_B(t) \begin{bmatrix} \overbrace{\Sigma_1(t)}^{k} & \overbrace{0}^{\ell} & \overbrace{0}^{m-k-\ell} \\ 0 & \Sigma_2(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} V_B^T(t), \tag{4.14}$$

where $\Sigma_1(t) \neq 0$ for all $t \in [t_0, t_f]$ and $\Sigma_2(t)$ contains the singular values which are zero for some $t \in [t_0, t_f]$ but do not vanish identically ($\Sigma_1(t), \Sigma_2(t)$ are diagonal and square). Set

$$\widetilde{A}_{22}(t) = U_B^T(t) A_{22}(t) = \begin{matrix} \\ k\{ \\ \ell\{ \\ n-r-k-\ell\{ \end{matrix} \begin{bmatrix} \overbrace{D_{11}(t)}^{k} & \overbrace{D_{12}(t)}^{\ell} & \overbrace{D_{13}(t)}^{m-r-k-\ell} \\ D_{21}(t) & D_{22}(t) & D_{23}(t) \\ D_{31}(t) & D_{32}(t) & D_{33}(t) \end{bmatrix}. \tag{4.15}$$

Perform an ASVD of $\begin{bmatrix} D_{21}(t) & D_{22}(t) & D_{23}(t) \\ D_{31}(t) & D_{32}(t) & D_{33}(t) \end{bmatrix}$,

$$U_D^T(t) \begin{bmatrix} D_{21}(t) & D_{22}(t) & D_{23}(t) \\ D_{31}(t) & D_{32}(t) & D_{33}(t) \end{bmatrix} V_D(t) = \begin{bmatrix} 0 & \Delta_2(t) & 0 \\ 0 & 0 & \Delta_3(t) \end{bmatrix}. \tag{4.16}$$

Let

$$[\widetilde{D}_{11}(t), \widetilde{D}_{12}(t), \widetilde{D}_{13}(t)] := [D_{11}(t), D_{12}(t), D_{13}(t)]\ V_D(t), \quad \begin{bmatrix} G_{22}(t) & G_{23}(t) \\ G_{32}(t) & G_{33}(t) \end{bmatrix} := V_D \tag{4.17}$$

and let $\alpha(t)$ be the maximum singular value of

$$\begin{bmatrix} \Delta_2(t) & 0 \\ 0 & \Delta_3(t) \end{bmatrix} + U_D^T(t) \begin{bmatrix} \Sigma_2(t) & 0 \\ 0 & 0 \end{bmatrix} U_D(t) =: \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix} = \Phi(t), \tag{4.18}$$

e.g. $\alpha(t) = \max\limits_{i=1,\ldots,n-r-k} |\delta_i(t)|$, where $\delta_i(t)$ are the singular values of $\Phi(t)$. Observe that since $[A_{22}(t), B_2(t)]$ had full rank independent of t, it follows that

$$\left[\begin{bmatrix} \Delta_2(t) & 0 \\ 0 & \Delta_3(t) \end{bmatrix}, U_D^T(t) \begin{bmatrix} \Sigma_2(t) & 0 \\ 0 & 0 \end{bmatrix} \right] \tag{4.19}$$

has full rank independent of t, and hence the symmetric matrix $\Phi(t)$ is nonsingular for all $t \in [t_0, t_f]$. Let

$$[G_{11}(t)\ G_{12}(t)\ G_{13}(t)] = \Sigma(t)^{-1}[\alpha(t)I_k - \widetilde{D}_{11}(t), -\widetilde{D}_{12}(t), -\widetilde{D}_{13}(t)] \tag{4.20}$$

and set

$$F_2(t) = V_B(t) \begin{bmatrix} G_{11}(t) & G_{12}(t) & G_{13}(t) \\ 0 & G_{22}(t) & G_{23}(t) \\ 0 & G_{32}(t) & G_{33}(t) \end{bmatrix} V_D^T(t). \tag{4.21}$$

Then

$$\begin{aligned} &A_{22}(t) + V_B(t)F_2(t) \\ &\quad = U_B(t)^T \left[\begin{bmatrix} D_{11}(t) & D_{12}(t) & D_{13}(t) \\ D_{21}(t) & D_{22}(t) & D_{23}(t) \\ D_{31}(t) & D_{32}(t) & D_{33}(t) \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} \Sigma_1(t) & 0 & 0 \\ 0 & \Sigma_2(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G_{11}(t) & G_{12}(t) & G_{13}(t) \\ 0 & G_{22}(t) & G_{23}(t) \\ 0 & G_{32}(t) & G_{33}(t) \end{bmatrix} V_D^T(t) \right] \\ &\quad = U_B(t)^T \begin{bmatrix} I & 0 \\ 0 & U_D(t) \end{bmatrix} \begin{bmatrix} \alpha(t)I_k & 0 & 0 \\ 0 & \Phi_{11}(t) & \Phi_{12}(t) \\ 0 & \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix} V_D(t)^T, \end{aligned} \tag{4.22}$$

which is clearly nonsingular, has maximum singular value $\alpha(t)$ and minimum singular value equal to the minimum singular value of $\Phi(t)$.

There may be better ways to improve the conditioning of $\Phi(t)$ by choosing a different matrix $\begin{bmatrix} G_{22} & G_{23} \\ G_{32} & G_{33} \end{bmatrix}$, but we do not know how to achieve this in general in the time varying case. The optimal conditioning of the (1,1)-block, however, is fixed by the condition of $\Phi(t)$. We now set

$$\widehat{A}(t) := \begin{bmatrix} A_{11}(t) & A_{12}(t) + B_1(t)F_2(t) \\ A_{21}(t) & A_{22}(t) + B_2(t)F_2(t) \end{bmatrix} = \begin{bmatrix} \widehat{A}_{11}(t) & \widehat{A}_{12}(t) \\ \widehat{A}_{21}(t) & \widehat{A}_{22}(t) \end{bmatrix} \tag{4.23}$$

and have achieved a system of the form

$$\begin{aligned} &\begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} \\ &= \left(\begin{bmatrix} \widehat{A}_{11}(t) & \widehat{A}_{12}(t) \\ \widehat{A}_{21}(t) & \widehat{A}_{22}(t) \end{bmatrix} + \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \dot{V}(t)V(t) \right) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} v(t), \end{aligned} \tag{4.24}$$

$$\widetilde{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = V(t_0)^T x^0. \tag{4.25}$$

The cost functional (1.8) transforms to

$$J(\widetilde{x},v) = \frac{1}{2}\left\{(\widetilde{x}(t)^T V(t) M V(t)^T \widetilde{x}(t))\Big|_{t_0}^{t_f} + \right.$$

$$\left. \int_{t_0}^{t_f} \begin{bmatrix} \widetilde{x}(t) \\ v(t) \end{bmatrix}^T \begin{bmatrix} V(t) & 0 \\ F(t) & I \end{bmatrix}^T \begin{bmatrix} Q(t) & S(t) \\ S(t)^T & R(t) \end{bmatrix} \begin{bmatrix} V(t) & 0 \\ F(t) & I \end{bmatrix} \begin{bmatrix} \widetilde{x}(t) \\ v(t) \end{bmatrix} dt \right\} \tag{4.26}$$

where now

$$\alpha \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} - \beta \left(\begin{bmatrix} \widehat{A}_{11}(t) & \widehat{A}_{12}(t) \\ \widehat{A}_{21}(t) & \widehat{A}_{22}(t) \end{bmatrix} + \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \dot{V}(t) V(t) \right) \tag{4.27}$$

is regular and has index 1 .

In the following we may therefore assume that this property holds.

Observe that the numerical procedures to achieve ASVD's in (4.14), (4.16) can be carried out by a local step to step method, so this preprocessing can be adapted into any solution method.

§ 5 Numerical Treatment of DAREs.

On the basis of the above considerations a code DAREX was implemented for the solution of equations of the form (1.3), which satisfy (1.14) and (1.15), i.e., we assume that the preprocessing of Section 4 has already been performed. The basic steps of DAREX are described in the following algorithm.

5.1 Algorithm DAREX (cf. Kunkel/Mehrmann [KM]).

Given $(n \times n)$ -matrix valued functions $A(t), E(t), \dot{E}(t), Q(t), W(t)$ with $Q(t)^T = Q(t)$, $W(t)^T = W(t)$, $t_0, t_f \in \mathbb{R}$, $M \in \mathbb{R}^{n,n}$, $M^T = M$ and a rank tolerance $\epsilon \in \mathbb{R}$, with $E(t), A(t)$ satisfying (1.14), (1.15), this algorithm solves the Riccati differential algebraic equation (1.3) in the interval $[t_0, t_f]$ starting at $t = t_f$.

Step 1 Perform a singular value decomposition of $E(t_f)$ using the LINPACK DSVDC routine, i.e. determine $U(t_f), V(t_f) \in \mathbb{R}^{n,n}$ orthogonal such that $U(t_f)^T E(t_f) V(t_f) = \operatorname{diag}(\sigma_1(t_f), \ldots, \sigma_n(t_f))$, $\sigma_1(t_f) \geq, \ldots, \geq \sigma_n(t_f) \geq 0$.

Step 2 Determine the rank r of $E(t_f)$ as the largest index $i \in \{1, \ldots, n\}$ such that $\sigma_i(t_f) \geq \epsilon \sigma_1(t_f)$ and set $\sigma_{r+1}(t_f) = \ldots = \sigma_n(t_f) = 0$.

Step 3 Determine $\dot{U}(t_f)$ according to Section 2 and transform (1.3) as

$$E(t_f) := \begin{bmatrix} \Sigma(t_f) & 0 \\ 0 & 0 \end{bmatrix}, \ \Sigma(t_f) = \operatorname{diag}(\sigma_1(t_f), \ldots, \sigma_r(t_f))$$

$$A(t_f) := U(t_f)^T A(t_f) V(t_f) + U(t_f)^T \dot{U}(t_f) \begin{bmatrix} \Sigma(t_f) & 0 \\ 0 & 0 \end{bmatrix}$$

$$Q(t_f) := V(t_f)^T Q(t_f) V(t_f)$$

$$W(t_f) := U(t_f)^T W(t_f) U(t_f)$$

$$M := V(t_f)^T M \ V(t_f)$$

$$X(t_f) := U(t_f)^T X(t_f) U(t_f).$$

Partition these matrices analogous to the rank deficiency of $E(t_f)$ and denote the blocks by the same letter with indices $i, j \in \{1, 2\}$.

Step 4 Check consistency of the equations and initial values, i.e. check whether $Q_{22}(t_f) = 0,\ M_{12} = 0,\ M_{22} = 0$. If these conditions do not hold, the system is not solvable, cf. [KM], then

STOP.

Step 5 Solve the algebraic part of the transformed Riccati differential algebraic equation, i.e., compute

$$X_{12}(t_f) = \ -\Sigma(t_f)^{-1}\Big(M_{11}\Sigma(t_f)^{-1}A_{12}(t_f) + Q_{12}(t_f)\Big)A_{22}(t_f)^{-1}.$$

Step 6 Rewrite the relevant part of $X(t_f) = \begin{bmatrix} X_{11}(t_f) & X_{12}(t_f) \\ X_{21}(t_f) & X_{22}(t_f) \end{bmatrix}$, i.e. the upper triangle of $X_{11}(t_f)$ including the diagonal and $X_{12}(t_f)$ in vector form by going rowwise through $[X_{11}(t_f), X_{12}(t_f)]$. Accordingly rewrite the corresponding parts of (1.3) in vector form to obtain a vectorized system

$$\widehat{E}(t)\dot{z}(t) = \widehat{f}(t, z(t)) \tag{5.2}$$

of size $\frac{r(2n-r+1)}{2}$.

Step 7 Solve (5.2) with LIMEX due to Deuflhard et al. [DHZ], where

a) in the beginning of each step, rewrite the actual solution vector $z(t_i)$ into matrix form $X(t_i)$ according to the process of step 6 completing by 0 in $X_{22}(t_i)$ and $X_{21}(t_i) = X_{12}(t_i)^T, X_{11}(t_i) = X_{11}(t_i)^T$ and transform $X(t_i)$ back by the actual $U(t_i)$. Compute a new reference matrix U^* following (4.4), (4.5), set $U(t_i) = U^*$ and determine $\dot{U}(t_i)$. Transform $X(t_i)$ by the new actual $U(t_i)$.

b) for the evaluation of the vectorized functions $\widehat{E}(t_i)$ and $\widehat{f}(t_i, z(t_i))$ determine locally smoothly varying matrices $U(t_i), V(t_i)$, and $\dot{U}(t_i)$ by the techniques of Section 2 and transform according to Step 3.

Step 8 Transform $X(t_0)$ back by $X(t_0) := U(t_0)X(t_0)U(t_0)^T$ and compute $X(t_0)E(t_0))$ for output.

Consider now the following example.

5.3 Example.

$$E(t) = \begin{bmatrix} 1-t^2 & -t+t^3 \\ 2t & -2t^3 \end{bmatrix},\ \dot{E}(t) = \begin{bmatrix} -2t & -1+3t^2 \\ 2 & -4t \end{bmatrix}$$

$$A(t) = \begin{bmatrix} 3t-t^3 & t^4-t^2+2t \\ 2t^2+t^3-t & -2t^3+t^2-1 \end{bmatrix},\ W(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$Q(t)$ chosen such that

$$X(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{-t} \end{bmatrix}$$

solves (1.3), i.e.

$$Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix} e^{-t} \quad \text{with}$$

$$\begin{aligned} Q_{11}(t) &= 4t^2 - 4t(2t^2 + t^3 - t) \\ Q_{12}(t) &= -4t^3 - 2t(-2t^3 + t^2 - 1) + 2t^2(2t^2 + t^3 - t) \\ Q_{21}(t) &= Q_{12}(t) \\ Q_{22}(t) &= 4t^4 + 4t^2(-2t^3 + t^2 - 1), \end{aligned}$$

$$M = E(t_f)^T X(t_f) E(t_f) = \begin{bmatrix} 4t_f^2 & -4t_f^3 \\ -4t_f^3 & 4t_f^4 \end{bmatrix} e^{-t_f}.$$

Starting point was $t_f = 1$, local tolerance for the DAE–solver LIMEX was 10^{-5}. The computations were done an a Comparex 8/83 under VM/XA SP in Fortran double precision with 56 bit mantissa.

Recall that the solution $X(t)$ of the problem is not unique but that in a control problem (1.8), (1.9), $X(t)$ appears together with a factor $E(t)$ in the control $u(t)$, see (1.11), and the nonuniqueness in $X(t)$ is in kernel $(E(t))$.

We obtained the following results:

(a) final point $t_0 = 0$.

After 0.93 sec CPU–time and 16 integration steps of LIMEX, the code DAREX obtained

$$X(t_0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; X(t_0)E(t_0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

subject to an error of 10^{-6}.

(b) final point $t_0 = -1$.

After 3.46 sec CPU–time and 54 integration steps of LIMEX, the code DAREX obtained

$$X(t_0) = \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}, \; X(t_0)E(t_0) = \begin{bmatrix} 0 & 0 \\ -2e & -2e \end{bmatrix}$$

subject to an error of 10^{-5}. solution.

In addition, the results of Kunkel/Mehrmann [KM] were reproduced. In no cases, trouble occured due to possibly non–smooth functions $\widehat{E}, \widehat{f}$. Note that for the present length of the mantissa the error introduced by standard finite differences is of order 10^{-7}.

§ 6 Conclusion.

In this paper we have discussed smooth orthogonal transformations of time varying matrix pencils and we have shown how these transformations can be carried out numerically. In

particular we have given an explicit formula for the derivative of the transforming matrix function.
We have applied this approach to the numerical solution of differential algebraic Riccati equations arising from linear quadratic optimal control problems with descriptor equation constraints.

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