Concurrence, Tangle and Entanglement Entropy of 1-Qubit Maps

Meik Hellmund[1] Armin Uhlmann[2]

[1]Uni Leipzig, Math. Inst.

[2]Uni Leipzig, ITP

Workshop Regensburg, 26 October 09

UNIVERSITÄT LEIPZIG

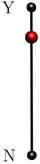
- Setting the stage: qubits, convex roofs, channels
- Concurrence of 1-qubit maps
- Examples
- Entanglement entropy of axially symmetric 1-qubit channels
- Outlook

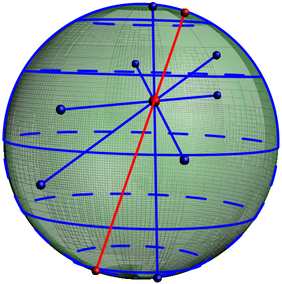
Main results

- general expression for concurrence (and tangle) of 1-qubit maps
- includes solutions with non-flat roofs
- applicable to $2 \times n$ bipartite systems provided the input state has rank 2
- numerical results for entanglement entropy: bifurcation pattern

M. Hellmund and A. Uhlmann, Phys Rev **A 79** (2009) 052319 = arXiv:0903.1340,
see also arXiv:0802.2092

Decomposition of mixed states





Classical bit

$$P = p\,\text{Ⓨ} + q\,\text{Ⓝ}, \quad p + q = 1$$
$$S_{\mathsf{class}}(P) = -p \log p - q \log q$$

Qbit

$$\rho = p\,|\psi_1\rangle\langle\psi_1| + q\,|\psi_2\rangle\langle\psi_2|$$
$$= p'\,|\psi_1'\rangle\langle\psi_1'| + q'\,|\psi_2'\rangle\langle\psi_2'|$$

v. Neumann:

$$S(\rho) = \min_{\mathsf{decomp}} S_{\mathsf{class}}(p, q)$$

optimum: orthogonal decomposition

$$S(\rho) = -\operatorname{Tr}(\rho \log \rho)$$

Entanglement entropy of $\Phi : \rho \mapsto \rho'$

$$E_\Phi(\rho) = \min_{\rho=\sum p_j \pi_j} \sum p_j \, S(\Phi(\pi_j))$$

$$\rho = \sum p_j \, \pi_j, \quad \pi_j \text{ pure, i.e. } \pi_j = |\psi_j\rangle\langle\pi_j|$$

$$\sum p_j = 1, \quad p_j > 0$$

Examples:

- Entanglement of formation: $\Phi =$ partial trace $\mathrm{Tr}_A : A \otimes B \to B$
- One-shot Holevo capacity of a channel Φ:

$$\chi^*_\Phi(\rho) = S(\Phi(\rho)) - E_\Phi(\rho)$$

$$\chi_\Phi = \max_\rho \chi^*_\Phi(\rho)$$

- replace S by another unitary invariant non-linear function G on state space: Φ-concurrence where $G(\rho) = 2\sqrt{\det \rho}$

Optimization/extension problem

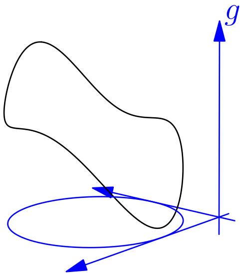
The global optimization problem

$$G(\rho) = \inf_{\rho = \sum p_j \, \pi_j} \sum p_j \, g(\pi_j).$$

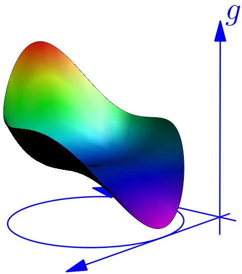
can be considered as an extension problem:

Given $g(\pi)$ on the set of pure states, extend it in an unique way to a function $G(\rho)$ on all states.

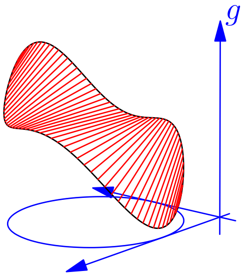
Convex Roofs



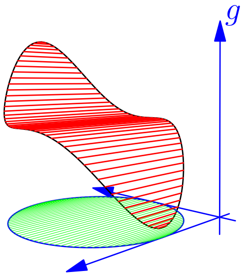
$g(\pi)$



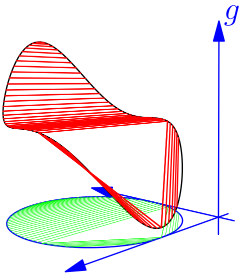
convex extension $\leq$



roof extension



a flat roof extension



the convex roof

largest convex extension
$=$ smallest roof extension
$=$ the convex roof

Convex Roofs

Theorem (Uhlmann)

Let $g(\pi)$ be a continuous real-valued function on the set of pure states $\partial_e \Omega$. Then there exists exactly one function $G(\rho)$ on Ω which can be characterized uniquely by each one of the following four properties:

1. *$G(\rho)$ is the solution of the optimization problem*

$$G(\rho) = \inf_{\rho = \sum p_j \, \pi_j} \sum p_j \, g(\pi_j).$$

2. *G is the unique convex roof extension of g.*
3. *G is the largest convex extension of g*
4. *G is the smallest roof extension of g.*

Convex Roofs (cont.)

- Let $G(\rho) : \Omega \to \mathbb{R}$ be convex and $h(x) : \mathbb{R} \to \mathbb{R}$ convex and non-decreasing. Then the composition $F(\rho) = h(G(\rho))$ is convex.
- Let $G(\rho)$ be a flat roof. Then $F(\rho) = h(G(\rho))$ is a flat roof, too.

  $\Longrightarrow$

  If the convex roof G of $g(\pi)$ is flat, then the convex roof of $f(\pi) = h(g(\pi))$ is given by $h(G)$ and is also flat with the same optimal decompositions.

  Examples:

  - $S(\rho) = h(2\sqrt{\det(\rho)})$ with $h(C) := \eta(\frac{1}{2}(1+\sqrt{1-C^2})) + \eta(\frac{1}{2}(1-\sqrt{1-C^2}))$ $\Rightarrow E_F(\rho) = h(C(\rho))$ for 2×2 system (Wootters)
  - Tangle $\tau(\rho) = C^2(\rho)$ for flat concurrence
- Furthermore, if G is a non-flat convex roof, then $h(G(\rho)$ provides a convex extension of f and therefore a lower bound for the convex roof of f.

Qbit state space

$\mathcal{M}_2$ (Hermitean 2×2 matrices) $\cong$ Minkowski space $\mathbb{R}^{1,3}$

$$\rho \;=\; \frac{1}{2}(x_0 I + \vec{x}\cdot\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix} \quad \Longleftrightarrow \quad \mathbf{x} = (x_0, \vec{x})$$

- $\rho = \rho^\dagger$ $\Rightarrow$ 4-dim real vector space

  $\det\rho = \frac{1}{4}(x_0^2 - x_1^2 - x_2^2 - x_3^2) = \frac{1}{4}\mathbf{x}\cdot\mathbf{x}$

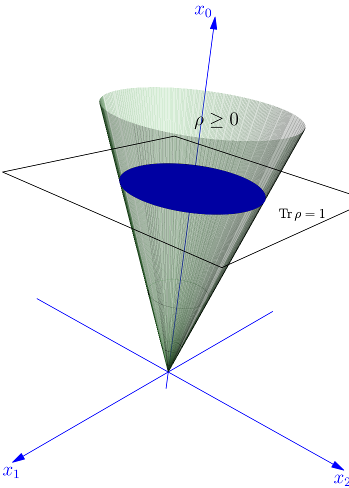
  Minkowski scalar product
- $\rho \geq 0$ $\Leftrightarrow$ $\det\rho \geq 0$ & $x_0 \geq 0$

  positive cone

  mixed states = timelike vectors

  pure states = lightlike vectors
- $\mathrm{Tr}\,\rho = 1$ $\Leftrightarrow$ $x_0 = 1$

  Bloch ball



Stochastic maps

- Linear map from one quantum system to another: $\Phi : \rho \mapsto \rho'$
- Trace-preserving: $\mathrm{Tr}\,\rho = \mathrm{Tr}\,\rho' = 1$
- Positive: $\rho \geq 0 \Rightarrow \rho' \geq 0$

  $\Longrightarrow$ Stochastic map
- Completely positive: $\Phi \otimes I_n$ positive $\forall n$

  $\Longrightarrow$ Quantum channel

Kraus representation $\Phi(\rho) = \sum_i^K A_i^\dagger \rho A_i, \quad \sum A_i A_i^\dagger = I$

1-qubit stochastic map

$$\Phi(\rho) = \Phi\left(\frac{1}{2}(x_0 I + \vec{x}\cdot\vec{\sigma})\right) = \frac{1}{2}\left(x_0 I + (x_0\vec{t} + \mathbf{\Lambda}\vec{x})\cdot\vec{\sigma}\right)$$

where $\mathbf{\Lambda}$ is a 3×3 matrix and $\vec{t}$ a 3-vector
affine map of Bloch ball

$$\vec{r} \quad \mapsto \quad \Lambda\vec{r} + \vec{t}$$

Concurrence of 1-qubit maps

Qubit map $\Phi \to$ quadratic q form on $\mathcal{M}_2 = \mathbb{R}^{1,3}$

$$q_w^\Phi(\mathbf{x}) = 4(\det \Phi(\rho) - w \det \rho) = \Phi(\mathbf{x}) \cdot \Phi(\mathbf{x}) - w\, \mathbf{x} \cdot \mathbf{x} = \sum_{i,j=0}^{4} (Q_w^\Phi)_{ij}\, x_i x_j$$

where

$$Q_w^\Phi = Q_0^\Phi - w\,\eta = \begin{pmatrix} 1 - |\vec{t}|^2 - w & -\vec{t}\boldsymbol{\Lambda} \\ -(\vec{t}\boldsymbol{\Lambda})^T & w\,\mathbf{I} - \boldsymbol{\Lambda}^T\boldsymbol{\Lambda} \end{pmatrix} \quad \text{with } \eta = \operatorname{diag}(+1,-1,-1,-1)$$

Theorem

For every positive trace-preserving 1-qubit map Φ, let w be the second largest eigenvalue of $\eta Q_0^\Phi = \begin{pmatrix} 1-|\vec{t}|^2 & -\vec{t}\boldsymbol{\Lambda} \\ (\vec{t}\boldsymbol{\Lambda})^T & \boldsymbol{\Lambda}^T\boldsymbol{\Lambda} \end{pmatrix}$.
Then $\sqrt{q_w^\Phi(\rho)}$ is a convex roof and provides the concurrence for this map

$$C_\Phi(\rho) \;=\; \min_{\textit{decomp}} \;\sum p_j\, C(\Phi(\pi_j)) \;=\; \sqrt{q_w^\Phi(\rho)}.$$

Sketch of proof

- $q^{1/2}(\pi) = 2\sqrt{\det \Phi(\pi)} = C(\pi)$ since $\det \pi = 0$.
  So, $q^{1/2}$ is an extension of C. We have to proof that, for the right choice of the parameter w, it is convex and a roof.

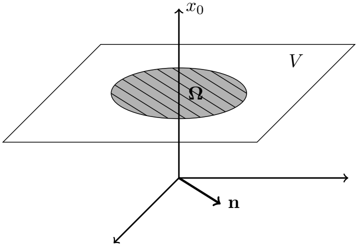
Proposition (1)

Let the quadratic form q on $\mathbb{R}^{1,3}$ and therefore the matrix Q be positive semi-definite and degenerate, i.e., $Q \geq 0$ and $\dim \operatorname{Ker} Q > 0$.
If $\operatorname{Ker} Q$ contains a non-zero vector $\mathbf{n}$ which is space-like or light-like, $\mathbf{n} \cdot \mathbf{n} \leq 0$, then $q^{1/2}$ is a *convex roof*.
Furthermore, this roof is *flat* if such an $\mathbf{n}$ exists with $n_0 = 0$.

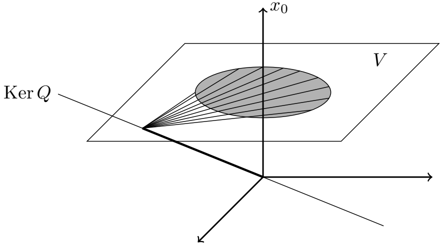
- q positive semi-definite $\Longrightarrow$ $\sqrt{q}$ provides a semi-norm and hence is convex
- $\sqrt{q}$ is a roof if there is a foliation s.t. $q^{1/2}$ is linear on the leaves

Let $\mathbf{n} = (n_0, \vec{n}) \in \operatorname{Ker} Q$. Then $q(\mathbf{m}+\mathbf{n}) = (\mathbf{m}+\mathbf{n})Q(\mathbf{m}+\mathbf{n}) = \mathbf{m}Q\mathbf{m} = q(\mathbf{m})$

- case I: $n_0 = 0$, hence q=const along $\vec{n}$, roof is flat



$\mathcal{M}_2$



- case II: no $\mathbf{n}$ with $n_0 = 0$ exists, hence $\dim \operatorname{Ker} Q = 1$
  $q^{1/2}$ is linear along the half-line $\mathbb{R}^+ \ni s \mapsto s\mathbf{m} + (1-s)\mathbf{n}$ since $q\,(s\mathbf{m} + (1-s)\mathbf{n}) = s^2 q(\mathbf{m})$

Proposition (2)

For every positive trace-preserving map Φ there exists a unique value for the parameter w such that the conditions of Proposition (1) are fulfilled, i.e., $Q_w^\Phi \geq 0, \ \mathrm{Ker}\, Q_w^\Phi \neq 0$ and $\mathrm{Ker}\, Q_w^\Phi$ not time-like.

Idea of proof: study flow of signature of $q_w = q_0 - w\eta$ as w runs from $-\infty$ to ∞

- signature changes at roots w_i of $\det(Q_0 - w\eta) = 0$
- Yakubovich's S-lemma: ensures existence of w s.t. $q_w \geq 0$
- Hence, flow pattern is (up to degeneracies) unique:

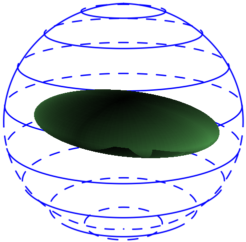
$$(+,-,-,-) \xrightarrow{w_4} (+,+,-,-) \xrightarrow{w_3} (+,+,+,-) \xrightarrow{w_2} (+,+,+,+) \xrightarrow{w_1} (+,+,+,-)$$

- q_w is positive semi-definite $(+,+,+,0)$ at $w = w_1$ and $w = w_2$.
- $\mathrm{Ker}\, Q$ is time-like at w_1 and space-like at w_2 □

R. Hildebrand, J. Math.Phys 48(2007)102108

Example: Bistochastic maps or unital channels

- $\vec{t} = 0; \mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$



- $w = \max(\lambda_1^2, \lambda_2^2, \lambda_3^2)$
- $C_\Phi(\rho) = q_\Phi^{1/2}(\rho) = \sqrt{1 - w + \sum_{i=1}^3 (w - \lambda_i^2)\, x_i^2}$ is flat
- $S_\Phi(\rho) = h(C_\Phi(\rho))$
- $\chi_\Phi = \log(2) - \eta(\frac{1+\sqrt{w}}{2}) - \eta(\frac{1-\sqrt{w}}{2})$ (where $\eta(x) = -x \log x$)

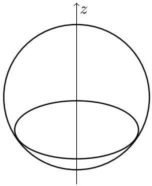
Example: Axially symmetric channels

modulo unitary transformations we can write

$$\begin{aligned}\Phi : \rho = \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} &\mapsto \begin{pmatrix} \alpha x_{00} + (1-\gamma)x_{11} & \beta x_{01} \\ \beta x_{10} & \gamma x_{11} + (1-\alpha)x_{00} \end{pmatrix} \\ \vec{r} &\mapsto \Lambda \vec{r} + \vec{t} \\ \text{where } \Lambda &= \operatorname{diag}(\beta, \beta, \alpha+\gamma-1), \quad \vec{t} = (0, 0, \alpha-\gamma)\end{aligned}$$

Positivity:

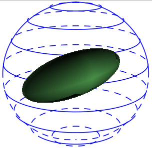
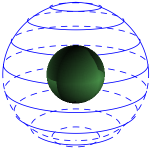
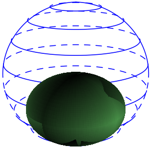
$$\begin{aligned} 0 &\le \alpha, \gamma \le 1, \\ \beta^2 &\le \beta^2_{\mathsf{max}} = 1 + 2\alpha\gamma - \alpha - \gamma + 2\sqrt{\alpha(1-\alpha)\gamma(1-\gamma)} \end{aligned}$$



Complete positivity:

$$\beta^2 \le \alpha\gamma$$

Axially symmetric channels II

		Kraus length	unital	
phase-damping	$\alpha = \gamma = 1$	2	✓	
depolarizing	$\alpha = \gamma, \beta = 2\alpha - 1$	4	✓	
amplitude-damping	$\gamma = 1, \beta^2 = \alpha$	2	×	

Axially symmetric channels III - Concurrence

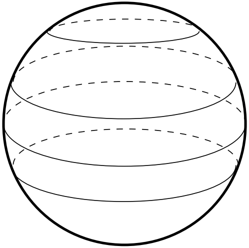
$$C_\Phi^2(\rho) = 4(\det \Phi(\rho) - w \det(\rho))$$

with

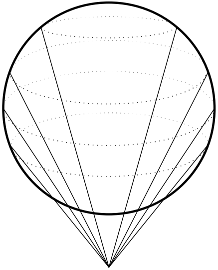
$$w = \max(\beta^2, \beta_c^2)$$

where $$\beta_c^2 = 1 + 2\alpha\gamma - \alpha - \gamma - 2\sqrt{\alpha(1-\alpha)\gamma(1-\gamma)}.$$

case A: $\beta \geq \beta_c$: flat roof



case B: $\beta < \beta_c$:



non-flat

$\operatorname{Ker} Q = \lambda \mathbf{n}$,

$\mathbf{n} = (1, 0, 0, z_0)$,

$z_0 = \frac{\sqrt{\gamma(1-\gamma)} + \sqrt{\alpha(1-\alpha)}}{\sqrt{\gamma(1-\gamma)} - \sqrt{\alpha(1-\alpha)}}$

$2 \times n$ bipartite system

- $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$, $\dim \mathcal{H}^A = 2$
- $\mathcal{H}_2$: a 2-dimensional subspace of $\mathcal{H}$, V: unitary mapping $\mathcal{H}^A$ onto $\mathcal{H}_2$.
- $\rho \mapsto \Phi(\rho) = \mathrm{Tr}_B(V\rho V^\dagger)$ is a 1-qubit channel for all density operators ρ supported by $\mathcal{H}_2$
- Hence, $C(\rho)^2 = 4(\det \mathrm{Tr}_B\,\rho - w \det \rho)$ where w depends on $\mathcal{H}_2$.

Example: 3 qubits $\mathcal{H} = \mathcal{H}^a \otimes \mathcal{H}^b \otimes \mathcal{H}^c$, $\Phi = \mathrm{Tr}_{bc}$

- $\mathcal{H}_2$ spanned by $|W\rangle = 3^{-1/3}(|001\rangle + |010\rangle + |100\rangle)$ and
  $|GHZ\rangle = 2^{-1/2}(|000\rangle + |111\rangle)$
- $\Phi : \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} \mapsto \begin{pmatrix} \frac{2}{3}x_{00} + \frac{1}{2}x_{11} & \frac{1}{\sqrt{6}}x_{01} \\ \frac{1}{\sqrt{6}}x_{10} & \frac{1}{3}x_{00} + \frac{1}{2}x_{11} \end{pmatrix}$
- We get $w = 1/6$ and

$$C(\rho)^2 = \frac{8}{9}\langle GHZ|\rho|GHZ\rangle^2 + \langle W|\rho|W\rangle^2 + \frac{4}{3}\langle GHZ|\rho|GHZ\rangle\,\langle W|\rho|W\rangle$$

Tangle of 1-qubit map

- We look for an convex roof extension of

$$\tau_\Phi(\pi) = C_\phi^2(\pi) = 4 \det \Phi(\pi)$$

- Ansatz:

$$\begin{aligned} \tau_\Phi(\rho) &= 4 \det \Phi(\rho) - 4\, v \det \rho \\ &= 1 - |\vec{t}|^2 - v - \vec{t}(\Lambda + \Lambda^T)\vec{r} + \vec{r}(vI - \Lambda^T \Lambda)\vec{r} \end{aligned}$$

  This is a convex roof if the quadratic part $(vI - \Lambda^T \Lambda)$ is positive semi-definite and degenerate, i.e., for

$$v = \lambda_{\mathsf{max}}(\Lambda^T \Lambda) \qquad \text{(max. eigenvalue)}$$

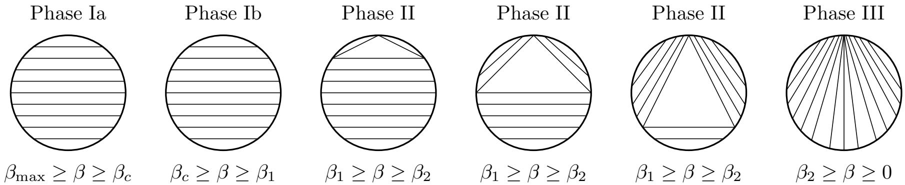
- Therefore,

$$\begin{aligned} \chi_2^\Phi(\rho) &:= 4 \left(\det \Phi(\rho) - \min_{\text{decomp}} \sum p_j \det \Phi(\pi_j)) \right) \\ &= 4 \lambda_{\mathsf{max}}(\Lambda^T \Lambda) \det \rho \end{aligned}$$

  (Osborne, Verstraete 2006)
- The roof is flat if $\vec{t} \cdot \vec{e} = 0$ with $\vec{e}$ eigenvector to λ_{max}.

Entanglement entropy for axially symmetric maps

$$E_\Phi(\rho) = \min_{\text{decomp}} \sum p_j \; S(\Phi(\pi_j))$$



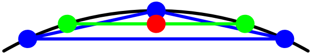
Leaves of the foliation of the entanglement entropy. The z axis points upwards.

- Phase Ia: C_Φ flat $\Rightarrow$ E_Φ flat
- Phase Ib: E_Φ still flat
- Phase II: solid cone with apex at N pole appears, states in cone have optimal decompositions of length 3
- phase III: leaves form cones with apex at the z axis outside the ball, no common apex

Bifurcation parameters

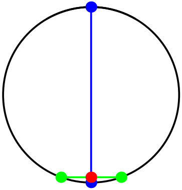
compare competing decompositions: $s(\phi) := S(\Phi(\pi)),\ \pi = (\sin\phi, 0, \cos\phi)$

Bifurcation β_1



$E_1 = \frac{1}{3}s(1) + \frac{2}{3}s(\cos(\phi))$
$E_2 = s(\frac{1}{3} + \frac{2}{3}\cos(\phi))$

Bifurcation β_2

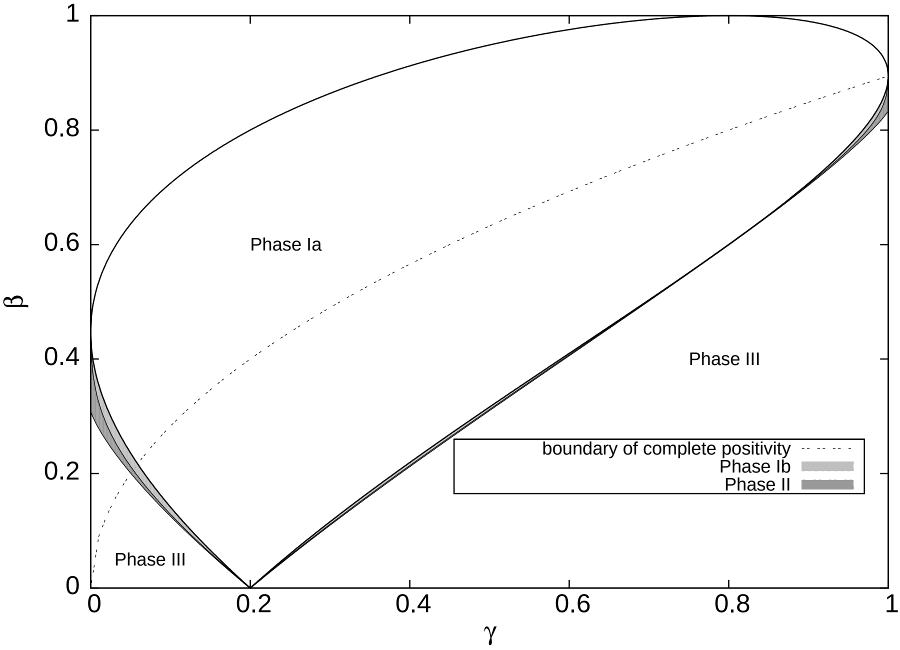


$E_1 = \frac{1+\cos(\phi)}{2}s(1) + \frac{1-\cos(\phi)}{2}s(-1)$
$E_2 = s(\cos(\phi))$

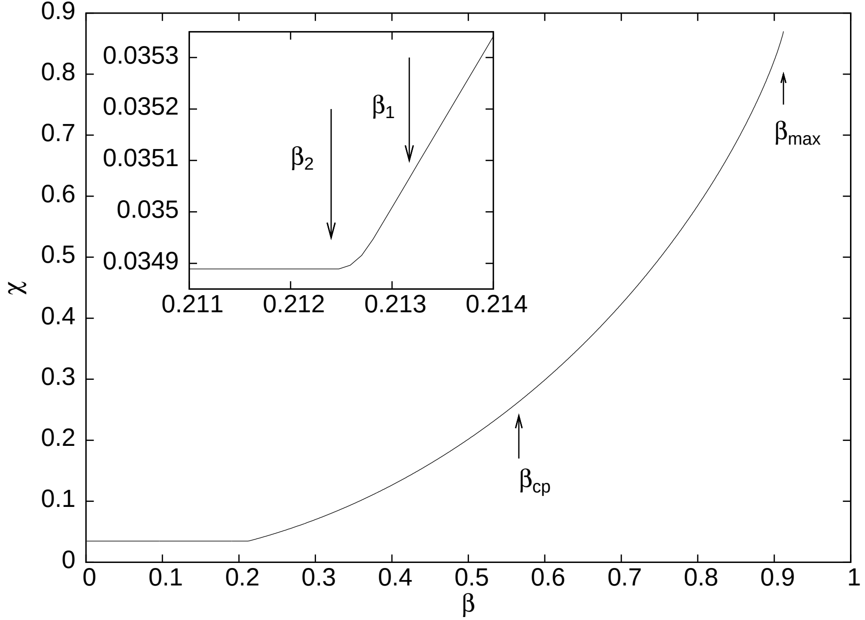
$E_1 - E_2 = g(\alpha,\beta,\gamma)\phi^2 + O(\phi^3)$, solve $g(\alpha,\beta,\gamma) = 0$:

$$\beta_1^2 = \frac{x}{2(x + (x^2-1)\operatorname{arctanh}(x))}\left\{x^2 + xy + (x^2-1)y\operatorname{arctanh}(x) - \sqrt{(1-x^2)\operatorname{arctanh}(x)(x^3 - xy^2 - (x^2-1)y^2\operatorname{arctanh}(x))}\right\}$$ with $x = 2\alpha - 1, y = 2\gamma - 1;$

$$\beta_2^2 = y\,\frac{(1+x)\log(1-y) + (1-x)\log(1+y) - (1+x)\log(1+x) - (1-x)\log(1-x)}{2(\log(1-y) - \log(1+y))}$$



Phase diagram in the (γ, β)-plane for $\alpha = 0.8$



The HSW capacity as a function of the channel parameter β at $\alpha = 0.8,\ \gamma = 0.4$. The inset shows the small region where the transition between phases I, II and III takes place.

Outlook

- Channels with higher output rank: $\det \Rightarrow e_2$ (Rungta et al.)

$$C_\Phi(\pi) = 2\sqrt{e_2(\Phi(\pi))}$$
$$e_2(A) := \prod_{i<j} \lambda_i \lambda_j$$

$2\sqrt{e_2(\Phi(\rho)) - w e_2(\rho)}$ is a convex extension but not a roof $\Rightarrow$ lower bound

- e.g., Choi map of 3×3 system:

$$\rho \;\mapsto\; \Phi[\mu](\rho) = \frac{1}{1+\mu} \begin{pmatrix} x_{00} + \mu x_{22} & -x_{01} & -x_{02} \\ -x_{10} & x_{11} + \mu x_{00} & -x_{12} \\ -x_{20} & -x_{21} & x_{22} + \mu x_{11} \end{pmatrix}$$
$$C_\Phi(\rho)^2 \geq \frac{4\mu}{(1+\mu)^2} \left[(x_{00} + x_{11} + x_{22})^2 + (\mu - 1)\left(|x_{01}|^2 + |x_{02}|^2 + |x_{12}|^2\right) \right]$$

Main results

- general expression for concurrence (and tangle) of 1-qubit maps
- includes solutions with non-flat roofs
- applicable to $2 \times n$ bipartite systems provided the input state has rank 2
- numerical results for entanglement entropy: bifurcation pattern