A LATTICE CALCULATION IN (1 + 1)-DIMENSIONAL QCD BY THE HAMILTONIAN ENSEMBLE PROJECTOR MONTE CARLO METHOD

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The SU(2) gauge theory coupled to one unflavored fermion in two dimensions is studied on the lattice by a local hamiltonian Monte Carlo method. The vacuum energy, the quark condensate and the correlation function of the lightest meson are measured. For massless quarks the chiral symmetry is unbroken as expected in the continuum. A vanishing condensate is found and the meson mass is determined.

Monte Carlo calculations allow studying nonperturbative properties of gauge theories. Inclusion of dynamical fermions in a local way is possible in the hamiltonian formulation, but a suitable Hilbert space basis (local and, at best, gauge invariant) has to be found. We use the ensemble projector method [1] to search for the ground state of the hamiltonian. This avoids the appearance of large fluctuating scores as in the simple projector method [2–4].

After the massless [5] and massive [6] Schwinger model were successfully studied by local hamiltonian methods, we consider a nonabelian model, SU(2) gauge theory coupled to one color doublet fermion in two dimensions [‡1]. Similar to QCD_4 this theory shows asymptotic freedom and color confinement. As a special property of the SU(2) group, complex conjugation is equivalent to a unitary transformation, so no independent anticolor representations exist. Due to this, the qq, $q\bar{q}$ and $\bar{q}\bar{q}$-states (baryon, meson and antibaryon) build a mass degenerate triplet [7]. In the case of massless quarks the discussion [8,9] has led to the conclusion, that this triplet appears as massless free particles and that the chiral symmetry is not broken [9].

The ensemble projector method [1] relies on the fact, that during the evolution of a state in imaginary time excited modes are damped out. If $|\Psi\rangle$ has the same conserved quantum number (fermion number) as the ground state, then

$$\lim_{\tau\to\infty} e^{-\tau H}|\Psi\rangle = |\text{phys. vac.}\rangle\,. \tag{1}$$

Contrary to the hamiltonian H, the operator $\exp(-\tau H)$ is not local, but H can be divided into $H = H_1 + H_2$, so that the H_i are sums of mutually commuting local terms and $\exp(-\tau H_i)$ is separable in space into products of local matrix elements enclosing two lattice points. The relation

$$e^{-\tau H} = \lim_{\substack{\Delta\tau\to 0 \\ n\Delta\tau=\tau}} [e^{-\Delta\tau H_1} e^{-\Delta\tau H_2}]^n \tag{2}$$

is used to simulate the time development in small time steps. For one step and one local matrix element

$$e^{-\Delta\tau H}|\Psi_i\rangle = \sum_j a_{ij}|\Psi_j\rangle \tag{3}$$

a final state $|\Psi_j\rangle$ is chosen according to the probability $P_{ij} = |a_{ij}|/\Sigma_j|a_{ij}|$ and it gets the score $S_j = \mathrm{sgn}(a_{ij}) \times \Sigma_j|a_{ij}|$. This is done for all matrix elements of H_1 and after that for all of H_2. The quantum state is represented by an ensemble of N lattices with scores S_i ($i = 1, ..., N$), the products of the local scores.

Negative matrix elements result in negative scores. According to our experience the subset of states with negative scores is as representative as the subset with positive ones. This is reasonable because a very small

‡1 During the course of our calculations we received a preprint [4] where this model is studied by the simple projector method using essentially the same basic states.

local change of a state can move it from one subset to the other. In what follows we will therefore ignore the sign of the matrix elements.

In the ensemble method the scores are used to replicate the states after each time step. Every lattice remains in the ensemble with the multiplicity

$$n_i = \mathrm{Int}[S_i/A + \text{random number}] \,. \tag{4}$$

We choose the average score A at every time step using the requirement that the number of states in the ensemble remains constant. In this way A converges to an equilibrium value A_0 which characterizes the ground state energy,

$$A_0 = \mathrm{e}^{-\Delta\tau E_0} \,. \tag{5}$$

If A_0 is stable we can start to measure observables. The one-dimensional lattice hamiltonian for SU(2) gauge theory using temporal gauge and Susskind fermions is ($i, j = 1, 2$)

$$H = \tfrac{1}{2}aq^2 \sum_{n=1}^{L} E_n^2 + \frac{1}{2a}\sum_{n=1}^{L} [\chi_{n,i}^{+} U_{n,ij} \chi_{n+1,j} + \text{h.c.}]$$

$$+ m \sum_{n=1}^{L} (-1)^n \chi_{n,i}^{+} \chi_{n,i} - m$$

$$= H_{\mathrm{g}} + H_{\mathrm{ia}} + H_{\mathrm{m}} \,. \tag{6}$$

Periodic boundary conditions will be assumed in spatial direction. The anticommuting algebra of the fermionic operators χ can be realized by introducing occupation numbers zero or one for each color on each lattice point. The field strength E_a is canonically conjugate to the potential, E^2 is the Casimir operator of the group. The eigenstates of E^2, the group representations, are suitable local basic states of the gauge field, invariantly characterized by the quantized electric flux $j \in \mathbb{Z}_+/2$, they carry

$$E^2|j\rangle = j(j+1)|j\rangle \,. \tag{7}$$

Out of these local states we have to construct gauge invariant lattice states. Due to Gauss' law the local quantum numbers have to fulfill nonlocal constraints concerning neighbouring links. Gauge invariant fermion states are described by an occupation number 0, 1 or 2, where "2" represents the color singlet state of two quarks. If on one site sit 0 or 2 quarks, the invariant states are

$$\overset{j}{\bullet}\!\!-\!\!-\!\!\overset{j}{\bullet} \;\hat{=}\; \mathrm{Tr}[\ldots D^{j}_{\alpha\beta}(U_n) D^{j}_{\beta\gamma}(U_{n+1}) \ldots] \,. \tag{8}$$

The constraint $j_n = j_{n+1}$ is flux conservation, Gauss' law. One quark on the site generates a flux jump by $\frac{1}{2}$. Clebsch–Gordan coefficients couple the link states in a gauge invariant way,

$$\overset{j}{\bullet}\!\!-\!\!-\!\!\overset{j\pm 1/2}{\bullet}$$

$$\hat{=}\; \mathrm{Tr}\Big(\ldots \sum_{i=\pm 1/2} D^{j}_{\alpha\beta} \chi_{n,i} (j\beta\tfrac{1}{2} i | j \pm \tfrac{1}{2}\delta)\, D^{j\pm 1/2}_{\delta\epsilon} \ldots\Big) \,. \tag{9}$$

We decompose $H = H_1 + H_2$ in such a way that H_1 (H_2) acts on all even (odd) links and includes the mass term with positive (negative) sign and approximate

$$\exp(-\Delta\tau H) = \exp(-\tfrac{1}{2}\Delta\tau H_{\mathrm{g}}) \exp(-\tfrac{1}{2}\Delta\tau H_{\mathrm{m}})$$

$$\times \exp(-\Delta\tau H_{\mathrm{ia}}) \exp(-\tfrac{1}{2}\Delta\tau H_{\mathrm{m}}) \exp(-\tfrac{1}{2}\Delta\tau H_{\mathrm{g}})$$

$$+ \mathrm{O}(\Delta\tau^3) \,. \tag{10}$$

H_{g} and H_{m} are diagonal in our basis. The matrix elements of the interaction term $\exp(-\Delta\tau H_{\mathrm{ia}})$ can be calculated exactly using the orthonormality of the CG coefficients and the relation

$$(j_1^{\alpha} j_2^{\beta} | j_3^{\gamma}) = (-1)^{j_2+\gamma+\alpha} [(2j_3+1)/(2j_1+1)]^{1/2}$$

$$\times (j_3 \gamma j_2 \beta | j\alpha) \,. \tag{11}$$

We have taken into account only transitions where at most one fermion jumps. Jumps of 2 fermions are suppressed by a factor $\Delta\tau$.

Expectation values of operators diagonal in the chosen basis can easily be measured,

$$\langle\Psi|\mathrm{e}^{-\Delta\tau H}\mathrm{O}|\Phi\rangle = \langle\Psi|\mathrm{e}^{-\Delta\tau H}|\varphi\rangle\langle\varphi|\mathrm{O}|\varphi\rangle \,. \tag{12}$$

The quark condensate

$$F = L^{-1} \sum_{n=1}^{L} (-1)^n \chi_{n,i}^{+} \chi_{n,i} \tag{13}$$

and the mesonic correlation function are of this type. If after the instant of measurement a configuration is replicated n times, the measured value gets in the ensemble and time average the weight n. In the computer program this is realized by replicating also the vector of length N in which the measured values of the N lat-

tices are accumulated in the course of time. To measure a correlation function

$$K(\tau) = \langle Q(t)\, Q^*(t+\tau)\rangle \tag{14}$$

a vector including $Q(t)$ is generated for every time slice t and is replicated from that time on. The mass of lightest meson is given by the exponential slope of the correlation function of the vector current $j_\mu = \bar{\Psi}_i \gamma^\mu \Psi_i$. (In two dimensions, the spatial component of a vector is a pseudoscalar.) In our canonical ensemble, we can get no information from zero momentum correlation, we have to determine the dispersion law by considering [5]

$$K(k,t) = \langle 0|j_0(k,\tau)\, j_0^*(k,\tau+t)|0\rangle ,$$

$$j_\mu(k,t) = \int e^{ikx} j_\mu(x,t)\, dx . \tag{15}$$

At large times $t \gg 1$ this is dominated by the lightest meson

$$K(k,t) \sim |\langle 0|j_0|M, p=k\rangle|^2 \exp[-E_M(k)t] , \tag{16}$$

and for large wavelengths the relativistic dispersion law $E^2 = k^2 + m^2$ is expected. On the lattice this reads:

$$j_0(k,t) = \sum \cos kn\, \chi_n^+ \chi_n + \mathrm{i} \sum \sin kn\, \chi_n^+ \chi_n$$

$$= C(k,t) + \mathrm{i}S(k,t) ,$$

$$K(k,t) = \frac{1}{NT} \sum_{i,j}^{N} \sum_{\tau}^{T} [C(k,\tau)_{i(j)} C(k,\tau+t)_j$$

$$+ S(k,\tau)_{i(j)} S(k,\tau+t)_j] , \tag{17}$$

where the notation $i(j)$ indicates that i is the predecessor in time of the configuration j. The imaginary part

$$\mathrm{Im}\, K(k,t) = \frac{1}{NT} \sum_{i,j}^{N} \sum_{\tau}^{T} [C(k,\tau)_{i(j)} S(k,\tau+t)_j$$

$$- C(k,\tau+t)_j S(k,\tau)_{i(j)}] \tag{18}$$

should of course be zero because of time symmetry and the ratio $\mathrm{Im}\, K/\mathrm{Re}\, K$ is therefore used as a measure of the statistical error. The ensemble has in time the structure of a tree with dying and ramifying branches. For the measuring process a dying branch has never existed and a ramified branch is treated as if there have ever been two branches. So in equilibrium time symmetry will be guaranteed by the measuring prescription.

The numerical calculations were done on a ESER computer with extensive use of a vector coprocessor. Our ensemble included 200 configurations each with 100 spatial lattice points. The initial state was chosen invariant under discrete chiral transformations, 100 lattices with quark occupation numbers 2–0–2–0– and 100 with 0–2–0–2–. The links were empty. Since we neglected matrix elements with two simultaneous quark jumps a small time step $\Delta\tau = 0.02$ ... 0.05 in units of the lattice spacing was used. Each time step included a replication step. After 200 ... 300 such iterations equilibrium was reached.

According to eq. (5) the averaged score provides us with the ground state energy (fig. 1). Our results for this are in agreement with ref. [4] where the measurement was done using the expectation value of the hamiltonian. The exactly known value E_0 at $g = 0$, $m = 0$ is reproduced. The measured values of the quark condensate (fig. 2) have a statistical error of about ±0.05. For massless quarks they are scattered around zero whereas for massive quarks the nonzero condensate is clearly visible.

Fig. 3 shows an example of the measured correlation function and fig. 4a the corresponding fit of the relativistic law which provides a value $m = 0.15$ for the meson mass at $m/g = 2, g = 0.3$. A small negative mass square is obtained in the case of massless quarks

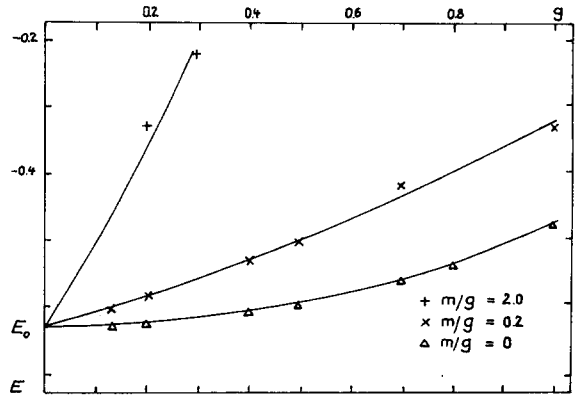

Fig. 1. Ground state energy per site as a function of the coupling constant. E_0 is the exact energy of two free massless fermions.

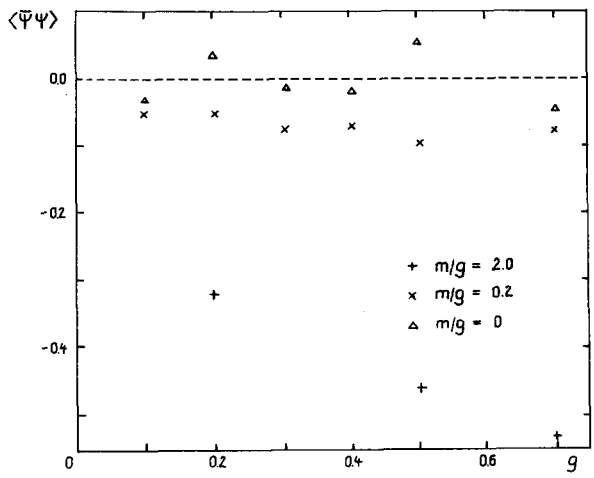

Fig. 2. Quark condensate as a function of the coupling constant.

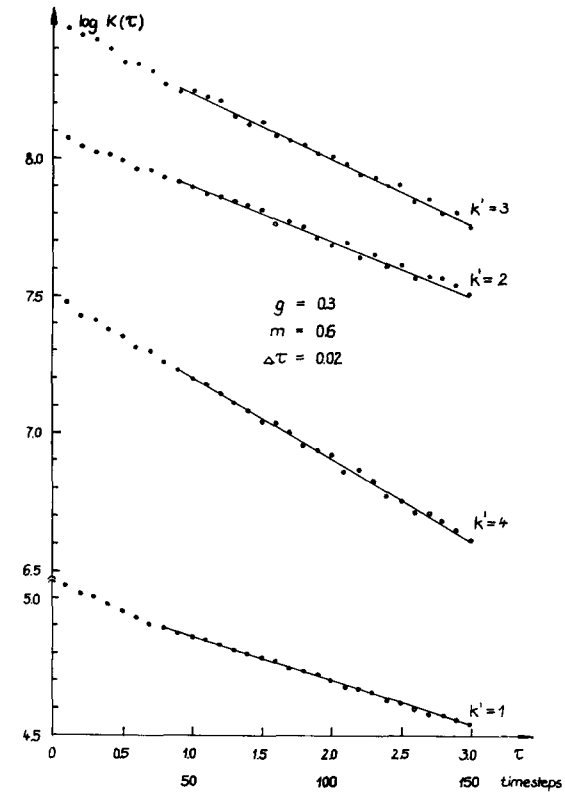

Fig. 3. Mesonic correlation function versus time difference for the momenta $k = 2\pi k'/L$, $k' = 1,2,3,4$ at $g = 0.3$ and $m = 0.6$. Indicated are the fits used to extract the energies.

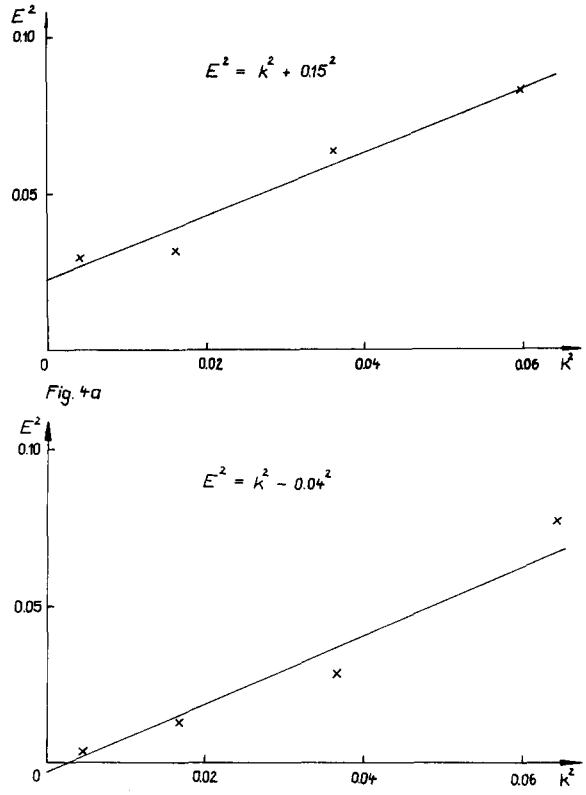

Fig. 4. The square of the meson energy as a function of k^2 and a linear fit with slope 1. (a) $g = 0.3$, $m = 0.6$; (b) $g = 0.3$, $m = 0$.

(fig. 4b). This should be a hint to say the meson is massless and confirms the continuum result of no spontaneous breaking of chiral symmetry.

In conclusion, the ensemble projector method is well suited to handle QCD_2. Expectation values of local operators as well as correlation functions have been measured. The replication trick avoids large fluctuating scores and makes runs over many time slices possible. At least in the case of QCD_2 neglecting the sign of the scores is admissible. The advantages of the method include that a configuration is described by quantum numbers, small integers and not complex matrices, and that the method is local even in the presence of dynamical fermions and well implementable on a vector processor. Therefore the extension to higher dimensions would be interesting.

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