AUGMENTED SYSTEMS FOR GENERALIZED TURNING POINTS

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Abstract. We consider augmented systems for singular points with a given rank drop which shall satisfy some further conditions. Regularity results are given and applied to a family of problems which include Bogdanov-Takens points.

1 Introduction

In studying parametrized systems of nonlinear equations

$$f(x,\tau) = 0, \quad f\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \tag{1.1}$$

or related dynamical problems, interest has grown in evaluating singular points, i.e. solutions (x^*,τ^*) of (1.1) with

$$\operatorname{rank} L = n-k, \quad L = f_x(x^*,\tau^*), \quad k \geq 1. \tag{1.2}$$

In addition, they are often required to satisfy some further conditions. Presumably, the best-known singular points of this kind are the so-called Bogdanov-Takens points where we have $k = 1$, i.e. simple turning points, and a second zero eigenvalue is required. See, e.g., [2,3,8] for analytical and numerical details.
A standard technique to compute singular points numerically is by means of augmented systems where (1.1) is embedded into a larger system such that the desired point (x^*,τ^*) becomes part of a regular solution of the new system. In the case of the Bogdanov-Takens points, such a system can be found in [8].
In [1], more complicated singular points fitting into the above general problem type are reported to occur in a Brusselator model but no suitable augmented system was given. It is the aim of the present paper to show the applicability of a family of augmented systems first introduced in [5] for the numerical treatment of this problem type.

2 Generalized Turning Points

There are two basic concepts in analyzing singular points, namely Lyapunov-Schmidt reduction and contact equivalence, see, e.g., [4]. For simplicity of notation, let all functions be C^∞ in the corresponding neighborhoods where they are used.

The Lyapunov-Schmidt reduction is applied to eliminate the regular part of (1.1) in a neighborhood of (x^*, τ^*). To introduce the necessary notation, we sketch this technique in the following. Let

$$U^T L V = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma \text{ regular}, \quad U, V \text{ orthogonal} \tag{2.1}$$

be a singular value decomposition of L. We split the orthogonal matrices according to the rank drop k of L into

$$U = (\overline{z}, z), \quad V = (\overline{t}, t). \tag{2.2}$$

Consequently, the columns of z and t form orthogonal bases of corange L and kernel L, respectively. Defining the orthogonal projectors $P = tt^T$ and $Q = zz^T$, we can split (1.1) into

$$(I - Q)f(v + w, \tau) = 0, \quad Qf(v + w, \tau) = 0 \tag{2.3}$$

with $x = v + w$, $v = Px$. By construction, the first part of (2.3) is uniquely solvable in a neighborhood of (x^*, τ^*) by $w = w^*(v, \tau)$ due to the implicit function theorem and we remain with the so-called reduced function

$$g(\xi, \tau) = z^T f(t\xi + w^*(t\xi, \tau), \tau), \quad g: \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^k. \tag{2.4}$$

By contact equivalence, these reduced functions are divided into classes due to their local behavior at (ξ^*, τ^*) where $\xi^* = t^T x^*$ by the following equivalence relation. Two functions g and h are called contact equivalent at (ξ^*, τ^*) if there is a neighborhood of (ξ^*, τ^*) such that

$$h(\xi, \tau) = T(\xi, \tau) g(\varrho(\xi, \tau), \lambda(\tau)) \tag{2.5.a}$$

with

$$\begin{array}{llll} T: \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^{k,k}, & T(\xi^*, \tau^*) \text{ regular}, & & \\ \varrho: \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^k \times \mathbb{R}^p, & \varrho_\xi(\xi^*, \tau^*) \text{ regular}, & \varrho(\xi^*, \tau^*) = (\xi^*, \tau^*), & \\ \lambda: \mathbb{R}^p \to \mathbb{R}^p, & \lambda_\tau(\tau^*) \text{ regular}, & \lambda(\tau^*) = \tau^*. & \end{array} \tag{2.5.b}$$

We then have the important property that two different reduced functions of the same function are contact equivalent. Thus, we have a well-defined equivalence class $[g]$ connected with a singular point (x^*, τ^*) of f.

As first step towards the treatment of our general problem, we define the notion of a generalized turning point.

Definition 1. A singular point (x^*, τ^*) is called generalized turning point if and only if it satisfies

$$z^T f_x(x^*, \tau^*)t = 0 \tag{2.6}$$

and

$$B = \begin{pmatrix} 0 & z^T f_\tau(x^*, \tau^*) \\ z^T f_{xx}(x^*, \tau^*)(t_1, t) & z^T f_{x\tau}(x^*, \tau^*)(t_1) \\ \vdots & \vdots \\ z^T f_{xx}(x^*, \tau^*)(t_k, t) & z^T f_{x\tau}(x^*, \tau^*)(t_k) \end{pmatrix} \text{ regular}, \quad t = (t_1, \ldots, t_k). \tag{2.7}$$

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Note that (2.6) restates the rank drop of k in L and that in the case $k = 1$ these conditions simplify to the known definition of a simple turning point. The following theorem shows that the above definition fits into the concept of contact equivalence.

Theorem 2. The properties (2.6) and (2.7) imply $h_\xi(\xi^*, \tau^*) = 0$ and

$$B_h = \begin{pmatrix} 0 & h_\tau(\xi^*, \tau^*) \\ h_{\xi\xi}(\xi^*, \tau^*)(e_1, \cdot) & h_{\xi\tau}(\xi^*, \tau^*)(e_1) \\ \vdots & \vdots \\ h_{\xi\xi}(\xi^*, \tau^*)(e_k, \cdot) & h_{\xi\tau}(\xi^*, \tau^*)(e_k) \end{pmatrix} \text{ regular} \tag{2.8}$$

where $e_1, \ldots, e_k$ is the canonical basis of $\mathbb{R}^k$ for all h being contact equivalent to a reduced function g of f.

Proof. Since g is a reduced function of f, we have

$$\begin{aligned} g_\xi(\xi^*, \tau^*)(a_1) &= z^T f_x(x^*, \tau^*)(ta_1), \\ g_{\xi\xi}(\xi^*, \tau^*)(a_1, a_2) &= z^T f_{xx}(x^*, \tau^*)(ta_1, ta_2), \\ g_{\xi\tau}(\xi^*, \tau^*)(a_1)(b_1) &= z^T f_{x\tau}(x^*, \tau^*)(ta_1)(b_1) + \\ &\quad + z^T f_{xx}(x^*, \tau^*)(ta_1, -L^+ f_\tau(x^*, \tau^*)(b_1)). \end{aligned}$$

Therefore (2.6) is equivalent to $g_\xi(\xi^*, \tau^*) = 0$. Furthermore, by elementary row and column operations we have rank B_g = rank B.
Now, let h be contact equivalent to g. By the relations

$$\begin{aligned} h_\xi(\xi^*, \tau^*)(a_1) &= T(\xi^*, \tau^*) g_\xi(\xi^*, \tau^*)(\varrho_\xi(\xi^*, \tau^*)(a_1)), \\ h_\tau(\xi^*, \tau^*)(b_1) &= T(\xi^*, \tau^*) g_\tau(\xi^*, \tau^*)(\lambda_\tau(\tau^*)(b_1)), \\ h_{\xi\xi}(\xi^*, \tau^*)(a_1, a_2) &= T(\xi^*, \tau^*) g_{\xi\xi}(\xi^*, \tau^*)(\varrho_\xi(\xi^*, \tau^*)(a_1), \varrho_\xi(\xi^*, \tau^*)(a_2)), \\ h_{\xi\tau}(\xi^*, \tau^*)(a_1)(b_1) &= T(\xi^*, \tau^*) g_{\xi\tau}(\xi^*, \tau^*)(\varrho_\xi(\xi^*, \tau^*)(a_1))(\lambda_\tau(\tau^*)(b_1)) + \\ &\quad + T(\xi^*, \tau^*) g_{\xi\xi}(\xi^*, \tau^*)(\varrho_\xi(\xi^*, \tau^*)(a_1), \varrho_\tau(\xi^*, \tau^*)(b_1)) + \\ &\quad + T_\xi(\xi^*, \tau^*)(a_1) g_\tau(\xi^*, \tau^*)(\lambda_\tau(\tau^*)(b_1)), \end{aligned}$$

we immediately have $h_\xi(\xi^*, \tau^*) = 0$. Moreover, inserting these relations into B_h we can transform B_h into B_g by a sequence of elementary row and column operations thus yielding $\operatorname{rank} B_h = \operatorname{rank} B_g$. With this, the proof is finished.

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3 Augmented Systems

For the numerical treatment of generalized turning points, we introduce the augmented systems

$$
\begin{aligned}
f_x(x,\tau)t_i - z\beta_i = 0, \quad & t_i^T t_{i'} = \delta_{ii'}, \quad 1 \le i' \le i \le k, \\
f_x(x,\tau)^T z_j - t\gamma_j = 0, \quad & z_j^T z_{j'} = \delta_{jj'}, \quad 1 \le j' \le j \le k, \\
f(x,\tau) &= 0, \\
b(x,\tau,t,z) &= 0
\end{aligned}
\tag{3.1}
$$

which are due to [5,7]. Here b must map into $\mathbb{R}^p$. Because of Definition 1, we want b to consist of the k^2 scalar equations (2.6). Thus, we must require $p = k^2$ which is indeed the codimension of the manifold of (n,n)-matrices with rank deficiency k in $\mathbb{R}^{n,n}$. Note that this b also guarantees that $\beta_i = 0$ and $\gamma_j = 0$ at a solution. The next theorem states regularity for the augmented systems (3.1).

Theorem 3. A generalized turning point (x^*, τ^*) is part of a regular solution of (3.1) with b consisting of (2.6), i.e. it is part of a solution of (3.1) where the corresponding Jacobian has full row rank.

Proof. The only nontrivial part is to show regularity. But this holds due to a more general result in [7] which in this case says that regularity of the solution is equivalent to (2.7).

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For every further scalar condition we want to impose on a generalized turning point, we need a further parameter. Let us therefore assume that $b = (b_1, b_2)$ in (3.1) with b_1 consisting of (2.6) and b_2 mapping into $\mathbb{R}^{\overline{p}}$, i.e. we have $\overline{p}$ additional conditions. Then the following theorem holds.

Theorem 4. Let (x^*, τ^*, t^*, z^*) be part of a solution of (3.1) with $p = k^2 + \overline{p}$ and $b = (b_1, b_2)$ as above. In addition, let b_2 satisfy

$$
\begin{aligned}
\frac{\partial b_2}{\partial t_i}(x^*, \tau^*, t^*, z^*)(t_{i'}^*) = 0, \quad & \frac{\partial b_2}{\partial z_j}(x^*, \tau^*, t^*, z^*)(z_{j'}^*) = 0, \\
i, i' = 1, \ldots, k, \quad & j, j' = 1, \ldots, k,
\end{aligned}
\tag{3.2}
$$

i.e. b_2 is independent of the special computed bases t^* and z^*. Accordingly to b, let $\tau = (\tau_1, \tau_2)$ with $\tau_2 \in \mathbb{R}^{\overline{p}}$. Assume that in a neighborhood of τ_2^*, the above solution extends to

a smooth manifold of generalized turning points given by $(x(\tau_2), \tau_1(\tau_2), t(\tau_2), z(\tau_2))$. Then, this solution is part of a regular solution of (3.1) if and only if

$$\det\Big(\frac{db_2}{d\tau_2}(x(\tau_2), \tau_1(\tau_2), \tau_2, t(\tau_2), z(\tau_2))\Big|_{\tau_2=\tau_2^*}\Big) \neq 0, \tag{3.3}$$

i.e. if and only if b_2 has a simple root on the above defined manifold.

Proof. The above result is an immediate consequence of a generalization of Schur's complement to the elimination with respect to a matrix with full row rank instead of one which is regular. For validity of this generalization, assumption (3.2) is needed. Due to space limitations, we must refer to [7] for details.

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4 Applications

For the given general problem, the above result immediately yields a suitable augmented system and states the corresponding regularity condition for its solutions. We want to demonstrate this by applying it to the examples mentioned in the introduction.
A Bogdanov-Takens point is a simple turning point with a second zero eigenvalue. Generically, we then have a nontrivial Jordan block for the zero eigenvalue. This can be obtained by requiring $z^T t = 0$, i.e. we have $k = 1$, $\overline{p} = 1$, and

$$b_2(x, \tau, t, z) = z^T t. \tag{4.1}$$

Theorem 4 states that b_2 must have a simple root on the manifold of simple turning points. Note that compared with the augmented system of [8] we have the same regularity condition but the system given there has a totally different structure.
In [1], they found Bogdanov-Takens points with an additional rank drop of the Jacobian L. Thus, we have $k = 2$. To obtain a nontrivial Jordan block as before, we require the determinant of the (2,2)-matrix $z^T t$ to be zero, i.e.

$$b_2(x, \tau, t, z) = \det(z^T t) \tag{4.2}$$

which in this form can immediately be generalized to an arbitrary rank drop k.

5 Concluding Remarks

The above results can easily be extended to functions $f: \mathbb{R}^{n+l} \times \mathbb{R}^p \to \mathbb{R}^n$ with $l, p \geq 0$. Due to the techniques in [6,7], the augmented systems (3.1) can be solved by a superlinearly convergent modified Gauß-Newton method requiring in leading order one decomposition of $f_x(x, \tau)$ per iteration step which is the same as for the computation of regular solutions of (1.1).

6 References

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