High-Temperature Series for random bonds and random clusters

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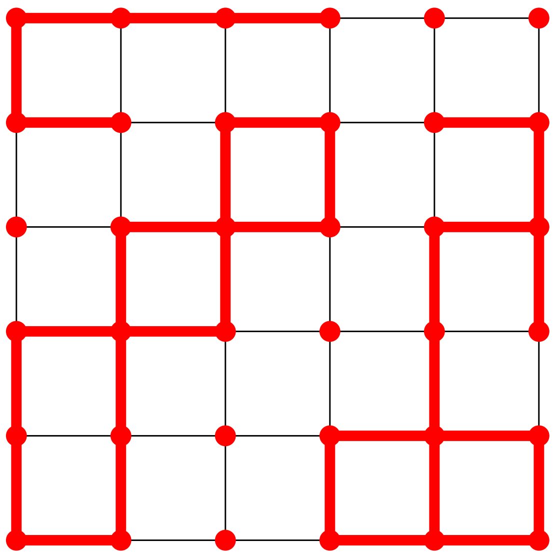
- Random bond Ising model in 3, 4 and 5 dimensions
  Phys. Rev. B74, 144201(2006) = cond-mat/0606320
- q-state Potts model and bond percolation on $\mathbb{Z}^d$
  Phys. Rev. E74, 051113(2006) = cond-mat/0607423

Random cluster model

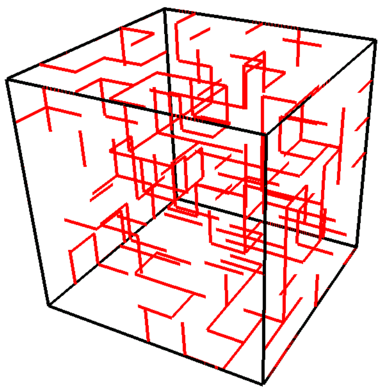


- graph $G = (V, E)$, cluster $C = (V, e)$ where $e \subseteq E$
- $k(C)$ connected components (incl. single vertices)
- state space = set of $2^{|E|}$ clusters
- $Z(q, v) = \sum_C q^{k(C)} v^{|e|}$
  - $q = 2, 3, \ldots$: Potts model
  - $q = 1$: bond percolation
  - $q, v \to 0$, $\frac{v}{q} = const.$: tree percolation

Ising model on $\mathbb{Z}^D$ with quenched disorder

$$\mathcal{Z}[\{J_{ij}\}] = \text{Tr} \exp\left(-\beta \sum_{<ij>} J_{ij}\delta(S_i, S_j)\right)$$

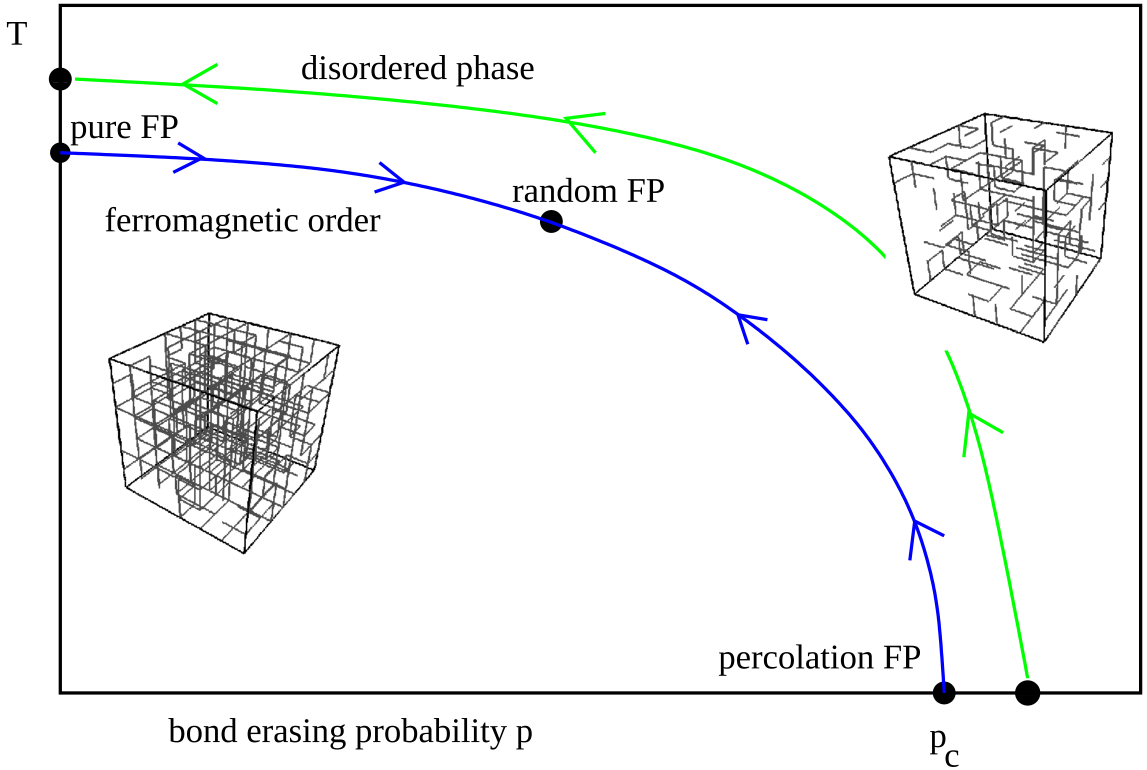
$$-\beta F = [\log \mathcal{Z}]_{P(J)}$$



- random couplings: J_{ij} choosen according to some probability distribution $P(J)$
  - spin glasses $J = +1/-1$
  - random bond models $J = +1/+5$
  - correlated disorder
- geometric disorder: site dilution, bond dilution:

$$P(J_{ij}) = (1-p)\delta(J_{ij} - J_0) + p\delta(J_{ij})$$

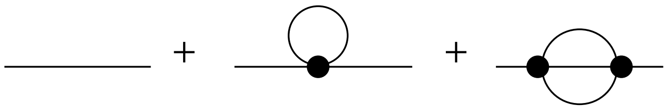
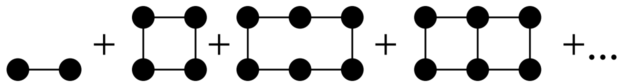
Effect of disorder on critical behaviour



Harris criterion:

- 3D: relevant ⟶ new fixed point
- 4D: marginal ⟶ log corrections
- 5D: irrelevant ⟶ only non-universal quantities (T_c) change

High-temperature series expansion

	weak coupling expansion	high-T expansion
starting point:	decoupled d.o.f. in p-space	decoupled d.o.f. in x-space
	Feynman graphs 	lattice graphs 

- Series coefficients are exact
- Quenched disorder average can be treated exactly
- Easily extendable to different dimensions D
- Large parameter spaces $(D, p, \dots)$ can be scanned

But: Expansion around one FP shall give information about another FP $\Rightarrow$ We need

- sophisticated extrapolation/resummation methods (Padé, differential approximants, etc.) to extract critical behaviour
- input from renormalization group
- many orders

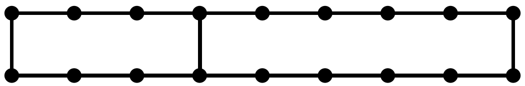
Series generation techniques - Star graph expansion

Disordered Ising/Potts models: $\log Z$ and $1/\chi$ have star-graph expansions, i.e. expansions including only biconnected graphs (*no* articulation points)

- Construct all star graphs embeddable in $\mathbb{Z}^D$ up to a given order (number of edges E):

order E	8	9	10	11	12	13	14	15	16	17	18	19	20	21
#graphs	2	3	8	9	29	51	142	330	951	2561	7688	23078	55302	165730

- Count the (weak) embedding numbers $E(G; \mathbb{Z}^D)$:



$$8\binom{D}{2} + 275\,184\binom{D}{3} + 18\,763\,392\binom{D}{4} + 208\,611\,840\binom{D}{5} + 645\,442\,560\binom{D}{6} + 559\,964\,160\binom{D}{7}$$

- Calculate Z and correlations $G_{ij} = \langle \delta_{s_i,s_j} \rangle$ for every graph with symbolic couplings $v_1, ..., v_E$ where $v = \tanh(\beta J)$ is our expansion variable (using a cluster representation)
- Calculate disorder averages $[\log Z]$, $C_{ij} = [G_{ij}/Z]$ up to $O(v^N)$
- Inversion of correlation matrix and subgraph subtraction
  $W_\chi(G) = \sum_{i,j} (C^{-1})_{ij} - \sum_{g \subset G} W_\chi(g)$
- Collect the results from all graphs
  $1/\chi = \sum_G E(G; \mathbb{Z}^d)\, W_\chi(G)$

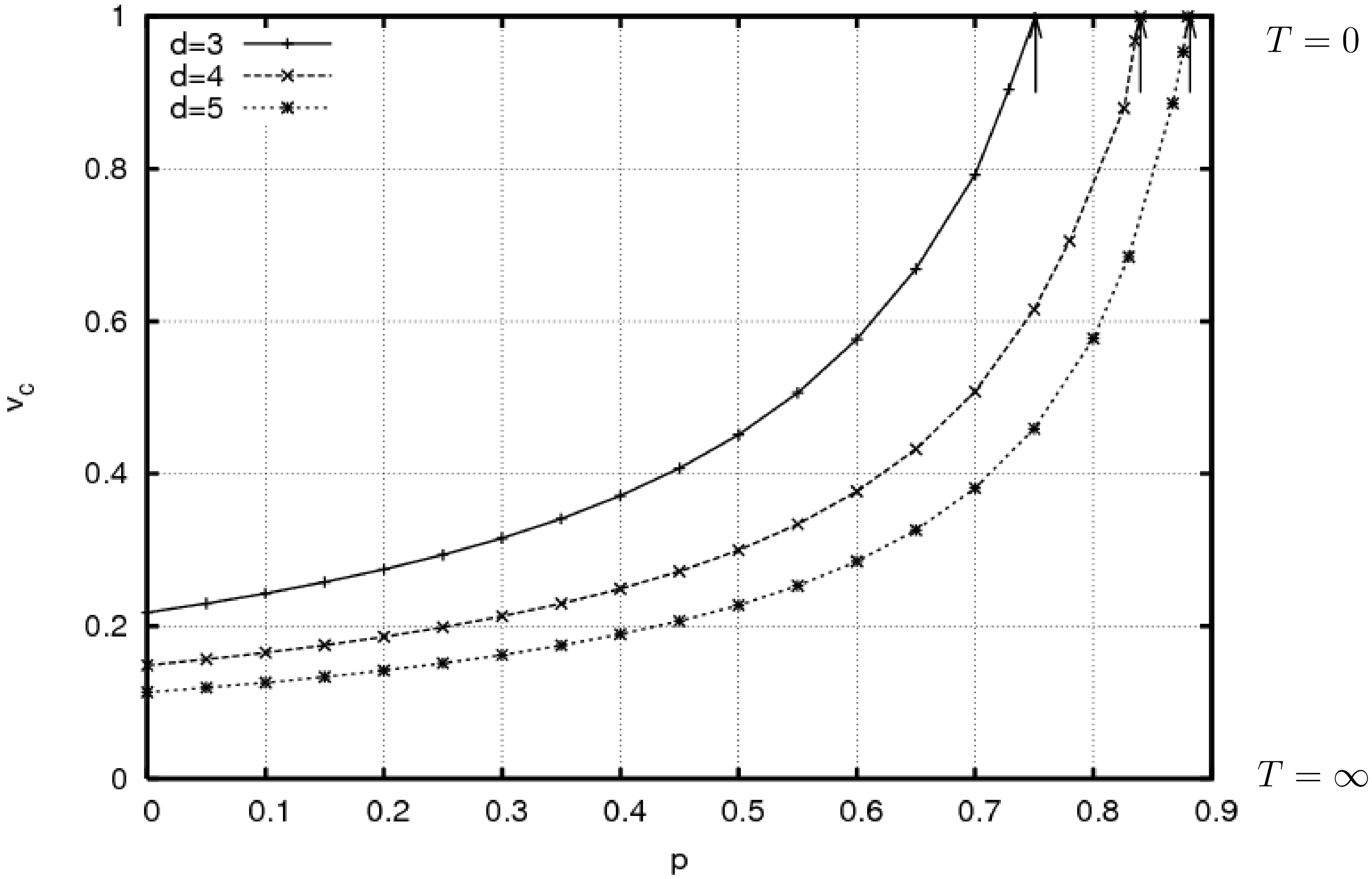
Result: susceptibility of the bond-diluted 3D Ising model

two CPU years later ...

$$\chi(p,v) = 1 + 6pv + 30p^2v^2 + 150p^3v^3 + 726p^4v^4 - 24p^4v^5 + 3534p^5v^5 - 24p^4v^6 - 192p^5v^6 + 16926p^6v^6 - 24p^4v^7 - 192p^5v^7 - 1608p^6v^7 + 81318p^7v^7 - 192p^5v^8 - 1608p^6v^8 - 10464p^7v^8 + 387438p^8v^8 + 24p^4v^9 - 1608p^6v^9 - 10536p^7v^9 - 67320p^8v^9 + 1849126p^9v^9 + 24p^4v^{10} + 192p^5v^{10} - 264p^6v^{10} - 9744p^7v^{10} - 67632p^8v^{10} - 395328p^9v^{10} + 8779614p^{10}v^{10} + 24p^4v^{11} + 192p^5v^{11} + 1080p^6v^{11} - 240p^7v^{11} - 60912p^8v^{11} - 397704p^9v^{11} - 2299560p^{10}v^{11} + 41732406p^{11}v^{11} + 192p^5v^{12} + 1344p^6v^{12} + 8400p^7v^{12} - 1440p^8v^{12} - 339936p^9v^{12} - 2295744p^{10}v^{12} - 12766944p^{11}v^{12} + 197659950p^{12}v^{12} - 24p^4v^{13} + 1608p^6v^{13} + 10296p^7v^{13} + 51480p^8v^{13} + 26544p^9v^{13} - 1886928p^{10}v^{13} - 12680496p^{11}v^{13} - 70404720p^{12}v^{13} + 936945798p^{13}v^{13} - 24p^4v^{14} - 192p^5v^{14} + 264p^6v^{14} + 10032p^7v^{14} + 64560p^8v^{14} + 341568p^9v^{14} + 259656p^{10}v^{14} - 9915696p^{11}v^{14} - 69162048p^{12}v^{14} - 377522064p^{13}v^{14} + 4429708830p^{14}v^{14} - 24p^4v^{15} - 192p^5v^{15} - 1080p^6v^{15} - 1704p^7v^{15} + 60024p^8v^{15} + 427920p^9v^{15} + 2062368p^{10}v^{15} + 2482464p^{11}v^{15} - 51644200p^{12}v^{15} - 367148472p^{13}v^{15} - 2014331904p^{14}v^{15} + 20955627110p^{15}v^{15} - 192p^5v^{16} - 1080p^6v^{16} - 12144p^7v^{16} - 14448p^8v^{16} + 360192p^9v^{16} + 2493600p^{10}v^{16} + 12550416p^{11}v^{16} + 17926128p^{12}v^{16} - 259622976p^{13}v^{16} - 1931961792p^{14}v^{16} - 10550435184p^{15}v^{16} + 98937385374p^{16}v^{16} + 24p^4v^{17} - 1080p^6v^{17} - 13440p^7v^{17} - 80928p^8v^{17} - 132024p^9v^{17} + 1840776p^{10}v^{17} + 14790144p^{11}v^{17} + 73051512p^{12}v^{17} + 126567264p^{13}v^{17} - 1293631728p^{14}v^{17} - 9980137536p^{15}v^{17} - 55050628008p^{16}v^{17} + 467333743110p^{17}v^{17} + 24p^4v^{18} + 192p^5v^{18} - 8544p^7v^{18} - 95040p^8v^{18} - 569760p^9v^{18} - 1214880p^{10}v^{18} + 9797904p^{11}v^{18} + 82573800p^{12}v^{18} + 420942768p^{13}v^{18} + 807789264p^{14}v^{18} - 6273975792p^{15}v^{18} - 51221501136p^{16}v^{18} - 283516855968p^{17}v^{18} + 2204001965006p^{18}v^{18} + 24p^4v^{19} + 192p^5v^{19} + 1080p^6v^{19} + 5832p^7v^{19} - 60888p^8v^{19} - 705216p^9v^{19} - 3910368p^{10}v^{19} - 8858136p^{11}v^{19} + 46862760p^{12}v^{19} + 457439184p^{13}v^{19} + 2361075624p^{14}v^{19} + 5069434800p^{15}v^{19} - 30136593768p^{16}v^{19} - 259361429784p^{17}v^{19} - 1455780298776p^{18}v^{19} + 10398318680694p^{19}v^{19} + 192p^5v^{20} + 1080p^6v^{20} + 18912p^7v^{20} + 36720p^8v^{20} - 437952p^9v^{20} - 4512600p^{10}v^{20} - 25012512p^{11}v^{20} - 63580104p^{12}v^{20} + 220823568p^{13}v^{20} + 2449336680p^{14}v^{20} + 13097561328p^{15}v^{20} + 30177202248p^{16}v^{20} - 141380350848p^{17}v^{20} - 1306684851840p^{18}v^{20} - 7403140259952p^{19}v^{20} + 48996301350750p^{20}v^{20} - 24p^4v^{21} + 1080p^6v^{21} + 24744p^7v^{21} + 129816p^8v^{21} + 283752p^9v^{21} - 2272584p^{10}v^{21} - 28419456p^{11}v^{21} - 158605280p^{12}v^{21} - 422656608p^{13}v^{21} + 936811968p^{14}v^{21} + 12971851368p^{15}v^{21} + 71258617752p^{16}v^{21} + 177314558064p^{17}v^{21} - 655481735280p^{18}v^{21} - 6514909866600p^{19}v^{21} - 37556614417032p^{20}v^{21} + 230940534213046p^{21}v^{21} + O(v^{22})$$

- 21th order in $3D$
- 19th order in $\geq 4D$ – This even extends known series for Ising model without disorder.

Results: Phase diagram



Critical coupling v_c as function of p for $D = 3, 4, 5$.
The arrows indicate the percolation thresholds in $D = 3, 4$ and 5, resp.

Random-bond Ising model in 5 Dimensions

- Critical point is a Gaussian fixed point: γ=1
- Disorder irrelevant
- Corrections to scaling:

$$\chi \sim (t - t_c)^{-\gamma}(A_0 + A_1(t - t_c)^{\Delta_1} + \ldots$$

Results from 19th order high-temperature series for susceptibility χ:

▶ w/o disorder:

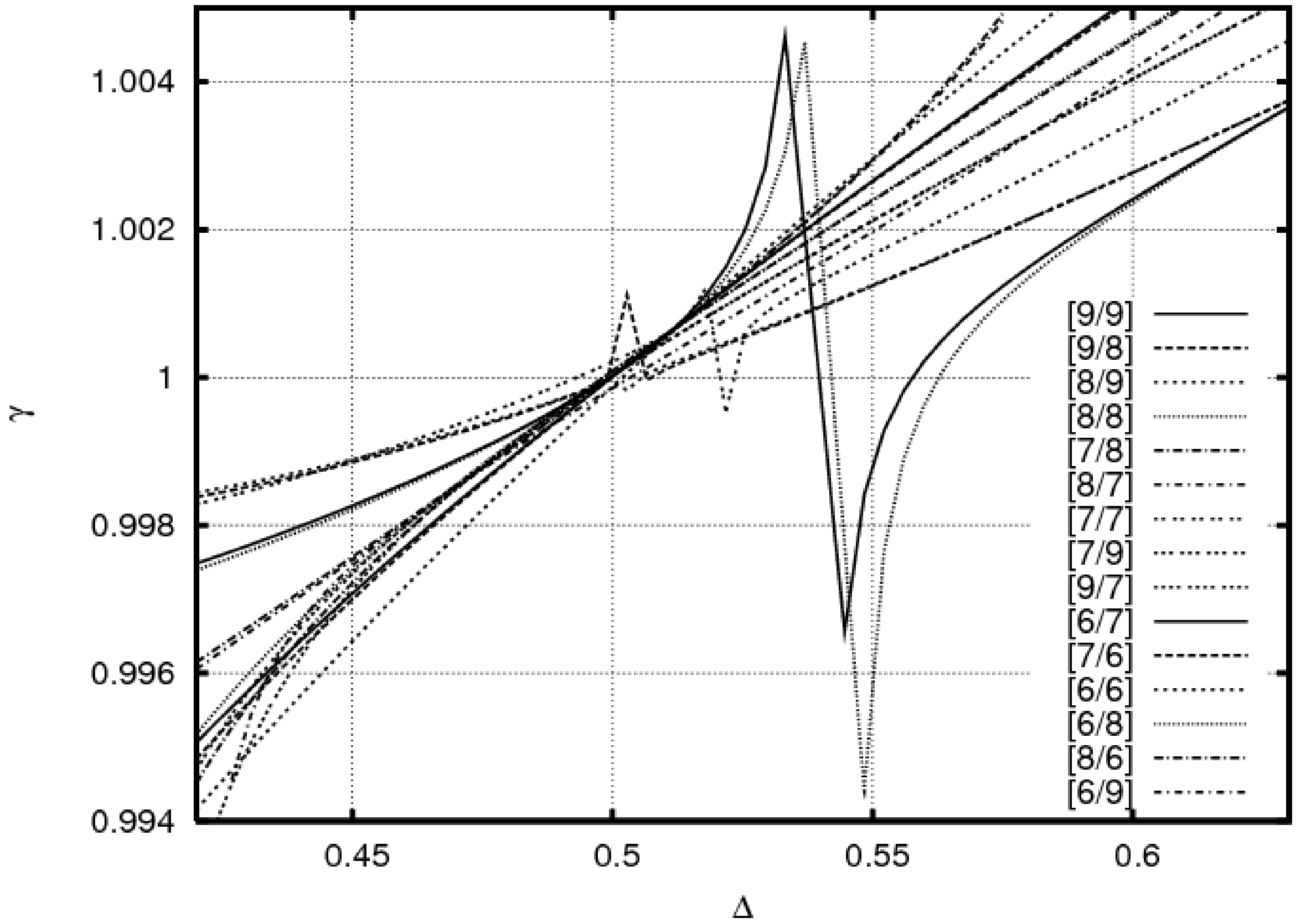
$$\begin{aligned} \gamma &= 1 \\ v_c &= .113425(3) \\ \Delta_1 &= 0.50(2) \end{aligned}$$

(compare MC data (*Binder et al.* 99): $v_c = 0.1134250(4)$)

▶ with disorder:

$$\gamma = 1$$
$$\Delta_1 = 0.50(5)$$

for large range of dilutions $p = 0 \ldots 0.7$ ($p_c = 0.8818$)



M2 analysis (γ as function of Δ_1) for the $p = 0.5$ diluted 5D Ising model at $v_c = .227498$

Random-bond Ising model in 4 Dimensions

Aharony 1976:

$$\chi \sim t^{-1} |\log t|^{\frac{1}{3}} \quad \xrightarrow{\text{disorder}} \quad \chi \sim t^{-1} \exp\left(\zeta \sqrt{|\log t|}\right), \quad \zeta = \sqrt{\frac{6}{53}} \approx 0.3364$$

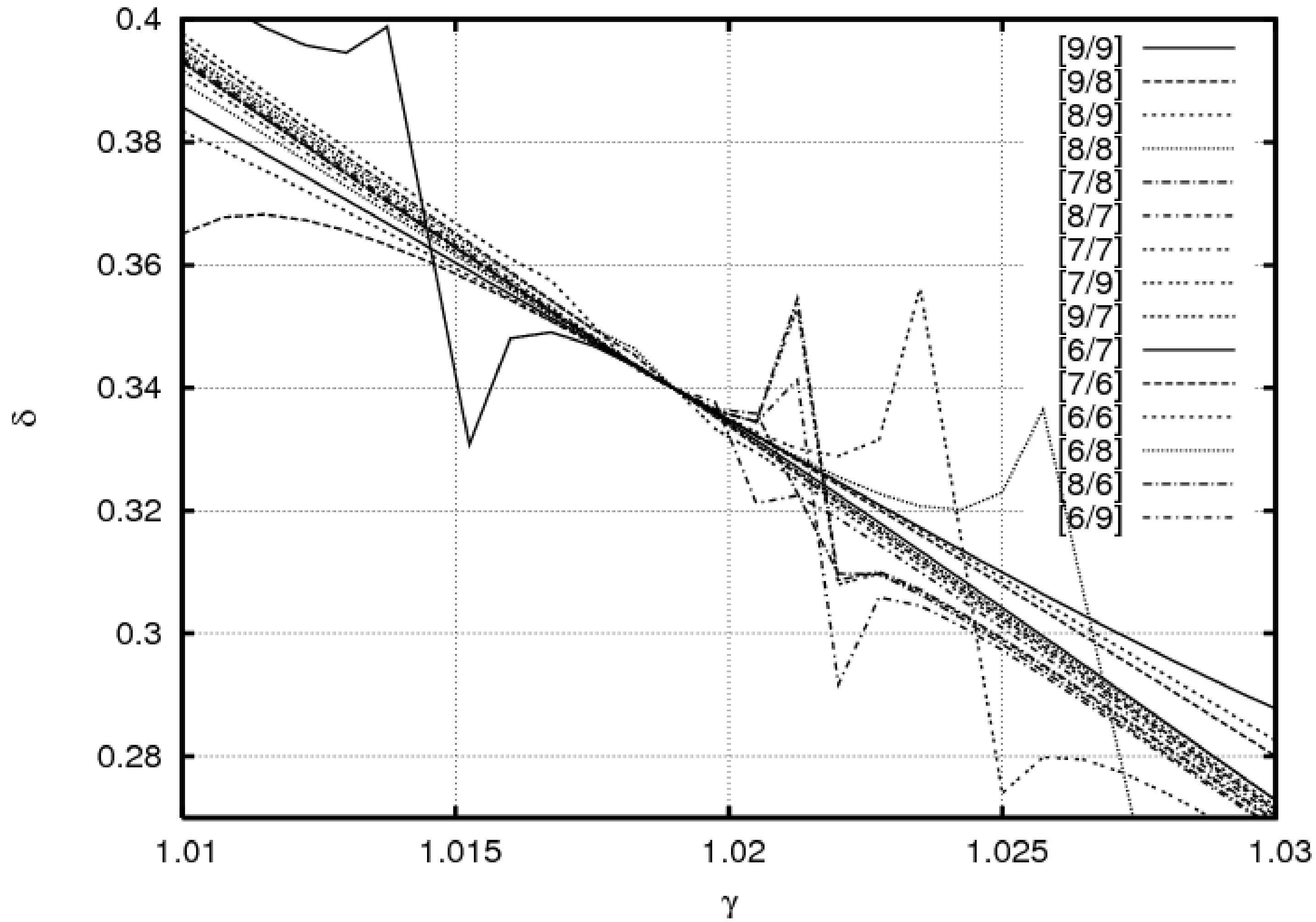
- pure case: We determine log correction (difficult to determine in MC simulations):

$$\begin{aligned} \delta &= 0.34(1) \\ v_c &= 0.148583(3) \end{aligned}$$

MC simulations *Bittner* et al.: $\quad v_c = \; 0.148589(2).$

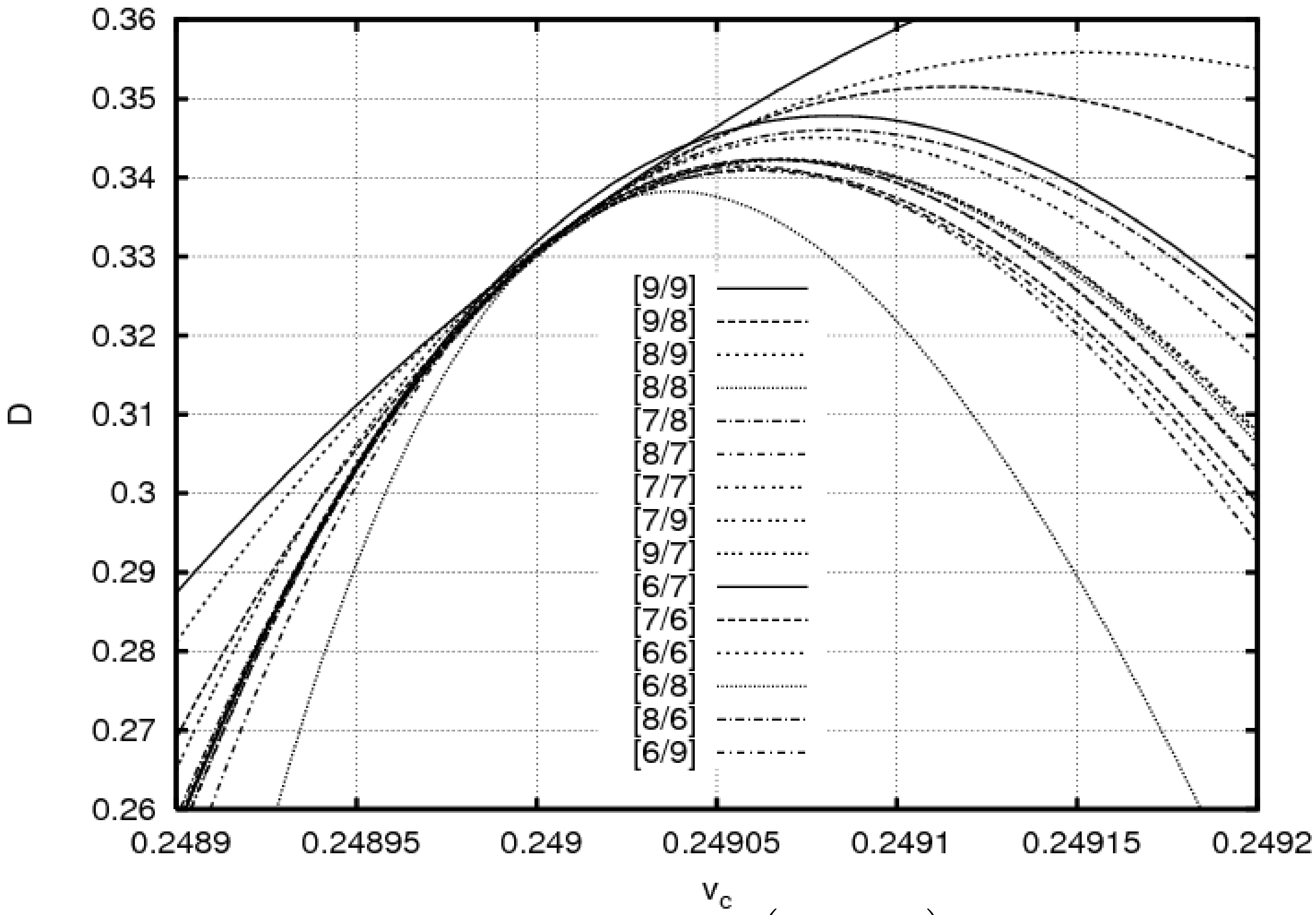
- disordered case:

successful fit with $t^{-1} \exp\left(\zeta \sqrt{|\log t|}\right)$ with p-dependent $\zeta = 0.2 \ldots 0.6$



$$\chi \sim t^{-\gamma} |\log t|^{\delta}$$

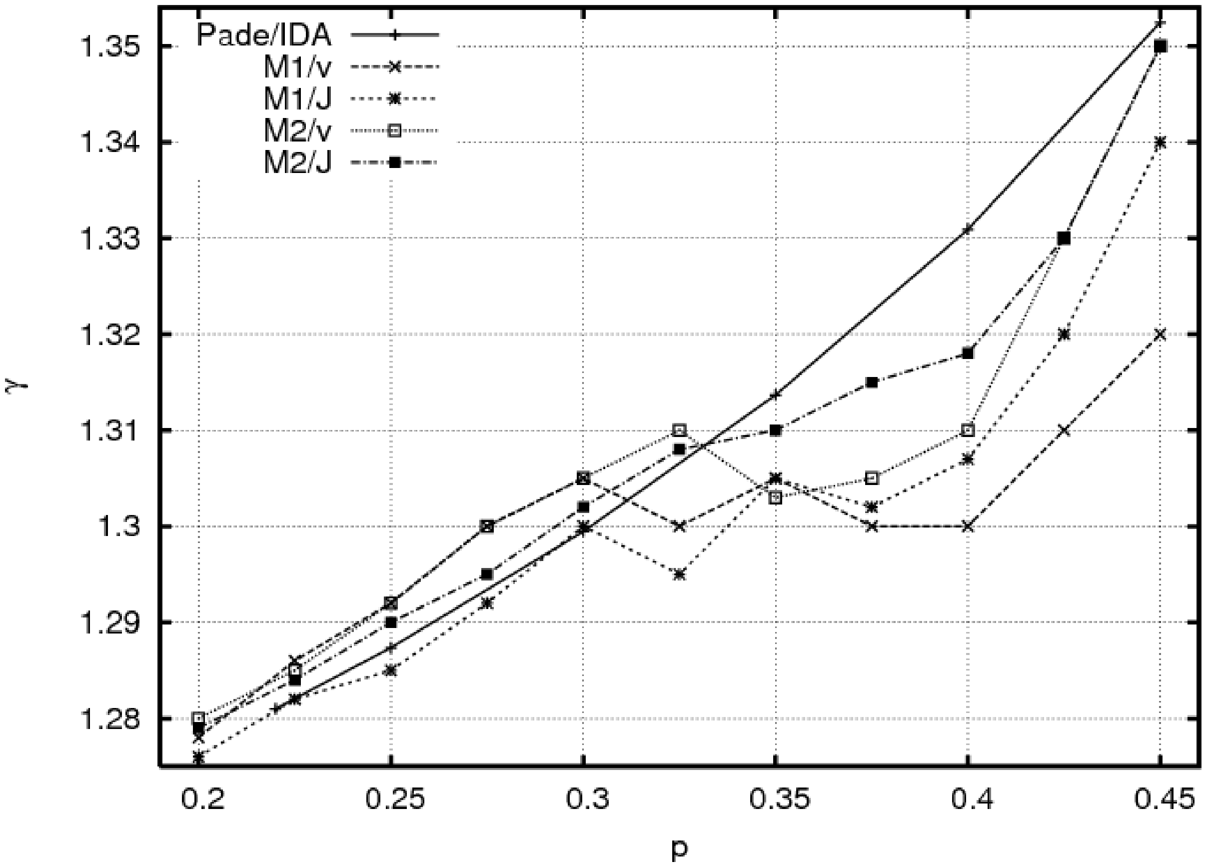
δ as function of γ for the pure 4D Ising model via unbiased M3 approximants at $v_c = 0.148607$



$$\chi \sim t^{-1} \exp\left(\zeta \sqrt{|\log t|}\right)$$

ζ as function of v_c for the $p = 0.4$ diluted 4D model

Random-bond Ising model in 3 Dimensions



dilution range $p \approx 0.3 \ldots 0.4$:

$\gamma \approx 1.305$

- strong crossover effects
- slightly smaller than MC results $\gamma \approx 1.34$

Ballesteros et al. (1998): site dil.

Berche et al. (2004): bond dil.

q-state Potts model without disorder

No quenched disorder but:

Local d.o.f. S_i can take q different values.

$$\begin{aligned} \mathcal{Z} &= \text{Tr}\exp\left(-\beta J \sum_{<ij>} \delta(S_i, S_j)\right) \\ &= \sum_{\text{cluster } C} q^{\#conn.comp.(C)}(e^{\beta J}-1)^{\#bonds(C)} \end{aligned}$$

$q \to 1$ limit describes bond percolation with $p = 1 - e^{-\beta J}$.

Scaling near p_c:

- cluster size distribution

$$n(s,p) \sim s^{-\tau} f((p_c - p)s^{\sigma})$$

- mean cluster size

$$\left\langle \sum_s s^2 n(s,p) \right\rangle \sim (p_c - p)^{-\gamma}, \qquad \gamma = \frac{3-\tau}{\sigma}$$

Percolation thresholds (bond percolation on $\mathbb{Z}^d$) and critical exponents:

Dim.	Methods	p_c	σ	τ	$\gamma = \frac{3-\tau}{\sigma}$
2	exact, CFT	$\frac{1}{2}$	$\frac{36}{91}$	$\frac{187}{91}$	$\frac{43}{18}$
3	MC (Ziff and Stell, 1988)	0.248 812(2)	0.453(1)	2.189(1)	1.795(5)
	HTS (Adler *et al.*, 1990)	0.248 8(2)			1.805(20)
	MC (Lorenz and Ziff, 1998)	0.248 812 6(5)			
	MCS (Ballesteros *et al.*, 1999)		0.4522(8)	2.18906(6)	1.7933(85)
	MCS (Deng and Blöte, 2005)		0.4539(3)	2.18925(5)	1.7862(30)
	present work	0.248 91(10)			1.804(5)
4	HTS (Adler *et al.*, 1990)	0.160 05(15)			1.435(15)
	MCS (Ballesteros *et al.*, 1997)				1.44(2)
	MC (Paul *et al.*, 2001)	0.160 130(3)		2.313(3)	
	MC (Grassberger, 2003)	0.160 131 4(13)			
	present work	0.160 08(10)			1.435(5)
5	HTS (Adler *et al.*, 1990)	0.118 19(4)			1.185(5)
	MC (Paul *et al.*, 2001)	0.118 174(4)		2.412(4)	
	MC (Grassberger, 2003)	0.118 172(1)			
	present work	0.118 170(5)			1.178(2)
6	RG (Essam *et al.*, 1978)		$\frac{1}{2}$	$\frac{5}{2}$	$\chi \sim t^{-1} \lvert \ln t \rvert^{\delta}$, $\delta = \frac{2}{7}$
	HTS (Adler *et al.*, 1990)	0.094 20(10)			
	MC (Grassberger, 2003)	0.094 201 9(6)			
	present work	0.094 202 0(10)			$\delta = 0.40(2)$
> 6	RG		$\frac{1}{2}$	$\frac{5}{2}$	1

MC=Monte Carlo, MCs = Monte Carlo, site percolation, HTS= High-temperature series

Large dimension expansion for q-state Potts model

Critical point equation $1/\chi(D, v_c) = 0$ can be iteratively solved:
Large-D expansion for v_c in terms of $\sigma = 2D - 1$

$$v_c(q,\sigma) = \frac{1}{\sigma}\left[1 + \frac{8-3q}{2\sigma^2} + \frac{3(8-3q)}{2\sigma^3} + \frac{3\left(68-31q+q^2\right)}{2\sigma^4} + \frac{8664-3798q-11q^2}{12\sigma^5}\right.$$
$$+\frac{78768-36714q+405q^2-50q^3}{12\sigma^6} + \frac{1476192-685680q-2760q^2-551q^3}{24\sigma^7}$$
$$\left.+\frac{7446864-3524352q-11204q^2-6588q^3-9q^4}{12\sigma^8} + \cdots\right]$$

Percolation thresholds p_c for hypercubic lattices

Dim.	Series exp.	MC data (Grassberger 2002)	$1/D$-expansion
5	0.118165(10)	0.118172(1)	0.118149
6	0.0942020(10)	0.0942019(6)	0.0943543
7	0.078682(2)	0.0786752(3)	0.0786881
8	0.067712(1)	0.06770839(7)	0.0677080
9	0.059497(1)	0.05949601(5)	0.0594951
10	0.0530935(5)	0.05309258(4)	0.05309213
11	0.0479503(1)	0.04794969(1)	0.04794947
12	0.0437241(1)	0.04372386(1)	0.04372376
13	0.0401877(1)	0.04018762(1)	0.04018757
14	0.0371838(1)		0.03718368

Ising model on $\mathbb{Z}^d$ with quenched uncorrelated random bond disorder

- High-T series for the susceptibility up to order 21 in 3D, order 19 in 4 and 5D and order 17 in arbitrary dimensions
- 3D: difficult analysis, $\gamma = 1.305(5)$
- 4D: logarithmic corrections
- 5D: disorder irrelevant

Random cluster (q-state Potts) model on $\mathbb{Z}^d$ and bond percolation

- High-T series for arbitrary q: up to order 20 in 3D, order 19 in 4 and 5D and up to order 17 for arbitrary dimensions
- $1/D$ expansion for critical coupling up to order D^{-9} for arbitrary q
- percolation thresholds and critical exponent γ in high dimensions
- mean cluster numbers and fluctuations