0 Introduction

In the numerical treatment of parametrized systems of nonlinear equations, points, where the corresponding Jacobian is singular, may be of great interest. But because of this singularity, the computation of such singular points cannot be performed in a standard manner. Up to now, the common way was to develop a so-called augmented system of equations, so that singular points of special type are regular solutions of this system.

In this way, quadratically convergent methods were developed for the computation of turning points (Seydel [23]), simple bifurcation points (Moore [20], Weber [25], and Pönisch [21]), and for Hopf bifurcations (Jepson [15,9], Roose/Hlavaček [22]).

Recently, efforts were increased to find an appropriate concept for the numerical treatment of higher degenerate singular points by generalizing the known approaches (see eg. Fink/Rheinboldt [10,11], Jepson/Spence [16,24], Menzel [18], Allgower/Böhmer [1]).

The present paper wants to go a way similar to that in [18], but in contrast to there, the aim is to develop a class of methods, which are quadratically convergent. Starting from ideas in Deuflhard [4] and Deuflhard/Fiedler/Kunkel [6], one looks for a possibly underdetermined system of equations for the numerical determination of the Lyapunov-Schmidt reduction, which allows the use of the Gauß-Newton method.

In a first step, the analytical and numerical basis of the new concept, which shall be introduced herein, are provided.

1 Preparations

Consider a system of nonlinear equations of the form

$$f(y) = 0, \quad f: \mathbb{R}^{n+1} \to \mathbb{R}^n \tag{1.1}$$

with f sufficiently smooth. Often (1.1) is used with y splitted into n state variables x and one parameter τ. We assume without loss of generality that the origin is solution of (1.1), ie. $f(0) = 0$. Because of the pure local analysis for singular points and convergence properties of iteration methods, the objects to examine are rather germs than functions. In this paper, both notions will be used synonymously and shall always imply a local range of definition. See eg. [14] for a detailed outline of the theory of germs. Here, we only need the following notations. We call rank $f_y(0)$ the rank of f. If the rank of f is equal to n, then we call f regular, else singular. The set of germs is denoted by $\mathcal{F}$.

For an efficient determination of singular points, two aspects are essential. Firstly, one needs an augmented system, which has the desired point as a solution, secondly, one needs a quadratically convergent method for solving this augmented system. But this cannot be done independent of each other, as the following consideration shows.

The convergence of the method belonging to the corresponding augmented system depends on the regularity of the Jacobian in the solution point (cp. chapter 1.3). If one has a regular solution of the systems, it is stable against perturbations, ie. if one perturbs the given problem, the perturbed problem is still solvable and the solution depends

continuously on the perturbation. So necessarily, the desired augmented systems must include unfolded singular points as solutions. From the theory of perturbed singularities (see Golubitsky/Schaeffer [13]) we know, that for a large class of singular points, one can give the number of parameters describing all possible unfoldings. These parameters must be present somehow in the desired system to have regular solutions. In the approach presented below, the necessary unfolding parameters appear in an elementary way.

First, the necessary tools on the analysis of singular points and on quadratically convergent methods for solving nonlinear equations are summarized.

For that, let $f \in \mathcal{F}$ be a singular continuously differentiable germ with $f(0) = 0$ and

$$\operatorname{rank} L = n - k, \quad L = f_y(0), \quad k \geq 1. \tag{1.2}$$

In the following part, the Lyapunov-Schmidt reduction for f shall be briefly described to introduce the notation needed. The presentation follows that in Golubitsky/Schaeffer [14]. Proofs are therefore omitted, for a more detailed discussion see [17].

1.1 Lyapunov-Schmidt Reduction

Aim of the Lyapunov-Schmidt reduction is to reduce the given problem (1.1) in dimension by separating the regular part of the germ f from the singular part. For that, one defines a projector P on $\operatorname{kernel} L$ and a projector Q on $\operatorname{corange} L$. With these projectors, one decomposes $y \in U$ into

$$y = v + w, \quad v = Py, \quad w = (I - P)y \tag{1.3}$$

and the equation $f(y) = 0$ into

$$\begin{aligned} (I - Q)f(v + w) &= 0, \\ Qf(v + w) &= 0. \end{aligned} \tag{1.4}$$

The first part of the equations can be solved with respect to w by application of the implicit function theorem.

$$(I - Q)f(v + w) = 0 \quad \Longleftrightarrow \quad w = w^*(v) \tag{1.5}$$

Now, one can define a reduced function $\overline{f}$ of f on $\operatorname{kernel} L$ and a function y^*, which, given any v from $\operatorname{kernel} L$, provides a value on the whole pre-image space.

$$\begin{aligned} y^*(v) &= v + w^*(v) \\ \overline{f}(v) &= Qf(y^*(v)) = f(y^*(v)) \end{aligned} \tag{1.6}$$

The germ $\overline{f}$ is completely degenerate with respect to v in the sense, that the derivatives of $\overline{f}$ with respect to v vanish. Summarizing the above results, one has

Theorem 1.1 (Lyapunov-Schmidt reduction)

The statements

$$(1.7.a) \qquad Py = v \quad \wedge \quad f(y) = 0$$

and

$$(1.7.b) \qquad y = y^*(v) \quad \wedge \quad \overline{f}(v) = 0$$

are equivalent.

∎

By this theorem, there is a one-to-one correspondence between the solution of the original problem and the solutions of the reduced problem. For the analysis of the singular part of the given equations, one can restrict oneself to the reduced equations $\overline{f}(v) = 0$.

For a further investigation of the reduced equation, one introduces local coordinates. Following again [14], we define bases for kernel and corange of L by

$$(1.8) \qquad [t_1, \ldots, t_{k+1}] = \text{kernel}\, L, \qquad [z_1, \ldots, z_k] = \text{corange}\, L,$$

where the brackets denote the linear space spanned by the corresponding vectors. Each $v \in \text{kernel}\, L$ can be written as

$$(1.9) \qquad v = t\xi, \qquad t = (t_1, \ldots, t_{k+1}), \qquad \xi \in \mathbb{R}^{k+1}.$$

In the above defined coordinates, the reduced equation then has the form

$$(1.10) \qquad g\colon \mathbb{R}^{k+1} \to \mathbb{R}^k, \qquad g(\xi) = z^T \overline{f}(t\xi), \qquad z = (z_1, \ldots, z_k).$$

To analyse the singularity of g in the origin, one needs the derivatives of g with respect to ξ. One observes that the reduction of f to g is given only by information on the first derivative L. Consequently, higher derivatives of the reduced function $\overline{f}$ may not correspond to derivatives of f.

1.2 Derivatives of the Reduced Equation

Differentiating the vector valued function $\overline{f}$ with respect to the vector valued quantity ξ, one gets tensors of successively higher order, which are symmetric in the arising arguments. To keep track of this symmetry, all arguments are written out explicitly. As before, we proceed as in [14].

Starting from

$$(1.11) \qquad \overline{f}(t\xi) = Qf(y^*(t\xi)), \qquad y^*(t\xi) = t\xi + w^*(t\xi),$$

one gets up to the third derivative

$$
\begin{aligned}
\overline{f}_{\xi}(t\xi)(a_1) &= Qf_y(y^*(t\xi))(\overline{a}_1(\xi)),\\
\overline{f}_{\xi\xi}(t\xi)(a_1,a_2) &= Q[f_{yy}(y^*(t\xi))(\overline{a}_1(\xi),\overline{a}_2(\xi))+\\
&\qquad + f_y(y^*(t\xi))(w^*_{\xi\xi}(t\xi)(a_1,a_2))],\\
\overline{f}_{\xi\xi\xi}(t\xi)(a_1,a_2,a_3) &= Q[f_{yyy}(y^*(t\xi))(\overline{a}_1(\xi),\overline{a}_2(\xi),\overline{a}_3(\xi))+\\
&\qquad + f_{yy}(y^*(t\xi))(\overline{a}_1(\xi),w^*_{\xi\xi}(t\xi)(a_2,a_3))+\\
&\qquad + f_{yy}(y^*(t\xi))(\overline{a}_2(\xi),w^*_{\xi\xi}(t\xi)(a_3,a_1))+\\
&\qquad + f_{yy}(y^*(t\xi))(\overline{a}_3(\xi),w^*_{\xi\xi}(t\xi)(a_1,a_2))+\\
&\qquad + f_y(y^*(t\xi))(w^*_{\xi\xi\xi}(t\xi)(a_1,a_2,a_3))],
\end{aligned}
\tag{1.12}
$$

where $\overline{a}_i(\xi) = ta_i + w^*_{\xi}(t\xi)(a_i)$ was used as an abbreviation. Evaluating at the origin and observing $L = f_y(0)$ and $QL = 0$, the last term of each derivative vanishes. Next, we have to compute the derivatives of w^* with respect to ξ from the implicit definition (1.5) of w^*. Differentiating once, one gets the relation

$$(I-Q)L(ta_1 + w^*_{\xi}(0)(a_1)) = 0 \tag{1.13}$$

for the origin. Since $Lt = 0$ and $w^*_{\xi}(0)(a_1) \in \operatorname{cokernel} L$ and since L restricted to $\operatorname{cokernel} L$ is invertible, it follows that

$$w^*_{\xi}(0)(a_1) = 0. \tag{1.14}$$

Differentiating once more leads to

$$(I-Q)f_{yy}(0)(ta_1,ta_2) + (I-Q)Lw^*_{\xi\xi}(0)(a_1,a_2) = 0. \tag{1.15}$$

As above, this implies

$$w^*_{\xi\xi}(0)(a_1,a_2) = -L^{-1}(I-Q)f_{yy}(0)(ta_1,ta_2). \tag{1.16}$$

Summarizing, one has

Lemma 1.2

Let f be sufficiently smooth. Then for the derivatives of the reduced function g, the following relations are valid.

$$
\begin{aligned}
g_{\xi}(0)(a_1) &= 0,\\
g_{\xi\xi}(0)(a_1,a_2) &= z^T f_{yy}(0)(ta_1,ta_2),\\
g_{\xi\xi\xi}(0)(a_1,a_2,a_3) &= z^T[f_{yyy}(0)(ta_1,ta_2,ta_3)-\\
&\qquad - f_{yy}(0)(ta_1,L^{-1}(I-Q)f_{yy}(0)(ta_2,ta_3))-\\
&\qquad - f_{yy}(0)(ta_2,L^{-1}(I-Q)f_{yy}(0)(ta_3,ta_1))-\\
&\qquad - f_{yy}(0)(ta_3,L^{-1}(I-Q)f_{yy}(0)(ta_1,ta_2))]
\end{aligned}
\tag{1.17}
$$

∎

If we take the Moore-Penrose pseudoinverse L^+ of L as an inverse on the range of L and $Q = I - LL^+$, we especially have

$$L^{-1}(I - Q) = L^+. \tag{1.18}$$

Thus, we have reduced the analysis of the $\mathbb{R}^n$-valued function f to the analysis of the $\mathbb{R}^k$-valued function g.

1.3 Convergence Properties of the Gauß-Newton Method

Before we go to concrete methods for the numerical realization of the Lyapunov-Schmidt reduction, we have to discuss the convergence behaviour of the Gauß-Newton method, which shall be used below. One considers a system of nonlinear equations

$$F(u) = 0, \qquad F: U \to \mathbb{R}^{\overline{m}}, \qquad U \subseteq \mathbb{R}^m. \tag{1.19}$$

Let the function F be continuously differentiable. Starting from an estimation $u_0 \in U$, the Gauß-Newton method then evaluates the iterates

$$\begin{aligned} u_{l+1} &= u_l + \Delta u_l \\ \Delta u_l &= -J^+(u_l)F(u_l), \end{aligned} \tag{1.20}$$

where $J(u)$ denotes the Jacobian of F at u and the superscript $+$ the Moore-Penrose pseudoinverse. By Deuflhard/Heindl [7], we have the following theorem for the convergence of this iteration method.

Theorem 1.3

Let $F: U \to \mathbb{R}^{\overline{m}}$ be continuously differentiable, $U \subseteq \mathbb{R}^m$ convex, and

$$\begin{aligned} &(a) && \|\Delta u_0\| \le \alpha, \quad u_0 \in U \\ &(b) && \|J^+(u)(J(v) - J(w))\| \le \omega \|v - w\|, \quad u, v, w \in U \\ &(c) && \|J^+(u)(I - J(v)J^+(v))F(v)\| \le \kappa(v)\|u - v\|, \quad u, v \in U \\ &(d) && \kappa(u) \le \overline{\kappa} < 1, \quad u \in U \\ &(e) && h = \frac{1}{2}\alpha\omega < 1 - \overline{\kappa} \\ &(f) && \overline{S}(u_0, \varrho) \subset U, \quad \varrho = \frac{\alpha}{1 - h - \overline{\kappa}} \end{aligned} \tag{1.21}$$

Then, the sequence $\{u_l\}$ of iterates defined by the Gauß-Newton method (1.20), satisfies

$$\begin{aligned} &(a) && \{u_l\} \subseteq \overline{S}(u_0, \varrho) \\ &(b) && \text{The } u_l \text{ converge to a } u^* \in \overline{S}(u_0, \varrho) \text{ with} \\ & && \qquad J^+(u^*)F(u^*) = 0. \end{aligned} \tag{1.22}$$

(c) The convergence rate is given by

$$\|u_{l+1} - u_l\| \leq \frac{\omega}{2}\|u_l - u_{l-1}\|^2 + \kappa(u_{l-1})\|u_l - u_{l-1}\|.$$

∎

The local convergence rate is therefore only linear in general. But if one has $\kappa(u^*) = 0$ for the solution u^*, the method converges superlinearly. In the case $\overline{\kappa} = 0$, the second term of the estimation (1.22.c) disappears identically and one has local and quadratic convergence. This condition, for example, is satisfied for the Newton method with regular Jacobian, if it is regarded as a special case of the Gauß-Newton method. In [4], it is emphasized, that $\overline{\kappa} = 0$ is satisfied for $m > \overline{m}$, when the corresponding Jacobian has full rank in the whole of U (cp. [6]).

Corollary 1.4

In the case $m > \overline{m}$ of an underdetermined system (1.19), the Gauß-Newton method converges locally and quadratically to a solution of (1.19), if $J(u)$ has full rank in the whole of U.

Proof

Because of the full rank of $J(u)$, we have

$$J(u)J^+(u) = I \quad \forall\, u \in U \tag{1.23}$$

and therefore $\overline{\kappa} = 0$. Further, it follows

$$F(u^*) = J(u^*)J^+(u^*)F(u^*) = 0$$

from (1.22.b) by (1.23) for the limit u^*.

∎

We have now shown, that we need not restrict ourselves to quadratic systems, ie. to systems consisting of as many equations as unknowns, searching for a quadratically convergent method, but we may allow them to be underdetermined. The only essential property is that relation (1.23) is satisfied in a neighbourhood of the desired solution. Because of the provided continuity of the first derivative, it suffices to guarantee (1.23) at the solution. However, one has to observe in the case of underdetermined systems that the solution is not isolated in general. It remains to analyse, what the set of solutions, denoted by $\mathcal{M}$ in the following, looks like and how it is connected with the given problem.

With these preparations, an underdetermined system for the evaluation of the Lyapunov-Schmidt reduction shall be introduced in the next chapter.

2 Numerical Determination of the Lyapunov-Schmidt Reduction

2.1 An Underdetermined Augmented System

To determine the Lyapunov-Schmidt reduction for a given $L = f_y(0)$, which satisfies the rank condition (1.2), Menzel proposed in [18] the following system of equations.

$$
\begin{aligned}
Lt_i &= 0 \\
t_i^T t_{i'} &= \delta_{i,i'} \qquad\qquad i,i' = 1,\ldots,k+1, \quad i \geq i' \\
L^T z_j &= 0 \\
z_j^T z_{j'} &= \delta_{j,j'} \qquad\qquad j,j' = 1,\ldots,k, \quad j \geq j'
\end{aligned}
\tag{2.1}
$$

This consists of two independent systems, where in one part the quantities t_i and in the other part the quantities z_j are determined. In general, each system is overdetermined. Since one obtains new solutions by orthogonal transformation of a given solution, the set of solutions is highly degenerate. Indeed, Menzel showed, that the corresponding Jacobians have the right rank, but because of the overdetermination one has superlinear convergence at best using the Gauß-Newton method. If the singularity is slightly unfolded, ie. if the rank condition (1.2) is slightly perturbed, which may arise for example by discretizing differential equations, the system (2.1) has no solution at all. One can only solve it in the sense of least squares and the Gauß-Newton method only converges linearly (if there are no perturbations of the Jacobian). If one looks closer at the Jacobian, one notices, that the equations $Lt_i = 0$ and $L^T z_j = 0$ lead directly to the rank deficiency of the Jacobian, because there the rank condition (1.2) is not taken into account.

Therefore, it is proposed to couple both systems by adding certain unfolding parameters. With

$$
\begin{aligned}
Lt_i - \sum_{j=1}^{k} \beta_{i,j} z_j &= 0 \\
L^T z_j - \sum_{i=1}^{k+1} \gamma_{j,i} t_i &= 0 \\
t_i^T t_{i'} &= \delta_{i,i'} \qquad\qquad i,i' = 1,\ldots,k+1, \quad i \geq i' \\
z_j^T z_{j'} &= \delta_{j,j'} \qquad\qquad j,j' = 1,\ldots,k, \quad j \geq j',
\end{aligned}
\tag{2.2}
$$

one gets an underdetermined system of equations, which seems to be more appropriate for the following reasons. It shall be shown that the Jacobian belonging to (2.2) satisfies the condition (1.23) of full rank for each solution. Continuity of the first derivative then yields (1.23) for a whole neighbourhood of the solution set $\mathcal{M}$ and for problems slightly unfolded relative to the quantities $\beta_{i,j}$ and $\gamma_{j,i}$. That is, the Gauß-Newton method applied to (2.2) converges quadratically even for slightly perturbed problems.

Theorem 2.1

Let $f \in \mathcal{F}$ and $\operatorname{rank} L = n-k$, $L = f_y(0)$, $k \geq 1$. Then the augmented system (2.2) of $(2k+1)(n+k+1) - k^2$ equations in $(2k+1)(n+k+1)$ unknowns is nondegenerate at a perfect solution

$$\beta_{i,j} = \gamma_{j,i} = 0 \quad \forall\, i,j, \tag{2.3}$$

ie. the Jacobian of the system has full rank there.

Proof

For a solution of (2.2) with (2.3), one gets

$$\operatorname{rank} J = \operatorname{rank} \begin{pmatrix} D_t & & -I_z & \\ & D_z & & -I_t \\ M_t & & & \\ & M_z & & \end{pmatrix},$$

where

$$\begin{aligned} D_t &= \operatorname{diag}(L,\ldots,L), & I_t &= \operatorname{diag}(t,\ldots,t), \\ D_z &= \operatorname{diag}(L^T,\ldots,L^T), & I_z &= \operatorname{diag}(z,\ldots,z) \end{aligned}$$

with an appropriate ordering of the equations and unknowns. In a first step, one transforms to a singular value decomposition of L, which is given by

$$\begin{aligned} L = U\Sigma V^T, \quad \Sigma &= \begin{pmatrix} \Sigma' & 0 \\ 0 & 0 \end{pmatrix}, \quad (n, n+1)\text{-matrix}, \\ \Sigma' &= \operatorname{diag}(\sigma_1,\ldots,\sigma_{n-k}), \quad (n-k, n-k)\text{-matrix}, \\ &\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n-k} > 0, \\ U &\quad \text{orthogonal } (n,n)\text{-matrix}, \\ V &\quad \text{orthogonal } (n+1, n+1)\text{-matrix}. \end{aligned}$$

U and V can be chosen, such that

$$\overline{t}_i = V^T t_i = e^{(n+1)}_{n-k+i}, \quad \overline{z}_j = U^T z_j = e^{(n)}_{n-k+j}, \tag{2.4}$$

where $e^{(n)}_j$ denotes the j-th unit vector of length n. This yields

$$\operatorname{rank} J = \operatorname{rank} \begin{pmatrix} \operatorname{diag}(\Sigma,\ldots,\Sigma) & -\overline{I}_z \\ \overline{M}_t & \end{pmatrix} + \operatorname{rank} \begin{pmatrix} \operatorname{diag}(\Sigma^T,\ldots,\Sigma^T) & -\overline{I}_t \\ \overline{M}_z & \end{pmatrix}$$

with

$$\overline{I}_t = (\overline{t}_1,\ldots,\overline{t}_{k+1}), \quad \overline{M}_t = \begin{pmatrix} 2\overline{t}_1^T & & & \\ \overline{t}_2^T & \overline{t}_1^T & & \\ \vdots & & \ddots & \\ \overline{t}_{k+1}^T & & & \overline{t}_1^T \\ & 2\overline{t}_2^T & & \\ & \vdots & \ddots & \\ & \overline{t}_{k+1}^T & & \overline{t}_2^T \\ & & \cdots & \cdots \\ & & & 2\overline{t}_{k+1}^T \end{pmatrix}, \quad \overline{I}_z, \overline{M}_z \text{ similarly.}$$

In the rows and columns, where Σ has a nonzero entry, all other entries vanish, because the $\overline{t}_i$ and $\overline{z}_j$ only have zero components there. The same is valid for the entries of $\overline{I}_t$ and $\overline{I}_z$. Since the remaining entries form a echelon matrix, the proof is finished. ∎

It should be remarked, that theorem 2.1 is valid for any matrix L. The proof for matrices with more columns than rows, as they appear for example in multi-parameter embeddings, is analogous. For the reverse case, observe the symmetry of (2.2) with respect to

$$L \to L^T, \quad t \to z, \quad z \to t, \quad \beta \to \gamma, \quad \gamma \to \beta.$$

Further, we always have

$$\beta_{i,j} = \gamma_{j,i} = z_j^T L t_i \quad \forall\, i,j, \tag{2.5}$$

at a solution.

Having shown quadratic convergence of systems (2.2), we now investigate the solution set $\mathcal{M}$. Note that we have an underdetermined system, where k^2 equations are missing for an isolated solution. But to each solution with (2.3), one can construct other solutions by orthogonal transformations in the spaces

$$\text{kernel } L = [t_1, \ldots, t_{k+1}], \quad \text{corange } L = [z_1, \ldots, z_k]. \tag{2.6}$$

If one counts the possible degrees of freedom in choosing a basis, one gets k degrees of freedom in kernel L for the first vector because of the normalization condition, $k-1$ degrees of freedom for the second one because of the orthonormalization condition, and so on. Summarizing, we then have $\frac{1}{2}k(k+1)$ degrees of freedom for choosing a basis in kernel L, and $\frac{1}{2}k(k-1)$ degrees of freedom for choosing a basis in corange L, respectively. Altogether, there are k^2 degrees of freedom, which exactly corresponds to the number of missing equations to have a regular system. This shows that the system (2.2) exactly reflects the underdetermination of the given problem.

Like any other nondegenerate underdetermined system of equations, one can extend (2.2) to a regular system consisting of as many unknowns as equations by adding artificial equations. A possible choice is

$$\begin{aligned} \hat{t}_i^T t_{i'} &= 0, \qquad i < i', \\ \hat{z}_j^T z_{j'} &= 0, \qquad j < j', \end{aligned} \tag{2.7}$$

where $\hat{t}_i$ and $\hat{z}_j$ are approximations to the corresponding solutions, for example the initial estimates.

2.2 Preliminary Remarks on Augmented Systems for Singularities

For the evaluation of singular points, the system for the determination of the Lyapunov-Schmidt reduction becomes dependent on y. Therefore, the number of unknowns is increased by $n+1$. We have to add the n equations $f(y) = 0$, which must be

extended by k unfolding parameters for a full rank of the Jacobian. Thus, there are $k+1$ equations b_l missing to be able to compute all quantities observing the given degeneration of the solution set (from now on, the singular point y^* of f is assumed to be isolated, ie. outside y^*, the Jacobian of the germ f shall have full rank).

The additional equations will depend on the type of the singular point to be computed and must guarantee full rank of the corresponding Jacobian without destroying the natural degeneration of the solution. In general, such a system will have the form

$$
\begin{aligned}
f_y(y)t_i - \sum_{j=1}^{k} \beta_{i,j} z_j &= 0 \\
f_y^T(y)z_j - \sum_{i=1}^{k+1} \gamma_{j,i} t_i &= 0 \\
t_i^T t_{i'} &= \delta_{i,i'} \qquad i,i' = 1,\ldots,k+1, \qquad i \geq i' \\
z_j^T z_{j'} &= \delta_{j,j'} \qquad j,j' = 1,\ldots,k, \qquad j \geq j' \\
f(y) - \sum_{j=1}^{k} \alpha_j z_j &= 0 \\
b_l(y,t,z) &= 0 \qquad l = 1,\ldots,k+1,
\end{aligned}
\tag{2.8}
$$

where the nonindexed quantities shall denote the collection of all indexed quantities. Keeping the solution, the variation of the additional equations b_l has the form

$$
db_l = \sum_{i=1}^{k+1} \frac{\partial b_l}{\partial t_i}(y,t,z)(dt_i) + \sum_{j=1}^{k} \frac{\partial b_l}{\partial z_j}(y,t,z)(dz_j).
$$

Because the additional equations shall not distinguish certain solutions of the solution set $\mathcal{M}$, one has to demand

$$
\begin{aligned}
\frac{\partial b_l}{\partial t_i}(y,t,z)(t_{i'}) &= 0, \qquad i,i' = 1,\ldots,k+1 \\
\frac{\partial b_l}{\partial z_j}(y,t,z)(z_{j'}) &= 0, \qquad j,j' = 1,\ldots,k
\end{aligned}
\tag{2.9}
$$

for all solutions in $\mathcal{M}$, at least in the perfect case. The next theorem will give the conditions on the b_l, under which the Jacobian J of the augmented sytem (2.8) has full rank. For the sake of simplicity, the argument (y,t,z) will be omitted.

Theorem 2.2

Let y^* be a perfect solution of the augmented system (2.8), ie.

$$
\alpha = 0, \quad \beta = 0, \quad \gamma = 0. \tag{2.10}
$$

Further, let $L = f_y(y^*)$ and $\operatorname{rank} L = n - k$, $k \geq 1$. The Jacobian J of (2.8) with (2.9) has full rank in y^*, if and only if the $(k+1, k+1)$-matrix satisfies

$$B = \left(\frac{\partial b_l}{\partial y}(t) - \sum_{i=1}^{k+1} \frac{\partial b_l}{\partial t_i}(L^+ f_{yy}(y^*)(t_i, t)) - \right. \\ \left. - \sum_{j=1}^{k} z_j^T f_{yy}(y^*)(L^+(\frac{\partial b_l}{\partial z_j})^T, t) \right) \quad \text{is nonsingular.} \tag{2.11}$$

Proof

Without loss of generality, $y^* = 0$ is a solution of (2.8). Starting from

$$\operatorname{rank} J = \operatorname{rank} \begin{pmatrix} T & D_t & & & -I_z & \\ Z & & D_z & & & -I_t \\ & M_t & & & & \\ & & M_z & & & \\ L & & & -z & & \\ \frac{\partial b}{\partial y} & \frac{\partial b}{\partial t} & \frac{\partial b}{\partial z} & & & \end{pmatrix}$$

with the block columns

$$T = \operatorname{col}\left(f_{yy}(0)(t_1), \ldots, f_{yy}(0)(t_{k+1})\right)$$
$$Z = \operatorname{col}\left(z_1^T f_{yy}(0), \ldots, z_k^T f_{yy}(0)\right)$$

and the notations of theorem 2.1, we again transform to a singular value decomposition of L. After transformation, there is exactly one nonzero entry in the columns corresponding to the unfolding parameters. Additionally, as shown in theorem 2.1, the matrices $\overline{M}_t$ and $\overline{M}_z$ have full rank and their nonzero entries are the only ones in the corresponding columns due to the separation condition (2.9). Further, the entry Σ' of the second but last block row is the only nonzero one in the corresponding rows. By these, full rank of J is equivalent to regularity of the quadratic matrix

$$\begin{pmatrix} \overline{z}^T f_{yy}(0)(t_1, t) & \Sigma' & & & & & \\ \vdots & & \ddots & & & & \\ \overline{z}^T f_{yy}(0)(t_{k+1}, t) & & & \Sigma' & & & \\ z_1^T f_{yy}(0)(\overline{t}, t) & & & & \Sigma' & & \\ \vdots & & & & & \ddots & \\ z_k^T f_{yy}(0)(\overline{t}, t) & & & & & & \Sigma' \\ \frac{\partial b}{\partial y}(t) & \frac{\partial b}{\partial t_1}(\overline{t}) & \cdots & \frac{\partial b}{\partial t_{k+1}}(\overline{t}) & \frac{\partial b}{\partial z_1}(\overline{z}) & \cdots & \frac{\partial b}{\partial z_k}(\overline{z}) \end{pmatrix}$$

with $\overline{t} = V\overline{P}$ and $\overline{z} = U\overline{Q}$ where $\overline{P}$ and $\overline{Q}$ are orthogonal projectors chosen such that

$$\Sigma' = \overline{Q}^T \Sigma \overline{P}.$$

Since the entries Σ' are assumed to be nonsingular, we can use them as pivots of a block Gauß decomposition and eliminate the last block row with the exception of the first entry. Together with

$$L^+ = \bar{t}\, {\Sigma'}^{-1} \bar{z}^T,$$

full rank of J is then equivalent to the regularity of B which completes the proof.

∎

One may use the proceeding of this proof to compute the Moore-Penrose pseudoinverse of J explicitly. This suggests a Gauß-Newton like algorithm which only needs a representation of L^+ and B^{-1} reducing the costs considerably. Details and comparisons will be given elsewhere.

How the additional equations have to be chosen, will depend on the type of the singular point y^* and will somehow be expressed by higher derivatives of the germ. Dividing the singular germs into classes relative to the so-called contact equivalence, as it shall be treated in chapter 3, possible choices of equations b_l will result from an invariance and covariance analysis.

3 Singularities

Looking for appropriate additional conditions b_l, one would like to use similar equations for as large classes of singular points as possible. Following Golubitsky/Schaeffer [13], we define

Definition 3.1

Two germs $f, \tilde{f} \in \mathcal{F}$ are called contact equivalent, if and only if in a neighbourhood U of the origin, there is a family T(y) of invertible (n, n)-matrices continuously parametrized by y and a diffeomorphism ϱ on U with $\varrho(0) = 0$, such that

$$\tilde{f}(y) = T(y)\, f(\varrho(y)) \quad \forall\, y \in U. \tag{3.1}$$

The corresponding equivalence classes are called singularities.

In contrast to the common treatment eg. in [13], there is no distinguished embedding parameter here. It follows, that now more germs are equivalent than otherwise. So, for example, a turning point here is a regular point, a pitchfork here is a simple bifurcation point, and an asymmetric cusp and a winged cusp are here equivalent (cf. Golubitsky/Schaeffer [14], but also Jepson/Spence [16]).

The procedure of the Lyapunov-Schmidt reduction is compatible with contact equivalence, which means, that starting from two contact equivalent germs, we get contact equivalent reduced germs. For a more detailed discussion see eg. [17].

Using this compatibility, we now search for suitable additional equations $b_l = 0$ to make the Gauß-Newton method for (2.8) converge locally and quadratically to a singularity of given type. For that, especially such quantities are relevant, which are invariant or covariant under contact equivalence. The latter means, that the quantity is indeed not conserved under transformation (3.1), but the structure of it is conserved.

Now let g be a germ obtained by Lyapunov-Schmidt reduction and let h be contact equivalent to g , ie.

$$h(\xi) = T(\xi)g(\varrho(\xi)). \tag{3.2}$$

Then together with (1.17), we have by differentiating and evaluating at the origin

$$\begin{aligned} h(0) &= 0, \\ h_\xi(0) &= 0, \\ h_{\xi\xi}(0)(a_1, a_2) &= T(0)g_{\xi\xi}(0)(\varrho_\xi(0)(a_1), \varrho_\xi(0)(a_2)). \end{aligned} \tag{3.3}$$

Starting from this, we want to develop suitable equations for several types of singularities, which shall be given by model equations. For methods classifying singularities, we refer to [3,12,14].

3.1 Simple Bifurcation

A simple bifurcation is given by the model equation

$$g(\xi_1, \xi_2) = \xi_1^2 - \xi_2^2, \qquad k = 1. \tag{3.4}$$

Because of $k = 1$, $T(0)$ is a scalar and the tensor of the second derivatives is equivalent to a matrix. The third relation of (3.3) then says, that the rank of this matrix is invariant under contact equivalence.

$$\operatorname{rank} h_{\xi\xi}(0) = \operatorname{rank} g_{\xi\xi}(0) \tag{3.5}$$

For the model equation (3.4) of a simple bifurcation,

$$\operatorname{rank} g_{\xi\xi}(0) = \operatorname{rank} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = 2 \tag{3.6}$$

is valid. So, we have already found the desired nonsingular $(2,2)$-matrix. The corresponding additional equations are

$$g_\xi(\xi) = 0 \tag{3.7}$$

or, rewritten for the nonreduced function f,

$$b(y, t, z) = z_1^T f_y(y)(t) = 0. \tag{3.8}$$

Since we have

$$f_y(0)(t_1) = 0, \quad f_y(0)(t_2) = 0, \quad z_1^T f_y(0) = 0 \tag{3.9}$$

in the case of a perfect solution by the reduction part of (2.8), the separation condition (2.9) is trivially satisfied. For (2.11), we further have

$$B = \frac{\partial b}{\partial y}(t) = z_1^T f_{yy}(0)(t,t), \tag{3.10}$$

and B, exactly corresponding to $g_{\xi\xi}(0)$ by the relations (1.27) for the derivatives, is nonsingular for all simple bifurcations. This gives the desired augmented system consisting of (2.8) for $k = 1$ and (3.8). In this case, the system obtained can be simplified as follows. From the equations (3.8), one immediately computes $\beta = 0$ and $\gamma = 0$. This separates the evaluation of t_1 and t_2 from the actual computation of the singularity. We thus obtain the reduced system

$$\begin{aligned} f_y^T(y) z_1 &= 0 \\ z_1^T z_1 &= 1 \\ f(y) - \alpha_1 z_1 &= 0, \end{aligned} \tag{3.11}$$

which is exactly the augmented system of Moore [20] specialized to the treatment of nonlinear equations. In this sense, Moore's system for the computation of simple bifurcation points may be seen as most suitable.

3.2 Triple Point Bifurcation

As an example for a singularity with $k = 1$, which needs additional equations with second derivatives, we discuss the triple point bifurcation, given by the model equation

$$g(\xi_1, \xi_2) = \xi_1^3 - \xi_1 \xi_2^2. \tag{3.12}$$

Because of (3.3) and

$$g_{\xi\xi}(0) = 0, \tag{3.13}$$

we have in this class, that the matrix of second derivatives vanishes. If we differentiate (3.2) once more and evaluate at the origin using (1.17) and (3.13), we get

$$h_{\xi\xi\xi}(0)(a_1, a_2, a_3) = T(0) g_{\xi\xi\xi}(0)(\varrho_\xi(0)(a_1), \varrho_\xi(0)(a_2), \varrho_\xi(0)(a_3)). \tag{3.14}$$

Setting

$$a_3 = \varrho_\xi^{-1}(0)(w), \quad w^T = (w_1, w_2), \quad w \in \mathbb{R}^2 \setminus \{0\} \tag{3.15}$$

to obtain a (2,2)-matrix from the (2,2,2)-tensor $h_{\xi\xi\xi}(0)$, the relation (3.14) says, that the expression $g_{\xi\xi\xi}(0)(w)$ transforms covariantly under contact equivalence. That is in this case, that for each given w, there is a $\overline{w} = \varrho_\xi^{-1}(0)(w)$, such that

$$\operatorname{rank} h_{\xi\xi\xi}(0)(\overline{w}) = \operatorname{rank} g_{\xi\xi\xi}(0)(w). \tag{3.16}$$

In the special case of the triple point bifurcation treated here, the model equation (3.12) satisfies

$$g_{\xi\xi\xi}(0)(w) = \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix} w_1 + \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} w_2 = \begin{pmatrix} 6w_1 & -2w_2 \\ -2w_2 & -2w_1 \end{pmatrix}. \tag{3.17}$$

Analysis of the determinant gives

$$\operatorname{rank} g_{\xi\xi\xi}(0)(w) = 2 \qquad \forall\, w \neq 0. \tag{3.18}$$

Since $\varrho_\xi(0)$ is nonsingular, this property is valid for each function h being contact equivalent to g. Thus, we have found

$$g_{\xi\xi}(\xi)(w) = 0, \qquad w \neq 0 \quad \text{arbitrary} \tag{3.19}$$

as possible additional equations or, rewritten for the nonreduced function f,

$$b(y,t,z) = z_1^T f_{yy}(y)(t,tw) = 0. \tag{3.20}$$

Now, we have to verify the separation condition (2.9), for which we get

$$\begin{aligned} \frac{\partial b}{\partial t}(0,t,z)(t) &= 2z_1^T f_{yy}(0)(t,tw) = 2g_{\xi\xi}(0)(w) = 0, \\ z^T \frac{\partial b}{\partial z}(0,t,z) &= z_1^T f_{yy}(0)(t,tw) = g_{\xi\xi}(0)(w) = 0. \end{aligned} \tag{3.21}$$

For the matrix B, we compute

$$\begin{aligned} B = z_1^T f_{yyy}(0)(t,tw,t) &- z_1^T f_{yy}(0)(L^+ f_{yy}(0)(t,t),tw) - \\ &- z_1^T f_{yy}(0)(t, L^+ f_{yy}(0)(tw,t)) - \\ &- z_1^T f_{yy}(0)(L^+ f_{yy}(0)(t,tw),t) = g_{\xi\xi\xi}(0)(w), \end{aligned} \tag{3.22}$$

which is nonsingular by (3.18). At this point, one sees the correspondence between requirements of the augmented system to have full rank and the invariances given by the theory, which here carry over directly. Quite essential for this is the correspondence of the condition (2.11) with the derivatives (1.17) of the reduced function, which appears here nontrivially for the first time.

It must be remarked, that the additional equations neither demand, that the first derivatives of the reduced function vanish, nor, that the part of the second derivatives orthogonal to w vanishes. The former is already considered in the part of the augmented system, which determines the Lyapunov-Schmidt reduction. For the latter, there are two more unfolding parameters belonging to the above singularity. But, they are not relevant for the computation of the singularity. After having computed y, they are explicitly given by

$$\begin{aligned} \alpha_2 &= z_1^T f_{yy}(0)(t_1, t\overline{w}), \\ \alpha_3 &= z_1^T f_{yy}(0)(t_2, t\overline{w}), \end{aligned} \tag{3.23}$$

where $\overline{w} \neq 0$ is orthogonal to w. Because of the symmetry of the second derivatives, there remains a dependence between these two unfolding parameters and the equations (3.20). For triple point bifurcations, we thus end up with the augmented system consisting of (2.8) for $k = 1$ together with (3.20). The following example, where $w^T = (1, 0)$ is chosen, shall demonstrate the efficiency of the related method.

Example 3.2

Consider the problem

$$\begin{aligned} x_1^2 - x_2 + 1 &= 0 \\ (x_1 - 1)x_2 - 2(x_2 - x_1 - 1)\cosh\tau &= 0 \end{aligned}$$

with initial values

$$\begin{aligned} x^T &= (0.8, 2.2), \quad \tau = -0.3 \\ t_1^T &= \frac{1}{10}(4, 8, 0), \quad t_2^T = (0, 0, 1) \\ z_1^T &= \frac{1}{10}(8, -4). \end{aligned}$$

All unfolding parameters are set to 0. Table 1 gives the results of the derived augmented system consisting of 17 equations in 18 unknowns, if one adds the equations (3.23) for simplicity. The table includes the following quantities. The entries `LEVELX` and `LEVELXQ` give the values of the natural monotonicity test due to Deuflhard [5], and the entries `RANK`, `COND`, and `SENS` give the numerical pseudorank, subcondition, and sensitivity for the Jacobian due to Deuflhard/Sautter [8]. The entry `NEW` is not relevant here, because no rank-1 updates are performed in order to show a possible quadratic convergence. For all tests, the required accuracy is `EPS=1.D-5`.

TABLE 1

ITER	LEVELX	LEVELXQ	NEW	RANK	COND	SENS
1	0.2038D 00	0.2367D-01	0	17	0.24D 02	0.51D 01
2	0.3289D-01	0.2638D-03	0	17	0.17D 02	0.50D 01
3	0.2597D-03	0.5713D-08	0	17	0.19D 02	0.51D 01
4	0.5722D-08	0.5197D-17	0	17	0.20D 02	0.51D 01

One can observe the rapid convergence of the Gauß-Newton method to compute the solution $y^T = (1, 2, 0)$ and the very convenient subcondition. The size of the unfolding parameters (absolute values less than 10^{-14}) allows the conjecture, that the computed singularity is perfect (which is right in this case).

3.3 Simple Bifurcation from a Double Eigenvalue

In this section, we want to consider a singularity that occurs generically in problems with D_3-symmetrie. Here, we have $k = 2$. Due to [13], a model equation is given by

$$g(\xi_1, \xi_2, \xi_3) = \begin{pmatrix} \xi_1^2 - \xi_2^2 - \xi_1\xi_3 \\ -2\xi_1\xi_2 - \xi_2\xi_3 \end{pmatrix}. \tag{3.24}$$

We want to have a nonsingular $(3,3)$-matrix in terms of derivatives of g, which transforms covariantly. Because of the special shape of the model equation, only second derivatives come into question. To get a (3,3)-matrix from the (2,3,3)-tensor of the second derivatives, we have to multiply the corresponding argument of the tensor with a (2)-vector. Thus, we have to analyse the matrix

$$w^T g_{\xi\xi}(0) = w_1 \begin{pmatrix} 2 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & 0 \end{pmatrix} + w_2 \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{3.25}$$

where $w \in \mathbb{R}^2$ is normalized to $w^T w = 1$. For the determinant, we get

$$\det\left(w^T g_{\xi\xi}(0)\right) = 2w_1^3 - 6w_1 w_2^2. \tag{3.26}$$

Hence, the matrix (3.25) is singular, iff

$$w \in \left\{ \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \pm\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \pm\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \right\}. \tag{3.27}$$

As for the above treated singularity, the expression $w^T g_{\xi\xi}(0)$ transforms covariantly under contact equivalence, but this time by the nonsingular matrix $T(0)$. In the class of singular points given by g, there are, up to a sign, precisely three pairwise linearly independent normalized vectors w, such that the matrix $w^T h_{\xi\xi}(0)$ is singular. For all other w, this matrix is regular. Let w be chosen, such that the corresponding matrix is nonsingular. Then, we have

$$w^T g_\xi(\xi) = 0 \tag{3.28}$$

as additional equations or, rewritten for the nonreduced function f,

$$b(y,t,z) = w^T z^T f_y(y)(t) = 0. \tag{3.29}$$

As in section 3.1 for simple bifurcations, the separation condition is trivially satisfied because of conditions only on first derivatives. Accordingly, the matrix B is nonsingular.

Example 3.3

In [2], Beyn describes a problem of seven reaction cells in a linear array, of which three are additionally connected with an outer reservoir.

$$\begin{aligned}
(1+p_1)x_1 + \frac{3}{10}(x_1 - x_2) + p_3 x_4 - R(4 - x_1) &= 0 \\
\frac{3}{10}(2x_2 - x_1 - x_3) - R(4 - x_2) &= 0 \\
\frac{3}{10}(2x_3 - x_2 - x_4) - R(4 - x_3) &= 0 \\
(2+p_2)x_4 + \frac{3}{10}(2x_4 - x_3 - x_5) + p_4 x_7 - 2R(4 - x_4) &= 0 \\
\frac{3}{10}(2x_5 - x_4 - x_6) - R(4 - x_5) &= 0 \\
\frac{3}{10}(2x_6 - x_5 - x_7) - R(4 - x_6) &= 0 \\
x_7 + \frac{3}{10}(x_7 - x_6) + p_5 x_1 - R(4 - x_7) &= 0
\end{aligned} \tag{3.30}$$

The reaction rate R models an inhibited Michaelis-Menten process.

$$R(c) = \frac{10^{1.4}c}{1 + c + \tau c^2}$$

Let us here omit the parameters p. Treating then this problem with the continuation method of [6], the stepsize control breaks down near the point

$$x^T = (3.6, 4, 4, 3.6, 4, 4, 3.6), \quad \tau = 9.3,$$

because the determinant tends to zero more than linearly. By QR-decomposition with column pivoting, one finds, that a degeneration with $k = 2$ is probable. Stopping the QR-decomposition according to this condition, one can get initial values for t and z. Because of

$$f_y(y) = Q \begin{pmatrix} R & S \\ 0 & 0 \end{pmatrix} \Pi$$

with a nonsingular upper $(n-k, n-k)$-matrix R, one obtains

$$z_j = Q e^{(n)}_{n-k+j}, \quad j = 1, \ldots, k$$

and

$$(\tilde{t}_1 \quad \cdots \quad \tilde{t}_{k+1}) = \Pi^T \begin{pmatrix} -R^{-1}S \\ I_{k+1} \end{pmatrix},$$

where the $\tilde{t}_i$ only have to be orthonormalized.

The initial values chosen here are rounded off roughly, since the procedure would else converge too rapidly and the interesting properties would not be visible.

$$t_1^T = \frac{1}{10}(-4, 0, 0, 8, 0, 0, -4, 0) \qquad z_1^T = \frac{1}{10}(6, 0, 0, -6, 0, 0, 6)$$
$$t_2^T = \frac{1}{10}(-7, 0, 0, 0, 0, 0, 7, 0) \qquad z_2^T = \frac{1}{10}(-7, 0, 0, 0, 0, 0, 7)$$
$$t_3^T = \frac{1}{10}(-6, 0, 0, -6, 0, 0, -6, 1)$$

For the additional equations, $w^T = (1, 0)$ is taken. The Gauß-Newton method applied to the augmented system (2.8) with (3.29) consisting of 56 equations in 60 unknowns then yields the following solution.

$$x_1 = x_4 = x_7 = 3.6116 \qquad \tau = 9.2930$$
$$x_2 = x_3 = x_5 = x_6 = 3.9954$$

The convergence behaviour is listed in Table 2.

TABLE 2

ITER	LEVELX	LEVELXQ	NEW	RANK	COND	SENS
1	0.1767D-01	0.4384D-03	0	56	0.14D+03	0.36D+02
2	0.4775D-03	0.1707D-06	0	56	0.14D+03	0.41D+02
3	0.1743D-06	0.7172D-13	0	56	0.14D+03	0.40D+02

Again, the good convergence behaviour of the method confirms the derived results. The computed singularity is unfolded relative to $\beta_{22} \doteq -4 \cdot 10^{-5}$. In agreement with [2], one can show that this corresponds to an unfolding into three simple bifurcation points.

We end this chapter with two remarks. Firstly, we emphasize that one can choose a similar approach for parametrized systems of nonlinear equations. This causes a somewhat more complicated structure in the derivatives of the reduced function because additional cases have to be treated. For turning points, for example, we would get the augmented system

$$\begin{aligned} f_x^T(x,\tau)z_1 &= 0 \\ z_1^T z_1 &= 1 \\ f(x,\tau) &= 0 \end{aligned} \tag{3.31}$$

after a reduction similar to that for simple bifurcations. The difference to the most common systems (cp. the overview of Mittelmann/Weber in [19], and especially Seydel [23]) lies in the fact, that here a kernel vector of f_x^T and not of f_x is to be computed. In addition, the system for simple bifurcations can be extended in the case of parametrized equations to an augmented system for pitchforks. Here, we only have to add another unfolding parameter of the form

$$\alpha_2 = z_1^T f_{xx}(x,\tau)(t_1,t_1) \tag{3.32}$$

describing, how much the tangent differs from the vertical. Secondly, we note that the presented methods allow to vary the additional parameters in multi-parameter embeddings in such a way, that special unfolding parameters vanish. This can be done by just adding equations of the form

$$\varepsilon = 0, \tag{3.33}$$

where ε denotes an appropriate collection of unfolding parameters α, β, and γ. If there are sufficiently many parameters, we can determine parameter values, such that the corresponding singularity is perfect. For example, one needs one additional parameter for a simple bifurcation, two parameters for a pitchfork, and five parameters for a simple bifurcation from a double eigenvalue (cp. the method of Jepson/Spence [16]).

Example 3.4

Following [2], we now let the parameters p of example 3.3 vary. Choosing $p = 0$ as initial values and $\varepsilon = (\alpha_1, \alpha_2, \beta_{12}, \beta_{22}, \beta_{32})$, the iteration methods behaves as indicated in table 3.

TABLE 3

ITER	LEVELX	LEVELXQ	NEW	RANK	COND	SENS
1	0.1756D-01	0.8168D-03	0	61	0.29D+03	0.36D+02
2	0.1286D-02	0.1532D-05	0	61	0.11D+04	0.41D+02
3	0.1546D-05	0.1712D-11	0	61	0.11D+04	0.40D+02

Comparing tables 2 and 3, we now have a slightly worse condition number, which says, that the chosen parameters p are not optimal compared with the set ε of unfolding parameters. But still the quadratic convergence of the Gauß-Newton method is evident.

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Mit freundlichen Gruessen des Autors

Quadratically Convergent Methods for the Computation of Unfolded Singularities*

P. Kunkel

Abstract

An underdetermined system of equations for the determination of the Lyapunov-Schmidt reduction is presented. This system can easily be extended to an augmented system for the the computation of nontrivial singularities by adding suitable equations, which characterize the singularity. Necessary and sufficient conditions for these additional equations are given, so that the corresponding Gauß-Newton method converges locally and quadratically. For several types of singularities possible equations are derived from invariance conditions. Numerical examples illustrating the results are included.

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