BACKGROUND FIELD EQUATIONS FOR THE HETEROTIC STRING AT FINITE TEMPERATURE

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The breaking of supersymmetry at $T \neq 0$ leads to non-vanishing massless particle tadpoles and therefore to BRST anomalies in the degeneration limit of one-loop amplitudes. Cancellation of those anomalies gives the loop-corrected background field equations where the thermal energy–momentum tensor appears as source term for the graviton and dilaton field equations.

The behaviour of string theories at finite temperature has attracted much attention, especially the singular behaviour of the free energy of a gas of free strings at the Hagedorn temperature. The partition function is calculated using an imaginary periodic time with period β [1]. Therefore, soliton states have to be included. The condition of antiperiodicity for space–time fermions leads to a mixing of the sum over soliton winding numbers with the sum over spin structures, breaking the space–time supersymmetry [2–4]. We are interested in the question of how this changes the background field equations. At tree level, these field equations are most easily derived from the absence of conformal anomalies, which depend on the short-distance behaviour only. Non-zero temperature effects first arise at the one-string-loop level, however. As first observed by Fischler and Susskind [5], integration over the moduli of the world sheet produces additional singularities, arising from the boundaries of modular space. The most appropriate way to study string loop contributions to the equations of motion seems to be to consider BRST invariance. For the bosonic string at zero temperature this has been carried out by Polchinski [6] and Rey [7], and in a somewhat different approach [8] for the open superstring. In this paper we essentially follow refs. [6,7] and apply these methods to the case of the heterotic string, taking into account the one-loop non-zero temperature contribution.

For definiteness, we consider a one-loop heterotic string amplitude with N external Neveu–Schwarz (bosonic) states. Only the three even spin structures can give non-zero tadpole, two- or three-point functions. Coordinates appropriate for the degeneration limit on the heterotic world sheet are given by glueing a punctured (heterotic) sphere with global coordinates (z', θ', z') with punctured (even heterotic) torus with coordinates $(z'', \theta'', \bar{z}'')$, $z'' = z'' + \tau$. The glueing prescription is [9]

$$z' z'' = -t^2, \quad \theta'' z' = t\theta', \quad \theta' z'' = -t\theta'' . \tag{1}$$

The factorization relation with the appropriate ghost insertions (to get non-zero expectation values despite of zero modes) then reads

$$\left\langle \prod_i^{M+N} V_i(z_i,\theta_i) \prod_j^{M+N} b(z_j) \prod_k^{M+N} \delta(\beta(z_k)) \right\rangle_\Sigma$$

$$= \sum_a (-t^2)^{h_a}(-\bar{t}^2)^{\bar{h}_a} \left\langle \Phi_a(z_0',\theta_0') \prod_i^{N} V_i(z_i',\theta_i') \prod_j^{N-2} b(z_j') \prod_k^{N-1} \delta(\beta(z_k')) \right\rangle_{\Sigma_1}$$

$$\times \left\langle \bar{\Phi}_a(z_0'',\theta_0'') \prod_i^{M} V_i(z_i'',\theta_i'') \prod_j^{M+2} b(z_j'') \prod_k^{M+1} \delta(\beta(z_k'')) \right\rangle_{\Sigma_2} . \tag{2}$$

Here, the Φ_a are a complete set of orthonormal states

$$\langle \Phi_a(0)\bar{\Phi}_b(\infty)\rangle_{\text{sphere}} = \frac{4\pi^3}{8\pi G_N}\delta_{ab} , \tag{3}$$

where the coordinate frame on the glueing sphere is given by

$$z=-t/z'=z''/t , \quad \theta=t^{1/2}\theta'/z'=-t^{-1/2}\theta'' . \tag{4}$$

As usual, b and c are the reparametrization ghosts, β and γ the superghosts and ϕ is the scalarized superghost [9–12]. The ghost parts of the Φ fields are given by

$$\langle :c\partial c\, \bar{c}\bar{\partial}\bar{c}\, \mathrm{e}^{-\phi}(0):\, :c\bar{c}\, \mathrm{e}^{-\phi}(\infty): \rangle_{\text{sphere}} = 1 . \tag{5}$$

The V represent BRST-invariant vertex operators of the NS sector

$$V(z,\theta) = c\bar{c}\, \mathrm{e}^{-\phi} \times \text{polynomial of matter superfields} . \tag{6}$$

Now we have to integrate over the (super)moduli of the punctured surface. The (super)Beltrami differentials μ_i, $\hat{\mu}_k$ provide (together with the ghost zero modes) the correct measure for the chosen parametrization (τ_i, θ_k) of the (super)moduli space. We choose as even moduli

on Σ_1: the positions of $(N+1)-3$ insertions,

on Σ_2: the positions of $(M+1)-1$ insertions, the modulus τ of the torus,

and $q=-t^2$,

and as odd moduli

on Σ_1: the supercoordinates of $(N+1)-2$ insertions,

on Σ_2: the supercoordinates of $M+1$ insertions.

Polchinski [6] emphasizes that a change of the moduli generally leads to a change of the local coordinate frame which describes the factorization. In our case, however, the (super)Beltrami differentials of the punctures are localized in the vicinity of those punctures. The τ integration also does not introduce additional terms, because the torus is flat. The θ_i integration ($i\neq 0$) eliminates the ghost term $\mathrm{e}^{-\phi}$ for the corresponding vertex generating the F_1-picture for that vertex operator. The q integration yields

$$\int \mathrm{d}q\, \mathrm{d}\bar{q} |(\mu_q, b)|^2 = (q\bar{q})^{-1} \oint \frac{\mathrm{d}z}{2\pi} z\, b(z) \oint \frac{\mathrm{d}\bar{z}}{2\pi} \bar{z}\, \bar{b}(\bar{z}) = (q\bar{q})^{-1} b_0 \bar{b}_0 . \tag{7}$$

The contribution from the supermodular parameter θ_0'' is

$$\int d\theta_0'' \, \theta_0''(2\pi)^{-1} \int d^2z \, \delta(\beta(z)) \, \hat{\mu}(z) \, e^{-\phi(0)} = (2\pi)^{-1} \int d^2z \frac{z}{2|z|} \delta(\beta(z)) \, \delta(|z|-r_0) \, e^{-\phi}$$
$$= (2\pi i)^{-1} \oint dz \, \delta(\beta(z)) \, \delta(\gamma(0)) = 1 \,. \tag{8}$$

The three remaining positions of vertex operators on the sphere are fixed at 0, 1 and ∞, and the two free odd coordinates are put equal to 0. In this way we get two vertex operators in the F_2-picture.

The vertex V_N is supposed to represent a spurious state, i.e. it is given as a pure BRST commutator. Such states decouple due to BRST invariance. In the antiholomorphic sector we have only bosonic fields and

$$c\bar{c} \, e^{-\phi} V_{\text{spurious}} = [\bar{Q}, c \, e^{-\phi} V^{s}] \,. \tag{9}$$

If the BRST current of the holomorphic sector

$$j_{\text{BRST}} = cT_{\text{B}} + \gamma T_{\text{F}} \tag{10}$$

would act on an operator including the ξ-field it would change not only the ghost charge but also the superghost charge. This is not the case, however. Therefore, we also have

$$c\bar{c} \, e^{-\phi} V_{\text{spurious}} = [Q, \bar{c} \, e^{-\phi} V^{s}] \,. \tag{11}$$

In the following, we will consider only the holomorphic part of Q_{BRST}, since the anomaly and the counterterms are left–right symmetric.

Now, by the usual arguments [6,7,9] we can deform the contour, which defines the commutator, and get surface terms in the moduli space due to

$$\langle [Q, b_0] \hat{O} \rangle = \frac{\partial}{\partial \log q} \langle \hat{O} \rangle \,, \tag{12}$$

where $\hat{O}$ is an arbitrary operator. For a tachyon-free theory the boundary $\tau \to 0$ gives no contribution.

We now study the tadpole configuration $M=0$, $q \to 0$:

$$A_{\text{T}} = \frac{8\pi G_{\text{N}}}{4\pi^3} \int dq \, d\bar{q} \frac{\partial}{\partial \log q} (q\bar{q})^{k^2+m^2-1}$$
$$\times \int d\tau \, d\bar{\tau} \langle |(\mu_\tau, b)|^2 \Phi_{F_1}(z_0'') \rangle \left\langle \bar{c} V_N^{s}(0) c \, \bar{c} V_1(1) c \, \bar{c} \partial c \, \Phi_{F_2}(\infty) \prod_{i=2}^{N-1} V_i(z_i) \right\rangle \prod_{i=2}^{N-1} \frac{2d^2 z_i}{\pi} \,. \tag{13}$$

This amplitude is non-zero for massless intermediate states

$$\int dq \, d\bar{q} \frac{\partial}{\partial \log q} (q\bar{q})^{k^2+m^2-1} \quad = 4\pi \quad \text{if } k^2+m^2=0 \,,$$
$$= 0 \quad \text{if } k^2+m^2>0 \,. \tag{14}$$

The construction of a string tree level counterterm, i.e. a BRST anomalous local operator insertion closely follows the analysis in ref. [7]. The bosonic part of the action

$$S_{\text{b}} = \int \frac{2d^2z}{\pi} \mathscr{L}(z) = \int \frac{2d^2z}{\pi} [\tfrac{1}{2} \bar{\partial} X^a \partial X^b G_{ab}(X) + \tfrac{8}{3} \bar{\partial}(\Phi(X) bc) + \text{c.c.}] \tag{15}$$

leads, in a weak field expansion, to the insertion of local operators $\delta\mathscr{L}(z)$. After BRST contour deformation, the corresponding change $\delta A_{g=0}$ is

$$\delta A_{g=0} = \mathrm{e}^{-2\Phi} \int \left\langle \bar{c} V_N^{\mathrm{s}}(0) c\, \bar{c} V_1(1) [Q, c\bar{c}\delta\mathscr{L}(\infty)] \prod_{i=2}^{N-1} V_i(z_i) \right\rangle \prod_{i=2}^{N-1} \frac{2\mathrm{d}^2 z_i}{\pi}$$

$$= \mathrm{e}^{-2\Phi} \int \left\langle \bar{c} V_N^{\mathrm{s}}(0) c\, \bar{c} V_1(1) c\, \partial c\, \bar{c} \frac{2}{\pi} \{ -\tfrac{1}{2}\beta^{\mathrm{G}}_{ab} \bar{\partial} X^a \partial X^b + \beta^{\Phi} [\tfrac{8}{3} \bar{\partial}(bc) + \text{c.c.}] \} \prod_{i=2}^{N-1} V_i(z_i) \right\rangle \prod_{i=2}^{N-1} \frac{2\mathrm{d}^2 z_i}{\pi} , \tag{16}$$

with

$$\beta^{\mathrm{G}}_{ab} = R_{ab} + 2\nabla_a \nabla_b \Phi , \tag{17}$$

and

$$\beta^{\Phi} = -(\nabla\Phi)^2 - \tfrac{1}{4} R . \tag{18}$$

We find as loop-corrected equations of motion

$$\beta^{\mathrm{G}}_{ab} = -\frac{1}{\pi\beta V_{d-1}} \langle \langle \bar{\partial} X_a \partial X_b \rangle \rangle , \tag{19}$$

$$\beta^{\Phi} = 0 , \tag{20}$$

where the double bracket means that modular integration as well as integration over the X^0 zero mode is included.

In order to evaluate the tadpole expectation value we now continue to imaginary periodic time. The sum over the right-handed spin structures contains a phase factor depending on the imaginary-time winding numbers n, m corresponding to the cycles of the torus [2–4, 13]. As pointed out in ref. [1], the string partition function is the logarithm of the thermodynamic partition function (i.e. it generates only connected string diagrams)

$$\log Z(\beta) = \beta F = \int \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^2} \sum_{n,m}{}' \beta \exp\left(\frac{-\beta^2}{2\pi \operatorname{Im}\tau} |n + m\tau|^2 \right) K(n, m) ,$$

$$K(n, m) = \sum_{\text{right spin structures } s} \chi(n, m, s)\, k(s) , \tag{21}$$

where χ is a phase factor and k includes the usual zero temperature contributions of the bosonic, fermionic and left-moving internal degrees of freedom. The tadpole is

$$\frac{1}{\beta V_{d-1}} \langle\!\langle \bar{\partial} X^i \partial X^i \rangle\!\rangle = \frac{-1}{\beta V_{d-1}} \beta F = \frac{-1}{\beta} \frac{\partial}{\partial\beta} (\beta F) = p \quad (i \neq 0) ,$$

$$\frac{1}{\beta V_{d-1}} \langle\!\langle \bar{\partial} X^0 \partial X^0 \rangle\!\rangle = \frac{1}{\beta V_{d-1}} \left(\langle\!\langle \bar{\partial} X^0_{\mathrm{qu}} \partial X^0_{\mathrm{qu}} \rangle\!\rangle + \langle\!\langle \partial \bar{X}^0_{\mathrm{cl}} \partial X^0_{\mathrm{cl}} \rangle\!\rangle \right) = \frac{-F}{V_{d-1}} - \frac{\beta}{V_{d-1}} \frac{\partial}{\partial\beta} F = -\rho . \tag{22}$$

As expected, it agrees with the thermal energy–momentum tensor of an ideal string gas. In this way we find the equations of motion for the heterotic string at finite temperature

$$R_{ab} + 2\nabla_a \nabla_b \Phi = 8\pi G_{\mathrm{N}} \exp(2\Phi)\, T_{ab} , \quad \Box\Phi = -(\nabla\Phi)^2 + \tfrac{1}{4} R = 0 . \tag{23}$$

It can be obtained from the effective lagrangian

$$\mathscr{L} = (16\pi G_{\mathrm{N}})^{-1} \exp(-2\Phi) [R + 4(\nabla\Phi)^2] + F(g_{00}^{1/2}\beta) . \tag{24}$$

After the field redefinition

$$g_{ab} \to \exp[4/(d-2)\Phi] g_{ab} \tag{25}$$

we obtain the field equations

$$\Box\Phi = -4\pi G_{\rm N}\exp(2\Phi)\,{\rm tr}\,T\,,\quad R_{ab}-\tfrac{1}{2}g_{ab}R+\frac{2}{d-2}\,[g_{ab}(\nabla\Phi)^2-2\nabla_a\nabla_b\Phi]=8\pi G_{\rm N}\exp(2\Phi)\,T_{ab}\,, \tag{26}$$

and the new effective lagrangian

$$\mathscr{L}=(16\pi G_{\rm N})^{-1}\left(R-\frac{4}{d-2}\,(\nabla\Phi)^2\right)+\exp(2\Phi)\,F(g_{00}^{1/2}\beta)\,. \tag{27}$$

The results found so far are those which one could have guessed immediately. Once it is understood, however, how to obtain these background field equations in a consistent and systematic manner, it becomes possible to investigate less trivial questions, in particular thermal corrections to two- and three-point functions. Also of interest would be the inclusion of higher order terms in the background field. Work on these problems is in progress.

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