Arithmetic properties of a theta lift from $GU(2)$ to $GU(3)$

Tobias Finis

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1 Introduction

The main subject of this thesis is the study of some finer arithmetic properties of a certain theta lift from $GU(2,\mathbb{Q})$ (or equivalently $\mathrm{GL}_2(\mathbb{Q})$) to $GU(3,\mathbb{Q})$, where $GU(n)$ denotes the quasi-split unitary similitude group in n variables with respect to a fixed imaginary quadratic field extension $K/\mathbb{Q}$. The lifting considered here was first studied by Kudla [Ku1, Ku2]: it takes holomorphic elliptic modular forms of level D (the negative of the discriminant of K) and character $\omega_{K/\mathbb{Q}}$ (the quadratic Dirichlet character associated to the extension $K/\mathbb{Q}$) to holomorphic, in general vector valued, modular forms of level one on $GU(3)$.

After integrality (away from the discriminant) of the suitably normalized lifting is demonstrated, it makes sense to reduce it mod ℓ for a prime ℓ, unramified in K. The main result is a precise determination of the kernel of the reduction mod ℓ in the case where ℓ splits in K, and the lifting goes to scalar modular forms, under the weak technical restriction $\ell \not| 2h_K$. Since Eisenstein series go to Eisenstein series under the lifting, as an application a criterion on congruences between Eisenstein series and cusp forms is obtained.

These results depend mainly on a careful study of the Fourier-Jacobi expansion of the lifting. A closed expression for the Fourier-Jacobi expansion is derived from Kudla's work, and its coefficients are then decomposed into primitive components as defined by Shintani [Shin]. The resulting formula may be of some independent interest, but it also allows to prove the cruical non-vanishing (modulo ℓ) of the lifting away from the expected kernel. As a second main ingredient a characteristic ℓ non-vanishing result on theta functions is proved, which is an analogue of a theorem of Washington [W2, Si2] on non-divisibility of Bernoulli numbers (special values of Dirichlet L-functions). In fact, by the work of Yang [Y] our result implies corresponding non-divisibility statements for special values of anticyclotomic L-functions of the field K.

To give a more detailed account, let $K = \mathbb{Q}(\sqrt{-D}) \subseteq \mathbb{C}$ be the imaginary quadratic field of discriminant $-D$ and $\delta = \sqrt{-D}$ fixed to have positive imaginary part. The skew-hermitian matrix

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \delta & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

defines a hermitian form $(\cdot,\cdot)$ of signature $(1,2)$ on $V = K^3$ by $(x,y) = -\delta^{-1}\bar{x}^{\mathrm{tr}}Ry$ for $x, y \in V$. Let $\mathfrak{o}_K$ be the ring of integers of K and fix the standard $\mathfrak{o}_K$-lattice $L = \mathfrak{o}_K^3$ in V.

We consider the group $G = GU(V) = GU(R)$ of unitary similitudes with respect to $(\cdot,\cdot)$, an algebraic group defined over $\mathbb{Q}$ with group of $\mathbb{Q}$-points

$$G(\mathbb{Q}) = \{g \in \mathrm{GL}_3(K) \,|\, \bar{g}^{\mathrm{tr}}Rg = \mu(g)R,\ \mu(g) \in \mathbb{Q}^\times\}.$$

For simplicity let us assume (only for this introduction) that K has class number one, such that we may easily work in an entirely classical setting. Also we consider for the moment only scalar modular forms on G. The group $G(\mathbb{R})$ operates on the symmetric domain

$$\mathfrak{D} = \{\mathfrak{z} = (z,w) \in \mathbb{C}^2 \,|\eta(\mathfrak{z}) = (z-\bar{z})/\delta - |w|^2 > 0\},$$

if we identify $\mathfrak{D}$ with the space of positive lines in $\mathbb{P}^2(\mathbb{C})$ by $\mathfrak{z} \mapsto (z : w : 1)$.

The function

$$j_1(g,\mathfrak{z}) = g_{31}z + g_{32}w + g_{33}, \quad g = (g_{ij}) \in G(\mathbb{R}),\ \mathfrak{z} \in \mathfrak{D},$$

is a holomorphic factor of automorphy for G. An automorphic form of weight k with respect to the level one discrete subgroup $\Gamma = \{g \in G(\mathbb{Q}) \mid gL = L\}$ is defined as a holomorphic function $F : \mathfrak{D} \to \mathbb{C}$ fulfilling the usual relation

$$F(\gamma(\mathfrak{z})) = j_1(\gamma, \mathfrak{z})^k F(\mathfrak{z}), \quad \mathfrak{z} \in \mathfrak{D}, \gamma \in \Gamma.$$

The space of these automorphic forms is denoted by $A(k, \Gamma)$. Obviously, it can be non-zero only if w_K, the number of units in K, divides k.

Kudla defined for each $k \geq 6$ a lifting $\mathcal{L}$ from the space $M_{k-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ of elliptic modular forms to the space $A(k, \Gamma)$ [Ku1]. We have

$$\mathcal{L}(f)(\mathfrak{z}) = \int_{\Gamma_0(D)\backslash\mathfrak{H}} f(\tau)\overline{\theta(\tau, \mathfrak{z})} y^{k-3} dx dy$$

(here $\tau = x + iy$) with the theta kernel function

$$\theta(\tau, \mathfrak{z}) = 2^{k-1} D^{-k/2} y^2 \sum_{X \in L} (X, P_+(\mathfrak{z})\eta(\mathfrak{z})^{-1})^k e^{2\pi i (2iy\eta(\mathfrak{z})^{-1}|(X, P_+(\mathfrak{z}))|^2 + \bar{\tau}(X,X))},$$

where we put

$$P_+(\mathfrak{z}) = \begin{pmatrix} z \\ w \\ 1 \end{pmatrix}$$

for $\mathfrak{z} = (z, w) \in \mathfrak{D}$.

Each automorphic form $F \in A(k, \Gamma)$ has a Fourier expansion whose coefficients are theta functions, a so-called Fourier-Jacobi expansion. The first part of our work, contained in Chapter 2, deals with the computation of the Fourier-Jacobi expansion of a lifted form $\mathcal{L}(f)$ in terms of f. Based on Kudla's work we obtain in Theorem 2.5 a first explicit formula for the Fourier coefficients. In the special case treated here it looks as follows: $\mathcal{L}(f)(\mathfrak{z}) = \sum_{r=0}^{\infty} g_r(w) q^r$, $q = e^{2\pi i z}$, with

$$\begin{aligned} g_0 &= \frac{(k-1)!}{2(2\pi i)^k} w_K L(\chi_k, k/2) \operatorname{Tr}_{\Gamma_0(D)\backslash \mathrm{SL}_2(\mathbb{Z})}(f(\tau)\vartheta(0,\tau))\big|_{\tau \to i\infty}, \\ g_r(w) &= T_r(\operatorname{Tr}_{\Gamma_0(D)\backslash \mathrm{SL}_2(\mathbb{Z})}(f(\tau)\vartheta(r\delta w, \tau)))\big|_{\tau = \tau_0}, \end{aligned}$$

where $\tau_0 \in \mathfrak{H}$ is such that $\mathfrak{o}_K = \mathbb{Z} + \mathbb{Z}\tau_0$, and ϑ is the theta function

$$\vartheta(w, \tau) = \sum_{a \in \mathfrak{o}_K} e^{2\pi i (\mathrm{N}(a)\tau + aw)}$$

associated to $\mathfrak{o}_K$. The Hecke character χ_k of K is defined by $\chi_k((x)) = (x/|x|)^k$.

An integral structure on $M_{k-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}})$, which also allows the consideration of modular forms mod ℓ for a prime ℓ, is given by looking at the lattice of forms with integral Fourier coefficients. The "q-expansion principle" explains why this is a good notion: the integral forms correspond to sections of a suitable sheaf on the moduli scheme, at least if we disregard primes dividing D. For $A(k, \Gamma)$ the corresponding notion is given by considering modular forms whose Fourier-Jacobi coefficients are integral theta functions. Here, in a way analogous to Shimura's definition of arithmetic theta functions [Shim4], a theta function g_r is declared to be integral if $e^{-\pi r\sqrt{D}|w|^2} g_r(w)$ is an algebraic integer for all $w \in K$. The existence of a good compactified moduli scheme (away from primes dividing D) in the unitary group case is known from work of Larsen [Lar1, Lar2], and we have therefore an analogous geometric interpretation of integrality.

Integrality of the lifting and reduction mod ℓ are discussed in the second part, Chapter 3. From our expression for g_r it may be deduced that the arithmetic variant $\mathcal{L}^{\mathrm{ar}} = \Omega_0^{-k}\mathcal{L}$ of the lifting, where $\Omega_0 = \eta(\tau_0)^2$ is a period associated to the lattice $\mathfrak{o}_K$, is integral away from D except for the constant term, i. e. for $f \in M_{k-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}[1/D])$ the Fourier-Jacobi coefficients g_r^{ar} of $\mathcal{L}^{\mathrm{ar}}(f)$ are integral away from D for all positive r. The corresponding general result is Theorem 3.5. Fixing a prime $\ell \not| D$, and embeddings $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_\ell : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_\ell$, we may (at least for $\ell \geq 5$) consequently consider the theta lift on the level of modular forms mod ℓ:

$$\bar{\mathcal{L}} : M_{k-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell) \to A(k, \Gamma; \bar{\mathbb{F}}_\ell),$$

in case $(\ell - 1) \not| k$; otherwise we have to restrict to cusp forms because the constant term may have a power of ℓ in its denominator.

In a first step it is not very difficult to determine a subspace of the space $M_{k-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell)$ contained in the kernel of $\bar{\mathcal{L}}$. This is the object of Section 3.2. Introducing the Hecke involution W_D on the modular curve $\Gamma_0(D)\backslash\mathfrak{H}$, we have $\bar{\mathcal{L}}(f) = 0$ for every modular form $f \in M_{k-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell)$, such that the Fourier coefficients a_n of $f|W_D$ with $\omega_{K/\mathbb{Q}}(n) = -1$ vanish. This is due to the fact that, roughly speaking, the trace $\mathrm{Tr}_{\Gamma_0(D)\backslash \mathrm{SL}_2(\mathbb{Z})}(f(\tau)\vartheta(r\delta w, \tau))$ already vanishes under these conditions.

It is much harder to show that for $\ell \geq 5$ split in K (in the general case we have to assume in addition $\ell \not| h_K$) this subspace is already the entire kernel. The proof occupies the remainder of Chapter 3 along with Chapter 4. The strategy is to look more closely at the Fourier-Jacobi expansion and to use Shintani's theory of primitive theta functions. This is the object of Sections 2.3 and 2.4. Shintani introduced an action of the group of ideals prime to rD on the space of all Fourier-Jacobi-coefficients (of degree $r > 0$, say), which leads to a decomposition of this space into eigenspaces parametrized by Hecke characters κ^* of K, whose restriction to $\mathbb{Q}$ is the quadratic character $\omega_{K/\mathbb{Q}}$. The eigenspaces contained in the subspace of primitive theta functions are one-dimensional. The Fourier-Jacobi coefficients of a Hecke eigenform are determined by the primitive components and the Hecke eigenvalues.

Proceeding from our closed formula for the Fourier coefficients g_r, we can compute the component of g_r in each of these subspaces. We give the result in the simplest case of a primitive theta function ϑ_κ representing the eigenspace of a character κ^* with exact conductor rD. Consider the form f as a function of pairs (L, x), where L is a lattice in $\mathbb{C}$ and $x \in L/DL$ of order D. Then we have

$$\langle \vartheta_\kappa, g_r \rangle = \frac{\overline{\vartheta_\kappa(0)}}{D} r^{k-2} \sum_{(L,x)} f(L, x + DL) \left(\frac{x}{|x|}\right)^{2k-1} \kappa^*((x))^{-1}.$$

Here L ranges over all lattices of index r in $\mathfrak{o}_K$ and for each L one considers all sublattices L' with $L/L' \simeq \mathbb{Z}/D\mathbb{Z}$ and $\mathfrak{o}_K L' = \mathfrak{o}_K$ and chooses a vector $x \in L'$ prime to rD. A more general statement can be found in Theorem 2.12. By the result of Yang mentioned above, the value $\vartheta_\kappa(0)$ is connected to the L-function of $\kappa^* \chi_k^{-2}$, see Proposition 2.11.

Getting back to the main task, how is it possible to show that for a modular form f over $\bar{\mathbb{F}}_\ell$ not in the "trivial kernel" described above, we have $\bar{g}_r \neq 0$ for some $r > 0$, where $\bar{g}_r$ are the Fourier-Jacobi coefficients of $\bar{\mathcal{L}}(f)$? Assuming $\ell \not| r$, from the canonical scalar product $\langle \cdot, \cdot \rangle$ one may easily construct a symmetric bilinear form b_{ar}, which is ℓ-integral for ℓ-integral arguments, and may therefore be reduced mod ℓ. Rewriting our formula yields (letting bars denote reduction mod ℓ via $i_\ell i_\infty^{-1}$):

$$\bar{b}_{\mathrm{ar}}(\bar{\vartheta}_\kappa, \bar{g}_r) = \frac{r^{k-2}}{D} \bar{\vartheta}_\kappa(0) \bar{P}(f, \bar{\eta}_f),$$

where ϑ_κ is an integral representative of the κ-eigenspace, and the character η_f of $(\mathfrak{o}_K/rD\mathfrak{o}_K)^\times$ (with reduction $\bar{\eta}_f$) is given by

$$\eta_f(x) = \kappa^*((x)) \left(\frac{x}{|x|}\right)^{1-2k}.$$

The sum $\bar{P}$ is defined by reducing the sum above:

$$\bar{P}(f, \bar{\eta}_f) = \sum_{H'=\langle Q\rangle} f(\bar{E}/(DH'), \hat{\pi}^*\bar{\omega}, \pi(Q))\bar{\eta}_f(x_Q)^{-1}.$$

Here $\bar{E}$ is the reduction mod ℓ of the elliptic curve $E \simeq \mathbb{C}/2\pi i\Omega_0\mathfrak{o}_K$ and $\bar{\omega}$ the invariant differential obtained by reducing $\omega = dz$. The sum ranges over all cyclic subgroups $H' = \langle Q\rangle$ of order rD of $\bar{E}[rD]$ which generate $\bar{E}[rD]$ as an $\mathfrak{o}_K$-module; for each H' we have the canonical isogeny $\pi : \bar{E} \to \bar{E}/(DH')$ of degree r (with dual $\hat{\pi}$), and to $Q \in \bar{E}[rD]$ corresponds $x_Q \in \mathfrak{o}_K/rD\mathfrak{o}_K$ by the analytic parametrization of E.

Take now an "auxilary prime" p, split in K, with $\ell \not| p(p-1)$, and consider $r = p^m$ with $m \geq 0$ and the set of all primitive Shintani eigenfunctions ϑ_κ in this infinite tower of spaces corresponding to characters κ^* of conductor $p^m D$. Equivalently, we have finitely many choices for the D-component and are varying the p-component of κ^* freely (among anticyclotomic characters). Because of our conditions on ℓ, the correspondence between a character κ^* and the $\bar{\mathbb{F}}_\ell$-valued Dirichlet character $\bar{\eta}_f$ is one-to-one, and Shintani theory works mod ℓ, i. e. there exists a decomposition into eigenspaces compatible with reduction mod ℓ. If we show the statements,

1. that $\bar{P}(f, \bar{\eta}_f) \neq 0$ for infinitely many $\bar{\eta}_f$ corresponding to characters κ^* in our set, and
2. that there exists a representative $\bar{\vartheta}_{\bar{\kappa}}$ (in the space of theta functions mod ℓ) with $\bar{\vartheta}_{\bar{\kappa}}(0) \neq 0$ for almost all (all but finitely many) primitive eigenspaces,

the non-vanishing of some $\bar{g}_r$ with r equal to a power of p follows from the above.

As for the first (rather weak) statement, its violation would imply an infinite number of strong relations for the values of a modular form closely related to f at elliptic curves with complex multiplication by an order (of p-power conductor) in K. This yields easily to a contradiction. See Theorem 3.21 below for more details.

The second statement is proved by applying ideas of Sinnott [Si1, Si2], who gave an algebraic proof of the theorem of Washington mentioned above, to the different situation considered here. Chapter 4 is devoted to this topic. Washington's theorem says that for a prime $p \neq \ell$, an integer $n \geq 1$ and a Dirichlet character χ (of the rational integers) the set of all Dirichlet characters ψ of p-power conductor, such that $\chi\psi(-1) = (-1)^n$ and

$$v(i_\ell i_\infty^{-1}(\frac{1}{2}L(1-n, \chi\psi))) > 0$$

(i_ℓ and i_∞ as above, and v denoting a valuation on $\bar{\mathbb{Q}}_\ell$) is finite. Instead of dealing with rational functions like Sinnott, the proof of our statement depends on making use of the algebro-geometric nature of theta functions. Assuming the existence of some character κ^* of large conductor contradicting the statement, by considering all Galois conjugates we finally arrive at an algebraic relation. An infinite set of points would have to be contained in a certain subvariety $\mathcal{D}$ of the abelian variety $\bar{E}^{(p-1)/w_K}$. It follows by a density argument that $\mathcal{D}$ has to contain a translate of a large abelian variety, from which it is possible to derive a contradiction. Combining our theorem with the work of Yang, we get as Corollary 4.7 a non-vanishing statement for anticyclotomic L-functions in characteristic ℓ.

Having obtained this result, it may be applied to the case of Eisenstein series, since the lifting of the standard Eisenstein series $E_{k-1,\omega_{K/\mathbb{Q}}}$ is the Eisenstein series $\mathcal{E}_k$ in $A(k,\Gamma)$. This is done in Chapter 5. Assuming ℓ odd, split in K, $(\ell-1) \not| k$ (this is not a restriction) as above, and in addition $(\ell-1) \not| (k-2)$, it is easy to see that $E_{k-1,\omega_{K/\mathbb{Q}}}$ modulo ℓ does not lie in the kernel of $\bar{\mathcal{L}}$, as described above. Therefore, the arithmetic variant $\Omega_0^{-k}\mathcal{E}_k$ is non-trivial mod ℓ. Using the geometry of the moduli surface, we see the existence of an ℓ-integral form h in $A(k,\Gamma)$ with constant term one. Consequently, if now the constant term $\Omega_0^{-k}g_0$ of the arithmetic Eisenstein series (which from the above is essentially a product of the L-functions $L(\chi_k, k/2)$ and $L(2-k, \omega_{K/\mathbb{Q}})$) is divisible by ℓ, the Eisenstein series is congruent mod ℓ to a cusp form, since we may subtract a suitable multiple of h. By the Deligne-Serre lemma, there exists then a Hecke eigenform in $A(k,\Gamma)$, whose Hecke eigenvalues are congruent to those of the Eisenstein series modulo a prime above ℓ.

To conclude, let us mention other possible applications and open problems which arise in this context. As a first application, the explicit Fourier coefficient formulas and the integrality theorem should allow us to interpolate the lifting ℓ-adically, i. e. to construct a lifting of ℓ-adic and Λ-adic modular forms.

Another topic is the ℓ-adic behaviour of a generalized lifting with additional level structure. Here new difficulties are to expected, since the lifting vanishes (even in characteristic zero) for certain supersingular local components by Gelbart-Rogawski-Soudry [GeRS]. Also, it would certainly be interesting to consider the case of inert primes ℓ. The kernel behaves here rather differently, and I do not have a precise result as in the split case (see the remarks at the end of Section 3.2). For Eisenstein series it seems that a different normalization of the lifting $\mathcal{L}^{\mathrm{ar}}$ is needed. Explicit computations indicate the existence of modular forms mod ℓ different from the Eisenstein series but with the same Hecke eigenvalues.

A very interesting problem is to obtain a criterion on congruences between stable and endoscopic forms on $GU(3)$. Let us sketch a possible strategy: we consider $\mathcal{L}$ and the "lift back" $\mathcal{L}^\dagger$ going from $A(k,\Gamma)$ to $M_{k-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ (or better to a certain subspace, or "trivial image") constructed from the same theta kernel. A Fourier coefficient formula and arithmeticity and integrality properties for this lift may be established (in fact it is much easier in this case, since we do not have to deal with theta functions). Now for a normalized Hecke eigenform $f \in M_{k-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}})$, by means of the Rallis inner product formula it seems to be not difficult to express $\mathcal{L}^\dagger(\mathcal{L}f)$ in terms of the twisted L-value $L(f_K \otimes \chi_k^{-1}, k/2)$ (see [Tan1, Tan2]). The non-triviality (under natural conditions) of the reduction of $\mathcal{L}(f)$ modulo a split prime ℓ is shown here, but to complete the scheme we need to know the surjectivity (mod ℓ) of $\mathcal{L}^\dagger$. This would allow us to conclude the existence of a congruence in the case where the cruical L-value is divisible by ℓ, or equivalently $\bar{\mathcal{L}}^\dagger(\bar{\mathcal{L}}\bar{f}) = 0$, by the same method as in the case of Eisenstein series. Unfortunately, a Fourier coefficient formula seems to be unsuited to demonstrate a surjectivity property of this kind.

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Notation We keep the notation introduced above but drop the class number one assumption. For each prime p (or generally prime ideal $\mathfrak{p}$ of a number field) we let v_p (resp. $v_\mathfrak{p}$) be the associated additive valuation. $\mathbb{A}$ (resp. $\mathbb{A}_K$) will denote the ring of adeles of $\mathbb{Q}$ (resp. K) and $\mathbb{A}_f$ (resp. $\mathbb{A}_{K,f}$) the ring of finite adeles. We write an idele $a \in \mathbb{A}^\times$ like $a = a_\infty a_f$ as a product of its infinite and finite part.

By $|\cdot|_\mathbb{A}$ we denote the norm on $\mathbb{A}^\times$. I_K denotes the group of fractional ideals of K. For each idele $a \in \mathbb{A}_{K,f}^\times$ let $(a_f) \in I_K$ be the associated ideal. In this way $\mathbb{A}_{K,f}^\times/\hat{\mathfrak{o}}_K \simeq I_K$. The ideal class group of K is denoted by Cl_K and its order, the class number of K, by h_K. The subgroup of ideal classes invariant under complex conjugation (the genus class group) is denoted by $\mathrm{Cl}_K^{\mathrm{inv}}$; by genus theory $\mathrm{Cl}_K = \mathrm{Cl}_K^2 \times \mathrm{Cl}_K^{\mathrm{inv}}$, and $\mathrm{Cl}_K^{\mathrm{inv}}$ has order $2^{\nu-1}$ if ν is the number of different prime divisors of D. Let $h_K' = h_K/2^{\nu-1}$ be the order of Cl_K^2. w_K is the number of units in K.

We have $\mathbb{A}_K^\times/\hat{\mathfrak{o}}_K K_\infty^\times K^\times \simeq \mathrm{Cl}_K$ as above. $I_K(\mathfrak{a})$ for an integral ideal $\mathfrak{a}$ of K denotes the group of ideals of K prime to $\mathfrak{a}$. By $P_{K,\mathfrak{a}} \subseteq I_K(\mathfrak{a})$ we mean the principal congruence subgroup of level $\mathfrak{a}$. We say a (unitary) Hecke character χ of K has weight k, if it has infinity type $\chi_\infty(x) = (x/|x|)^{-k}$.

2 Fourier-Jacobi coefficients of the theta lift

2.1 Automorphic forms on $GU(3)$ and theta lifts

This section sets up the context we will work in: we define holomorphic automorphic forms on G in a semi-classical way following Shintani and formulate Kudla's lifting in this context. For more details see [Shim4, Shin, Ku1]. In general we use Shintani's conventions.

The group G Let us first describe some standard subgroups of the group G. Let B be the Borel subgroup of all upper triangular matrices in G, which is also the stabilizer of the point $(1:0:0)$ in $\mathbb{P}^2(K)$. Its unipotent radical is denoted by H and is isomorphic to the Heisenberg group of a two-dimensional $\mathbb{Q}$-vector space. We write the elements of H as (w, u) with $w \in \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_a$ and $u \in \mathbb{G}_a$ by

$$(w,u) = \begin{pmatrix} 1 & \delta\bar{w} & u + \delta w\bar{w}/2 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

The center of H is the group of all elements $(0, u)$, and therefore isomorphic to $\mathbb{G}_a$.

The maximal torus T of G contained in B (the group of diagonal matrices in G) may be written as a direct product $T = Z(G)A$, where $Z(G) \simeq \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$ is the center of G and A may be identified with $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$ by $a \mapsto \mathrm{diag}(\bar{a}, 1, a^{-1})$. We have $B = Z(G)AH$ and set $M := AH = HA$.

Automorphy factors We now explain some functions associated to the action of $G(\mathbb{R})$ on its symmetric domain $\mathfrak{D}$. For each point $\mathfrak{z} \in \mathfrak{D}$ we define in addition to $P_+(\mathfrak{z})$ the matrix

$$P_-(\mathfrak{z}) = \begin{pmatrix} -\bar{w} & \bar{z} \\ -\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

and set $P(\mathfrak{z}) = (P_-(\mathfrak{z}), P_+(\mathfrak{z})) \in \mathbb{C}^{3\times 3}$. The columns of $P_-(\mathfrak{z})$ span the two-dimensional subspace of $V_\mathbb{C} = V \otimes_K \mathbb{C}$ orthogonal to the line spanned by $P_+(\mathfrak{z})$ and $(\cdot,\cdot)$ is negative, resp. positive definite on these spaces. The canonical holomorphic factors of automorphy for G are then given by

$$gP(\mathfrak{z}) = P(g\mathfrak{z}) \begin{pmatrix} \overline{\kappa(g,\mathfrak{z})} & 0 \\ 0 & j_1(g,\mathfrak{z}) \end{pmatrix}, \quad \mathfrak{z} \in \mathfrak{D}, \quad g \in G(\mathbb{R}). \tag{3}$$

Explicitly

$$\kappa(g,\mathfrak{z}) = \begin{pmatrix} -\delta\bar{g}_{21}w + \bar{g}_{22} & \delta(\bar{g}_{21}z + \bar{g}_{23}) \\ -\bar{g}_{31}w + \delta^{-1}\bar{g}_{32} & \bar{g}_{31}z + \bar{g}_{33} \end{pmatrix}.$$

We follow Shintani in setting $j_2(g,\mathfrak{z}) = (\mu(g)j_1(g,\mathfrak{z}))^{-1}(\det g)\kappa(g,\mathfrak{z})$ for the factor of automorphy with values in GL_2. We have $\det j_2(g,\mathfrak{z}) = (\det g)j_1(g,\mathfrak{z})^{-1}$.

Automorphic forms An automorphic form on G may be considered as a function on the coset space $G(\mathbb{Q})\backslash G(\mathbb{A})$. We consider forms of level one, i. e. require them to be right-invariant under the stabilizer $G(L)_f$ of $L\otimes\hat{\mathbb{Z}}$ in $G(\mathbb{A}_f)$, which is an open compact subgroup of this group. To fix the infinity type let ρ be an irreducible rational representation of $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times$ on a complex vector space $\mathcal{V}$. Such a representation determines a factor of automorphy J with values in $\mathrm{GL}(\mathcal{V})$ by

$$J(g,\mathfrak{z}) := \rho(j_2(g,\mathfrak{z}), \det g).$$

Since the center of G is the group of scalar matrices, i. e. isomorphic to $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$, by taking g to be an element z of the center of $G(\mathbb{R})$ we get from J a character c_ρ of $\mathbb{C}^\times$. Finally, fix a point $\mathfrak{z}_0 \in \mathfrak{D}$.

Now let $A(\rho, L)$ be the space of $\mathcal{V}$-valued functions F on the double coset space $G(\mathbb{Q})\backslash G(\mathbb{A})/G(L)_f$ such that for each $g_f \in G(\mathbb{A}_f)$ the function

$$J(g_\infty, \mathfrak{z}_0)c_\rho(\mu(g_\infty)^{-1/2})F(g_\infty g_f)$$

of $g_\infty \in G(\mathbb{R})$ depends only upon $\mathfrak{z} = g_\infty\mathfrak{z}_0$ and is holomorphic in $\mathfrak{z}$. (We denote this function on $\mathfrak{D}$ by F_{g_f}.) For a Hecke character χ of K, let $A(\rho, L, \chi)$ be the space of functions in $A(\rho, L)$ with central character χ. For this to be non-zero, χ has to be unramified and have infinity type $\chi(z_\infty) = c_\rho(|z_\infty|/z_\infty)$ for $z_\infty \in \mathbb{C}^\times$. For a description of the space $A(\rho, L)$ in classical language, see [Shin, p. 18].

An automorphic form $F \in A(\rho, L)$ is called a cusp form, if for every $g_f \in G(\mathbb{A}_f)$ the function F_{g_f} goes to zero for $\mathfrak{z} = (z, w)$, $z \to i\infty$. The space of cusp forms is denoted by $A_0(\rho, L)$ (resp. by $A_0(\rho, L, \chi)$ if we consider forms with central character χ).

Definition of the lifting We consider now theta liftings $\mathcal{L}$ from the spaces $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ to certain $A(\rho, L, \chi)$. Take $m \geq 5$ and choose a pair of integers (ν, μ) with $m = \mu - \nu - 1$ which determines a representation ρ as follows: let S_ν be the space of homogeneous polynominals of degree ν in two variables with GL_2 acting by $(g\psi)(v) = \psi(vg)$ for $\psi \in S_\nu$. Then $\rho = \rho_{\nu\mu}$ is defined on the dual space $\mathcal{V} = S_\nu^*$ (which carries a GL_2-action by the contragredient representation) by

$$(\rho(x,y))\psi^* = y^\mu(\det x)^{\nu-\mu}x\psi^*, \tag{4}$$

and therefore

$$J(g,\mathfrak{z})\psi^* = j_1(g,\mathfrak{z})^\mu(\mu(g)^{-1}\kappa(g,\mathfrak{z}))\psi^*.$$

To fix the lifting, choose an unramified Hecke character χ of weight $\nu + \mu$ and an additional unramified character ε of weight zero (i. e. a character of the ideal class group Cl_K). We define the theta kernel $\theta = \theta_{\nu,\mu;\varepsilon,\chi}$ on $\mathfrak{H} \times G(\mathbb{A})$ by

$$\overline{\theta(\tau, g; \psi)} = \sum_{(b)\in\mathrm{Cl}_K} \varepsilon(\det g)|\mathrm{N}(b)\mu(g_f)|_{\mathbb{A}}^{(\nu+\mu)/2}(\varepsilon^3\chi^{-1})(b)\overline{\theta_b(\tau, g; \psi)} \tag{5}$$

(b goes over a system of representatives for Cl_K in $\mathbb{A}_{K,f}$) with

$$\begin{aligned}\theta_b(\tau, g; \psi) &= 2^{\mu-1} D^{-\mu/2} y^{2+\nu} \eta(\mathfrak{z}_0)^{-\mu} \mu(g_\infty)^{-(\nu+\mu)/2} \\ &\quad \sum_{X \in b g_f L} (X, g_\infty X_0)^\mu \bar{\psi}((X, g_\infty Y_0)) \\ &\quad e^{2\pi i |\mathrm{N}(b)\mu(g_f)|_{\mathbb{A}} (2iy \frac{|(X, g_\infty X_0)|^2}{\mu(g_\infty)\eta(\mathfrak{z}_0)} + \bar{\tau}(X,X))} \end{aligned} \tag{6}$$

for $\psi \in S_\nu$. Here $(X, Y) = -\delta^{-1} \bar{X}^{\mathrm{tr}} R Y$ for a 3×2-matrix Y, $X_0 = P_+(\mathfrak{z}_0)$, $Y_0 = P_-(\mathfrak{z}_0)$. We may easily check that $\theta_{\lambda b}(\tau, g) = \bar{\lambda}^{\nu+\mu} \theta_b(\tau, g)$, $\lambda \in K^\times$, so that $\theta(\tau, g)$ is well-defined. The function θ takes values in the antilinear dual $\bar{S}_\nu^*$ and $\bar{\theta}$ takes values in S_ν^*.

The following proposition is nothing more than a translation of Kudla's results in [Ku1] to our slightly different setting.

Proposition 2.1. *1. The function θ is automorphic in the argument τ, i. e. for $\tau \in \mathfrak{H}$ and $\gamma \in \Gamma_0(D)$ we have*

$$\theta(\gamma(\tau), g) = \omega_{K/\mathbb{Q}}(\gamma) j(\gamma, \tau)^m \theta(\tau, g).$$

Moreover, it is rapidly decreasing in this argument.

2. θ is a smooth function on $G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(L)_f$ in the argument g.

3. Define the lifting $\mathcal{L} = \mathcal{L}_{\nu,\mu;\varepsilon,\chi}$ by

$$\mathcal{L}(f)(g) = \int_{\Gamma_0(D)\backslash \mathfrak{H}} f(\tau) \overline{\theta(\tau, g)} y^{m-2} dx dy$$

for $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$. Then $\mathcal{L}(f)$ is an element of $A(\rho_{\nu\mu}, L, \chi)$. If f is a cusp form, so is $\mathcal{L}(f)$.

Proof. The first assertion follows from results of Shintani (see [Ku1, p. 7][1]). The second assertion is easily checked from the definitions. The remaining assertions may without difficulties be reduced to the results of Kudla: let $F = \mathcal{L}(f)$ and $g_f \in G(\mathbb{A}_f)$. Then

$$F_{g_f}(\mathfrak{z}) = \varepsilon(\det g_f) |\mu(g_f)|_{\mathbb{A}}^{(\nu+\mu)/2} \sum_{\mathfrak{b}} \mathrm{N}(\mathfrak{b})^{-(\nu+\mu)/2} (\varepsilon^3 \chi^{-1})(\mathfrak{b}) F_{g_f, \mathfrak{b}}(\mathfrak{z}), \tag{7}$$

where $F_{g_f,\mathfrak{b}}$ is the scalar product of f and $\Theta_{g_f,\mathfrak{b}}$, which is defined by

$$\begin{aligned}\Theta_{g_f,\mathfrak{b}}(\tau, \mathfrak{z}; \psi) &= 2^{\mu-1} D^{-\mu/2} y^{2+\nu} \sum_{X \in \mathfrak{b} g_f L} (X, P_+(\mathfrak{z}))^\mu \eta(\mathfrak{z})^{-\mu} \bar{\psi}((X, P_-(\mathfrak{z}))) \\ &\quad e^{2\pi i \mathrm{N}(\mathfrak{b})^{-1} |\mu(g_f)|_{\mathbb{A}} (2iy\eta(\mathfrak{z})^{-1} |(X, P_+(\mathfrak{z}))|^2 + \bar{\tau}(X,X))}. \end{aligned} \tag{8}$$

This theta kernel function is of the type considered by Kudla. The holomorphy of F_{g_f} and the cusp form property follow from his results.

Eisenstein series The theta lifting takes Eisenstein series to Eisenstein series. To be more precise, let

$$e_{m,\omega_{K/\mathbb{Q}}}(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(D)} \omega_{K/\mathbb{Q}}(\gamma) j(\gamma, \tau)^{-m}$$

[1] One only has to check that the restriction $c \equiv 0\,(2)$ made in Theorem 2.2 there is actually unnecessary.

(Γ_∞ denotes the stabilizer of $i\infty$ in $\Gamma_0(D)$) be the standard Eisenstein series for $\Gamma_0(D)$ of character $\omega_{K/\mathbb{Q}}$, normalized to have constant term one. Then the lifting $F = \mathcal{L}_{\nu,\mu;\varepsilon,\chi}(e_{m,\omega_{K/\mathbb{Q}}})$ is given by

$$F(g) = \frac{(\mu-1)!}{2(2\pi\sqrt{D})^\mu} w_K L(\chi\varepsilon^{-3}, (\mu-\nu)/2) \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g),$$

with

$$\begin{aligned} f(g;\psi) &= \varepsilon(\det g)|\mu(g)|_{\mathbb{A}}^{(\nu-\mu)/2}(g_\infty^{-1}e_1, X_0)^{-\mu}\psi(\overline{(g_\infty^{-1}e_1, Y_0)}) \\ &\quad (\chi\varepsilon^{-3})(\mathfrak{c}(g_f))\mathrm{N}(\mathfrak{c}(g_f))^{(\nu-\mu)/2}, \end{aligned}$$

where $\mathfrak{c}(g_f)$ is the inverse of the ideal generated by the entries of $g_f^{-1}e_1$. The function f, which may easily be written as a product of local factors, has the property

$$f(bg) = \chi(z)(\chi^{-1}\varepsilon)(a)|\mathrm{N}(a)|_{\mathbb{A}}^{(\mu-\nu)/2} f(g)$$

for $b = z\mathrm{diag}(\bar{a}, 1, a^{-1})u \in B(\mathbb{A})$. So our results contain the computation of Fourier-Jacobi coefficients of Eisenstein series as a special case.

We shortly remark on how to prove our statement. The integral over $\Gamma_0(D)\backslash\mathfrak{H}$ of $e_{m,\omega_{K/\mathbb{Q}}}$ against $\overline{\theta(\tau,g)}$ may be unfolded to get

$$F(g) = \int_{\Gamma_\infty\backslash\mathfrak{H}} \overline{\theta(\tau,g)} y^{m-2} dx dy$$

and the easy evaluation of this integral yields

$$\begin{aligned} F(g) &= \frac{(\mu-1)!}{2(2\pi\sqrt{D})^\mu}\varepsilon(\det g)\mu(g_\infty)^{(\mu-\nu)/2}\sum_b |\mathrm{N}(b)\mu(g_f)|_{\mathbb{A}}^{(\nu-\mu)/2}(\chi^{-1}\varepsilon^3)(b) \\ &\quad \sum_{X\in bg_f L,\, X\neq 0,\, (X,X)=0} (X, g_\infty X_0)^{-\mu}\psi(\overline{(X, g_\infty Y_0)}). \end{aligned}$$

From this we may deduce the final result by using the transitivity of the operation of $G(\mathbb{Q})$ on the cone of all $X \in K^3\backslash\{0\}$ with $(X,X) = 0$.

Hecke operators We may compute the action of Hecke operators on the lifting. Let us define for each prime ideal $\mathfrak{p}$ of $\mathfrak{o}_K$ (lying above the prime p of $\mathbb{Q}$) a Hecke operator $T_\mathfrak{p}$ acting on $A(\rho, L, \chi)$ by

$$T_\mathfrak{p} F(g) = \int_{S_\mathfrak{p}} F(gx)dx$$

where $S_\mathfrak{p}$ is a subset of $G(\mathbb{Q}_p)$ defined as follows (see [Shin, p. 19]): if $\bar{\mathfrak{p}} = \mathfrak{p}$,

$$S_\mathfrak{p} = \{g \in G(\mathbb{Q}_p) \,|\, g(L\otimes\mathbb{Z}_p) \subseteq (L\otimes\mathbb{Z}_p),\ v_p(\mu(g)) = v_p(\mathrm{N}(\mathfrak{p}))\},$$

and if p splits in K,

$$S_\mathfrak{p} = \{g \in G(\mathbb{Q}_p) \,|\, g(L\otimes\mathbb{Z}_p) \subseteq (L\otimes\mathbb{Z}_p),\ \det g \in \mathfrak{p}\bar{\mathfrak{p}}^2 \otimes \mathbb{Z}_p\}.$$

If F is a Hecke eigenform with central character χ and eigenvalues $\lambda(\mathfrak{p})$, Shintani defined an associated L-function $\zeta(F,\xi,s)$ for a Hecke character ξ of K [Shin, p. 81]: $\zeta(F,\xi,s) = \prod_\mathfrak{p} P_\mathfrak{p}(\xi(\mathfrak{p})\mathrm{N}(\mathfrak{p})^{-(s+1)})^{-1}$ with

$$P_\mathfrak{p}(x) = \begin{cases} 1 - \lambda(\bar{\mathfrak{p}})\chi(\mathfrak{p})^{-1}N^{-1}x + \lambda(\mathfrak{p})\chi(\mathfrak{p})^{-1}N^{-1}x^2 - \chi(\bar{\mathfrak{p}}\mathfrak{p}^{-1})x^3, & \mathfrak{p} \neq \bar{\mathfrak{p}}, \\ (1-x)(1 + (N^{1/2} - \lambda(\mathfrak{p})\chi(\mathfrak{p})^{-1})N^{-1}x + x^2), & \mathfrak{p} = p\mathfrak{o}_K, \\ (1-x)(1 + (N - \lambda(\mathfrak{p})\chi(\mathfrak{p})^{-1})N^{-1}x + x^2), & \mathfrak{p} \,|\, \delta\mathfrak{o}_K. \end{cases}$$

(Here $N = \mathrm{N}(\mathfrak{p})$.) For a description of ζ as an automorphic L-function with respect to a representation of the L-group of G, see [Ku2, p. 340].

The following proposition is again essentially due to Kudla [Ku2]. A general result was obtained by Gelbart and Rogawski [GeR, p. 469]. Let W_D be the Hecke involution on $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ given by the action of the matrix $\begin{pmatrix} 0 & 1 \\ -D & 0 \end{pmatrix}$ (cf. Section 3.2 below).

Proposition 2.2. *If $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ is a Hecke eigenform with eigenvalues a_p, $f|W_D$ has eigenvalues $\tilde{a}_p$, and the lifting $\mathcal{L}_{\nu,\mu;\varepsilon,\chi}(f)$ is non-zero, it is again a Hecke eigenform with eigenvalues*

$$\lambda(\mathfrak{p}) = \begin{cases} p^{2-(\mu-\nu)/2}\varepsilon(\bar{\mathfrak{p}})a_p + p(\chi\varepsilon^{-2})(\bar{\mathfrak{p}}), & \mathfrak{p} \neq \bar{\mathfrak{p}}, \\ p^{2-(\mu-\nu)/2}\varepsilon(\bar{\mathfrak{p}})(a_p + \tilde{a}_p) + p(\chi\varepsilon^{-2})(\bar{\mathfrak{p}}), & \mathfrak{p}|\delta\mathfrak{o}_K, \\ p^{4-\mu+\nu}a_p^2 + 2p^2 + p, & \mathfrak{p} = p\mathfrak{o}_K, \end{cases}$$

and we have the relation

$$\zeta(\mathcal{L}(f), \xi, s) = L(f_K \otimes \varepsilon\chi^{-1}\xi, s + (\mu - \nu)/2)L(\varepsilon^{-2}\xi, s + 1),$$

where f_K denotes the base change of f to $\mathrm{GL}(2, K)$.

If f is a cusp form, it is well known that $\tilde{a}_p = \bar{a}_p$. We omit the proof of this proposition, since it is completely analogous to Kudla's arguments in [Ku2] for the special case $K = \mathbb{Q}(i)$.

2.2 Fourier-Jacobi expansions

In this section we compute the Fourier-Jacobi expansion of a theta lift of the type defined above. Let us first recall the relevant definitions from Shintani.

Fourier-Jacobi coefficients Let λ be the additive character of $\mathbb{A}/\mathbb{Q}$ normalized by $\lambda(x_\infty) = e^{2\pi i x_\infty}$. Then the Fourier-Jacobi coefficients of a form $F \in A(\rho, L, \chi)$ are given by integration against the center of H:

$$F_r(g) = \int_{\mathbb{A}/\mathbb{Q}} F((0, u)g)\lambda(-ru)du, \quad F(g) = \sum_{r \in \mathbb{Q}} F_r(g)$$

[Shin, p. 27], [GeR, p. 451]. The form F is determined by the restriction of the F_r to $M(\mathbb{A})$.

The "classical" Fourier-Jacobi coefficients may be extracted out of this functions as follows: set for $w_\infty \in \mathbb{C}$ and $a \in \mathbb{A}_K^\times$

$$g_{r,a}(w_\infty) = \rho\left(\begin{pmatrix} \bar{a}_\infty & \delta\bar{w}_\infty \\ 0 & 1 \end{pmatrix}, \bar{a}_\infty/a_\infty\right) F_r((w_\infty, 0)a)e^{-\pi i r\delta(|a_\infty|^2 + |w_\infty|^2)}. \tag{9}$$

This depends only on $\mathfrak{a} = (a_f)$ (in particular, not on a_∞ at all) and w_∞ and is a holomorphic $\mathcal{V}$-valued function in this variable. Indeed it belongs to a space of generalized theta functions in the sense of Shimura [Shim4]. More precisely, we may define for a (fractional) ideal $\mathfrak{a}$ of $\mathfrak{o}_K$ and an element $r \in \mathbb{Q}^{\geq 0}$, such that $r\mathrm{N}(\mathfrak{a})$ is integral, the space $T_{r,\mathfrak{a};\rho}$ as the space of $\mathcal{V}$-valued holomorphic functions ϑ on $\mathbb{C}$ satisfying the functional equation

$$\vartheta(w + l) = \psi(l)e^{-2\pi i r\delta\bar{l}(w+l/2)}\rho(\begin{pmatrix} 1 & \delta\bar{l} \\ 0 & 1 \end{pmatrix}, 1)\vartheta(w), \quad l \in \mathfrak{a} \tag{10}$$

where $\psi(l) = (-1)^{rD|l|^2}$ is a semi-character on $\mathfrak{a}$. Then $g_{r,a} \in T_{r,\mathfrak{a};\rho}$ whenever $r \geq 0$ and $r\mathrm{N}(\mathfrak{a})$ is integral, and vanishes otherwise. Moreover, replacing a by λa with $\lambda \in K^\times$ we have the relation

$$g_{r/\mathrm{N}(\lambda),\lambda a}(\lambda w) = \rho(\mathrm{diag}(\bar{\lambda}, 1), \bar{\lambda}/\lambda) g_{r,a}(w). \tag{11}$$

We will often write $g_{r,\mathfrak{a}} = g_{r,a}$ for $\mathfrak{a} = (a_f)$.

Our Fourier-Jacobi coefficients are therefore parametrized by pairs $(\mathfrak{a}, d)$ consisting of a fractional ideal $\mathfrak{a}$ of $\mathfrak{o}_K$ and a non-negative integer $d = r\mathrm{N}(\mathfrak{a})$. We may define the space $\mathcal{T}_{d,\rho}$ of Fourier-Jacobi coefficients of degree d as the space of all vectors $(t_\mathfrak{a}) \in \prod_{\mathfrak{a} \in I_K} T_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}$, such that

$$t_{\lambda\mathfrak{a}}(\lambda w) = \rho(\mathrm{diag}(\bar{\lambda}, 1), \bar{\lambda}/\lambda) t_\mathfrak{a}(w).$$

Obviously always $g_d := (g_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}) \in \mathcal{T}_{d,\rho}$. After choosing a system of representatives $\mathcal{A}$ for the ideal classes of K we get an isomorphism

$$\mathcal{T}_{d,\rho} \simeq \bigoplus_{\mathfrak{a} \in \mathcal{A}} T^1_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho},$$

where $T^1_{r,\mathfrak{a};\rho} \subseteq T_{r,\mathfrak{a};\rho}$ denotes the subspace of theta functions ϑ invariant under the action of the roots of unity:

$$\vartheta(\omega w) = \rho(\mathrm{diag}(\omega^{-1}, 1), \omega^{-2})\vartheta(w), \quad \omega \in \mathfrak{o}_K^\times.$$

The coefficient $g_{0,\mathfrak{a}}$ is the constant term of F at the cusp corresponding to $\mathfrak{a}$. It is an element of the one-dimensional space $T_{0,\mathfrak{a};\rho}$; for $\rho = \rho_{\nu\mu}$ this space consists of the constant functions which are multiples of the functional $\psi \mapsto \psi(0,1)$ on S_ν. Consequently, $\mathcal{T}_{0,\rho}$ has dimension h_K.

It is easy to extend these considerations to automorphic forms $F \in A(\rho, L)$ without central character. In this case we define F_r as above, and set for $a, b \in \mathbb{A}_K^\times$:

$$g_{r,a,b}(w_\infty) = \rho\left(\begin{pmatrix} \bar{a}_\infty & \delta\bar{w}_\infty \\ 0 & 1 \end{pmatrix}, \bar{a}_\infty/a_\infty\right) F_r(b(w_\infty, 0)a) e^{-\pi i r\delta(|a_\infty|^2 + |w_\infty|^2)},$$

where we regard b as an element of the center of $G(\mathbb{A})$. In b this function depends only upon b_∞ and $\mathfrak{b} = (b_f)$, and the dependence on the infinity component is described by $g_{r,a,b_\infty b} = c_\rho(|b_\infty|/b_\infty) g_{r,a,b}$ for $b_\infty \in \mathbb{C}^\times$. We may therefore restrict to $b \in \mathbb{A}_{K,f}^\times$ and write $g_{r,a,b} = g_{r,\mathfrak{a},\mathfrak{b}}$ for $\mathfrak{a} = (a_f)$, $\mathfrak{b} = (b_f)$. Except for the additional parameter $\mathfrak{b}$ there are no changes, and we get Fourier-Jacobi coefficients $g_{d;\mathfrak{b}} \in \mathcal{T}_{d,\rho}$. If F has central character χ, we have $g_{r,\mathfrak{a},\mathfrak{b}} = \chi(\mathfrak{b}) g_{r,\mathfrak{a}}$.

For some purposes (operation of Hecke and Shintani operators) it is important to split the spaces $\mathcal{T}_{d,\rho}$ into naturally defined subspaces. We have the natural exact sequence of genus theory

$$1 \longrightarrow \mathrm{Cl}_K^2 \longrightarrow \mathrm{Cl}_K \longrightarrow \mathrm{N}(I_K)/\mathrm{N}(K^\times) \longrightarrow 1,$$

so choosing $C \in \mathrm{N}(I_K)/\mathrm{N}(K^\times)$ yields a subspace $\mathcal{V}_{d,C;\rho}$ of $\mathcal{T}_{d,\rho}$ by restricting the index $\mathfrak{a}$ to the preimage of C. Obviously $\mathcal{T}_{d,\rho}$ is the direct sum of this subspaces. It is not difficult to see that $\mathcal{V}_{d,C;\rho}$ is canonically isomorphic to Shintani's adelically defined space $V_{d/c}(\rho, c)$ for any representative c of C in $\mathbb{Q}^\times$ [Shin, p. 29].

For later use we recall the action of the Heisenberg group on $\mathcal{V}$-valued functions and the scalar products on $T_{r,\mathfrak{a};\rho}$ and $\mathcal{T}_{d,\rho}$. For $l \in \mathbb{C}$ and $f : \mathbb{C} \to \mathcal{V}$ define

$$(A_l f)(w) = e^{2\pi i r\delta\bar{l}(w+l/2)} \rho(\begin{pmatrix} 1 & -\delta\bar{l} \\ 0 & 1 \end{pmatrix}, 1) f(w + l). \tag{12}$$

For $l \in \mathfrak{a}^* = (r\mathrm{N}(\mathfrak{a})D)^{-1}\mathfrak{a}$, the dual lattice, A_l is an endomorphism of $T_{r,\mathfrak{a};\rho}$; it acts by multiplication with $\psi(l)$, if $l \in \mathfrak{a}$.

To define a scalar product on $T_{r,\mathfrak{a};\rho}$, take a scalar product $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ on $\mathcal{V}$ such that $\rho(\mathrm{diag}(t,1),1)$ acts as a unitary operator for $|t| = 1$ and set

$$\langle\vartheta_1,\vartheta_2\rangle = \frac{2}{\sqrt{D}\mathrm{N}(\mathfrak{a})}\int_{\mathbb{C}/\mathfrak{a}}\langle(A_u\vartheta_1)(0),(A_u\vartheta_2)(0)\rangle_{\mathcal{V}}du. \tag{13}$$

(Actually, a semi-definite hermitian product on $\mathcal{V}$ may suffice, see below.) The A_l are unitary operators with respect to this scalar product.

To get a scalar product on $\mathcal{T}_{d,\rho}$, fix a scalar product $\langle\cdot,\cdot\rangle_1$ fulfilling the condition above and set $\langle x,y\rangle_\alpha := \langle\rho(\mathrm{diag}(\alpha,1),1)x,\rho(\mathrm{diag}(\alpha,1),1)y\rangle_1$. Then set

$$\langle g,h\rangle = \sum_{\mathfrak{a}}\langle g_{\mathfrak{a}},h_{\mathfrak{a}}\rangle_{\mathrm{N}(\mathfrak{a})^{-1/2}}$$

for $g = (g_{\mathfrak{a}})$, $h = (h_{\mathfrak{a}}) \in \mathcal{T}_{d,\rho}$, where $\mathfrak{a}$ runs over a system of representatives for the ideal classes of K. In the case of scalar modular forms we always choose $\langle x,y\rangle_1 = \bar{x}y$, of course.

Ordinary and generalized theta functions The generalized theta functions introduced in the last paragraph are closely connected to ordinary (scalar valued) theta functions (cf. [Shim4, p. 580], [Shim3]). Assume (without any loss of generality) that the representation ρ is such that $\mathcal{V} = S_\nu^*$, and $\rho(n,1)$ acts on $\mathcal{V}$ by the canonical GL_2-action for all $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ (this is certainly the case for the representation defined in Section 2.1). For $\psi^* \in \mathcal{V}$ and $0 \le l \le \nu$ write $\psi_l^* = \psi^*(X^{\nu-l}Y^l)$. Fix a rational number $r > 0$. Define now for every λ with $0 \le \lambda \le \nu$ a linear differential operator $d_{\nu\lambda}$ from holomorphic functions on $\mathbb{C}$ to $\mathcal{V}$-valued holomorphic functions on $\mathbb{C}$ by

$$(d_{\nu\lambda}f)_l(u) = \begin{cases} r^{-l}\binom{l}{\lambda}\left(\frac{1}{2\pi i}\frac{d}{du}\right)^{l-\lambda}f(u), & l \ge \lambda, \\ 0, & \text{otherwise.}\end{cases} \tag{14}$$

For a fractional ideal $\mathfrak{a}$ such that $r\mathrm{N}(\mathfrak{a})$ is integral, we consider the spaces $T_{r,\mathfrak{a}}$ and $T_{r,\mathfrak{a};\rho}$ (here, and in the following, we skip ρ in $T_{r,\mathfrak{a};\rho}$, if we are dealing with scalar modular forms).

Proposition 2.3. *The operators $d_{\nu\lambda}$ have the following properties.*

1. *$d_{\nu\lambda}$ commutes with the action of the Heisenberg group, i. e. $d_{\nu\lambda}A_a = A_{a;\rho}d_{\nu\lambda}$ for $a \in \mathbb{C}$.*

2. *$d_{\nu\lambda}$ is an injection of $T_{r,\mathfrak{a}}$ into $T_{r,\mathfrak{a};\rho}$.*

3. *$T_{r,\mathfrak{a};\rho}$ is the orthogonal direct sum of the images of the $d_{\nu\lambda}$.*

4. *$d_{\nu\lambda}$ is compatible with the natural inner products:*

$$\langle d_{\nu\lambda}\vartheta_1, d_{\nu\lambda}\vartheta_2\rangle = C_{\nu\lambda}(r)\langle\vartheta_1,\vartheta_2\rangle.$$

Here the constant $C_{\nu\lambda}(r)$ is determined in terms of the inner product $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ on $\mathcal{V}$ as

$$C_{\nu\lambda}(r) = r^{-2\lambda}\sum_{l=\lambda}^{\nu}\langle e_l,e_l\rangle_{\mathcal{V}}\frac{l!}{\lambda!}\binom{l}{\lambda}\left(\frac{\sqrt{D}}{2\pi r}\right)^{l-\lambda},$$

where $e_l \in \mathcal{V}$ is defined by $e_l(X^{\nu-k}Y^k) = \delta_{lk}$.

We remark that it suffices from the above to take a semi-definite hermitian product $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ with $\langle e_\nu, e_\nu\rangle_{\mathcal{V}} > 0$ to obtain a positive definite inner product on $T_{r,\mathfrak{a};\rho}$.

Proof. The first assertion is just a computation from the definitions and the second is an immediate consequence (take $a \in \mathfrak{a}$). To show that $T_{r,\mathfrak{a};\rho}$ is the direct sum of the images of $d_{\nu\lambda}$, we prove by descending induction on l the statement: the sum of $\mathrm{im} d_{\nu\lambda}$ for $l \le \lambda \le \nu$ is the space of all $\Theta \in T_{r,\mathfrak{a};\rho}$ with $\Theta_n = 0$ for $n < l$. To begin, the case $l = \nu + 1$ is trivial, and taking Θ with all components below l zero, the l-th component is an element ϑ of $T_{r,\mathfrak{a}}$. So subtracting $r^l d_{\nu l}\vartheta$ deletes the l-th component and we have reduced the statement for l to the statement for $l+1$.

It remains to show the orthogonality of the $\mathrm{im} d_{\nu\lambda}$ and the scalar product formula. For this purpose, let $\Theta \in T_{r,\mathfrak{a};\rho}$ and $\vartheta \in T_{r,\mathfrak{a}}$. We compute

$$(A_u d_{\nu\lambda}\vartheta)_l(0) = r^{-l}\binom{l}{\lambda} e^{-\pi i r\delta|u|^2}\left(\frac{1}{2\pi i}\frac{\partial}{\partial u}\right)^{l-\lambda}(\vartheta(u)e^{2\pi i r\delta|u|^2})$$

for $l \ge \lambda$. Setting $\varphi(u) = \rho(\begin{pmatrix} 1 & -\delta\bar{u} \\ 0 & 1 \end{pmatrix}, 1)\Theta(u)$, we obtain

$$\begin{aligned}\langle\Theta, d_{\nu\lambda}\vartheta\rangle &= \frac{2}{\sqrt{D}\mathrm{N}(\mathfrak{a})}\sum_{l=\lambda}^{\nu}\langle e_l, e_l\rangle_{\mathcal{V}} r^{-l}\binom{l}{\lambda} \\ &\quad \int_{\mathbb{C}/\mathfrak{a}}\overline{\varphi_l(u)}\left(\frac{1}{2\pi i}\frac{\partial}{\partial u}\right)^{l-\lambda}(\vartheta(u)e^{2\pi i r\delta|u|^2})du \\ &= \frac{2}{\sqrt{D}\mathrm{N}(\mathfrak{a})}\sum_{l=\lambda}^{\nu}\langle e_l, e_l\rangle_{\mathcal{V}} r^{-l}\binom{l}{\lambda} \\ &\quad \int_{\mathbb{C}/\mathfrak{a}}\overline{\left(\frac{1}{2\pi i}\frac{\partial}{\partial \bar{u}}\right)^{l-\lambda}\varphi_l(u)}\vartheta(u)e^{2\pi i r\delta|u|^2}du\end{aligned}$$

using integration by parts. It follows from the definition of φ that

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial \bar{u}}\right)^{l-\lambda}\varphi_l(u) = \frac{l!}{\lambda!}\left(\frac{\sqrt{D}}{2\pi}\right)^{l-\lambda}\varphi_\lambda(u),$$

and therefore we get

$$\langle\Theta, d_{\nu\lambda}\vartheta\rangle = r^\lambda C_{\nu\lambda}(r)\frac{2}{\sqrt{D}\mathrm{N}(\mathfrak{a})}\int_{\mathbb{C}/\mathfrak{a}}\overline{\varphi_\lambda(u)}\vartheta(u)e^{2\pi i r\delta|u|^2}du.$$

From this we may read off that for $\Theta \in \mathrm{im} d_{\nu\lambda'}$ with $\lambda' > \lambda$ the scalar product vanishes, since we have $\varphi_\lambda(u) = 0$ then. For $\Theta = d_{\nu\lambda}\vartheta'$ we have $\varphi_\lambda(u) = r^{-\lambda}\vartheta'(u)$ and the scalar product formula is proven.

Considering $\rho = \rho_{\nu\mu}$, as defined in Section 2.1, we may also define operators

$$d_{\nu\lambda} : \mathcal{T}_{d,\rho_{0,\mu+\lambda}} \to \mathcal{T}_{d,\rho_{\nu\mu}}$$

using the operation of $d_{\nu\lambda}$ on the components. We have

$$\langle d_{\nu\lambda}\vartheta_1, d_{\nu\lambda}\vartheta_2\rangle = C_{\nu\lambda}(d)\langle\vartheta_1, \vartheta_2\rangle$$

for all $\vartheta_i \in \mathcal{T}_{d,\rho_{0,\mu+\lambda}}$.

Differential operators of Maass and Shimura In this paragraph we recall some properties of the differential operators D_k studied by Maass and Shimura. For more information see [Hid, p. 310ff.], for example.

The operator $D = D_k$ on C^∞-modular forms of weight k on the upper half-plane $\mathfrak{H}$ is defined as

$$D_k = \frac{1}{2\pi i}\left(\frac{\partial}{\partial \tau} + \frac{k}{2iy}\right). \tag{15}$$

It maps forms of weight k to forms of weight $k+2$ and its n-th power is therefore defined by $D_k^n := D_{k+2(n-1)} \cdots D_{n+2} D_n$. For these powers we have the formula

$$D_k^n = \sum_{j=0}^{n} \binom{n}{j} \frac{(k+n-1)!}{(k+j-1)!} (-4\pi y)^{j-n} \left(\frac{1}{2\pi i}\frac{\partial}{\partial \tau}\right)^j. \tag{16}$$

The adjoint of D_k with respect to the usual Petersson inner product $\langle \cdot, \cdot \rangle$ of modular forms is given by the operator $\epsilon = (2\pi i)^{-1} y^2 \partial/\partial\bar{\tau}$, which takes modular forms of weight k to forms of weight $k-2$:

$$\langle D_k f, g \rangle_{k+2} = \langle f, \epsilon g \rangle_k,$$

for f and g of weights k and $k+2$, respectively, if at least one of them is rapidly decreasing.

Intrinsic theta functions Before stating the main result of this section we need to define the intrinsic theta functions (in the sense of Shimura, cf. [Shim2, Shim4]) associated to K. For a fractional ideal $\mathfrak{a}$ of K and an integer $k \geq 0$ we set

$$\vartheta_{\mathfrak{a},k}(w,\tau) = \sum_{a \in \mathfrak{a}} \left(\frac{\bar{a}}{\mathrm{N}(\mathfrak{a})} + \frac{w}{2iy}\right)^k e^{2\pi i (\mathrm{N}(\mathfrak{a})^{-1}\mathrm{N}(a)\tau + aw)}, \tag{17}$$

for $w \in \mathbb{C}$ and $\tau \in \mathfrak{H}$. This defines a C^∞-function on $\mathbb{C} \times \mathfrak{H}$, holomorphic in the first variable. The special case $\mathfrak{a} = \mathfrak{o}_K$, $k = 0$ was considered in the introduction. The properties of these functions are summarized in the following proposition.

Proposition 2.4. *The functions $\vartheta_{\mathfrak{a},k}$ fulfill the following functional equations.*

1. *For $\lambda \in K^\times$ we have $\vartheta_{\lambda\mathfrak{a},k}(w,\tau) = \lambda^{-k}\vartheta_{\mathfrak{a},k}(\lambda w, \tau)$.*
2. *For $\gamma \in \Gamma_0(D)$ we have*

$$\vartheta_{\mathfrak{a},k}(j(\gamma,\tau)^{-1}w, \gamma(\tau)) = \omega_{K/\mathbb{Q}}(\gamma) j(\gamma,\tau)^{k+1} \vartheta_{\mathfrak{a},k}(w,\tau). \tag{18}$$

3. *For $\tau \in \mathfrak{H} \cap K$ the function $\vartheta_{\mathfrak{a},k}$ (considered as a function of the first variable w) is a theta function with respect to the ideal $L = (\tau - \bar{\tau})\mathfrak{a}^{-1}(\mathfrak{o}_K \cap \delta^{-1}\bar{\tau}^{-1}\mathfrak{o}_K)$. More precisely, $\vartheta_{\mathfrak{a},k}(\cdot, \tau) \in T_{r,L}$ with $r = \mathrm{N}(\mathfrak{a})\delta^{-1}(\bar{\tau} - \tau)^{-1}$.*

Proof. The first assertion is trivial. To prove the second assertion, we compute the Taylor expansion in w of $\vartheta_{\mathfrak{a},k}$. Define for $l \geq 0$ the theta functions

$$\vartheta_{\mathfrak{a}}^{(l)} = \sum_{a \in \mathfrak{a}} a^l e^{2\pi i \mathrm{N}(\mathfrak{a})^{-1}\mathrm{N}(a)\tau} \tag{19}$$

on $\mathfrak{H}$. By results of Hecke (see [P, p. 237, (B.19)]), $\vartheta_{\mathfrak{a}}^{(l)}$ is a modular form of weight $l+1$ and character $\omega_{K/\mathbb{Q}}$ for the group $\Gamma_0(D)$. Furthermore,

$$\left(\frac{\mathrm{N}(\mathfrak{a})}{2\pi i}\frac{\partial}{\partial\tau}\right)^n \vartheta_{\mathfrak{a}}^{(l)}(\tau) = \sum_{a \in \mathfrak{a}} a^{l+n}\bar{a}^n e^{2\pi i \mathrm{N}(\mathfrak{a})^{-1}\mathrm{N}(a)\tau}. \tag{20}$$

We may express $\vartheta_{\mathfrak{a},k}$ in terms of these modular forms as follows:

$$\begin{aligned}\vartheta_{\mathfrak{a},k}(w,\tau) &= \sum_{j=k}^{\infty}\frac{(2\pi i)^j}{j!}(D^k\vartheta_{\mathfrak{a}}^{(j-k)})(\tau)u^j \\ &\quad + \sum_{j=0}^{k-1}\frac{(2\pi i)^j}{j!}\mathrm{N}(\mathfrak{a})^{j-k}(D^j\vartheta_{\bar{\mathfrak{a}}}^{(k-j)})(\tau)u^j. \end{aligned} \tag{21}$$

This formula is easily proved by expanding the exponential in (17) with respect to w and using (16) and (20) to express the resulting Taylor coefficients in terms of the $\vartheta_{\mathfrak{a}}^{(l)}$. Now the second assertion results, since the j-th Taylor coefficient is a C^∞-modular form of weight $k+j+1$. The third assertion is proved by an easy argument involving changing the variable a to $a+c$ with $c \in \mathfrak{a}$ in (17). We skip the details here, since a similiar argument will appear in the proof of Lemma 2.14.

Fourier-Jacobi expansion of the lifting We give now our first result on the Fourier-Jacobi expansion of a theta lift. To give an expression for the constant term, we have to introduce the following partial L-functions: for an element c of the genus class group $\mathrm{Cl}_K^{\mathrm{inv}}$ set

$$L(\chi,s)_c = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s},$$

where the sum ranges over all non-zero integral ideals $\mathfrak{a}$ such that the ideal class of $\mathfrak{a}$ is c times a square, or equivalently $\mathrm{N}(\mathfrak{a}) \in \mathrm{N}(\mathfrak{c})\mathrm{N}(K^\times)$. This implies the relation $L(\chi\sigma,s) = \sum_{c\in\mathrm{Cl}_K^{\mathrm{inv}}}\sigma(c)L(\chi,s)_c$ for all ideal class characters σ of order two.

Theorem 2.5. *Let ν and μ be integers with $m=\mu-\nu-1\geq 5$, f a modular form in $M_m(\Gamma_0(D),\omega_{K/\mathbb{Q}})$ and ε and χ Hecke characters of weight zero and $\nu+\mu$, respectively. Then for the Fourier-Jacobi coefficients $g_{r,\mathfrak{a}}$ of $\mathcal{L}_{\nu,\mu;\varepsilon,\chi}(f)$ the following holds true:*

$$\begin{aligned} g_{0,\mathfrak{a}}(\psi) &= \psi(0,1)\delta^{-\nu}\frac{(\mu-1)!}{2(2\pi i)^\mu}w_K(\varepsilon\chi^{-1})(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{(\nu-\mu)/2} \\ &\quad \sum_{\mathfrak{c}\in\mathrm{Cl}_K^{\mathrm{inv}}} L(\chi\varepsilon^{-3},(\mu-\nu)/2)_{\mathfrak{c}}\ \mathrm{Tr}(f(\tau)\vartheta_{\mathfrak{ac},0}(0,\tau))\big|_{\tau\to i\infty} \end{aligned} \tag{22}$$

and

$$\begin{aligned} g_{r/N(\mathfrak{a}),\mathfrak{a}}(w) &= \varepsilon(\bar{\mathfrak{a}}/\mathfrak{a})\sum_{\mathfrak{b}\in\mathrm{Cl}_K}(\varepsilon^3\chi^{-1})(\mathfrak{b})\mathrm{N}(\mathfrak{b})^{(\nu+\mu)/2}\sum_{\lambda=0}^{\nu}\delta^{\lambda-\nu} \\ &\quad d_{\nu\lambda}\left(T_r(\mathrm{Tr}((D^\lambda f)(\tau)\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta w,\tau)))\big|_{\tau=\tau_0(\mathfrak{a}\bar{\mathfrak{b}})}\right). \end{aligned} \tag{23}$$

Here $\mathfrak{b}$ ranges over a system of representatives for the ideal classes of K such that $\mathfrak{a}\bar{\mathfrak{b}} = \mathbb{Z}+\mathbb{Z}\tau_0$ for some $\tau_0=\tau_0(\mathfrak{a}\bar{\mathfrak{b}})\in K$ (here choosen to lie in $\mathfrak{H}$) and Tr *denotes* $\mathrm{Tr}_{\Gamma_0(D)\backslash\mathrm{SL}_2(\mathbb{Z})}$.

We remark that in the case of Eisenstein series similar formulas were obtained by Shimura [Shim4].

The proof of the theorem depends heavily on Kudla's paper [Ku1]. We use Kudla's technique of restricting to the upper half-plane $\mathfrak{H}$ embedded into $\mathfrak{D}$ as the subset of points $(z,0)$ and making the connection to the lifting from $GU(2)$ to $GU(2)$. To get a complete formula, we have to consider all Taylor coefficients in w.

The following proposition is essentially a restatement of Kudla's Theorem 6.6. We include it here, since we need his result in a slightly modified form[2]. Consider the two-dimensional hermitian space $V^0 = K^2$ with hermitian form

$$(x,y) = -\delta^{-1}\bar{x}^{\mathrm{tr}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$$

and set $\eta^0(z) = ((z,1),(z,1)) = (2/\sqrt{D})\mathrm{Im}(z)$. For a fractional ideal $\mathfrak{c}$ of K a theta kernel on $\mathfrak{H} \times \mathfrak{H}$ is given by

$$\begin{aligned} \Theta^0_{\nu,\mu;\mathfrak{c}}(\tau,z) &= 2^{\mu-1}D^{-\mu/2}y^{1+\nu}\sum_{X\in\bar{\mathfrak{c}}^2}(X,(z,1))^\mu\eta^0(z)^{-\mu}(X,(\bar{z},1))^\nu \\ & \quad e^{2\pi i\mathrm{N}(\mathfrak{c})^{-1}(2iy|(X,(z,1))|^2\eta^0(z)^{-1}+\bar{\tau}(X,X))}. \end{aligned} \tag{24}$$

Denote the summand corresponding to X by $\Phi_{\nu,\mu;\mathfrak{c}}(X;\tau,z)$. Set

$$E_{\nu\mu}(L) := \sum_{\lambda\in L\setminus\{0\}} \bar{\lambda}^\nu\lambda^{-\mu}$$

for lattices $L \subseteq \mathbb{C}$.

Proposition 2.6. *Let $\mathfrak{c} = \mathbb{Z}+\mathbb{Z}\tau_0$, $\tau_0 \in \mathfrak{H}$ be a fractional ideal of $\mathfrak{o}_K$ and $\mu-\nu \geq 4$ even, $f \in M_{\mu-\nu}(\mathrm{SL}_2(\mathbb{Z}))$. Let $\mathcal{L}^0_{\nu,\mu;\mathfrak{c}}(f) \in M_{\mu-\nu}(\mathrm{SL}_2(\mathbb{Z}))$ be the scalar product (in τ) of f with the theta kernel $\Theta^0_{\nu,\mu;\mathfrak{c}}$. Writing $\mathcal{L}^0(f) = \sum_{r=0}^\infty a_r q^r$, we have*

$$a_0 = \frac{(\mu-1)!}{2(2\pi i)^\mu}\mathrm{N}(\mathfrak{c})^\mu\delta^{-\nu}E_{\nu\mu}(\mathfrak{c})f(i\infty), \tag{25}$$

$$a_r = \mathrm{N}(\mathfrak{c})^{\nu+\mu}\, D^\nu(T_r f)|_{\tau=\tau_0}. \tag{26}$$

The lifting $\mathcal{L}^0$ is trivially extended to C^∞-modular forms and then compatible with the differential operator D. More precisely, we have the following lemma.

Lemma 2.7. *For integers μ and $\nu \geq 1$ with $\mu-\nu \geq 6$ even, and any C^∞-modular form of weight $\mu-\nu$ for the group $\mathrm{SL}_2(\mathbb{Z})$, we have $\mathcal{L}^0_{\nu-1,\mu+1;\mathfrak{c}}(Df) = D(\mathcal{L}^0_{\nu,\mu;\mathfrak{c}}f)$.*

Proof. By the adjointness of D and ϵ,

$$\mathcal{L}^0_{\nu-1,\mu+1}(Df) = \langle\Theta^0_{\nu-1,\mu+1}, Df\rangle = \langle\epsilon_\tau\Theta^0_{\nu-1,\mu+1}, f\rangle.$$

Therefore we have to show that

$$\overline{\epsilon_\tau\Theta^0_{\nu-1,\mu+1}} = D_z\overline{\Theta^0_{\nu,\mu}}.$$

We leave the verification of this identity to the reader.

We now begin with the proof of the theorem. To compute the Fourier-Jacobi coefficients $g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$ of $F = \mathcal{L}(f)$ set $g_f = \mathrm{diag}(\bar{a}_f, 1, a_f^{-1})$ for $a_f \in \mathbb{A}^\times_{K,f}$ with $(a_f) = \mathfrak{a}$ and consider the Fourier-Jacobi expansion of F_{g_f}: it is not difficult to see that

$$F_{g_f}(\mathfrak{z}) = \sum_{r\in\mathbb{Q}} g_{r,\mathfrak{a}}(w)e^{2\pi irz}.$$

[2] The correct statement of Kudla's Theorem 6.6 should include an additional factor $2^{-\mu}$, if $M > 1$, and $2^{1-\mu}$, if $M = 1$, in the constant C_3. The mistake occurs on p. 16, where a factor $2^{-\mu}$ contained in (6.9) is forgotten afterwards, and on p. 17, where in (6.14) an additional factor 2 needs to be inserted in the case $M = 1$, since the group contains -1 then. Of course, the mistake is unimportant for Kudla's purposes, but here the exact result is needed.

From (7) we have

$$F_{g_f}(\mathfrak{z}) = \varepsilon(\bar{\mathfrak{a}}/\mathfrak{a}) \sum_{\mathfrak{b}} \mathrm{N}(\mathfrak{b})^{-(\nu+\mu)/2}(\varepsilon^3\chi^{-1})(\mathfrak{b}) F_{\mathfrak{a},\mathfrak{b}}(\mathfrak{z}),$$

where $F_{\mathfrak{a},\mathfrak{b}}$ is the lifting of f against the theta kernel $\Theta_{g_f,\mathfrak{b}} =: \Theta_{\mathfrak{a},\mathfrak{b}}$ defined in (8).

By considering Taylor coefficients at zero, we are reduced to computing the Fourier expansions of the scalar valued functions

$$\phi^{(n)}_{\mathfrak{a},\mathfrak{b};l}(z) := \frac{1}{n!}\frac{\partial^n}{\partial w^n} F_{\mathfrak{a},\mathfrak{b};l}(z,w)\Big|_{w=0}$$

defined on the upper half-plane, where we set $\psi^*_l := \psi^*(X^{\nu-l}Y^l)$ for $\psi^* \in \mathcal{V}$. We may assume $n+l-\nu$ even.

In a first step, we express these functions in terms of the two-dimensional lifting $\mathcal{L}^0$. Evidently $\phi^{(n)}_{\mathfrak{a},\mathfrak{b};l} = \langle \Theta^{(n)}_{\mathfrak{a},\mathfrak{b};l}, f\rangle$ with

$$\overline{\Theta^{(n)}_{\mathfrak{a},\mathfrak{b};l}(\tau,z)} = \frac{1}{n!}\frac{\partial^n}{\partial w^n}\overline{\Theta_{\mathfrak{a},\mathfrak{b};l}(\tau,(z,w))}\Big|_{w=0}.$$

Explicitly, we have

$$\begin{aligned}\overline{\Theta_{\mathfrak{a},\mathfrak{b};l}(\tau,\mathfrak{z})} &= 2^{\mu-1}D^{-\mu/2}y^{2+\nu}\delta^{l-\nu} \sum_{X\in\bar{\mathfrak{a}}\mathfrak{b}\oplus\mathfrak{b}\oplus\mathfrak{a}^{-1}\mathfrak{b}} (\overline{(X^0,(z,1))} - X_2\bar{w})^{\mu} \\ &\quad (\eta^0(z)-|w|^2)^{-\mu}(X_3w-X_2)^{\nu-l}\overline{(X^0,(\bar{z},1))}^{\,l} \\ &\quad e^{-2\pi i \mathrm{N}(\mathfrak{b})^{-1}(-2iy|(X^0,(z,1))-\bar{X}_2 w|^2(\eta^0(z)-|w|^2)^{-1}+\tau((X^0,X^0)-\mathrm{N}(X_2)))},\end{aligned}$$

with $X^0 = (X_1, X_3)$. Taking holomorphic derivatives with respect to w yields

$$\begin{aligned}\overline{\Theta^{(n)}_{\mathfrak{a},\mathfrak{b};l}(\tau,z)} &= 2^{\mu-1}D^{-\mu/2}(-\delta)^{l-\nu}\mathrm{N}(\mathfrak{a})^{-(\mu+n)} \\ &\quad \sum_{j=0}^{\min(\nu-l,n)} \binom{\nu-l}{j}(-1)^j y^{2+\nu+n-j}\frac{1}{(n-j)!}\left(\frac{4\pi}{\mathrm{N}(\mathfrak{b})}\right)^{n-j} \\ &\quad \left(\sum_{Y\in\bar{\mathfrak{c}}^2} Y_2^j \Phi_{l,\mu+n-j;\mathfrak{c}}(Y;\tau,z')\right) \\ &\quad \left(\sum_{X_2\in\mathfrak{b}} X_2^{\nu-l-j}\bar{X}_2^{n-j} e^{2\pi i \mathrm{N}(\mathfrak{b})^{-1}\mathrm{N}(X_2)\tau}\right),\end{aligned}$$

where we set $Y = (X_1, \mathrm{N}(\mathfrak{a})X_3)$, $z' = \mathrm{N}(a)^{-1}z$ and $\mathfrak{c} = \mathfrak{a}\bar{\mathfrak{b}}$.

To transform this further, note that

$$Y_2 = \eta^0(z')^{-1}(\overline{(Y,(z',1))} - \overline{(Y,(\bar{z'},1))}), \quad Y \in V^0.$$

Substituting this for Y_2 and expanding the power yields

$$\begin{aligned}\overline{\Theta^{(n)}_{\mathfrak{a},\mathfrak{b};l}(\tau,z)} &= \mathrm{N}(\mathfrak{a})^{-(\mu+n)}(-\delta)^{l-\nu}\sum_{0\le k\le j\le \nu-l}\binom{\nu-l}{j}\binom{j}{k}(-1)^k\frac{1}{(n-j)!} \\ &\quad \left(\frac{4\pi}{\mathrm{N}(\mathfrak{b})}\right)^{n-j}(\sqrt{D}/2)^{n+k-j}y^{\nu-l+n-2j+k+1}\eta^0(z')^{-(j-k)} \\ &\quad \overline{\Theta^0_{l+j-k,\mu+n+k-j;\mathfrak{c}}(\tau,z')}\left(\sum_{b\in\mathfrak{b}} b^{\nu-l-j}\bar{b}^{n-j}e^{2\pi i\mathrm{N}(\mathfrak{b})^{-1}\mathrm{N}(b)\tau}\right). \quad (27)\end{aligned}$$

Now we have (cf. equation (20))

$$\sum_{b\in\mathfrak{b}} b^{\nu-l-j}\bar{b}^{n-j}e^{2\pi i \mathrm{N}(\mathfrak{b})^{-1}\mathrm{N}(b)\tau} = \begin{cases} \left(\frac{\mathrm{N}(\mathfrak{b})}{2\pi i}\right)^{\nu-l-j} \frac{\partial^{\nu-l-j}}{\partial\tau^{\nu-l-j}}\vartheta_{\bar{\mathfrak{b}}}^{(n+l-\nu)}, & n\geq \nu-l, \\ \left(\frac{\mathrm{N}(\mathfrak{b})}{2\pi i}\right)^{n-j} \frac{\partial^{n-j}}{\partial\tau^{n-j}}\vartheta_{\mathfrak{b}}^{(\nu-l-n)}, & n\leq \nu-l, \end{cases}$$

and we need to distinguish the two cases. We deal only with the first case in the following, since the second is very similiar. From (16) we compute

$$D^{\nu-l-t}\vartheta_{\bar{\mathfrak{b}}}^{(n+l-\nu)} = \sum_{j=t}^{\nu-l}\binom{\nu-l-t}{\nu-l-j}\frac{(n-t)!}{(n-j)!}(-4\pi y)^{t-j}\left(\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\right)^{\nu-l-j}\vartheta_{\bar{\mathfrak{b}}}^{(n+l-\nu)},$$

and substituting $t=j-k$ in (27) we finally get (assuming $n\geq\nu-l$)

$$\begin{aligned}\overline{\Theta_{\mathfrak{a},\mathfrak{b};l}^{(n)}(\tau,z)} &= \mathrm{N}(\mathfrak{a})^{-(\mu+n)}\mathrm{N}(\mathfrak{b})^{\nu-l-n}(-\delta)^{l-\nu}\sum_{t=0}^{\nu-l}\binom{\nu-l}{t}\frac{(2\pi\sqrt{D})^{n-t}}{(n-t)!} \\ & \quad y^{\nu+n+1-l-2t}\eta(z')^{-t}\overline{\Theta_{l+t,\mu+n-t;\mathfrak{c}}^{0}(\tau,z')}D^{\nu-l-t}\vartheta_{\bar{\mathfrak{b}}}^{(n+l-\nu)}.\end{aligned}$$

Consequently we have for $\phi_{\mathfrak{a},\mathfrak{b};l}^{(n)}(z)$:

$$\begin{aligned}\phi_{\mathfrak{a},\mathfrak{b};l}^{(n)}(z) &= (-\delta)^{l-\nu}\mathrm{N}(\mathfrak{a})^{-(\mu+n)}\mathrm{N}(\mathfrak{b})^{\nu-n-l}\sum_{t=0}^{\nu-l}\binom{\nu-l}{t}\frac{(2\pi\sqrt{D})^{n-t}}{(n-t)!} \\ & \quad \eta^{0}(z')^{-t}\mathcal{L}_{l+t,\mu+n-t;\mathfrak{c}}^{0}(\mathrm{Tr}[(D^{\nu-l-t}\vartheta_{\bar{\mathfrak{b}}}^{(n+l-\nu)})f])(z').\end{aligned}$$

In a second step, we use our knowledge of the lifting $\mathcal{L}^0$ to compute the Fourier expansions of these functions. Write (cf. [Hid, p. 313])

$$(D^{\nu-l-t}\vartheta_{\bar{\mathfrak{b}}}^{(n+l-\nu)})f = \sum_{j=0}^{\nu-l-t}D^{j}f_{\nu-l-t,j}$$

with $f_{\nu-l-t,j}$ a holomorphic modular form of weight $\mu+n-l-2t-2j$. Then by Lemma 2.7

$$\begin{aligned}\phi_{\mathfrak{a},\mathfrak{b};l}^{(n)}(z) &= (-\delta)^{l-\nu}\mathrm{N}(\mathfrak{a})^{-(\mu+n)}\mathrm{N}(\mathfrak{b})^{\nu-n-l}\sum_{t=0}^{\nu-l}\binom{\nu-l}{t}\frac{(2\pi\sqrt{D})^{n-t}}{(n-t)!} \\ & \quad \eta^{0}(z')^{-t}\sum_{j=0}^{\nu-l-t}(D_{z'}^{j}\mathcal{L}_{l+t+j,\mu+n-t-j;\mathfrak{c}}^{0}(\mathrm{Tr}f_{\nu-l-t,j}))(z').\end{aligned}$$

Since we know a priori that $\phi^{(n)}$ is holomorphic, we can also take the holomorphic part of this expression (cf. [Hid, p. 310ff.]). This leaves only the terms with $t=0$ and gives

$$\begin{aligned}\phi_{\mathfrak{a},\mathfrak{b};l}^{(n)}(z) &= (-\delta)^{l-\nu}\mathrm{N}(\mathfrak{a})^{-(\mu+n)}\mathrm{N}(\mathfrak{b})^{\nu-n-l}\frac{(2\pi\sqrt{D})^{n}}{n!} \\ & \quad \sum_{j=0}^{\nu-l}\left(\frac{1}{2\pi i}\frac{\partial}{\partial z'}\right)^{j}\mathcal{L}_{l+j,\mu+n-j;\mathfrak{c}}^{0}(\mathrm{Tr}f_{\nu-l,j})(z').\end{aligned}$$

We know from Proposition 2.6 that

$$\mathcal{L}_{l+j,\mu+n-j}^{0}(\mathrm{Tr}f_{\nu-l,j})(z') = \mathrm{N}(\mathfrak{c})^{\mu+n+l}\sum_{r=0}^{\infty}a_{jr}e^{2\pi i r z'}$$

with

$$a_{jr} = D^{l+j}(T_r(\mathrm{Tr} f_{\nu-l,j}))(\tau_0(\mathfrak{c})) = r^{-(l+j)} T_r(\mathrm{Tr}(D^{l+j} f_{\nu-l,j}))(\tau_0(\mathfrak{c}))$$

for $r > 0$. Therefore $\phi^{(n)}_{\mathfrak{a},\mathfrak{b};l} = \sum_{r=0}^{\infty} a_r e^{2\pi i r z'}$ with

$$a_r = (-\delta)^{l-\nu} \mathrm{N}(\mathfrak{a})^l \mathrm{N}(\mathfrak{b})^{\nu+\mu} \frac{(2\pi\sqrt{D})^n}{n!} \sum_{j=0}^{\nu-l} r^j a_{jr}$$

for $r > 0$. Finally

$$a_r = (-\delta)^{l-\nu} \mathrm{N}(\mathfrak{a})^l \mathrm{N}(\mathfrak{b})^{\nu+\mu} \frac{(2\pi\sqrt{D})^n}{n!} r^{-l} T_r(\mathrm{Tr}[D^l((D^{\nu-l}\vartheta_{\mathfrak{b}}^{(n+l-\nu)})f)])(\tau_0(\mathfrak{c})).$$

Summing over all representatives $\mathfrak{b}$ we have computed the n-th Taylor coefficient of $g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a};l}$ at $w = 0$ (for $r > 0$ and $n \geq \nu - l$). Using (21) the reader may verify that the result is in accordance with the theorem.

The constant term may be handled in the same way: the analogous computation gives

$$\begin{aligned} g_{0,\mathfrak{a}}(\psi) \quad = \quad & \psi(0,1)\delta^{-\nu} \frac{(\mu-1)!}{2(2\pi i)^\mu} \varepsilon(\bar{\mathfrak{a}}/\mathfrak{a}) \\ & \sum_{\mathfrak{b}} (\varepsilon^3\chi^{-1})(\mathfrak{b}) \mathrm{N}(\mathfrak{b})^{(\mu-\nu)/2} E_{\nu\mu}(\mathfrak{a}\bar{\mathfrak{b}}) \ \mathrm{Tr}(f(\tau)\vartheta_{\mathfrak{b},0}(0,\tau))|_{\tau\to i\infty}. \end{aligned}$$

But the values of $\vartheta_{\mathfrak{b},0}(0,\tau)$ at the cusps depend only on the class of $\mathfrak{b}$ in $\mathrm{Cl}_K/\mathrm{Cl}_K^2$ (see Section 3.2 below). Consequently, summation over $\mathfrak{b}$ in these classes yields the asserted partial L-values and we get the desired result.

2.3 Theta functions and Shintani operators

In this section we review Shintani's theory of primitive theta functions, give some complements due mainly to Murase-Sugano [MS] (see also [Ro, HKS]), and explain the connection between values of certain linear functionals on Shintani eigenspaces and L-values, which is a reformulation of results of Yang [Y]. We do not present anything really original, but mainly translate known results to our situation.

Review of Shintani theory We give a short review of the theory of primitive theta functions. For more details see [Shin, GlR, MS]. To begin, we define the subspaces of primitive theta functions of the spaces $T_{r,\mathfrak{a};\rho}$ and $\mathcal{T}_{d,\rho}$ and construct certain operators $\mathcal{E}$ on the spaces $\mathcal{V}_{r,C;\rho}$.

For each pair of ideals $\mathfrak{b} \supseteq \mathfrak{a}$ such that $r\mathrm{N}(\mathfrak{b})$ is integral, there is a natural inclusion $T_{r,\mathfrak{b};\rho} \hookrightarrow T_{r,\mathfrak{a};\rho}$. Its adjoint with respect to the natural inner product is the trace operator

$$t_{\mathfrak{b}} : T_{r,\mathfrak{a};\rho} \longrightarrow T_{r,\mathfrak{b};\rho}$$

defined as

$$t_{\mathfrak{b}} = \sum_{l\in\mathfrak{b}/\mathfrak{a}} \psi(l) A_l.$$

We may therefore define for each integral ideal $\mathfrak{c}$ such that $\mathrm{N}(\mathfrak{c})$ divides $d > 0$, an inclusion

$$\iota_{\mathfrak{c}} : \mathcal{T}_{d/\mathrm{N}(\mathfrak{c}),\rho} \hookrightarrow \mathcal{T}_{d,\rho}$$

by using the inclusions $T_{d/\mathrm{N}(\mathfrak{a}\mathfrak{c}),\mathfrak{a};\rho} \hookrightarrow T_{d/\mathrm{N}(\mathfrak{a}\mathfrak{c}),\mathfrak{a}\mathfrak{c};\rho}$, and a trace operator

$$\tau_{\mathfrak{c}} : \mathcal{T}_{d,\rho} \longrightarrow \mathcal{T}_{d/\mathrm{N}(\mathfrak{c}),\rho}$$

built up from the $t_{\mathfrak{a}/\mathfrak{c}}$ on $T_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}$. The space of primitive theta functions $\mathcal{T}_{d,\rho}^{\mathrm{prim}}$ (resp. $T_{r,\mathfrak{a};\rho}^{\mathrm{prim}}$) is defined as the kernel of all trace operators $\tau_{\mathfrak{c}}$ (resp. $t_{\mathfrak{a}/\mathfrak{c}}$) with $\mathfrak{c}$ as above. These notions are compatible with the decomposition of $\mathcal{T}_{d,\rho}$ into $\mathcal{V}_{d,C;\rho}$. In addition, we define for each ideal $\mathfrak{d}$, which is a product of ramified primes, a projector $\Pi_{\mathfrak{d}}$ on $T_{r,\mathfrak{a};\rho}$ as

$$\Pi_{\mathfrak{d}} = \mathrm{N}(\mathfrak{d})^{-1} \sum_{l \in \mathfrak{a}\mathfrak{d}^{-1}/\mathfrak{a}} \psi(l) A_l$$

(note that ψ may be canonically extended to $\mathfrak{a}/\mathfrak{d}$ even if $r\mathrm{N}(\mathfrak{a}/\mathfrak{d})$ is not integral — and this case is the whole point, since otherwise we get a multiple of $t_{\mathfrak{a}/\mathfrak{d}}$). The definition extends to $\mathcal{T}_{d,\rho}$ and $\mathcal{V}_{d,C;\rho}$, too.

Now let $\mathfrak{b} \in I_K^1$, the group of norm one ideals of K, and take the unique integral ideal $\mathfrak{c}$ with $\mathfrak{c} + \bar{\mathfrak{c}} = \mathfrak{o}_K$ and $\mathfrak{b} = \bar{\mathfrak{c}}\mathfrak{c}^{-1}$. Then the composition

$$T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho} \hookrightarrow T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}\bar{\mathfrak{c}};\rho} \xrightarrow{t_{\mathfrak{a}\bar{\mathfrak{c}}/\mathfrak{c}}} T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}\bar{\mathfrak{c}}\mathfrak{c}^{-1};\rho}$$

is a linear operator called $\mathcal{E}(\mathfrak{b})$. One easily checks that it induces in fact an endomorphism of $\mathcal{V}_{r,C;\rho}$. Set $\mathcal{F}(\mathfrak{b}) = \mathrm{N}(\mathfrak{c})^{-1/2}\mathcal{E}(\mathfrak{b})$. We call the operators $\mathcal{E}$ and $\mathcal{F}$ Shintani operators.

The main results on these operators are the following: $\mathcal{F}$ defines a unitary representation of the group $I_K^1(r)$ (the group of ideals in I_K^1 prime to r) on $\mathcal{V}_{r,C;\rho}$. It decomposes into eigenspaces corresponding to characters κ of $I_K^1(r)$. The subspace $\mathcal{V}_{r,C;\rho}^{\mathrm{prim}}$ of primitive theta functions decomposes into one-dimensional eigenspaces [GlR]. If we associate to each character κ of $I_K^1(r)$ a character κ^* of $I_K(rD)$ by $\kappa^*(\mathfrak{a}) = \prod_{p\,\mathrm{inert}}(-1)^{v_p(\mathfrak{a})}\kappa(\mathfrak{a}\bar{\mathfrak{a}}^{-1})$ (p ranging over the prime numbers inert in K), then for each κ occuring in $\mathcal{F}$ the character κ^* is a Hecke character of K [Shin, GlR] of conductor dividing rD. For each primitive eigenfunction ϑ we define $\mathfrak{d}_\vartheta$ as the unique product of ramified primes such that $\Pi_{\mathfrak{d}}\vartheta = \vartheta$, if $\mathfrak{d} \,|\, \mathfrak{d}_\vartheta$, and $\Pi_{\mathfrak{d}}\vartheta = 0$, otherwise. We have always $\mathfrak{d}_\vartheta + r\mathfrak{o}_K = \mathfrak{o}_K$ (see below for proofs of the last assertions).

Taking $\rho = \rho_{\nu\mu}$, the Shintani operators commute with the differential operators

$$d_{\nu\lambda} : \mathcal{T}_{r,\rho_{0,\mu+\lambda}} \to \mathcal{T}_{r,\rho_{\nu\mu}}$$

which split $\mathcal{T}_{r,\rho}$ (or equivalently $\mathcal{V}_{r,C;\rho}$) into an orthogonal direct sum of $\nu+1$ subspaces. An eigenfunction $\vartheta \in \mathcal{T}_{r,\rho}$ lies in the image of $d_{\nu\lambda}$ precisely when the weight of the associated Hecke character κ^* equals $2(\mu+\lambda)-1$.

Classical and adelic theta functions In this paragraph we introduce some adelic function spaces isomorphic to the classically defined spaces $T_{r,\mathfrak{a};\rho}$ and $\mathcal{V}_{d,C;\rho}$. The content is more or less standard or contained in Shintani.

We begin with spaces of adelic theta functions. Let $\mathcal{V} = S_\nu^*$, and ρ be such that $\rho(n,1)$ acts on $\mathcal{V}$ by the canonical GL_2-action for all upper triangular unipotent matrices n. Let λ be the additive character of $\mathbb{A}/\mathbb{Q}$ normalized by $\lambda(x_\infty) = e^{2\pi i x_\infty}$.

To begin, we define two differential operators D_- and D_+ on smooth functions on $H(\mathbb{A})$ (the "lowering" and "raising" operator, resp.):

$$\begin{aligned}
(D_-\theta)((w,t)) &= \left(\frac{1}{2\pi i}\frac{\partial}{\partial \bar{w}_\infty}\right)(\theta((w,t))e^{-\pi i r\delta|w_\infty|^2})e^{\pi i r\delta|w_\infty|^2}, \\
(D_+\theta)((w,t)) &= \left(\frac{1}{2\pi i}\frac{\partial}{\partial w_\infty}\right)(\theta((w,t))e^{\pi i r\delta|w_\infty|^2})e^{-\pi i r\delta|w_\infty|^2}.
\end{aligned}$$

We define the space $T^{\mathbb{A}}_{r,\rho}$ for a non-negative rational number r as the space of all smooth functions $\theta : H(\mathbb{Q})\backslash H(\mathbb{A}) \to \mathcal{V}$ with $\theta((0,t)h) = \lambda(rt)\theta(h)$ for all $h \in H(\mathbb{A})$, $t \in \mathbb{A}$, such that

$$e^{-\pi i r\delta|w_\infty|^2}\rho(\begin{pmatrix} 1 & \delta\bar{w}_\infty \\ 0 & 1 \end{pmatrix}, 1)\theta((w_\infty, 0)h_f)$$

is holomorphic in $w_\infty \in K_\infty = \mathbb{C}$ for all $h \in H(\mathbb{A}_f)$. Given a fractional ideal $\mathfrak{a}$ of K, we define a subgroup $H(\mathfrak{a})_f$ of $H(\mathbb{A}_f)$ by

$$H(\mathfrak{a})_f = \{(w,t) \in H(\mathbb{A}_f) \mid w \in \hat{\mathfrak{a}},\ t + \delta w\bar{w}/2 \in \mathrm{N}(\mathfrak{a})\hat{\mathfrak{o}}_K\}$$

and we denote by $T^{\mathbb{A}}_{r,\rho}(\mathfrak{a})$ the subspace of $H(\mathfrak{a})_f$-right-invariant functions in the space $T^{\mathbb{A}}_{r,\rho}$.

Define the space $T^{\mathbb{A}}_{r,\nu}$ as the space of all smooth functions $\theta : H(\mathbb{Q})\backslash H(\mathbb{A}) \to \mathbb{C}$ with $\theta((0,t)h) = \lambda(rt)\theta(h)$ and $D_-^{\nu+1}\theta = 0$. The subspaces $T^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ are defined as above. The following simple lemma connects this spaces to the spaces $T_{r,\mathfrak{a};\rho}$.

Lemma 2.8. *1. The space $T^{\mathbb{A}}_{r,\rho}(\mathfrak{a})$ is canonically isomorphic to $T_{r,\mathfrak{a};\rho}$ by the map associating to $\theta \in T^{\mathbb{A}}_{r,\rho}(\mathfrak{a})$ the holomorphic $\mathcal{V}$-valued function*

$$\vartheta(w_\infty) = e^{-\pi i r\delta|w_\infty|^2}\rho(\begin{pmatrix} 1 & \delta\bar{w}_\infty \\ 0 & 1 \end{pmatrix}, 1)\theta((w_\infty, 0))$$

on $\mathbb{C}$.

2. The space $T^{\mathbb{A}}_{r,\nu}$ is canonically isomorphic to $T^{\mathbb{A}}_{r,\rho}$ as an $H(\mathbb{A}_f)$-module. The canonical isomorphism is obtained by taking $\theta \in T^{\mathbb{A}}_{r,\nu}$ to the vector Θ with

$$\Theta_j = \frac{(2\pi i\delta^{-1})^{\nu-j}}{\nu(\nu-1)\cdots(j+1)} D_-^{\nu-j}\theta, \quad 1 \le j \le \nu;$$

in the other direction associate to $\Theta \in T^{\mathbb{A}}_{r,\rho}$ just its ν-th component function.

The differential operators $d_{\nu\lambda}$ correspond under these isomorphisms to the maps

$$r^{-\nu}\binom{\nu}{\lambda} D_+^{\nu-\lambda} : T^{\mathbb{A}}_{r,0} \to T^{\mathbb{A}}_{r,\nu}.$$

We now introduce adelic counterparts of the spaces $\mathcal{V}_{d,C;\rho}$. Our definition is similiar to Shintani's definition of the spaces $V_{d/c}(\rho, c)$, $c \in \mathbb{Q}^\times$ a representative for the class C. First consider the algebraic group R over $\mathbb{Q}$ defined as the semidirect product of H with $U(1) < \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$ (the group of norm one elements), where $U(1)$ acts on H by $u(w,t)u^{-1} = (uw, t)$. Given parameters r and $\mathfrak{a}$ let $V^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ be the space of all smooth functions

$$\varphi : R(\mathbb{Q})\backslash R(\mathbb{A})/\hat{\mathfrak{o}}^1_K K^1_\infty H(\mathfrak{a})_f \to \mathbb{C}$$

with $\varphi((0,t)g) = \lambda(rt)\varphi(g)$ and $D_-^{\nu+1}\varphi = 0$.

To every $\varphi \in V^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ we may associate the functions $\varphi_g \in T^{\mathbb{A}}_{r,\nu}((g_f)\mathfrak{a})$ for $g \in \mathbb{A}^1_K$ by setting $\varphi_g(h) = \varphi(hg)$. By definition φ_g depends only on the norm one ideal (g_f) and $\varphi_{\lambda g}((\lambda w, t)) = \varphi_g((w,t))$ for $\lambda \in K^1$. Therefore we have in the case $\rho = \rho_{\nu\mu}$ for every unramified Hecke character γ of weight $\nu + \mu$ an isomorphism

$$v_\gamma : V^{\mathbb{A}}_{r,\nu}(\mathfrak{a}) \to \mathcal{V}_{r\mathrm{N}(\mathfrak{a}),\mathrm{N}(\mathfrak{a})\mathrm{N}(K^\times);\rho}, \quad \varphi \mapsto (\mathrm{N}(\mathfrak{a}')^{(\nu-\mu)/2}\gamma(\mathfrak{a}')^{-1}I_{\mathfrak{a}'}(\varphi_{g_f(\mathfrak{a}')}))_{\mathfrak{a}'},$$

where $\mathfrak{a}'$ ranges over all fractional ideals of the same norm as $\mathfrak{a}$, and $g_f(\mathfrak{a}') \in \mathbb{A}^1_{K,f}$ is such that $\mathfrak{a}' = (g_f)\mathfrak{a}$. For each $\mathfrak{a}'$ we let $I_{\mathfrak{a}'}$ denote the isomorphism between $T^{\mathbb{A}}_{r,\nu}(\mathfrak{a}')$ and $T_{r,\mathfrak{a}';\rho}$ constructed above.

Under these isomorphisms the Shintani operators on the "classical side" may be expressed directly on the "adelic side". First, we have for $\eta \in K^1$ a "classical" Shintani operator $\mathcal{F}(\eta)$ on $T_{r,\mathfrak{a};\rho}$ given by composing the operator $\mathcal{F}((\eta)) : T_{r,\mathfrak{a};\rho} \to T_{r,\mathfrak{a}\eta;\rho}$ with the isomorphism $T_{r,\mathfrak{a}\eta;\rho} \simeq T_{r,\mathfrak{a};\rho}$ associating to $t_{\eta\mathfrak{a}}$ the function $t_{\mathfrak{a}}(w) = \rho(\mathrm{diag}(\eta, 1), \eta^2)t_{\eta\mathfrak{a}}(\eta w)$.

It is not difficult to show (see [GlR, p. 92][3]), that $\mathcal{F}(\eta)$ corresponds to the operator $\eta^{\mu+\nu}L(\eta)$ on $T^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$, where $L(\eta) = \mathrm{N}(\mathfrak{c})^{1/2}P_{\mathfrak{a}}l(\eta)$,

$$P_{\mathfrak{a}} = \mathrm{vol}(H(\mathfrak{a})_f)^{-1} \int_{H(\mathfrak{a})_f} \rho(g)dg$$

is the projector on the space of $H(\mathfrak{a})_f$-invariants, and $(l(\eta)\theta)((w,t)) = \theta((\eta w, t))$ for $\theta \in T^{\mathbb{A}}_{r,\nu}$.

Using this, we may easily give a description of the operator $L(\mathfrak{b})$ on $V^{\mathbb{A}}_{r,\nu}(\mathfrak{a})$ corresponding under v_γ to $\gamma^{-1}(\mathfrak{b})\mathcal{F}(\mathfrak{b})$ on $\mathcal{V}_{r,C;\rho}$. It may analogously be written as $L(\mathfrak{b}) = \mathrm{N}(\mathfrak{c})^{1/2}P_{\mathfrak{a}}l(\mathfrak{b})$, $l(\mathfrak{b})$ denoting right translation by β^{-1} for some $\beta \in \mathbb{A}^1_K$ with $(\beta) = \mathfrak{b}$.

Weil representation and theta functions To construct theta functions in the adelic setting we use the Weil representation. By the Stone-von Neumann theorem there exists a unique irreducible smooth representation ρ of $H(\mathbb{A})$ on a space V such that $\rho((0,t))$ acts by the scalar $\lambda(rt)$. The representation may be written as a tensor product $V = \otimes_p V_p$ (p ranging over all places of $\mathbb{Q}$, including the infinite one).

A standard realization of V_p is the lattice model $V_p \subseteq S(K_p)$ considered (among others) by Murase-Sugano [MS]. At the infinite place it may be supplemented by the Fock representation (cf. [MS], [I, Ch. 1, §8]): $V_\infty \subseteq S(K_\infty)$ (the space of Schwartz functions on K_∞) is defined as

$$V_\infty = \{\phi : K_\infty \to \mathbb{C} \,|\, \phi(z)e^{-\pi i r\delta|z|^2} \text{ antiholomorphic}, \quad \int_{K_\infty} |\phi(z)|^2 dz < \infty\}.$$

It is a Hilbert space with the obvious scalar product. The space of "automorphic" vectors $V^{\mathrm{aut}}_\infty \subseteq V_\infty$ (i. e. of K-finite vectors under the Weil representation) is given by the space of all ϕ such that $\phi(z)e^{-\pi i r\delta|z|^2}$ is a polynominal in $\bar{z}$. Denote by $V^{(\nu)}_\infty \subseteq V^{\mathrm{aut}}_\infty$ the subspace obtained by restricting to polynominals of degree at most ν. The action of $H(\mathbb{R})$ on V_∞ is given by

$$(\rho((w,t))\phi)(z) = e^{2\pi i r(\delta(\bar{z}w - z\bar{w})/2 + t)}\phi(z+w).$$

Putting everything together we have a global lattice model $V \subseteq S(\mathbb{A}_K)$ with $H(\mathbb{A}_f)$-invariant subspaces $V^{(\nu)} \subseteq V^{\mathrm{aut}} \subseteq V$. The theta functional $V \to \mathbb{C}$ is given by $\theta(\phi) = \sum_{z\in K}\phi(z)$. To every $\phi \in V$ we associate the theta function $\theta = \theta_\phi : H(\mathbb{Q})\backslash H(\mathbb{A}) \to \mathbb{C}$ by $\theta(h) = \theta(\rho(h)\phi)$. Trivially $\theta((0,t)h) = \lambda(rt)\theta(h)$.

We may now define operators D_+ and D_- on V compatible with the map $\phi \mapsto \theta_\phi$: namely set

$$\begin{aligned} D_+(\phi)(z) &= r\delta\bar{z}_\infty\phi(z), \\ D_-(\phi)(z) &= (2\pi i)^{-1}(\partial/\partial\bar{z}_\infty)(\phi(z)e^{-\pi i r\delta|z|^2})e^{\pi i r\delta|z|^2}. \end{aligned}$$

From this it is easy to see that for $\phi \in V^{(\nu)}$ we have $\theta_\phi \in T^{\mathbb{A}}_{r,\nu}$. Moreover, the map $\phi \mapsto \theta_\phi$ is actually an $H(\mathbb{A}_f)$-equivariant isomorphism of these spaces.

[3]To be precise, the proof given there only considers the case $\nu = 0$, but carries over to the general case.

Murase-Sugano define a (modified) Weil representation $\mathcal{M}_p$ of $K_p^\times$ on V_p. This extends analogously to V_∞ and we get a representation $\mathcal{M} = \bigotimes_p \mathcal{M}_p$ of $\mathbb{A}_K^\times$ on V fulfilling the commutation rule $\mathcal{M}(z)\rho(h) = \rho((\bar{z}/z)h(\bar{z}/z)^{-1})\mathcal{M}(z)$. The structure of the representations $\mathcal{M}_p$ for finite p is described by Murase-Sugano. Consideration of the infinite place does not pose any problems. We obtain at the infinite place the eigenvectors

$$\phi_\infty^{(k)}(z) = \bar{z}^k e^{\pi i \delta r |z|^2}, \quad k \geq 0,$$

with eigencharacters

$$\mathcal{M}_\infty(z)\phi_\infty^{(k)} = \left(\frac{|z|}{z}\right)^{2k+1} \phi_\infty^{(k)}.$$

We are now able to relate the Shintani operators $\mathcal{F}(\eta)$ and $L(\eta)$ to the operation of $\mathcal{M}$ on V.

Proposition 2.9. *Under the isomorphism between $T_{r,\nu}^{\mathbb{A}}(\mathfrak{a})$ and the space $V^{(\nu)}(\mathfrak{a}) \simeq \bigotimes_{p|r\mathrm{N}(\mathfrak{a})D} V_p(\mathfrak{a}_p) \otimes V_\infty^{(\nu)}$ of $H(\mathfrak{a})_f$-invariants in $V^{(\nu)}$ given above the operator $L(z/\bar{z})$ for $z \in K^\times$ with $\eta = z/\bar{z} \in K^1(r\mathrm{N}(\mathfrak{a}))$ corresponds to*

$$\prod_{p \,\mathrm{inert}, p \nmid r\mathrm{N}(\mathfrak{a})} (-1)^{v_p(z)} \bigotimes_{p|r\mathrm{N}(\mathfrak{a})D} \mathcal{M}_p(z) \otimes \mathcal{M}_\infty(z).$$

Proof. Take $\phi \in V^{(\nu)}(\mathfrak{a})$ corresponding to $\theta_\phi \in T_{r,\nu}^{\mathbb{A}}$. Since the theta functional is invariant under $\mathcal{M}(z)$ for $z \in K^\times$, we have

$$\theta_{\mathcal{M}(z)\phi}(h) = \theta(\rho(h)\mathcal{M}(z)\phi) = \theta(\mathcal{M}(z)\rho((z/\bar{z})h(z/\bar{z})^{-1})\phi) = (l(z/\bar{z})\theta_\phi)(h),$$

and therefore to the operator $L(\eta)$ on $T_{r,\nu}^{\mathbb{A}}$ corresponds the operator $\mathrm{N}(\mathfrak{c})^{1/2} P_\mathfrak{a} \mathcal{M}(z)$ on $V^{(\nu)}(\mathfrak{a})$. We may write this as a local product over all places p of $\mathbb{Q}$; if p is non-split or z is a unit at p, the space $V_p(\mathfrak{a}_p)$ is invariant under $\mathcal{M}_p(z)$, and the factor $P_{\mathfrak{a}_p}$ is superfluous. If $p \nmid r\mathrm{N}(\mathfrak{a})D$ is inert, then $\mathcal{M}_p(z)$ acts by multiplication with $(-1)^{v_p(z)}$.

On the other hand, z can be a non-unit at a split p only if $p \nmid r\mathrm{N}(\mathfrak{a})D$, and the space $V_p(\mathfrak{a}_p)$ is then one-dimensional. It remains to verify that

$$p^{m_p/2} P_{\mathfrak{a}_p} \mathcal{M}(z_p)\phi_p = \phi_p$$

for such p, $z_p \in K_p^\times$, $m_p = |v_\mathfrak{p}(z_p) - v_\mathfrak{p}(\bar{z}_p)|$ and ϕ_p in the one-dimensional space $V_p(\mathfrak{a}_p)$. This follows from the trace formula of Murase-Sugano [MS, Prop. 7.3]. The formula given there holds actually for all $z_p \in K_p^\times$ and yields in our case

$$\mathrm{Tr} P_{\mathfrak{a}_p} \mathcal{M}(z_p)|_{V_p(\mathfrak{a}_p)} = |\mathrm{N}(z_p)|^{1/2} (\max(|x_p|, |y_p|))^{-1},$$

where x_p and y_p are the coefficients of the expression of z_p with respect to some basis of $K_p/\mathbb{Q}_p$. Since $V_p(\mathfrak{a}_p)$ is one-dimensional, this is exactly what we need.

It is now easy to transcribe the results of Murase-Sugano on the characters occuring in the representations $\mathcal{M}_p$ to our situation. We define local epsilon factors as in [MS] and [Tat, p. 17] (Langlands' conventions): if F is a local field, χ a character of $F^\times$ and ψ an additive character of F, we set

$$\varepsilon(\chi, \psi) = \chi(c)\frac{S}{|S|}, \quad S = \int_{\mathfrak{o}_F^\times} \chi^{-1}(u)\psi(u/c)du,$$

where c is an element of $F^\times$ of valuation $a(\chi) + n(\psi)$, $a(\chi)$ the exponent of the conductor of χ and $n(\psi)$ the largest integer n such that ψ is trivial on $\mathfrak{p}_F^{-n}$. We set $\lambda_K = \lambda \circ \mathrm{Tr}_{K/\mathbb{Q}}$.

Corollary 2.10. *A Hecke character γ of K with $\gamma|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$ appears in the representation $\mathcal{F}$ on $\mathcal{V}^{\text{prim}}_{d,C;\rho}$ (i. e., more precisely, the character κ of $I^1_K(d)$ with $\kappa^* = \gamma$ appears in this representation) if and only if the following conditions are satisfied.*

1. *The weight of γ equals $2(\mu+\lambda)-1$ with $0 \le \lambda \le \nu$.*
2. *The conductor $\mathfrak{f}_\gamma$ of γ is of the form $dD\mathfrak{d}_\gamma^{-1}$, where $\mathfrak{d}_\gamma$ is a product of ramified primes. (We then have automatically $\mathfrak{d}_\gamma + d\mathfrak{o}_K = \mathfrak{o}_K$.)*
3. *For each prime $p|D$ and a representative $c \in \mathbb{Q}^\times$ for the class C we have*

$$\varepsilon(\gamma_p^{-1}, \lambda_{K,p})\gamma_p^{-1}(\delta)\omega_{K/\mathbb{Q},p}(d/c) = +1.$$

The κ-eigenspace is then contained in $\text{imd}_{\nu\lambda}$, and for a non-zero element Θ we have $\mathfrak{d}_\Theta = \mathfrak{d}_\gamma$.

If we consider the whole space $\mathcal{V}_{d,C;\rho}$ instead of the primitive subspace, we have to change the second condition into: $\mathfrak{f}_\gamma = (dt^{-1})D\mathfrak{d}_\gamma^{-1}$, where $t|d$ is an ideal norm from K.

From this corollary, we see that for given ν and μ, a Hecke character γ of K with $\gamma|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$ occurs in some space $\mathcal{V}^{\text{prim}}_{d,C;\rho}$, if and only if $2\mu - 1 \le \text{wt}(\gamma) \le 2(\mu+\nu)-1$, and the global root number $\varepsilon(\gamma, 1/2) = +1$. Every character satisfying these conditions appears in precisely one space of primitive theta functions, and the corresponding values of d and C may be read off from the corollary.

Connection to L-values (results of Yang) We review some results of Yang [Y] connecting theta functions with complex multiplication to values of Hecke L-functions at the center of the critical strip. Our reason for including this material is the connection to our main theorem (see the next section).

Yang considers a different model (V, ρ, ω) for the Weil representation, the standard Schrödinger model: here $V = S(\mathbb{A})$ with the standard scalar product

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathbb{A}} \overline{\phi_1(x)}\phi_2(x)dx,$$

where we fix a Haar measure on $\mathbb{A}$ normalized by $\text{vol}(\mathbb{Q}\backslash\mathbb{A}) = 1$. He defines a Weil representation of $\mathbb{A}^1_K$ on V by taking a splitting of the metaplectic group over $U(1)$ (see [Ku3]). The splitting is determined by the choice of a Hecke character χ of K with $\chi|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$. We denote the resulting Weil representation by ω_χ. The normalized theta functional on V is given by $\theta(\phi) = \sum_{x\in\mathbb{A}} \phi(x)$.

We quote Yang's main result from [Y, p. 43, (2.19)]: choose local and global Haar measures on $U(1)$ in a compatible way (we do not require any normalization). For every character η of $\mathbb{A}^1_K/K^1$ "appearing" in ω_χ (i. e. the local components η_p appear in the spaces V_p for all non-split p) there is an explicit function $\phi = \prod_p \phi_p \in V$ with

$$\frac{2}{\text{vol}(K^1\backslash\mathbb{A}^1_K)^2}\left|\int_{K^1\backslash\mathbb{A}^1_K} \theta(\omega_\chi(g)\phi)\eta(g)dg\right|^2 = \text{Tam}(K^1)c(0)\frac{L(\chi\tilde{\eta}, 1/2)}{L(\omega_{K/\mathbb{Q}}, 1)}. \qquad (28)$$

Here $\tilde{\eta}$ is the "base change" of η to $\mathbb{A}^\times_K$ given by $\tilde{\eta}(z) = \eta(z/\bar{z})$,

$$\text{Tam}(K^1) = \text{vol}(K^1_\infty\hat{\mathfrak{o}}^1_K)/\text{vol}(K^1\backslash\mathbb{A}^1_K)$$

is the Tamagawa number of K^1, and

$$c(0) = \prod_{p\in S_1}(1+p^{-1})^{-1}\prod_{p\in S_2} p^{-n_p}(1-p^{-1})^{-2},$$

where S_1 (resp. S_2) is the set of inert (resp. split) primes for which $\chi\tilde{\eta}$ is ramified. For $p \in S_2$ set n_p to the maximum of the exponents of the conductors of χ and $\tilde{\eta}$ at p. Yang's choice of the function ϕ is as follows: at all non-split places p he takes ϕ_p to be a unitary eigenfunction of K_p^1 with eigencharacter $\bar{\eta}_p$. In the split case he defines ϕ_p in [Y, p. 48, (2.30)]: we have

$$\phi_p = \varrho(\mathrm{char}_{\mathbb{Z}_p}),$$

in case $\chi\tilde{\eta}$ is unramified at p, and

$$\phi_p = p^{n_p/2}\varrho(\mathrm{char}_{1+p^{n_p}\mathbb{Z}_p})$$

in the ramified case, where char_S is the characteristic function of the set S, and ϱ the intertwining isometry between the "natural" and the "standard" Schrödinger models at p given by [Y, p. 47, (2.28)][4].

It is not difficult to translate his results to our situation. Take $\rho = \rho_{\nu\mu}$ as above and consider for an unramified Hecke character γ of weight $\nu + \mu$ the linear functional ℓ_γ on $\mathcal{T}_{r,\rho}$ defined by

$$\ell_\gamma(\Theta) = \sum_{\mathfrak{a}} \gamma(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{(\mu-\nu)/2}\Theta_{\mathfrak{a}}(0; Y^\nu), \tag{29}$$

$\mathfrak{a}$ ranging over a system of representatives for the ideal classes of K.

Proposition 2.11. *Let $\langle\cdot,\cdot\rangle$ be the scalar product on $\mathcal{V}_{d,C;\rho}$ associated to the semi-definite form $\langle x, y\rangle_{\mathcal{V}} = \bar{x}_\nu y_\nu$ on $\mathcal{V}$. Let Θ be a primitive eigenfunction of the Shintani operators with associated Hecke character κ^* and γ be an unramified Hecke character of weight $\nu + \mu$. Then*

$$\frac{|\ell_\gamma(\Theta)|^2}{\langle\Theta,\Theta\rangle} = \frac{w_K^2\sqrt{D}}{4\pi h_K}\prod_{p|d}(1-\omega_{K/\mathbb{Q}}(p)p^{-1})^{-1}L(\kappa^*\gamma^{-2}, 1/2). \tag{30}$$

Proof. Consider the Weil representation (V, ρ, ω_χ) as above. It is equivalent to the representation of $R(\mathbb{A})$ on V obtained by combining ρ and ω_χ. We denote this representation also by ω_χ. Take a character η of $\mathbb{A}_K^1/K^1$ appearing in V. Assume we are given a function $\phi' \in V$ which is an eigenfunction for $\mathcal{K} = K_\infty^1\hat{\mathfrak{o}}_K^1$ with eigencharacter $\bar{\eta}|_{\mathcal{K}}$. Consider the function $\varphi(g) = \eta(g)\theta(\omega_\chi(g)\phi')$ on $R(\mathbb{A})$ (here we extend η to $R(\mathbb{A})$ by the canonical map $R(\mathbb{A}) \to \mathbb{A}_K^1$). From the definition we see that φ is a non-zero element of $V_{r,\nu}^{\mathbb{A}}$ (for a suitable ν).

We define ϕ' as the projection of Yang's function ϕ on the $\bar{\eta}|_{\mathcal{K}}$-eigenspace of $\mathcal{K}$. We have $\phi' = \prod_p \phi'_p$, and ϕ'_p differs from ϕ_p only for $p \in S_2$. The integral in (28) remains unchanged if we replace ϕ by ϕ'.

On the other hand, by condition [Y, p. 43, (2.18)] for ϕ we have $\langle\phi,\phi\rangle = 1$. Using the description of the Weil representation at split places given in [Y, p. 44-48], we may easily verify that projection on the $\mathfrak{o}_{K,p}^1$-eigenspace for a split prime $p \in S_2$ induces multiplication of the scalar product with a factor $p^{-n_p}(1-p^{-1})^{-1}$. Therefore

$$\langle\phi',\phi'\rangle = \prod_{p\in S_2} p^{-n_p}(1-p^{-1})^{-1}.$$

Choosing a measure on $H(\mathbb{A})$ subject to $\mathrm{vol}(H(\mathbb{Q})\backslash H(\mathbb{A})) = 1$, we obtain easily

$$\int_{R(\mathbb{Q})\backslash R(\mathbb{A})} |\varphi(g)|^2 dg = \mathrm{vol}(K^1\backslash\mathbb{A}_K^1)\langle\phi',\phi'\rangle.$$

[4]The printing error $|x_0^3\alpha|^{1/3}$ in this equation should be corrected to $|x_0^3\alpha|^{1/2}$.

Putting this together with (28) we get

$$\frac{\left|\int_{K^1\backslash\mathbb{A}_K^1}\varphi(g)dg\right|^2}{\int_{R(\mathbb{Q})\backslash R(\mathbb{A})}|\varphi(g)|^2dg}=\frac{\mathrm{vol}\mathcal{K}}{2}\prod_{p\in S_1\cup S_2}(1-\omega_{K/\mathbb{Q}}(p)p^{-1})^{-1}\frac{L(\chi\tilde{\eta},1/2)}{L(\omega_{K/\mathbb{Q}},1)},$$

where now φ or ϕ' may be multiplied by an arbitrary non-zero complex number.

Let $\mathfrak{a}$ be a fractional ideal such that φ or ϕ' are $H(\mathfrak{a})_f$-invariant. Using the isomorphism $v_\gamma : V_{r,\nu}^{\mathbb{A}}(\mathfrak{a}) \to \mathcal{V}_{r\mathrm{N}(\mathfrak{a}),\mathrm{N}(\mathfrak{a})\mathrm{N}(K^\times);\rho}$ we get from φ a theta function $\Theta = v_\gamma(\varphi) \in \mathcal{V}_{r\mathrm{N}(\mathfrak{a}),\mathrm{N}(\mathfrak{a})\mathrm{N}(K^\times);\rho}$. It is easily verified that

$$\int_{K^1\backslash\mathbb{A}_K^1}\varphi(g)dg=\frac{\mathrm{vol}\mathcal{K}}{w_K}\ell_\gamma(\Theta)$$

and

$$\int_{R(\mathbb{Q})\backslash R(\mathbb{A})}|\varphi(g)|^2dg=\frac{\mathrm{vol}\mathcal{K}}{w_K}\langle\Theta,\Theta\rangle.$$

Furthermore, Θ is an eigenfunction of the Shintani operators $\mathcal{F}$ with eigencharacter κ satisfying $\kappa^*\gamma^{-2}=(\chi\tilde{\eta})^{-1}$. To prove this, we have to show that $L(\mathfrak{p}/\bar{\mathfrak{p}})$ acts on φ as multiplication by $(\chi\tilde{\eta})(\mathfrak{p})^{-1}$ for all but finitely many split prime ideals $\mathfrak{p}$ of K. Assuming that $\chi\tilde{\eta}$ is unramified at $\mathfrak{p}$, and that the space of $H(\mathfrak{a}_p)$-invariants in V_p is one-dimensional, we are reduced to proving (analogously as above), that $p^{1/2}P_{\mathfrak{a}_p}\omega_\chi(\beta)^{-1}\phi'_p=\chi(\mathfrak{p})^{-1}\phi'_p$ for $\beta=(p,p^{-1})\in K_p\simeq\mathbb{Q}_p\oplus\mathbb{Q}_p$. This may easily be checked using the description of $\phi_p=\phi'_p$ cited above and the description of the "natural" Schrödinger model given at [Y, p. 44-45, esp. Cor. 2.10].

Putting everything together equation (30) follows for our Θ, since $L(1,\omega_{K/\mathbb{Q}})=\frac{2\pi h_K}{w_K\sqrt{D}}$ by Dirichlet (cf. [Hid, p. 66]). If we take $\eta=1$, and choose χ accordingly, $\mathfrak{a}$ may be choosen of norm d/r, where d is the unique positive integer such that the conductor $\mathfrak{f}$ of χ is equal to $dD\mathfrak{d}^{-1}$ for a product of ramified primes $\mathfrak{d}$. This may be seen again by considering the definition of the "natural" Schrödinger model [Y, p. 44]. It follows that Θ has to belong to the primitive subspace in this case. Corollary 2.10 implies that every primitive eigenfunction may be constructed in this way, and we are done.

2.4 Primitive coefficients of liftings

We now turn to our main result and determine the decomposition into primitive theta functions of the Fourier-Jacobi coefficients computed above. By the main result of Shintani's paper [Shin, p. 68] the Fourier-Jacobi expansion of a Hecke eigenform is completely determined by the knowledge of the Hecke eigenvalues and the primitive coefficients.

Statement of the result We interpret a modular form f of weight m and character ω for $\Gamma_0(D)$ as a function of pairs (L,x) consisting of a lattice $L\subseteq\mathbb{C}$ and an element $x\in L/DL$ of order D such that

$$f(\lambda L,\lambda x)=\lambda^{-m}f(L,x),\quad \lambda\in\mathbb{C}^\times$$

and

$$f(L,tx)=\omega(t)f(L,x),\quad t\in(\mathbb{Z}/D\mathbb{Z})^\times.$$

We may translate f back to a function on $\mathfrak{H}$ by setting $f(\tau)=f(\mathbb{Z}+\tau\mathbb{Z},1+D(\mathbb{Z}+\tau\mathbb{Z}))$.

Theorem 2.12. *Let $m \geq 5$, $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$, ν and μ integers with $m = \mu - \nu - 1$ and ε and χ Hecke characters of weight zero and $\nu + \mu$, respectively. Set $\rho = \rho_{\nu\mu}$ and let $g_r \in \mathcal{T}_{r,\rho}$ for $r > 0$ be the Fourier-Jacobi coefficients of $\mathcal{L}_{\nu,\mu;\varepsilon,\chi}(f)$ and Θ be an element of the κ-eigenspace of $\mathcal{T}_{r,\rho}^{\mathrm{prim}}$. Let $0 \leq \lambda \leq \nu$ be such that Θ lies in the image of $d_{\nu\lambda}$. Then the Θ-part of g_r is given by*

$$\begin{aligned}\langle \Theta, g_r \rangle &= C'_{\nu\lambda}(r)\overline{\ell_{\chi\varepsilon^{-1}}(\Theta)} \sum_{\mathfrak{c}} (\chi\varepsilon)^{-1}(\mathfrak{c})\mathrm{N}(\mathfrak{c})^{(m+2\lambda)/2} \\ &\quad \sum_{(L,x)} (D^\lambda f)(L, x + DL) \left(\frac{x}{|x|}\right)^{2(\mu+\lambda)-1} \overline{\kappa^*(x\mathfrak{c}^{-1}\mathfrak{d}_{L,x}^{-1})} \mathrm{N}(\mathfrak{d}_{L,x})^{1/2}. \end{aligned} \tag{31}$$

Here $\mathfrak{c}$ runs over a system of representatives for the ideal classes of K and L over all lattices $L \subseteq \mathfrak{c}$ of index r in $\mathfrak{c}$. For each lattice L consider the lattices $L' \subseteq L$ with $L/L' \simeq \mathbb{Z}/D\mathbb{Z}$ and $(\mathfrak{o}_K L')\mathfrak{c}^{-1} = \mathfrak{d}_{L,x} \,|\, \mathfrak{d}_\Theta$ and take for each such L' an element $x \in L'$ such that $x\mathfrak{c}^{-1}\mathfrak{d}^{-1}$ is prime to rD. The constant $C'_{\nu\lambda}(r)$ is given by

$$C'_{\nu\lambda}(r) = (-1)^{\nu+\lambda} \binom{\nu}{\lambda}^{-1} D^{\lambda-\nu-1} r^{\mu+2\lambda-2} C_{\nu\lambda}(r).$$

Because κ^* is a Hecke character of weight $2(\mu+\lambda)-1$ and conductor $rD\mathfrak{d}_\Theta^{-1}$, the summands do not depend on the choice of x. We may also describe the summation condition as: sum over lattices $L' \subseteq \mathfrak{c}$ with $\mathfrak{c}/L' \simeq \mathbb{Z}/rD\mathbb{Z}$ and $(\mathfrak{o}_K L')\mathfrak{c}^{-1} = \mathfrak{d}|\mathfrak{d}_\Theta$, and set L to the unique lattice of index r in $\mathfrak{c}$ containing L'.

For a primitive eigenfunction Θ with associated eigencharacter κ the value $\ell_{\chi\varepsilon^{-1}}(\Theta)$ vanishes by Proposition 2.11 if and only if $L(\kappa^*\varepsilon^2\chi^{-2}, 1/2) = 0$. As a consequence we obtain a direct explanation of the result of Gelbart-Rogawski on the vanishing of certain Fourier-Jacobi coefficients of endoscopic forms [GeR, p. 468] (in the special case treated here).

We note that in the case of Eisenstein series the primitive coefficients may easily be expressed in terms of L-values. For scalar valued Eisenstein series this result was obtained by Hickey [Hic2] (by a different method).

Corollary 2.13. *Let $E_{m,\omega_{K/\mathbb{Q}}}$ be the (unnormalized) standard Eisenstein series of weight m and character $\omega_{K/\mathbb{Q}}$ given by*

$$E_{m,\omega_{K/\mathbb{Q}}}(\tau) = \sum_{(a,b)\in\mathbb{Z}^2\setminus\{0\}} \frac{\omega_{K/\mathbb{Q}}(b)}{(Da\tau + b)^m}$$

and g_r be the Fourier-Jacobi coefficients of $\mathcal{E}_{\nu,\mu;\varepsilon,\chi} = \mathcal{L}_{\nu,\mu;\varepsilon,\chi}(E_{m,\omega_{K/\mathbb{Q}}})$. Then

$$\begin{aligned} g_{0,\mathfrak{a}}(\psi) &= \psi(0,1)\delta^{-\nu}\frac{(\mu-1)!}{(2\pi i)^\mu} w_K(\varepsilon\chi^{-1})(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{(\nu-\mu)/2} \\ &\quad L(\chi\varepsilon^{-3}, (\mu-\nu)/2)L(\omega_{K/\mathbb{Q}}, m), \end{aligned}$$

and the Θ-part of g_r (assumptions as above) is given by

$$\begin{aligned} \langle \Theta, g_r \rangle &= C'_{\nu\lambda}(r)(-2\pi r\sqrt{D})^{-\lambda} w_K \frac{(m+\lambda-1)!}{(m-1)!} \\ &\quad \alpha_{\chi\varepsilon}(\mathfrak{d}_\Theta) L(\kappa^*(\chi\varepsilon)^{-1}, m/2)\overline{\ell_{\chi\varepsilon^{-1}}(\Theta)}, \end{aligned} \tag{32}$$

with

$$\alpha_{\chi\varepsilon}(\mathfrak{d}) = \prod_{\mathfrak{p}|\mathfrak{d}} (1 + (\chi\varepsilon)(\mathfrak{p})p^{(1-m-2\lambda)/2}).$$

Proof of the primitive coefficient formula We now turn to the proof of the theorem. It relies on the connection between the intrinsic theta functions $\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}$ and the Shintani operators $\mathcal{E}$. We begin by "expanding" Hecke operator and trace in Theorem 2.5.

We denote by $g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$ the Fourier-Jacobi coefficients of $\mathcal{L}_{\nu,\mu;\varepsilon,\chi}(f)$ for an integer $r>0$ and a fractional ideal $\mathfrak{a}$ of K. Then an easy calculation shows

$$\begin{aligned} g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}(w) &= \varepsilon(\bar{\mathfrak{a}}/\mathfrak{a}) \sum_{\mathfrak{b}\in\mathrm{Cl}_K} (\varepsilon^3\chi^{-1})(\mathfrak{b})\mathrm{N}(\mathfrak{b})^{(\nu+\mu)/2} \\ &\qquad \sum_{\lambda=0}^{\nu} \delta^{\lambda-\nu} r^{\lambda+\mu-1} \sum_{(L,x)} x^{-(\nu-\lambda+1)}(D^\lambda f)(L, x+DL) \\ &\qquad d_{\nu\lambda}\left[\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta x^{-1}w, \gamma_{L,x}(\tau_0(\mathfrak{a}\bar{\mathfrak{b}})))\right]. \end{aligned} \tag{33}$$

Here L ranges over all lattices of index r in $\mathfrak{c}=\mathfrak{a}\bar{\mathfrak{b}}=\mathbb{Z}+\mathbb{Z}\tau_0$, and for each sublattice $L'<L$ with $L/L'\simeq\mathbb{Z}/D\mathbb{Z}$ one chooses a primitive element $x\in L'$ and takes $\gamma_{L,x}\in\mathbb{Z}^{2\times 2}$ with positive determinant such that $\gamma\binom{\tau_0}{1}=\binom{*}{x}$ and $\mathbb{Z}^2\gamma\binom{\tau_0}{1}=L$ (of course, this means that x has to be primitive in L, too). Because of Proposition 2.4 all the individual terms in this sum are elements of $T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}$ and do not depend on the choice of x.

Let $\Theta=(\Theta_\mathfrak{a})\in\mathcal{T}^{\mathrm{prim}}_{r,\rho}$ be as in the statement of the theorem. Write $\Theta=d_{\nu\lambda}(\vartheta)$ with $\vartheta=(\vartheta_\mathfrak{a})\in\mathcal{T}^{\mathrm{prim}}_{r,\rho_{0,\mu+\lambda}}$. Then

$$\begin{aligned} \langle\Theta, g_r\rangle &= C_{\nu\lambda}(r)\delta^{\lambda-\nu}r^{\mu+\lambda-1}\sum_{\mathfrak{a}}(\varepsilon\chi^{-1})(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{(\mu-\nu)/2+\lambda} \\ &\qquad \sum_{\mathfrak{c}}(\varepsilon^{-3}\chi)(\mathfrak{c})\mathrm{N}(\mathfrak{c})^{(\mu+\nu)/2}\sum_{(L,x)}x^{-(\nu-\lambda+1)} \\ &\qquad (D^\lambda f)(L,x+DL)\langle\vartheta_\mathfrak{a}, \vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta x^{-1}w, \gamma_{L,x}(\tau_0(\mathfrak{c})))\rangle. \end{aligned} \tag{34}$$

At this point we insert two intermediate lemmas. We begin with the calculation of the scalar products.

Lemma 2.14. *Let $\mathfrak{a}$, $\mathfrak{b}$ be non-zero ideals of $\mathfrak{o}_K$ with $\mathfrak{c}=\mathfrak{a}\bar{\mathfrak{b}}=\mathbb{Z}+\mathbb{Z}\tau_0$ and $\vartheta_\mathfrak{a}\in T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$. Let $\gamma\in\mathbb{Z}^{2\times 2}$ be of determinant r and set $x=j(\gamma,\tau_0)$. Let $0\le\lambda\le\nu$. Then*

$$\begin{aligned} \langle\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta x^{-1}w,\gamma(\tau_0)),\vartheta_\mathfrak{a}\rangle &= \frac{1}{Dr}\binom{\nu}{\lambda}^{-1}\left(\frac{r}{\mathrm{N}(\mathfrak{a})}\right)^{\lambda}\left(-\frac{x}{\delta\mathrm{N}(\mathfrak{c})}\right)^{\nu-\lambda} \\ &\qquad (d_{\nu\lambda}(P\vartheta_\mathfrak{a}))(0;Y^\nu) \end{aligned}$$

with the operator

$$P=\sum_{a\in\bar{x}^{-1}\bar{\mathfrak{c}}\mathfrak{a}/\mathfrak{a}} e^{-\pi i\mathrm{N}(x\mathfrak{c}^{-1})\mathrm{Tr}(\gamma(\tau_0))\mathrm{N}(a\mathfrak{a}^{-1})}A_a. \tag{35}$$

Proof. Write $\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta x^{-1}w,\gamma(\tau_0))=\sum_{a\in\bar{\mathfrak{b}}}f_a(w)$ with

$$f_a(w)=\left(\frac{\bar{x}}{\mathrm{N}(\mathfrak{c})}\right)^{\nu-\lambda}(w+\mathrm{N}(\mathfrak{a})\bar{a}\bar{x}^{-1})^{\nu-\lambda}e^{2\pi i(\gamma(\tau_0)\mathrm{N}(a\mathfrak{b}^{-1})+ar\delta x^{-1}w)},\quad a\in\bar{\mathfrak{b}}.$$

It is not difficult to see that

$$f_{a+x\bar{b}\mathrm{N}(\mathfrak{a})^{-1}}=A_b f_a,\quad b\in\mathfrak{a}$$

(where we take the A_a associated to $T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$, of course), which yields

$$\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta x^{-1}w,\gamma(\tau_0)) = \sum_{b\in\mathfrak{a}} A_b \left(\sum_{a\in\bar{\mathfrak{b}}/x\mathfrak{a}^{-1}} f_a \right).$$

Therefore we can unfold the integral over $\mathbb{C}/\mathfrak{a}$ in the scalar product to obtain:

$$\begin{aligned}\langle\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta x^{-1}w,\gamma(\tau_0)),\vartheta_{\mathfrak{a}}\rangle &= \frac{2}{\sqrt{D}\mathrm{N}(\mathfrak{a})}\int_{\mathbb{C}/\mathfrak{a}} \overline{\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta x^{-1}u,\gamma(\tau_0))}\vartheta_{\mathfrak{a}}(u) \\ &\quad e^{-2\pi\sqrt{D}r\mathrm{N}(\mathfrak{a})^{-1}|u|^2}du \\ &= \frac{2}{\sqrt{D}\mathrm{N}(\mathfrak{a})}\left(\frac{x}{\mathrm{N}(\mathfrak{c})}\right)^{\nu-\lambda} \sum_{a\in\bar{\mathfrak{b}}/x\mathfrak{a}^{-1}} e^{-2\pi i\gamma(\bar{\tau}_0)\mathrm{N}(a\mathfrak{b}^{-1})} \\ &\quad \int_{\mathbb{C}} e^{2\pi i r\delta\bar{x}^{-1}\bar{a}\bar{u}}(\bar{u}+\mathrm{N}(\mathfrak{a})x^{-1}a)^{\nu-\lambda}\vartheta_{\mathfrak{a}}(u) \\ &\quad e^{-2\pi\sqrt{D}r\mathrm{N}(\mathfrak{a})^{-1}|u|^2}du.\end{aligned}$$

The integral occuring here may be evaluated by using the formula

$$\int_{\mathbb{C}} \bar{w}^k f(w)e^{-\lambda|w|^2}dw = \frac{\pi}{\lambda^{k+1}}f^{(k)}(0)$$

for holomorphic functions f such that the integral converges absolutely. We get

$$\begin{aligned}&\langle\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta x^{-1}w,\gamma(\tau_0)),\vartheta_{\mathfrak{a}}\rangle = \\ &\frac{1}{Dr}\left(\frac{x\mathrm{N}(\mathfrak{a})}{2\pi\sqrt{D}r\mathrm{N}(\mathfrak{c})}\right)^{\nu-\lambda} \sum_{a\in\bar{\mathfrak{b}}/x\mathfrak{a}^{-1}} e^{-2\pi i\gamma(\bar{\tau}_0)\mathrm{N}(a\mathfrak{b}^{-1})} \\ &\left(\frac{d}{dw}\right)^{\nu-\lambda}\left(\vartheta_{\mathfrak{a}}(w-\mathrm{N}(\mathfrak{a})\bar{x}^{-1}\bar{a})e^{2\pi i\delta r\mathrm{N}(\mathfrak{a})^{-1}(-\mathrm{N}(\mathfrak{a})x^{-1}a)w}\right)\Bigg|_{w=0},\end{aligned}$$

and substituting $a \leftarrow -\mathrm{N}(\mathfrak{a})\bar{x}^{-1}\bar{a}$ gives the result.

We now make the connection to Shintani operators.

Lemma 2.15. *Given a primitive theta function $\vartheta_{\mathfrak{a}} \in T^{\mathrm{prim}}_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$ and two lattices $L' < L$, where L has index r in $\mathfrak{c}$ and $L/L' \simeq \mathbb{Z}/D\mathbb{Z}$, we may distinguish two cases. In the first case, where $\mathfrak{d} = (\mathfrak{o}_K L')\mathfrak{c}^{-1}$ is* not *a product of ramified primes, we have $P\vartheta_{\mathfrak{a}} = 0$ for any primitive $x \in L'$.*

In the remaining case, where $\mathfrak{d}$ is a product of ramified primes, after choosing x such that $x\mathfrak{c}^{-1}\mathfrak{d}^{-1}$ is prime to rD, we have $P\vartheta_{\mathfrak{a}} = \mathrm{N}(x\mathfrak{d}\mathfrak{c}^{-1})^{1/2}\mathcal{F}(\bar{\mathfrak{m}}_{\mathrm{pr}}\mathfrak{m}_{\mathrm{pr}}^{-1})\Pi_{\mathfrak{d}}\vartheta_{\mathfrak{a}}$ with $\mathfrak{m}_{\mathrm{pr}} = \bar{x}\bar{\mathfrak{c}}^{-1}\mathfrak{d}^{-1}$.

Proof. Notation as above, we set $\mathfrak{m} = \bar{x}\bar{\mathfrak{c}}^{-1}$, which is an integral ideal. We first show that $P\vartheta_{\mathfrak{a}} = 0$ if $\mathfrak{d}_0 = (\mathfrak{o}_K L)\mathfrak{c}^{-1} \neq \mathfrak{o}_K$. Certainly $\mathfrak{d}_0$ is an integral ideal the norm of which divides r. By definition $\bar{\mathfrak{d}}_0$ divides $\mathfrak{m}$.

Take $y \in \bar{\mathfrak{d}}_0^{-1}\mathfrak{a}$. Since $\sigma = \mathrm{N}(x\mathfrak{c}^{-1})\gamma(\tau_0) \in \mathfrak{d}_0\mathfrak{m}$, we may verify that

$$\begin{aligned}PA_y &= \sum_{a\in\mathfrak{m}^{-1}\mathfrak{a}/\mathfrak{a}} e^{-\pi i(\mathrm{Tr}(\sigma)\mathrm{N}(a\mathfrak{a}^{-1})+r\mathrm{N}(\mathfrak{a})^{-1}\mathrm{Tr}(\delta\bar{a}y))}A_{a+y} \\ &= e^{\pi i\mathrm{Tr}(\sigma)\mathrm{N}(y\mathfrak{a}^{-1})}P = \psi(y)P,\end{aligned}$$

and consequently $Pt_{\mathfrak{a}/\bar{\mathfrak{d}}_0} = \mathrm{N}(\mathfrak{d}_0)P$. Since $\vartheta_{\mathfrak{a}}$ was supposed primitive, we get $P\vartheta_{\mathfrak{a}} = 0$, if $\mathfrak{d}_0$ is nontrivial.

Assuming $(\mathfrak{o}_K L)\mathfrak{c}^{-1} = \mathfrak{o}_K$, we know that $\mathfrak{d} = (\mathfrak{o}_K L')\mathfrak{c}^{-1} \,|\, D\mathfrak{o}_K$. So $\mathfrak{d}$ is either a product of ramified prime ideals or there exists $p \,|\, D$ dividing $\mathfrak{d}$. Since $\mathfrak{c}/L'$ is of order rD with a cyclic subgroup of order D, and $L' \subseteq \mathfrak{d}\mathfrak{c} \subseteq p\mathfrak{c}$, p has to be a divisor of r. Arguing as above, we may conclude that $Pt_{\mathfrak{a}/\bar{\mathfrak{p}}} = pP$ for the prime ideal $\mathfrak{p}$ above p, and consequently $P\vartheta_{\mathfrak{a}} = 0$.

Consider now the case that $\mathfrak{o}_K L = \mathfrak{c}$ and $\mathfrak{d}$ is a product of ramified primes. Obviously it is possible to choose $x \in L'$, primitive as an element of L, such that $x\mathfrak{c}^{-1}\mathfrak{d}^{-1}$ is prime to rD. We get $\mathfrak{m} = \mathfrak{d}\mathfrak{m}_{\mathrm{pr}}$ with $\mathfrak{m}_{\mathrm{pr}} + \bar{\mathfrak{m}}_{\mathrm{pr}} = \mathfrak{o}_K$. By writing $a = a_1 + a_2$ with $a_1 \in \mathfrak{m}_{\mathrm{pr}}^{-1}\mathfrak{a}/\mathfrak{a}$, $a_2 \in \mathfrak{d}^{-1}\mathfrak{a}/\mathfrak{a}$ in the definition of P, we may factorize the operator P as $P = P_1 P_2$, where

$$P_1 = \sum_{a_1 \in \mathfrak{m}_{\mathrm{pr}}^{-1}\mathfrak{a}/\mathfrak{a}} e^{-\pi i \mathrm{Tr}(\sigma)\mathrm{N}(a_1\mathfrak{a}^{-1})} A_{a_1}, \quad P_2 = \sum_{a_2 \in \mathfrak{d}^{-1}\mathfrak{a}/\mathfrak{a}} e^{-\pi i \mathrm{Tr}(\sigma)\mathrm{N}(a_2\mathfrak{a}^{-1})} A_{a_2}.$$

It is easy to verify that $P_2 = \mathrm{N}(\mathfrak{d})\Pi_{\mathfrak{d}}$, since $\mathrm{Tr}(\sigma) \in \mathrm{N}(\mathfrak{d})$ and the exponential collapses to $\psi(a)$. By choosing representatives $a_1 \in \bar{\mathfrak{m}}_{\mathrm{pr}}\mathfrak{m}_{\mathrm{pr}}^{-1}\mathfrak{a}/\bar{\mathfrak{m}}_{\mathrm{pr}}\mathfrak{a}$, we see that $P_1 = \mathcal{E}(\bar{\mathfrak{m}}_{\mathrm{pr}}\mathfrak{m}_{\mathrm{pr}}^{-1})$ for the same reason. The lemma is proved.

We remark that the operators $P = P_{(L,x)}$ are connected to the local operators considered by Murase-Sugano [MS].

We now are able to finish the proof of the theorem. Take in (34) a summand corresponding to a choice of $\mathfrak{c}$, L and L'. From the lemmas above we see

$$\begin{aligned}
\langle \vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta x^{-1}w, \gamma(\tau_0)), \vartheta_{\mathfrak{a}} \rangle &= C d_{\nu\lambda}((P\vartheta_{\mathfrak{a}}))(0; Y^\nu) \\
&= C|x| \mathrm{N}(\mathfrak{d}\mathfrak{c}^{-1})^{1/2} \kappa^*(x\mathfrak{c}^{-1}\mathfrak{d}^{-1}) \\
&\quad d_{\nu\lambda}(\vartheta_{\mathfrak{a}\bar{\mathfrak{m}}\mathfrak{m}^{-1}})(0; Y^\nu) \\
&= C|x| \left(\frac{x}{|x|}\right)^{-2(\mu+\nu)} \mathrm{N}(\mathfrak{d}\mathfrak{c}^{-1})^{1/2} \kappa^*(x\mathfrak{c}^{-1}\mathfrak{d}^{-1}) \\
&\quad \Theta_{\mathfrak{a}\bar{\mathfrak{c}}\mathfrak{c}^{-1}}(0; Y^\nu)
\end{aligned}$$

with

$$C = \frac{1}{Dr} \binom{\nu}{\lambda}^{-1} \left(\frac{r}{\mathrm{N}(\mathfrak{a})}\right)^{\lambda} \left(-\frac{x}{\delta \mathrm{N}(\mathfrak{c})}\right)^{\nu-\lambda},$$

whenever $(\mathfrak{o}_K L')\mathfrak{c}^{-1} = \mathfrak{d} \,|\, \mathfrak{d}_\vartheta$ and $x \in L'$ is primitive with $x\mathfrak{c}^{-1}\mathfrak{d}^{-1}$ prime to rD. All terms with $(\mathfrak{o}_K L')\mathfrak{c}^{-1} = \mathfrak{d} \nmid \mathfrak{d}_\vartheta$ vanish. Putting everything together we get

$$\begin{aligned}
\langle \Theta, g_r \rangle &= C'_{\nu\lambda}(r) \sum_{\mathfrak{a}} (\varepsilon\chi^{-1})(\mathfrak{a}) \mathrm{N}(\mathfrak{a})^{(\mu-\nu)/2} \sum_{\mathfrak{c}} (\varepsilon^{-3}\chi)(\mathfrak{c}) \mathrm{N}(\mathfrak{c})^{(m+2\lambda)/2} \\
&\quad \sum_{(L,x)} (D^\lambda f)(L, x + DL) \left(\frac{x}{|x|}\right)^{2(\mu+\lambda)-1} \mathrm{N}(\mathfrak{d})^{1/2} \\
&\quad \overline{\kappa^*(x\mathfrak{c}^{-1}\mathfrak{d}^{-1}) \Theta_{\mathfrak{a}\bar{\mathfrak{c}}\mathfrak{c}^{-1}}(0; Y^\nu)}.
\end{aligned}$$

Making here the substitution $\mathfrak{a} \leftarrow \mathfrak{a}\bar{\mathfrak{c}}/\mathfrak{c}$ allows to write the sum as a product of the sums over $\mathfrak{a}$ and $\mathfrak{c}$ and yields the desired statement. (We may now discard the condition $\mathfrak{c} = \mathbb{Z} + \mathbb{Z}\tau$ and do not need to require x to be primitive.)

3 Integrality and reduction modulo ℓ

We now come to arithmetic applications of the Fourier coefficient formulas obtained in the previous chapter. In the first section the integrality of the lifting is established, followed by a study of the kernel of the resulting map on the level of "modular forms mod ℓ". After a subspace of the kernel, called the "trivial kernel", has been constructed, we state the central non-vanishing theorem for split ℓ in Section 3.3, and give the first part of its proof. The proof will be finished in the next chapter.

3.1 Integrality of the theta lift

After recalling some facts on elliptic modular forms and defining the basic notions of arithmetic and integral theta functions, the main object of this section is to prove the integrality (except for the constant term) of the (suitably normalized) theta lifting. This will be done by expressing the value of a Fourier-Jacobi coefficient at a point $x \in K$ in terms of elliptic modular forms, from which integrality may easily be deduced.

Geometric interpretation of elliptic modular forms We have to recall briefly the geometric interpretation of elliptic modular forms following Katz [Ka1, Ka2]. From this point of view, modular forms are sections of certain line bundles on moduli schemes (or stacks), or functorial applications on "test objects".

For an elliptic curve E over a ring R, let $E[N]$ be the kernel of multiplication by N and $e_N : E[N] \times E[N] \to \mu_N$ the canonical alternating pairing (Weil pairing). In Katz, for an integer N a (naive) level N structure (or $\Gamma(N)$-structure) on E/R is an isomorphism (of R-group schemes) $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$. The existence of such a level structure implies that N is invertible in R. It is obviously equivalent to consider isomorphisms $\alpha : \mathfrak{o}_K/N\mathfrak{o}_K \to E[N]$, as we will do later. To each pair (E, α) we may associate the discrete invariant $\det \alpha$, a primitive N-th root of unity, by $e_N(\alpha(x), \alpha(y)) = (\det \alpha)^{\mathrm{Tr}(\delta^{-1}\bar{x}y)}$. A (naive) $\Gamma_1(N)$-structure on E/R is an inclusion $i : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]$.

A $\Gamma_1(N)$-test object over R is a triple (E, ω, i) consisting of an elliptic curve E over R, a nowhere-vanishing invariant differential ω on E and a naive $\Gamma_1(N)$-structure on E/R. A modular form over a ring R, of weight m for the group $\Gamma_1(N)$, is now defined as a rule associating to each $\Gamma_1(N)$-test object (E, ω, i), defined over an R-algebra R', a value $f(E, \omega, i) \in R'$ depending only on the R-isomorphism class of the test object, subject to the conditions that the formation of $f(E, \omega, i)$ commutes with base change over R, and that it is homogeneous of degree $-m$ in ω:

$$f(E, \lambda\omega, i) = \lambda^{-m} f(E, \omega, i), \quad \lambda \in R'^{\times}.$$

In addition, we require holomorphy of f at the cusps (see below). We denote the R-module of these forms by $M_m(\Gamma_1(N); R)$. For a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$ the submodule $M_m(\Gamma_0(N), \chi; R)$ is the module of all f fulfilling the additional condition

$$f(E, \omega, i \circ x) = \chi(x) f(E, \omega, i),$$

for $x \in (\mathbb{Z}/N\mathbb{Z})^\times$. In the same manner, we define $\Gamma(N)$-test objects and modular forms for $\Gamma(N)$.

The definitions given above make (geometric) sense, since the corresponding functors (isomorphism classes of elliptic curves with $\Gamma(N)$- or $\Gamma_1(N)$-structure) are representable by schemes (more precisely, smooth affine curves) over $\mathbb{Z}[1/N]$ for all $N \geq 3$ (resp. $N \geq 4$ for $\Gamma_1(N)$). So geometric modular forms are sections of certain invertible sheaves on the compactified moduli scheme obtained by adding a finite number of points (cusps). The $\Gamma_1(N)$-moduli scheme is geometrically connected, while the $\Gamma(N)$-moduli scheme has $\varphi(N)$ connected components over $\mathbb{Z}[1/N, \zeta_N]$ corresponding to the values of the discrete invariant $\det \alpha$. We know that for an ideal I of R base change from $M_m(\Gamma_0(N), \chi; R)$ to $M_m(\Gamma_0(N), \chi; R/I)$ is surjective, if $6\,\mathrm{order}(\chi)$ is invertible in R/I (cf. [Ka1, p. 85])[5].

We have the important special elliptic curve $\mathrm{Tate}(q)$ (the "Tate curve") over the ring $\mathbb{Z}((q))$ of finite-tailed Laurent series over $\mathbb{Z}$. It may be viewed as an algebraic

[5] We may relax this requirement to $t\,\mathrm{order}(\chi)$ invertible, where t is the l. c. m. of the orders of the torsion elements in the subgroup $\Gamma_0(N) \supseteq \Gamma \supseteq \Gamma_1(N)$ defined by the kernel of χ.

version of $\mathbb{G}_m/q^{\mathbb{Z}}$, and carries a canonical invariant differential ω_{can} deduced from dq/q on $\mathbb{G}_m$. If the ground ring R contains the N-th roots of unity, we may give $\text{Tate}(q)$ a $\Gamma(N)$-structure α over $R((q^{1/N}))$, and evaluate $f \in M_m(\Gamma(N); R)$ on the test object $(\text{Tate}(q), \omega_{\text{can}}, \alpha)$ to obtain an element $f(q) \in R((q^{1/N}))$, the q-expansion of f at the cusp determined by α. Holomorphy of f at the cusps means that the q-expansions are already in $R[[q^{1/N}]]$. The same thing works for $\Gamma_1(N)$-level structures. In this case special level structures on $\text{Tate}(q)$ are given by the inclusions $i_{\zeta_N} : x \mapsto \zeta_N^x$ for $\zeta_N \in \mu_N^{\text{prim}}$. They give q-expansions in $R[[q]]$ depending on the choice of the primitive N-th root of unity ζ_N.

The q-expansion principle [Ka1, p. 83] says that modular forms are determined by their q-expansions: assuming again that R is a $\mathbb{Z}[1/N, \zeta_N]$-algebra, a modular form (of level N) having q-expansion zero at at least one cusp on each connected component of the moduli space is necessarily zero. In addition, if a modular form over R has one q-expansion on every component defined over a $\mathbb{Z}[1/N, \zeta_N]$-subalgebra S, it descends to S.

Let us describe the link with the situation over the complex numbers: to a lattice $L \subseteq \mathbb{C}$ and a point $l \in L/NL$ of order N associate the $\Gamma_1(N)$-test object (E, ω, i) over $\mathbb{C}$ by setting $E = \mathbb{C}/(2\pi i)L$, $\omega = dz$ and $i(x) = (2\pi i)xl/N + (2\pi i)L \in E[N]$. In this way every complex analytic modular form (viewed as a function of pairs (L, l)) corresponds to a geometric modular form over $\mathbb{C}$, and the correspondence preserves q-expansions if we make use of the canonical choice $\zeta_N = e^{2\pi i/N}$ of a primitive N-th root of unity (this property is the reason for the factor $2\pi i$ in the definition). By the q-expansion principle a modular form is already defined over a subring $\mathbb{Z}[1/N, \zeta_N] \subseteq R \subseteq \mathbb{C}$, if and only if it has its q-expansion coefficients at some cusp in R.

The (tautological) rationality and integrality property of modular forms defined over R may be phrased in the complex analytic context as follows: for $f \in M_m(\Gamma_1(N); R)$ we have $f(L, x+NL) \in R$, if L is the period lattice of a nowhere-vanishing differential ω on an elliptic curve E defined over R, and $x \in L/NL$ the point corresponding to $i(1)$ for a $\Gamma_1(N)$-level structure i defined over R. The following lemma (due to Katz [Ka2], [Ka4, Theorem 2.4.5]) extends this even to the non-holomorphic derivatives $D^\nu f$, if we restrict to elliptic curves E with complex multiplication (and disregard primes dividing the discriminant of the endomorphism ring).

Lemma 3.1. *Let (E, ω, i) be a test object for $\Gamma_1(N)$ defined over a ring $\mathbb{Z}[1/N] \subseteq R \subseteq \mathbb{C}$ and let E have complex multiplication by an imaginary quadratic number ring (order) $\mathfrak{o}$. Then $(D^\nu f)(L, x + NL)$ is an element of $R[1/\text{disc}(\mathfrak{o})]$, if L is the period lattice of ω, and $x \in L/NL$ corresponds to $i(1)$. The same assertion holds for $\Gamma(N)$-test objects.*

Arithmetic and integral theta functions We come now to the basic definitions of arithmetic and integral theta functions. The notion of an arithmetic theta function was introduced by Shimura [Shim4, Shim2]. Given a K-rational representation ρ of $\text{GL}_2(\mathbb{C}) \times \mathbb{C}^\times$ on a complex vector space $\mathcal{V} = \mathcal{V}_K \otimes \mathbb{C}$, we define for every field $\mathbb{C} \supseteq L \supseteq K^{\text{ab}}$, K^{ab} denoting the maximal abelian extension of K inside $\mathbb{C}$, and parameters r and $\mathfrak{a}$, the L-vector space $T_{r,\mathfrak{a};\rho}(L) \subseteq T_{r,\mathfrak{a};\rho}$ of L-arithmetic theta functions as the space of all $\vartheta \in T_{r,\mathfrak{a};\rho}$ such that

$$B(A_x\vartheta)(0) \in L\mathcal{V}_K, \quad x \in K.$$

Here $B \in \text{GL}(\mathcal{V})$ is a ρ_0-regulator in the sense of Shimura [Shim4, p. 577-580] for the $\mathbb{G}_m$-representation

$$\rho_0(x) = \rho(\text{diag}(1, x^{-1}), x^{-1})$$

(note that our ρ differs from Shimura's, since we are using Shintani's conventions). We know by [Shim4, p. 590, Prop. 4.7], that $T_{r,\mathfrak{a};\rho}(L) \otimes_L \mathbb{C} = T_{r,\mathfrak{a};\rho}$. The notion of arithmeticity may trivially be extended to define subspaces $\mathcal{T}_{d,\rho}(L) \subseteq \mathcal{T}_{d,\rho}$.

For the representations $\rho_{\nu\mu}$, which are certainly K-rational, a ρ-regulator may be constructed as follows: take a period $\Omega_0 \in \mathbb{C}^\times$ such that $\Delta(2\pi i \Omega_0 \mathfrak{o}_K)$ is a unit in K^{ab}, or $\Omega_0^{12} = (2\pi i)^{-12}\Delta(\mathfrak{o}_K)u$ for a unit $u \in K^{\mathrm{ab}}$ (the possible values of Ω_0 form a coset in $\mathbb{C}^\times/(\bar{\mathbb{Z}}^\times \cap K^{\mathrm{ab}})$). Then $B = B_0 := \mathrm{diag}((D\Omega_0)^\nu, \dots, D\Omega_0, 1)$.

Note that the Shintani operators $\mathcal{E}$ and $\mathcal{F}$ act on the space of arithmetic theta functions; consequently the decomposition of $\mathcal{T}_{d,\rho}$ into eigenspaces is arithmetic (takes place in $\mathcal{T}_{d,\rho}(\bar{\mathbb{Q}})$).

We now proceed to generalize this notion and define integral theta functions (with respect to a ring R). This notion was introduced (in the scalar case) independently by Hickey [Hic1], Larsen [Lar1, Lar3] and the present author [Fi]. We take $\rho = \rho_{\nu\mu}$ acting on $\mathcal{V} = \mathcal{V}_{\mathfrak{o}_K} \otimes_{\mathfrak{o}_K} \mathbb{C}$ (where $\mathcal{V}_{\mathfrak{o}_K}$ is the $\mathfrak{o}_K$-dual of the module of polynominals with $\mathfrak{o}_K$-coefficients) and define for every ring $\mathbb{C} \supseteq R \supseteq K^{\mathrm{ab}} \cap \bar{\mathbb{Z}}$ the R-module $T_{r,\mathfrak{a};\rho}(R) \subseteq T_{r,\mathfrak{a};\rho}$ of R-integral theta functions as the space of all $\vartheta \in T_{r,\mathfrak{a};\rho}$ with

$$B(A_x\vartheta)(0) \in R\mathcal{V}_{\mathfrak{o}_K}, \quad x \in K,$$

where $B = B(\mathfrak{a}) \in \mathrm{GL}(\mathcal{V})$ is now an integral regulator. In terms of a period $\Omega(\mathfrak{a}) \in \mathbb{C}^\times$ such that $\Delta(2\pi i \Omega(\mathfrak{a})\mathfrak{a})$ is a unit in K^{ab}, we have

$$B(\mathfrak{a}) = \mathrm{diag}((D\Omega(\mathfrak{a})\mathrm{N}(\mathfrak{a}))^\nu, \dots, D\Omega(\mathfrak{a})\mathrm{N}(\mathfrak{a}), 1).$$

The periods $\Omega(\mathfrak{a})$ and Ω_0 are related as follows.

Lemma 3.2. *Let $\mathfrak{a}$ be a fractional ideal of K and $\alpha(\mathfrak{a}) \in K^{\mathrm{ab}}$ with $\alpha\bar{\mathbb{Z}} = \mathfrak{a}\bar{\mathbb{Z}}$ (such an element exists certainly in the Hilbert class field of K). Then the number $\alpha(\mathfrak{a})\Omega(\mathfrak{a})\Omega_0^{-1}$ is a unit in K^{ab}.*

Proof. See [Lan, p. 165, Th. 5].

We now define the submodule $\mathcal{T}_{d,\rho}(R) \subseteq \mathcal{T}_{d,\rho}$ as the space of all $\vartheta \in \mathcal{T}_{d,\rho}$ with $\alpha(\mathfrak{a})^{\nu+\mu}\mathrm{N}(\mathfrak{a})^{-\nu}\vartheta_{\mathfrak{a}} \in T_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}(R)$, where $\alpha(\mathfrak{a}) \in K^{\mathrm{ab}}$ is as above. Equivalently,

$$B_0\rho(\mathrm{diag}(\overline{\alpha(\mathfrak{a})}^{-1}), 1), \alpha(\mathfrak{a})/\overline{\alpha(\mathfrak{a})})(A_x\vartheta_{\mathfrak{a}})(0) \in R\mathcal{V}_{\mathfrak{o}_K}, \quad x \in K.$$

Generalizing Shimura's aforementioned result on arithmetic theta functions, we have that even integral theta functions generate the full space over $\mathbb{C}$: $T_{r,\mathfrak{a};\rho}(R) \otimes_R \mathbb{C} = T_{r,\mathfrak{a};\rho}$. In the proof of the integrality theorem below, we will need the following density lemma.

Lemma 3.3. *If r, $\mathfrak{a}$ and ρ are as above, ℓ is a prime, and $g \in T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}$ with*

$$B(\mathfrak{a})(A_w g)(0) \in \bar{\mathbb{Z}}_{(\ell)}\mathcal{V}_{\mathfrak{o}_K}$$

for an infinite number of elements $w \in K/\mathfrak{a}$ of order prime to ℓ, then g is $\bar{\mathbb{Z}}_{(\ell)}$-integral. Furthermore, if $\mathfrak{L}$ is a prime ideal of $\bar{\mathbb{Z}}_{(\ell)}$, and

$$B(\mathfrak{a})(A_w g)(0) \in \mathfrak{L}\mathcal{V}_{\mathfrak{o}_K},$$

for infinitely many $w \in K/\mathfrak{a}$ of prime-to-ℓ order, we have $g \in \mathfrak{L}T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}(\bar{\mathbb{Z}}_{(\ell)})$.

To get a better picture of the situation, we interpret integral theta functions geometrically. Consider the elliptic curve $E_{\mathfrak{a}} = \mathbb{C}/2\pi i \Omega(\mathfrak{a})\mathfrak{a}$. It descends to $\bar{\mathbb{Q}}$ and even to $\bar{\mathbb{Z}}$, since CM elliptic curves have potentially good reduction everywhere. Our condition on $\Omega(\mathfrak{a})$ ensures that $\omega_{\mathfrak{a}} = dz$ is a nowhere-vanishing differential over $\bar{\mathbb{Z}}$. In the scalar valued case $\nu = 0$ we know that $T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}$ is the space of global sections of $L^{\otimes r}$ over $\mathbb{C}$ for a symmetric line bundle L of degree D on the elliptic curve $E_{\mathfrak{a}}$. Since L may be defined over $\bar{\mathbb{Z}}$, integral theta functions correspond just to sections of the bundle over $\bar{\mathbb{Z}}$ (cf. [Hic1]). In particular, we are able to reduce integral theta functions modulo a prime ideal $\mathfrak{L}$ of $\bar{\mathbb{Z}}$ to obtain sections of the reduction $\bar{L}$ on the reduced curve $\bar{E}_{\mathfrak{a}}$.

For vector valued theta functions, we have to change the vector bundle slightly (by the regulator $B(\mathfrak{a})$) to get an object defined over $\bar{\mathbb{Z}}$. More precisely, $T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}$ is the space of sections of the rank $\nu+1$ bundle $V_\nu \otimes L^{\otimes r}$, where V_ν is a ν-fold extension of $\mathcal{O}_E$ by itself; explicitly it is obtained as $V_\nu = (\mathcal{V}_\nu \times \mathbb{C})/\mathfrak{a}$, the action of $\mathfrak{a}$ being

$$l(v,w) = (\rho(\begin{pmatrix} 1 & \delta\bar{l} \\ 0 & 1 \end{pmatrix}, 1)v, w+l), \quad l \in \mathfrak{a}.$$

Since V_ν is the ν-th symmetric power of V_1, we have to look at the extension

$$0 \longrightarrow \mathcal{O}_E \longrightarrow V_1 \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

which is determined by its extension class in $\mathrm{Ext}^1(\mathcal{O}_E, \mathcal{O}_E) \simeq H^1(\mathcal{O}_E)$. An easy computation shows that it is equal to $D\mathrm{N}(\mathfrak{a})\Omega(\mathfrak{a})$ times the generator of $H^1(\mathcal{O}_E)$ dual to $\omega_{\mathfrak{a}}$ (via Serre duality). Changing everything by $B(\mathfrak{a})$ we get a vector bundle V_ν^{ar} defined over $\bar{\mathbb{Z}}$.

Turning to the assertions made above, it follows now almost trivially that $T_{r,\mathfrak{a};\rho}(R) \otimes_R \mathbb{C} = T_{r,\mathfrak{a};\rho}$. Lemma 3.3 is a consequence of the (Zariski) density of the set of torsion points of prime-to-ℓ order in the reduction of $E_{\mathfrak{a}}$ mod $\mathfrak{L}$.

Now we are able to define R-integral modular forms in the spaces $A(\rho, L)$ and $A(\rho, L, \chi)$: a form $F \in A(\rho, L, \chi)$ is called R-integral if $g_d \in \mathcal{T}_{d,\rho}(R)$ for all $d \geq 0$. The R-module of these forms is denoted by $A(\rho, L, \chi; R)$. In the same way, we have $F \in A(\rho, L; R)$ precisely when $\alpha(\mathfrak{b})^{-(\nu+\mu)}\mathrm{N}(\mathfrak{b})^{(\nu+\mu)/2} g_{d;\mathfrak{b}} \in \mathcal{T}_{d,\rho}(R)$ for all $d \geq 0$. Taking $R = K^{\mathrm{ab}}$, we recover Shimura's notion of arithmetic modular forms [Shim4]. We also have a naive notion of modular form over $\bar{\mathbb{F}}_\ell$ as a formal q-expansion obtained by reducing the q-expansion of a $\bar{\mathbb{Z}}_{(\ell)}$-integral modular form modulo a prime ideal $\mathfrak{L}$. In Chapter 5 we will sketch a geometric treatment of modular forms analogous to the theory of geometric elliptic modular forms summarized above.

More intrinsic theta functions In order to prove the integrality of the theta lift, we have to introduce certain theta functions on the upper half plane closely connected to the intrinsic theta functions considered so far, or more precisely to their values at points in K.

For a nonnegative integer k, a positive integer N, a fractional ideal $\mathfrak{a}$ of K and a function $\varphi : (N\delta)^{-1}\mathfrak{a}/\mathfrak{a} \to \mathbb{C}$ consider the theta function

$$\vartheta^{(k)}_{\mathfrak{a};\varphi}(\tau) = \sum_{\alpha \in (N\delta)^{-1}\mathfrak{a}} \alpha^k \varphi(\alpha) e^{2\pi i N \mathrm{N}(\mathfrak{a})^{-1}\mathrm{N}(\alpha)\tau} \tag{36}$$

on $\mathfrak{H}$.

Lemma 3.4. *Let k, N, $\mathfrak{a}$, φ be as above and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(ND)$. Then*

$$\vartheta^{(k)}_{\mathfrak{a};\varphi}(\gamma(\tau)) = \omega_{K/\mathbb{Q}}(\gamma) j(\gamma,\tau)^{k+1} \vartheta^{(k)}_{\mathfrak{a};\varphi'}(\tau)$$

with

$$\varphi'(\alpha) = \varphi(d\alpha) e^{2\pi i N \mathrm{N}(\mathfrak{a})^{-1}\mathrm{N}(\alpha) bd}, \quad \alpha \in N^{-1}\mathfrak{a}.$$

Consequently $\vartheta^{(k)}_{\mathfrak{a};\varphi}$ is a modular form of weight $k+1$ and character $\omega_{K/\mathbb{Q}}$ for the group $\Gamma(ND)$. If φ is supported on $N^{-1}\mathfrak{a}$, it is a modular form for the group $\Gamma_0(ND) \cap \Gamma(N)$.

Evidently, in case N is relatively prime to D, we have $\Gamma_0(ND)\cap\Gamma(N) = \Gamma_0(D)\cap\Gamma(N)$.

Proof. This is just a restatement of [P, p. 237, (B.19)] (the result is essentially due to Hecke).

Integrality theorem We now state the main result of this section.

Theorem 3.5. *Let ν and μ be integers with $m = \mu - \nu - 1 \geq 5$, and ε and χ Hecke characters of weight zero and $\nu + \mu$, respectively. Define an arithmetic variant of the lifting $\mathcal{L}_{\nu,\mu;\varepsilon,\chi} : M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}) \to A(\rho_{\nu\mu}, L, \chi)$ by*

$$\mathcal{L}^{\mathrm{ar}}_{\nu,\mu;\varepsilon,\chi} = \Omega_0^{-(\nu+\mu)}\mathcal{L}_{\nu,\mu;\varepsilon,\chi}.$$

Then the lifting $\mathcal{L}^{\mathrm{ar}}$ preserves integrality away from D, except for the constant term: for $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}[1/D])$ we have $F_{\mathrm{ar}} = \mathcal{L}^{\mathrm{ar}}_{\nu,\mu;\varepsilon,\chi}(f) \in A(\rho_{\nu\mu}, L, \chi; \bar{\mathbb{Q}})$ and for every $d > 0$ the degree d Fourier-Jacobi coefficient g^{ar}_d of F_{ar} lies in $\mathcal{T}_{d,\rho}(\bar{\mathbb{Z}}[1/D])$. Furthermore, for a split prime $\ell \not| D$ we have

$$\mathcal{L}^{\mathrm{ar}}_{\nu,\mu;\varepsilon,\chi}(M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}_{(\ell)})) \subseteq A(\rho_{\nu\mu}, L, \chi; \bar{\mathbb{Z}}_{(\ell)}),$$

i. e. the constant term is ℓ-integral too, except in the case where $\nu = 0$ and $(\ell-1)|\mu$. For inert ℓ the same statement holds if $(\ell - 1) \not| (\mu - \nu)$ or $\nu = 0$.

Arithmeticity of the lifting $\mathcal{L}^{\mathrm{ar}}$ (i. e. the assertion $F_{\mathrm{ar}} \in A(\rho_{\nu\mu}, L, \chi; \bar{\mathbb{Q}})$) was proven by Kudla [Ku1] in a more general context. The case of Eisenstein series had been treated earlier by Shimura in [Shim4]. A (somewhat weaker) result on integrality of scalar valued Eisenstein series was obtained by Larsen [Lar1, Lar3].

To prove this theorem, we need to evaluate the Fourier-Jacobi coefficients of the lifting at points of finite order. A convenient expression for these values is provided by the following proposition. As usual, we denote $v(X^{\nu-l}Y^l)$ by v_l for elements $v \in S^*_\nu$.

Proposition 3.6. *Keeping the notation of the theorem, let $r > 0$, $\mathfrak{a}$ a fractional ideal of K, and $w \in M^{-1}\mathfrak{a}$ with $(M, rD) = 1$. Then we have for the Fourier-Jacobi coefficient $g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$ of $\mathcal{L}_{\nu,\mu;\varepsilon,\chi}(f)$:*

$$\begin{aligned}(A_w g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}})(0)_l &= \left(\frac{M}{\delta}\right)^{\nu-l}(\varepsilon\chi^{-1})(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{(\nu-\mu)/2} \\ &\quad \sum_{\mathfrak{c}}(\chi\varepsilon^{-3})(\mathfrak{c})\mathrm{N}(\mathfrak{c})^{(\mu-\nu)/2+l} \\ &\quad D^l\left[T_r\mathrm{Tr}_{\Gamma_0(D)\cap\Gamma(M^2)\backslash\Gamma(M^2)}(f\vartheta^{(\nu-l)}_{\mathfrak{b};\varphi})\right]\Big|_{\tau=\tau_0(\mathfrak{c})},\end{aligned}$$

where $\mathfrak{c} = \mathbb{Z} + \tau_0(\mathfrak{c})\mathbb{Z}$, $\tau_0(\mathfrak{c}) \in \mathfrak{H}$, runs over a system of representatives for the ideal classes of K, $\mathfrak{b} = \bar{\mathfrak{c}}\bar{\mathfrak{a}}^{-1}$, and $\varphi : M^{-2}\mathfrak{b}/\mathfrak{b} \to \mathbb{C}$ is defined by

$$\varphi(\alpha) = \lambda_{\mathbb{C}}(-\mathrm{N}(\mathfrak{c})^{-1}M\alpha\bar{w}\tau_0)e^{\pi i r\mathrm{N}(\mathfrak{c})^{-1}\mathrm{Tr}(\tau_0)\mathrm{N}(w\mathfrak{a}^{-1})},$$

if $M^2\alpha \equiv r\mathrm{N}(\mathfrak{a})^{-1}(Mw)\,(M\mathfrak{b})$, and $\varphi(\alpha) = 0$, otherwise.

Here $\lambda_{\mathbb{C}}$ is the additive character of $\mathbb{C}$ defined by $\lambda_{\mathbb{C}}(x) = e^{2\pi i(x+\bar{x})}$.

Admitting this result for the moment, we may deduce the theorem as follows. Let $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}[1/D])$. Then for $r > 0$ the modular form

$$h = \alpha(\mathfrak{b})^{l-\nu}T_r\mathrm{Tr}_{\Gamma_0(D)\cap\Gamma(M^2)\backslash\Gamma(M^2)}(f\vartheta^{(\nu-l)}_{\mathfrak{b};\varphi})$$

is integral over the ring $\bar{\mathbb{Z}}[1/DM]$, since $\alpha(\mathfrak{b})^{l-\nu}\vartheta^{(\nu-l)}_{\mathfrak{b};\varphi}$ is (consider the Fourier expansion). Consequently, $(D^l h)(\tau_0(\mathfrak{c})) \in \Omega(\mathfrak{c})^{\mu+l}\bar{\mathbb{Z}}[1/DM]$.

Furthermore, it is elementary that for a Hecke character γ of K of weight w, and a fractional ideal $\mathfrak{a}$ prime to the conductor, $\gamma(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{w/2}\alpha(\mathfrak{a})^{-w}$ is a unit. Using this, we get easily from Proposition 3.6

$$B_0\rho(\mathrm{diag}(\overline{\alpha(\mathfrak{a})}^{-1}), 1), \alpha(\mathfrak{a})/\overline{\alpha(\mathfrak{a})})(A_w g^{\mathrm{ar}}_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}})(0) \in \bar{\mathbb{Z}}[1/DM]\mathcal{V}_{\mathfrak{o}_K},$$

for $w \in M^{-1}\mathfrak{a}$. Invoking the "density lemma" (Lemma 3.3) for each prime $\ell \not| D$, we conclude $g^{\mathrm{ar}}_r \in \mathcal{T}_{r,\rho}(\bar{\mathbb{Z}}[1/D])$ for all $r > 0$.

It remains to consider the constant term. Write

$$\begin{aligned} g^{\mathrm{ar}}_{0,\mathfrak{a}}(\psi) &= \psi(0,1)(\chi^{-1}\varepsilon)(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{(\nu-\mu)/2} \\ &\quad \Omega_0^{-(\mu+\nu)} \sum_{\mathfrak{c}\in \mathrm{Cl}_K} (\chi\varepsilon^{-3})(\mathfrak{c})\mathrm{N}(\mathfrak{c})^{(\mu+\nu)/2} D^\nu(E^{\mathrm{norm}}_{\mu-\nu})(\mathfrak{c})\mathrm{Tr}(f\vartheta_{\mathfrak{c}\mathfrak{a}})(i\infty), \end{aligned}$$

where E^{norm}_k is the Eisenstein series of weight k for $\mathrm{SL}_2(\mathbb{Z})$, normalized to have q-expansion $-B_k/2k + q + \ldots$ If now ℓ is a split prime, and $\nu = 0$, we see that E^{norm}_μ is ℓ-integral if not $(\ell-1)|\mu$, the case we excluded. This quickly gives the desired statement. If $\nu > 0$ (and μ arbitrary), let $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_\ell : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_\ell$ be embeddings. By Katz's comparison theorem [Ka2] there is an ℓ-adic modular form $\varphi = \theta^\nu(E^{\mathrm{norm}}_{\mu-\nu})$ (with Serre's differential operator $\theta = qd/dq$) taking the value $i_\ell i_\infty^{-1}(\Omega(\mathfrak{c})^{-(\mu+\nu)}D^\nu(E^{\mathrm{norm}}_{\mu-\nu})(\mathfrak{c}))$ on the suitably trivialized elliptic curve $\mathbb{C}/(2\pi i)\Omega(\mathfrak{c})\mathfrak{c}$. The derivation θ kills the constant term, and φ is integral, which gives what we want. In case ℓ is inert, we have to use the vanishing of the Hasse invariant to deduce integrality. We omit the proof, since we will not consider the inert case in the sequel.

Proof of the torsion point formula We now turn to the proof of Proposition 3.6. We begin by restating the result of Theorem 2.5 on the $g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$ in a slightly different way:

$$\begin{aligned} g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}(w) &= (\varepsilon\chi^{-1})(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-(\nu+\mu)/2}\sum_{\mathfrak{c}}(\chi\varepsilon^{-3})(\mathfrak{c})\mathrm{N}(\mathfrak{c})^{(\nu+\mu)/2}\sum_{\lambda=0}^{\nu}\delta^{\lambda-\nu} \\ &\quad d_{\nu\lambda}\left[T_r(\mathrm{Tr}_{\Gamma_0(D)\backslash\mathrm{SL}_2(\mathbb{Z})}((D^\lambda f)(\tau)\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta w,\tau)))\big|_{\tau=\tau_0(\mathfrak{c})}\right]. \end{aligned}$$

Here we may expand Hecke operator and trace to obtain:

$$\begin{aligned} &T_r(\mathrm{Tr}_{\Gamma_0(D)\backslash\mathrm{SL}_2(\mathbb{Z})}((D^\lambda f)(\tau)\vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta u,\tau)))\big|_{\tau=\tau_0} \\ &= r^{\mu+\lambda-1}\sum_{\gamma\in\Gamma_0(D)\backslash M_r} j(\gamma,\tau_0)^{-(\mu+\lambda)}(D^\lambda f)(\gamma(\tau_0)) \\ &\quad \vartheta_{\bar{\mathfrak{b}},\nu-\lambda}(r\delta j(\gamma,\tau_0)^{-1}u,\gamma(\tau_0)), \end{aligned} \tag{37}$$

where M_r denotes the set of all matrices in $\mathbb{Z}^{2\times 2}$ of determinant r.

To compute $(A_w g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}})(0)$, we look at the resulting terms separately. Most of the necessary computations are contained in the following lemma.

Lemma 3.7. *For a fractional ideal $\mathfrak{a}$ of K, $\tau \in \mathfrak{H}\cap K$, and $w \in \mathbb{C}$, we have*

$$A_w(d_{\nu\lambda}\vartheta_{\mathfrak{a},\nu-\lambda}(\cdot,\tau))(0)_l = \left(\frac{\delta(\bar{\tau}-\tau)}{\mathrm{N}(\mathfrak{a})}\right)^l \binom{l}{\lambda}\mathrm{N}(\mathfrak{a})^{l-\nu}(D^{l-\lambda}\vartheta^{(\nu-l)}_{\mathfrak{a};w,\tau})(\tau)$$

with

$$\vartheta^{(\nu-l)}_{\mathfrak{a};w,\tau}(\sigma) = \sum_{\alpha\in\bar{\mathfrak{a}}+\bar{a}_0} \alpha^{\nu-l}\lambda_{\mathbb{C}}(-(\alpha-\bar{a}_0/2)a_0\tau\mathrm{N}(\mathfrak{a})^{-1})e^{2\pi i\mathrm{N}(\mathfrak{a})^{-1}\mathrm{N}(\alpha)\sigma}$$

for $a_0 = \mathrm{N}(\mathfrak{a})\bar{w}/(\bar{\tau}-\tau)$.

Proof. By an easy calculation we have for $u\in\mathbb{C}$

$$\begin{aligned}(A_w\vartheta_{\mathfrak{a},k}(\cdot,\tau))(u) &= \sum_{\alpha\in\bar{\mathfrak{a}}+\bar{a}_0}\left(\frac{\alpha}{\mathrm{N}(\mathfrak{a})}+\frac{u}{\tau-\bar{\tau}}\right)^k \\ &\quad \lambda_{\mathbb{C}}(-(\alpha-\bar{a}_0/2)a_0\tau\mathrm{N}(\mathfrak{a})^{-1})e^{2\pi i(\mathrm{N}(\mathfrak{a})^{-1}\mathrm{N}(\alpha)\tau+\alpha u)}.\end{aligned}$$

From this the assertion follows as in the proof of Proposition 2.4.

Applying this lemma to the individual terms in (37) we conclude

$$\begin{aligned}(A_w g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}})_l(0) &= (\varepsilon\chi^{-1})(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-(\nu+\mu)/2}\sum_{\mathfrak{c}}(\chi\varepsilon^{-3})(\mathfrak{c})\mathrm{N}(\mathfrak{c})^{(\nu+\mu)/2} \\ &\quad \sum_{\lambda=0}^{l}\delta^{\lambda-\nu}\left(\frac{\mathrm{N}(\mathfrak{a})}{r}\right)^l\binom{l}{\lambda}\mathrm{N}(\mathfrak{b})^{l-\nu}r^{\mu+\lambda-1} \\ &\quad \sum_{\gamma\in\Gamma_0(D)\backslash M_r} j(\gamma,\tau_0(\mathfrak{c})))^{-(\mu+\lambda)}(D^\lambda f)(\gamma(\tau_0(\mathfrak{c}))) \\ &\quad (r\delta j(\gamma,\tau_0(\mathfrak{c}))^{-1})^{l-\lambda}(D^{l-\lambda}\vartheta^{(\nu-l)}_{\mathfrak{b};w,\gamma})(\gamma(\tau_0(\mathfrak{c}))) \\ &= \delta^{l-\nu}(\varepsilon\chi^{-1})(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{(\nu-\mu)/2}\sum_{\mathfrak{c}}(\chi\varepsilon^{-3})(\mathfrak{c})\mathrm{N}(\mathfrak{c})^{(\mu-\nu)/2+l} \\ &\quad D^l\left[r^{\mu-l-1}\sum_{\gamma\in\Gamma_0(D)\backslash M_r} j(\gamma,\tau)^{-(\mu-l)}\right. \\ &\quad \left.\left. f(\gamma(\tau))\vartheta^{(\nu-l)}_{\mathfrak{b};w,\gamma}(\gamma(\tau))\right]\right|_{\tau=\tau_0(\mathfrak{c})},\end{aligned} \tag{38}$$

with

$$\vartheta^{(\nu-l)}_{\mathfrak{b};w,\gamma}(\tau) = \sum_{\alpha\in\mathfrak{b}+\bar{a}_\gamma}\alpha^{\nu-l}\lambda_{\mathbb{C}}(-(\alpha-\bar{a}_\gamma/2)a_\gamma\gamma(\tau_0)\mathrm{N}(\mathfrak{b})^{-1})e^{2\pi i\mathrm{N}(\mathfrak{b})^{-1}\mathrm{N}(\alpha)\tau}$$

and $a_\gamma = -r\delta\mathrm{N}(\mathfrak{b})\bar{w}/(j(\gamma,\bar{\tau}_0)(\gamma(\bar{\tau}_0)-\gamma(\tau_0)))$.

It remains to write the sum over γ as a Hecke operator and trace again. Let us identify the individual terms with values of theta functions of the form $\vartheta^{(k)}_{\mathfrak{b};\varphi}$. Assume that $x\in M^{-1}\mathfrak{a}$ for a positive integer M and let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in M_r$, $x=j(\gamma,\tau_0(\mathfrak{c}))$. Then $a_\gamma = x\bar{w}/\mathrm{N}(\mathfrak{a})\in M^{-1}\bar{\mathfrak{b}}$. Furthermore,

$$\vartheta^{(\nu-l)}_{\mathfrak{b};w,\gamma}(\tau) = M^{\nu-l}\vartheta^{(\nu-l)}_{\mathfrak{b};\varphi_\gamma}(\tau)$$

with $N=M^2$, where $\varphi_\gamma : M^{-2}\mathfrak{b}/\mathfrak{b}\to\mathbb{C}$ is defined by

$$\varphi_\gamma(\alpha) = \lambda_{\mathbb{C}}(-\mathrm{N}(\mathfrak{c})^{-1}M\alpha\bar{w}(a\tau_0+b))e^{\pi i\mathrm{N}(\mathfrak{c})^{-1}\mathrm{Tr}((a\tau_0+b)(c\bar{\tau}_0+d))\mathrm{N}(w\mathfrak{a}^{-1})},$$

if $M^2\alpha\equiv\bar{x}\mathrm{N}(\mathfrak{a})^{-1}(Mw)\,(M\mathfrak{b})$, and $\varphi_\gamma(\alpha)=0$, otherwise. Let us now assume that $(M,rD)=1$. From Lemma 3.4, we know that the functions $\vartheta^{(\nu-l)}_{\mathfrak{b};\varphi_\gamma}$ are modular forms for the group $\Gamma(M^2) \cap \Gamma_0(D)$. By [Shim1] the Hecke operator T_r on forms of weight k on $\Gamma(M^2)$ is defined by

$$(T_r h)(\tau) = r^{k-1} \sum_{\gamma \in \Gamma(M^2) \backslash \Delta'_r} j(\gamma, \tau)^{-k} h(\gamma(\tau)),$$

where

$$\Delta'_r = \{g \in M_r \,|\, g \equiv \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} (M^2)\}.$$

Consequently, for a form h of weight k on $\Gamma(M^2) \cap \Gamma_0(D)$, we have

$$(T_r(\mathrm{Tr}_{\Gamma(M^2)\cap\Gamma_0(D)\backslash\Gamma(M^2)} h))(\tau) = r^{k-1} \sum_{\gamma \in \Gamma(M^2)\cap\Gamma_0(D)\backslash \Delta'_r} j(\gamma, \tau)^{-k} h(\gamma(\tau)). \quad (39)$$

Now under our restrictions on M the obvious map from $\Gamma(M^2) \cap \Gamma_0(D) \backslash \Delta'_r$ to $\Gamma_0(D)\backslash M_r$ is a bijection, and we may in (38) choose representatives $\gamma \in \Delta'_r$. It is clear that for such γ we have $\varphi_\gamma = \varphi$, where φ is as in the statement of the proposition. Using (39), the proposition is proved.

3.2 The trivial kernel

In the last section we constructed for any prime $\ell \not| D$ a map

$$\mathcal{L}^{\mathrm{ar}}_{\nu,\mu;\varepsilon,\chi} : M^*_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}_{(\ell)}) \to A(\rho_{\nu\mu}, L, \chi; \bar{\mathbb{Z}}_{(\ell)}),$$

where $M^*_m = M_m$ as long as we are not in the "exceptional cases" of Theorem 3.5, where we have to restrict to cusp forms by setting $M^*_m = S_m$.

Fixing embeddings $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_\ell : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_\ell$, we may extend this to coefficients in $\bar{\mathbb{Z}}_\ell$ and (assuming $\ell \geq 5$ to ensure surjectivity of base change) by reduction modulo ℓ consider

$$\bar{\mathcal{L}}_{\nu,\mu;\bar{\varepsilon},\bar{\chi}} : M^*_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell) \to A(\rho_{\nu\mu}, L, \bar{\chi}; \bar{\mathbb{F}}_\ell).$$

It is certainly interesting to investigate properties of this map on the level of "modular forms mod ℓ". Our goal will be a precise determination of its kernel, at least in the case when ℓ splits in K. In this section we will determine a certain subspace $\bar{\mathcal{K}}$ of $M^*_m(\Gamma_0(D), \omega_{K/Q}; \bar{\mathbb{F}}_\ell)$, which is certainly contained in the kernel. The lifting vanishes mod ℓ on this subspace for (in principle) rather simple reasons, although the proof contains tedious computations in some cases. On the other hand, a precise result on the kernel is much more difficult to obtain. We will show later that if ℓ is split in K (and does not divide $2h_K$) the kernel is in fact precisely equal to the subspace $\bar{\mathcal{K}}$. The case of ℓ inert in K is of a different nature. At least for $\nu = 0$ the kernel has to be "much bigger" than in the split case, essentially because the lifting of a modular form in $M_m(\Gamma_0(D), \omega_{K/Q}; \bar{\mathbb{F}}_\ell)$ depends only on its values at the finitely many supersingular elliptic curves in characteristic ℓ (see the remark at the end of this section).

W-Operators and associated idempotents In this section, we define the W-Operators of Atkin and Li [AL] acting on the spaces $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ and $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; R)$ and construct an associated family of idempotents if the prime 2 is invertible in R.

If t is a divisor of D with $(t, D/t) = 1$, we may set

$$W_t = \begin{pmatrix} tx & y \\ Dz & tw \end{pmatrix},$$

where $y \equiv 1\,(t)$, $x \equiv 1\,(D/t)$, and $\det W_t = t^2xw - Dyz = t$. Then the standard action of $\mathrm{GL}_2^+(\mathbb{R})$ on functions on $\mathfrak{H}$ induces an action of W_t on the space $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \mathbb{C})$ which is independent of the choice of x, y, z and w [AL, Prop. 1.1][6].

If we are considering geometric modular forms, it is better to look at the operators $W'_t = t^{-m/2}W_t$, which can be defined without worrying about the sign of the square root. Assume that the ground ring R is a $\mathbb{Z}[\zeta_D, 1/D]$-algebra. We fix a primitive D-th root of unity ζ_D playing the role of $e^{2\pi i/D}$ in the complex picture. (If $R = \bar{\mathbb{Z}}_\ell$ or $R = \bar{\mathbb{F}}_\ell$, choose ζ_D compatible to $e^{2\pi i/D}$ via $i_\ell i_\infty^{-1}$.) Giving a $\Gamma_1(D)$-level structure i on an elliptic curve E is equivalent to giving level structures $i_t : \mathbb{Z}/t\mathbb{Z} \hookrightarrow E[t]$ and $i_{D/t} : \mathbb{Z}/(D/t)\mathbb{Z} \hookrightarrow E[D/t]$ by $i_t(x) = i((D/t)x)$ and $i_{D/t}(x) = i(tx)$. To any $\Gamma_1(D)$-test object $(E, \omega, i) = (E, \omega, i_t, i_{D/t})$ we associate the test object $W'_t(E, \omega, i) = (E/i_t(\mathbb{Z}/t\mathbb{Z}), \hat{\pi}^*(\omega), i'_t, i'_{D/t})$, where $\pi : E \to E/i_t(\mathbb{Z}/t\mathbb{Z})$ is the projection, $i'_{D/t} = \hat{\pi}^{-1} i_{D/t}$ (π and $\hat{\pi}$ are bijections on the D/t-division points), and $i'_t(x) = [x]\pi(P)$ for $P \in E[t]$ with $e_t(P, i_t(1)) = \zeta_t = \zeta_D^{D/t}$. This map on test objects induces a corresponding map W'_t on modular forms in $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; R)$, and the reader may verify that for $R = \mathbb{C}$ the definition agrees with the one given above.

In addition to these operators we need the usual Hecke operators U_t for t dividing D. For a divisor t of D with $(t, D/t) = 1$ we have the factorization $\omega_{K/\mathbb{Q}} = \omega_D = \omega_t \omega_{D/t}$ of $\omega_{K/\mathbb{Q}}$ into Dirichlet characters mod t and D/t, respectively.

Recall some elementary properties of the operators W'_t from [AL, p. 223]. Although Atkin and Li state them over $\mathbb{C}$ they hold in general.

Lemma 3.8. *On $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; R)$ we have*

1. $(W'_t)^2 = \omega_t(-1)\omega_{D/t}(t)t^{-m}$ *for each* $t|D$, $(t, D/t) = 1$;
2. $U_s W'_t = \omega_t(s) W'_t U_s$ *for* $s|D$ *coprime to* t;
3. $W'_s W'_t = \omega_t(s)\omega_s(t) W'_t W'_s = \omega_t(s) W'_{st}$ *for coprime* s, $t|D$ *with* $(s, D/s) = (t, D/t) = 1$.

Define for $t|D$, $(t, D/t) = 1$ the Gauss sum g_t by

$$g_t = \sum_{x \bmod t} \omega_t(x)\zeta_t^x;$$

it is well known that $g_t^2 = \omega_t(-1)t$. Over the complex numbers we have $g_D = \delta$. We may now introduce an operator Y_t on $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; R)$ by

$$Y_t = g_t^{-1} t U_t W'_t.$$

Assuming 2 invertible in R, we set $X_{p,\varepsilon} = 1/2(1 + \varepsilon Y_{p^{v_p(D)}})$ for any prime p dividing D, and $\varepsilon = \pm 1$. For a positive integer n denote by $\nu(n)$ the number of different prime divisors of n.

Lemma 3.9. 1. *We have $Y_t^2 = 1$, and $Y_{t_1} Y_{t_2} = Y_{t_1 t_2}$ for coprime $t_1|D$ and $t_2|D$ with $(t_i, D/t_i) = 1$.*

2. *The operators $X_{p,\varepsilon}$ are commuting idempotents with $X_{p,1} + X_{p,-1} = 1$. Consequently, for each divisor t of D with $(t, D/t) = 1$ and signs ε_p for primes*

[6] In the special case $t = D$ the operator W_D is the well-known Hecke involution on the modular curve of level D. Our definition is in accordance with Atkin and Li but differs (in the odd weights considered here) in sign from the more usual definition by the matrix $H_D = \begin{pmatrix} 0 & -1 \\ D & 0 \end{pmatrix}$.

$p|t$ *the operator*

$$X_{t,\varepsilon} = 2^{-\nu(t)} \sum_{s|t,\,(s,D/s)=1} \varepsilon_s Y_s = \prod_{p|t} X_{p,\varepsilon_p},$$

where we set $\varepsilon_s = \prod_{p|s} \varepsilon_p$ *for all* $s|t$, *is an idempotent.*

Proof. The product rule $Y_{t_1}Y_{t_2} = Y_{t_1t_2}$ follows from Lemma 3.8. To show $Y_t^2 = 1$, it is enough to do this for $t = p^{v_p(D)}$, $p|D$. If p is odd, we have $v_p(D) = 1$ and this follows from [O, Theorem 4], using again Lemma 3.8. To deal with the case $p = 2$, by [AL, p. 226-227], the Fourier expansion of $f|Y_t$ is $\sum_{n=0}^{\infty} a_n\omega_t(n)q^n$ up to terms involving powers q^{pn}, if $f(q) = \sum_{n=0}^{\infty} a_n q^n$. From this we see that $f|(Y_t^2 - 1)$ has as q-expansion a power series in q^p. For $p = 2$ especially 2 is invertible in R, and we conclude $f|(Y_t^2 - 1) = 0$ as in the proof of the next proposition. The second assertion follows easily.

The importance of the idempotents $X_{t,\varepsilon}$ lies in the properties of their action on Fourier expansions.

Proposition 3.10. *1. Let* $t|D$ *with* $(t, D/t) = 1$ *and signs* ε_p *for prime divisors* p *of* t *be given. Then for* $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; R)$ *with q-expansion* $f(q) = \sum_{n=0}^{\infty} a_n q^n$ *we have for the q-expansion coefficients* b_n *of* $f|X_{t,\varepsilon}$ *for all* $n \geq 0$ *with* $(n,t) = 1$*:*

$$b_n = \begin{cases} a_n, & \omega_p(n) = \varepsilon_p\,\forall p|t, \\ 0, & \text{otherwise.} \end{cases}$$

2. The kernel of $X_{t,\varepsilon}$ *is the space of all* $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; R)$ *with* $f(q) = \sum_{n=0}^{\infty} a_n q^n$*, such that* $a_n = 0$ *in degrees* $n \geq 0$ *with* $\omega_p(n) = \varepsilon_p$ *for all* $p|t$*. The image of* $X_{t,\varepsilon}$ *is the space of all* f *with* $a_n = 0$ *in all degrees* n *with* $(n,t) = 1$ *and* $\omega_p(n) \neq \varepsilon_p$ *for some* $p|t$*.*

Proof. The first assertion follows easily from the Fourier coefficient formula of [AL, p. 226-227] cited above. The second assertion is a corollary of the first together with the fact that there is no non-zero modular form $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; R)$ which has all Fourier coefficients $a_n = 0$ for indices n with $(n, D) = 1$. This is well known over the complex numbers, and continues to hold over R, since 2 is invertible and the R-valued character $\omega_{K/\mathbb{Q}}$ has still conductor D. We omit the details.

Theta transformation formula We now restrict again to modular forms over the complex numbers. We will expand the trace in Proposition 3.6 by using the general transformation formula for the theta functions $\vartheta^{(k)}_{\mathfrak{a};\varphi}$, which we cite as the following proposition.

Proposition 3.11. *Let* $k \geq 0$, $N \geq 1$, $\mathfrak{a}$ *a fractional ideal of* K, $\varphi : (N\delta)^{-1}\mathfrak{a}/\mathfrak{a} \to \mathbb{C}$*. Then for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ *with* $c > 0$*, we have*

$$\vartheta^{(k)}_{\mathfrak{a};\varphi}(\gamma(\tau)) = j(\gamma,\tau)^{k+1}\vartheta^{(k)}_{\mathfrak{a};\varphi'}(\tau),$$

with

$$\varphi'(\alpha) = \sum_{\beta \in (N\delta)^{-1}\mathfrak{a}/\mathfrak{a}} C^{(N)}_{\gamma}(\beta,\alpha)\varphi(\beta),$$

and the transformation coefficients $C^{(N)}_{\gamma}(\beta,\alpha)$ *are given by*

$$C^{(N)}_{\gamma}(\beta,\alpha) = \frac{1}{cN\delta} \sum_{x \in (N\delta)^{-1}\mathfrak{a}/c\mathfrak{a},\, x \equiv \beta\ (\mathfrak{a})} e^{2\pi i N \mathrm{N}(\mathfrak{a})^{-1}(a\mathrm{N}(x) - \mathrm{Tr}(\alpha\bar{x}) + d\mathrm{N}(\alpha))/c}. \qquad (40)$$

Proof. See [P, p. 235, (B.15)].

We remark that by [P] we have the relation

$$C_\gamma^{(N)}(\beta,\alpha) = e^{-2\pi i N b \mathrm{N}(\mathfrak{a})^{-1}(d\mathrm{N}(\alpha)-\mathrm{Tr}(\alpha\bar\beta))} C_\gamma^{(N)}(\beta - d\alpha, 0). \tag{41}$$

In this general formula, we want to look at the special case where N is prime to D, and $\gamma \in \Gamma(N)$ runs over the left cosets mod $\Gamma_0(D) \cap \Gamma(N)$. A system of representatives is given by matrices in $\Gamma(N)$ whose second row is $(Nc\ d)$, where c is a positive divisor of D, and $d \equiv 1\,(N)$ ranges over all residue classes mod D/c subject to $(d,c)=1$. In this situation we can reduce everything to the case $N=1$ by means of the following easy lemma.

Lemma 3.12. *Let N be prime to D, c a positive divisor of D, and*

$$\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma(N).$$

Then we have

$$C_\gamma^{(N)}(\beta,\alpha) = \begin{cases} 0, & \beta \not\equiv \alpha\,(\delta^{-1}\mathfrak{a}), \\ C_{\gamma'}^{(1)}(\beta',\alpha'), & \beta \equiv \alpha\,(\delta^{-1}\mathfrak{a}), \end{cases}$$

where $\gamma' = \begin{pmatrix} a & Nb \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and $\alpha', \beta' \in \delta^{-1}\mathfrak{a}$ with $\alpha' \equiv \alpha\,(N^{-1}\mathfrak{a})$ and $\beta' \equiv \beta\,(N^{-1}\mathfrak{a})$.

Proof. We may apply (41) and decompose the Gauss sum in (40) as a product by writing $x = x_1 + x_2$ with $x_1 \in N\delta^{-1}\mathfrak{a}/Nt\mathfrak{a}$ and $x_2 \in N^{-1}t\mathfrak{a}/Nt\mathfrak{a}$. Under the condition $\gamma \in \Gamma(N)$ the sum over x_2 collapses to zero if $\beta \not\equiv \alpha\,(\delta^{-1}\mathfrak{a})$, and gives N^2 otherwise. The reader may easily check that this yields the assertion.

It remains to actually compute the values $C_\gamma^{(1)}(\beta,\alpha)$. Only the case $\beta = 0$ is necessary for the following. We summarize the result in the following lemma.

Lemma 3.13. *Let $\alpha \in \delta^{-1}\mathfrak{a}/\mathfrak{a}$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, where c is a positive divisor of D. For simplicity, let $\mathfrak{a}$ be such that $\mathrm{N}(\mathfrak{a})$ has no D-component. Write $c_\alpha = (D, D\mathrm{N}(\alpha\mathfrak{a}^{-1}))$, $t_\alpha = D/c_\alpha$, and $n_\alpha = \mathrm{N}(\alpha\mathfrak{a}^{-1})t_\alpha \in \mathbb{Z}$. Then $C_\gamma(0,\alpha) = 0$ unless $c|c_\alpha$ and $(c, c_\alpha/c) = 1$. Assuming these conditions, we may distinguish three cases.*

1. *In case $(c, D/c) = 1$,*

$$C_\gamma(0,\alpha) = \frac{g_c}{\delta}\omega_c(d\mathrm{N}(\mathfrak{a}))e^{2\pi i n_\alpha c_{t_\alpha}^{-1} d/t_\alpha}.$$

(Here $c_{t_\alpha}^{-1}$ denotes an inverse of c modulo t_α.)

2. *In case $v_2(c) = v_2(c_\alpha) = v_2(D) - 1$,*

$$C_\gamma(0,\alpha) = \frac{g_{2c}}{\delta}\omega_{2c}(d\mathrm{N}(\mathfrak{a}))\omega_{D/2c}(2)e^{2\pi i n_\alpha (2c)_{t_\alpha/2}^{-1} d/(t_\alpha/2)}.$$

3. *In case $v_2(c) = v_2(c_\alpha) = 1$,*

$$C_\gamma(0,\alpha) = 2\frac{g_{c/2}}{\delta}\omega_{c/2}(2d\mathrm{N}(\mathfrak{a}))e^{2\pi i n_\alpha (c/2)_{2t_\alpha}^{-1} d/(2t_\alpha)}.$$

Proof. From the definition (40) we have

$$C_\gamma(0,\alpha) = \frac{1}{c\delta} e^{2\pi i d\mathrm{N}(\alpha\mathfrak{a}^{-1})/c} \sum_{x\in\mathfrak{a}/c\mathfrak{a}} e^{2\pi i \mathrm{N}(\mathfrak{a})^{-1}(a\mathrm{N}(x)-\mathrm{Tr}(\alpha\bar{x}))/c}.$$

Let $\mathfrak{c}$ be the unique ideal of K of norm c. Then $\mathrm{N}(x\mathfrak{a}^{-1})$ is constant modulo c if x changes by an element of $\mathfrak{a}\mathfrak{c}$. Consequently $C_\gamma(0,\alpha) = 0$, if not $\alpha \in \delta^{-1}\mathfrak{a}\mathfrak{c}$, or $c|c_\alpha$. Under this assumption, we may rewrite the sum as c times the corresponding sum over $x \in \mathfrak{a}/\mathfrak{a}\mathfrak{c}$.

We consider now only the case $(c, D/c) = 1$. In this case there exists $x_0 \in \mathfrak{a}$ with $\delta a x_0 \equiv -\delta\alpha\,(c\mathfrak{a})$. Substituting $\mathrm{N}(x) = \mathrm{N}(x+x_0) - \mathrm{N}(x_0) - \mathrm{Tr}(x_0\bar{x})$, we get finally

$$C_\gamma(0,\alpha) = \frac{1}{\delta} e^{2\pi i d\mathrm{N}(\alpha\mathfrak{a}^{-1})/c} e^{-2\pi i a\mathrm{N}(x_0\mathfrak{a}^{-1})/c} \sum_{x\in\mathfrak{a}/\mathfrak{a}\mathfrak{c}} e^{2\pi i a\mathrm{N}(x\mathfrak{a}^{-1})/c}.$$

The remaining sum may be evaluated to $\omega_c(d\mathrm{N}(\mathfrak{a}))g_c$ (by standard results on Gauss sums), and since $\mathrm{N}(x_0\mathfrak{a}^{-1})a^2 t_\alpha \equiv n_\alpha\,(c)$, we are done.

If we do not have $(c, D/c) = 1$, this gcd must be a power of two, and we proceed by factorizing the Gauss sum into a factor corresponding to a power of the prime above two, and a second one corresponding to the odd part of $\mathfrak{c}$. The first factor can be computed directly, and the second one dealt with as above. We omit the details.

Computation of the trace Let now f be a modular form in $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$, M an integer prime to D, $N = M^2$, and $\varphi : N^{-1}\mathfrak{a}/\mathfrak{a} \to \mathbb{C}$. We want to develop a formula for $\mathrm{Tr}_{\Gamma_0(D)\cap\Gamma(N)\backslash\Gamma(N)}(f\vartheta^{(k)}_{\mathfrak{a};\varphi})$. (It would be easy to consider general N, but this is not necessary for our purpose.) Extend φ by zero to $(N\delta)^{-1}\mathfrak{a}/\mathfrak{a}$, and set $\varphi_\beta(\alpha) = \varphi(\alpha-\beta)$ for $\beta \in \delta^{-1}\mathfrak{a}/\mathfrak{a}$. Then we have

$$\mathrm{Tr}_{\Gamma_0(D)\cap\Gamma(M^2)\backslash\Gamma(M^2)}(f\vartheta^{(k)}_{\mathfrak{a};\varphi}) = \sum_{\alpha\in\delta^{-1}\mathfrak{a}/\mathfrak{a}} \lambda(f; M\alpha)\vartheta^{(k)}_{\mathfrak{a};\varphi_\alpha},$$

with

$$\lambda(f;\alpha) = \sum_{\gamma\in\Gamma_0(D)\backslash\mathrm{SL}_2(\mathbb{Z})} C^{(1)}_\gamma(0,\alpha) f|\gamma. \tag{42}$$

The $\lambda(f;\alpha)$ are obviously modular forms for the group $\Gamma(D)$.

Using Lemma 3.13 we now derive an expression for $\lambda(f;\alpha)$ which makes its vanishing (or vanishing mod ℓ) for certain f evident. To state the result, introduce the following notation. For a modular form $f = \sum_{n=0}^\infty a_n q^n$, a positive integer N, and an integer n_0 modulo N, write

$$[f]_{n_0;N} = \sum_{n\equiv n_0\,(N)} a_n q^{n/N}.$$

Define the ideal $\mathfrak{d}_{\mathrm{odd}}$ by $\delta\mathfrak{o}_K = \mathfrak{p}_2^{v_2(D)}\mathfrak{d}_{\mathrm{odd}}$.

Proposition 3.14. *Let $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ and $\alpha \in \delta^{-1}\mathfrak{a}/\mathfrak{a}$. Assume that $\mathrm{N}(\mathfrak{a})$ has no D-component. Write $c_\alpha = (D, D\mathrm{N}(\alpha\mathfrak{a}^{-1}))$, $t_\alpha = D/c_\alpha$, and $n_\alpha = \mathrm{N}(\alpha\mathfrak{a}^{-1})t_\alpha \in \mathbb{Z}$. Setting (independently of α)*

$$f' = f|W_D \prod_{p|D} X_{p,\omega_p(-\mathrm{N}(\mathfrak{a}))},$$

we distinguish three cases.

1. In case $(c_\alpha, D/c_\alpha) = 1$, *we have*

$$\lambda(f;\alpha) = -g_{t_\alpha}^{-1}\omega_{t_\alpha}(c_\alpha \mathrm{N}(\mathfrak{a}))t_\alpha^{1-m/2}2^{\nu(c_\alpha)}\,[f'|W_{c_\alpha}]_{-n_\alpha;t_\alpha}.$$

2. In case $v_2(c_\alpha) = v_2(D) - 1$, *we have*

$$\begin{aligned}\lambda(f;\alpha) &= -g_{t_\alpha/2}^{-1}\omega_{t_\alpha/2}(c_\alpha \mathrm{N}(\mathfrak{a}))(2t_\alpha)^{1-m/2}2^{\nu(c_\alpha)-1} \\ &\quad [f'|U_2W_{2c_\alpha}]_{-n_\alpha;t_\alpha/2}(\tau/2) - \lambda(f;\alpha'),\end{aligned}$$

where $\alpha' \in \mathfrak{d}_{\mathrm{odd}}^{-1}\mathfrak{a}$ *with* $\alpha' \equiv \alpha\,(\mathfrak{p}_2^{-1}\mathfrak{a})$ *(this implies* $c_{\alpha'} = 2c_\alpha$ *and* $2n_{\alpha'} \equiv n_\alpha\,(t_\alpha/2)$*).*

3. In case $v_2(c_\alpha) = 1$ *and* $8|D$, *we have*

$$\begin{aligned}\lambda(f;\alpha) &= -g_{2t_\alpha}^{-1}\omega_{2t_\alpha}(c_\alpha \mathrm{N}(\mathfrak{a})/2)(2t_\alpha)^{1-m/2}2^{\nu(c_\alpha)} \\ &\quad \left([f'|W_{c_\alpha/2}U_2]_{-n_\alpha;t_\alpha} + (1/2)\,[f'|W_{c_\alpha/2}U_4]_{-2^{-1}n_\alpha;t_\alpha/4}(\tau/2)\right) \\ &\quad - \lambda(f;\alpha')/2 - \lambda(f;\alpha'')/2,\end{aligned}$$

where $\alpha' \in \mathfrak{d}_{\mathrm{odd}}^{-1}\mathfrak{a}$ *with* $\alpha' \equiv \alpha\,(2^{-1}\mathfrak{a})$, *and* $\alpha'' - \alpha' \in \mathfrak{p}_2^{-1}\mathfrak{a}\backslash\mathfrak{a}$.

Proof. We give the proof only in the first case. The other two cases are very much analogous, and once the correct formula is known, its proof consists out of not very enlightening computations.

Assume $(c_\alpha, D/c_\alpha) = 1$. Using Lemma 3.13 write (42) as follows:

$$\lambda(f;\alpha) = \sum_{c|c_\alpha,\,(c,D/c)=1}\left(\frac{g_c}{\delta}\sum_{d \bmod D/c}\omega_c(d\mathrm{N}(\mathfrak{a}))e^{2\pi i n_\alpha c_{t_\alpha}^{-1}d/t_\alpha}f|\begin{pmatrix}* & *\\ c & d\end{pmatrix}\right).$$

It is easily seen that

$$f|\begin{pmatrix}* & *\\ c & d\end{pmatrix} = \omega_{D/c}(-c)\left(\frac{D}{c}\right)^{-m/2}(f|W_{D/c})\left(\frac{\tau+d'}{D/c}\right)$$

for $d \equiv cd'\,(D/c)$ and $d \equiv 1\,(c)$. Combining these formulas a short computation yields $\lambda(f;\alpha) = [g]_{-n_\alpha;t_\alpha}$ for

$$g = \sum_{c|c_\alpha,\,(c,D/c)=1}\frac{g_c}{\delta}\omega_{D/c}(-c\mathrm{N}(\mathfrak{a}))\left(\frac{D}{c}\right)^{1-m/2}f|W_{D/c}U_{c_\alpha/c}.$$

Changing the index of summation into $s = c_\alpha/c$, after some juggling with Lemma 3.8 we arrive at

$$\begin{aligned}g &= -g_{t_\alpha}^{-1}\omega_{t_\alpha}(c_\alpha\mathrm{N}(\mathfrak{a}))t_\alpha^{1-m/2} \\ &\quad \sum_{s|c_\alpha,\,(s,D/s)=1} g_s^{-1}s^{1-m/2}\omega_s(-\mathrm{N}(\mathfrak{a}))f|W_DU_sW_sW_{c_\alpha} \\ &= -g_{t_\alpha}^{-1}\omega_{t_\alpha}(c_\alpha\mathrm{N}(\mathfrak{a}))t_\alpha^{1-m/2}2^{\nu(c_\alpha)}f|W_D\prod_{p|c_\alpha}X_{p,\omega_p(-\mathrm{N}(\mathfrak{a}))}W_{c_\alpha}.\end{aligned}$$

From Proposition 3.10 it is clear that

$$[g]_{-n_\alpha;t_\alpha} = [g|\prod_{p|t_\alpha}X_{p,\omega_p(-n_\alpha)}]_{-n_\alpha;t_\alpha}.$$

Using the commutation rule $W_cX_{p,\varepsilon} = X_{p,\varepsilon\omega_p(c)}W_c$ for $p \not| c$, which follows again from Lemma 3.8, we may conclude

$$g|\prod_{p|t_\alpha}X_{p,\omega_p(-n_\alpha)} = -g_{t_\alpha}^{-1}\omega_{t_\alpha}(c_\alpha\mathrm{N}(\mathfrak{a}))t_\alpha^{1-m/2}2^{\nu(c_\alpha)}f'|W_{c_\alpha},$$

as desired.

Description of the trivial kernel After these preparatory considerations, we are in a position to deduce the following result on the kernel of our lifting $\mathcal{L}$.

Theorem 3.15. *Let ν and μ be integers with $m = \mu - \nu - 1 \geq 5$, and ε and χ Hecke characters of weight zero and $\nu + \mu$, respectively. Let $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \mathbb{C})$, and $f|W_D = \sum_{n=0}^{\infty} b_n q^n$ with $b_n = 0$ for all $n \geq 0$ with $\omega_{K/\mathbb{Q}}(n) = -1$. Then $\mathcal{L}_{\nu,\mu;\varepsilon,\chi}(f) = 0$.*

Furthermore, if $\ell \not| 2D$ the same statement holds in characteristic ℓ: if $f \in M^*_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell)$ (assumed to be liftable to $M^*_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}_\ell)$ in case $\ell = 3$), and $f|W'_D$ has q-expansion $\sum_{n=0}^{\infty} b_n q^n$ with $b_n = 0$ for all $n \geq 0$ with $\omega_{K/\mathbb{Q}}(n) = -1$, we have $\bar{\mathcal{L}}_{\nu,\mu;\bar{\varepsilon},\bar{\chi}}(f) = 0$.

We denote the subspaces of $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ (resp. $M^*_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell)$) defined above by $\mathcal{K}$ (resp. $\bar{\mathcal{K}}$).

Proof. Most of the work already being done, the proof is now quite easy. In the complex case, we get from Proposition 3.10 that

$$f|W_D \prod_{p|D} X_{p,\omega_p(-\mathrm{N}(\mathfrak{b}))} = 0$$

for all fractional ideals $\mathfrak{b}$ (such that $\mathrm{N}(\mathfrak{b})$ has no D-component). Therefore, by Proposition 3.14 we conclude

$$\mathrm{Tr}_{\Gamma_0(D)\cap\Gamma(M^2)\backslash\Gamma(M^2)}(f\vartheta^{(\nu-l)}_{\mathfrak{b};\varphi}) = 0$$

for all $M \geq 1$ and $\varphi : M^{-2}\mathfrak{b}/\mathfrak{b} \to \mathbb{C}$. By Proposition 3.6 we see that every Fourier-Jacobi coefficient $g_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$ of $F = \mathcal{L}_{\nu,\mu;\varepsilon,\chi}(f)$ for $r > 0$ has infinitely many zeroes in $\mathbb{C}/\mathfrak{a}$, i. e. is zero. But this means that $F = 0$.

In the characteristic ℓ case, we may introduce these arguments into the proof of Theorem 3.5. Lift $f \in M^*_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}, \bar{\mathbb{F}}_\ell)$ to $\bar{\mathbb{Z}}_\ell$ and even via $i_\infty i_\ell^{-1}$ to $f' \in M^*_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}, \bar{\mathbb{Z}}_{(\ell)})$. Looking at $F' = \mathcal{L}^{\mathrm{ar}}_{\nu,\mu;\varepsilon,\chi}(f')$ and its Fourier-Jacobi coefficients $g^{\mathrm{ar}}_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}}$, we see that (again by Propositions 3.10 and 3.14) for $\ell \not| M$ the modular form h in the proof of Theorem 3.5 is not only integral over $\bar{\mathbb{Z}}_{(\ell)}$, but even contained in $\mathfrak{L}M_{\mu-l}(\Gamma(M^2); \bar{\mathbb{Z}}_{(\ell)})$, where $\mathfrak{L}$ is the prime ideal of $\bar{\mathbb{Z}}_{(\ell)}$ determined by $i_\ell i_\infty^{-1}$. Consequently, $(D^l h)(\tau_0(\mathfrak{c})) \in \Omega(\mathfrak{c})^{\mu+l}\mathfrak{L}$. Proceeding as in the previous proof, we get (using the "density lemma") $g^{\mathrm{ar}}_r \in \mathfrak{L}\mathcal{T}_{r,\rho}(\bar{\mathbb{Z}}_{(\ell)})$ for all $r > 0$. Consideration of the constant term does not pose any difficulties; consequently $F' \in \mathfrak{L}A(\rho, L, \chi; \bar{\mathbb{Z}}_{(\ell)})$, and reduces to zero modulo $\mathfrak{L}$.

An immediate corollary is that theta series

$$\vartheta_\chi(\tau) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) e^{2\pi i \mathrm{N}(\mathfrak{a})\tau}$$

in $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ associated to Hecke characters χ of the field K (unramified of weight $m-1$) lift to zero, since they are eigenforms with all eigenvalues $a_q = 0$ for $\omega_{K/\mathbb{Q}}(q) = -1$. Therefore, everything congruent mod ℓ to such a CM form lifts to zero mod ℓ. On the other hand, by [LL] eigenforms f without complex multiplication by K have a non-zero eigenvalue for some prime q with $\omega_{K/\mathbb{Q}}(q) = -1$, and therefore do not belong to the (characteristic zero) "trivial kernel". In fact, it is easily seen that the two-dimensional subspace of $M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}})$ generated by f and its complex conjugate has a one-dimensional intersection with $\mathcal{K}$. We conclude that in characteristic zero we have

$$\dim \mathcal{K} = \begin{cases} 1/2(\dim M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}) + h_K), & m \equiv 1\,(w_K), \\ 1/2 \dim M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}), & m \not\equiv 1\,(w_K). \end{cases}$$

Let us finally remark that in the case of inert ℓ, we may easily give a much larger space contained in the kernel of $\bar{\mathcal{L}}$. Restricting to the scalar valued case for simplicity, if $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell)$ is such that for arbitrary $c \in \mathrm{N}(I_K(D))$ the level D modular form $f|W_D' \prod_{p|D} X_{p,\omega_p(-c)}$ vanishes at the finitely many supersingular points on the characteristic ℓ fiber of the moduli scheme, its lifting mod ℓ vanishes. The condition is true, for example, if f itself vanishes at the supersingular points. One may prove this statement easily by arguing as above, since for the computation of the lifting we only need to evaluate f at supersingular elliptic curves.

3.3 The non-vanishing theorem: overview

We continue to study the arithmetic properties of the theta lifting $\mathcal{L}$. Let as above $\bar{\mathcal{K}} \subseteq M_m^*(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell)$ be the "trivial kernel". We would like to show that for ℓ split in K the subspace $\bar{\mathcal{K}}$ is already the precise kernel of $\bar{\mathcal{L}}_{\nu,\mu;\bar{\varepsilon},\bar{\chi}}$, or equivalently that $\bar{\mathcal{L}}$ is injective on $M_m^*(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell)/\bar{\mathcal{K}}$. Our main result is the following theorem.

Theorem 3.16. *Let $\mu \geq 6$ be an integer, and ε and χ Hecke characters of weight zero and μ, respectively. Let $\ell \not| 2h_K$ be a prime split in K and f an element of $M_{\mu-1}^*(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell) \backslash \bar{\mathcal{K}}$ (assumed to be liftable to $M_{\mu-1}^*(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}_\ell)$ if $\ell = 3$). Then $\bar{\mathcal{L}}_{0,\mu;\bar{\varepsilon},\bar{\chi}}(f) \neq 0$.*

The proof of this theorem falls into several different steps, and will occupy several sections. We give a brief overview here. In the characteristic zero case, Gelbart, Rogawski and Soudry [GeRS] prove a much more general non-vanishing theorem for theta lifts, by completely different methods. Another possibility to show non-vanishing is the (regularized) Siegel-Weil formula [Tan1, Tan2]. Both of these methods do not seem to generalize well to the characteristic ℓ case.

Construction of a bilinear form We use the formula for primitive coefficients of $\mathcal{L}(f)$ given in Theorem 2.12. To write it in a slightly different way better suited to reduction mod ℓ, define for a $\mathbb{R}$-rational representation ρ of $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times$ on a complex vector space $\mathcal{V} = \mathcal{V}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ and parameters r and $\mathfrak{a}$ a complex-antilinear map $T_{r,\mathfrak{a};\rho} \to T_{r,\bar{\mathfrak{a}};\rho}$ by $\vartheta^\dagger(w) = \overline{\vartheta(-\bar{w})}$. (Rationality over $\mathbb{R}$ means that complex conjugation on $\mathcal{V}$ is compatible with ρ; for $\mathcal{V} = S_\nu^*$ this is certainly true.) These maps fit together to a map from $\mathcal{T}_{d,\rho}$ to itself, also denoted by $\vartheta \mapsto \vartheta^\dagger$. Define a non-degenerate symmetric bilinear form b on $\mathcal{T}_{d,\rho}$ by

$$b(\vartheta_1, \vartheta_2) = \langle \vartheta_1^\dagger, \vartheta_2 \rangle.$$

Sketch of the strategy If now Θ is a Shintani eigenfunction in $\mathcal{T}_{r,\rho}^{\mathrm{prim}}$, $r > 0$, and g_r the r-th Fourier-Jacobi coefficient of $\mathcal{L}_{\nu,\mu;\varepsilon,\chi}(f)$, we may rewrite the result of Theorem 2.12 as an explicit formula for $b(\Theta, g_r)$.

Define for a C^∞-modular form ϕ of weight m', a Hecke character η of K of weight $-m'$ and conductor $rD\mathfrak{d}_\eta^{-1}$ restricting to $\omega_{K/\mathbb{Q}}$ on $\mathbb{Q}$, and a character σ of the genus class group $\mathrm{Cl}_K^{\mathrm{inv}}$ of K (the subgroup of Cl_K of elements invariant under complex conjugation) the number ("discrete period") $P(\varphi, \eta; \sigma)$ by

$$\begin{aligned} P(\varphi, \eta; \sigma) \quad &= \sum_{\mathfrak{c},\,(L,x)} \varphi(L, x + DL) \mathrm{N}(\mathfrak{c})^{m'/2} \left(\frac{x}{|x|}\right)^{m'} \\ &\qquad \eta(x\mathfrak{c}^{-1}\mathfrak{d}_{L,x}^{-1}) \sigma(\mathfrak{d}_{L,x}) \mathrm{N}(\mathfrak{d}_{L,x})^{1/2}, \end{aligned}$$

where $\mathfrak{c}$ runs over a system of representatives for the ideal classes of K, L over the cyclic sublattices of index r in $\mathfrak{c}$, and x over generators mod DL of sublattices L'

with $L/L' \simeq \mathbb{Z}/D\mathbb{Z}$, subject to $(\mathfrak{o}_K L')\mathfrak{c}^{-1} = \mathfrak{d}_{L,x}|\mathfrak{d}_\eta$. With this definition we have

$$b(\Theta, g_r) = C'_{\nu\lambda}(r)\ell_{\chi\varepsilon^{-1}}(\Theta)P(D^\lambda f, \kappa^{*-1}\chi\varepsilon; \chi\varepsilon),$$

where $C'_{\nu\lambda}$ and ℓ_γ are as in Section 2.4, and κ is as usual the eigencharacter of Θ.

We will now "arithmetize" this formula by constructing an arithmetic and even ℓ-integral counterpart of b. Seeing that everything behaves well under reduction mod ℓ, we are left with the task of showing the existence of an integral primitive Shintani eigenfunction Θ for which the two factors $\ell_{\chi\varepsilon^{-1}}(\Theta)$ and $P(D^\lambda f, \kappa^{*-1}\chi\varepsilon; \chi\varepsilon)$ (this factor modified by a suitable period) are both non-zero mod ℓ. We will do this by considering the spaces of theta functions of degree $r = p^n$ for all n for a fixed "auxilary prime" $p \neq \ell$, and restricting to eigenfunctions Θ with $\mathfrak{d}_\Theta = \mathfrak{o}_K$. If we are able to show that the first factor vanishes mod ℓ only a finite number of times, while the second one is non-zero infinitely often, we are done. After the "arithmetization" of the bilinear form b in the following section, the second factor will be dealt with in Section 3.5, and the (more difficult) task of dealing with the first factor follows in the next chapter. See Theorem 3.21 and Corollary 3.22 below for precise statements.

3.4 Arithmetizing the canonical bilinear form

In this section we show that the symmetric bilinear form b has an arithmetic counterpart b_{ar}, which is ℓ-integral and non-degenerate at ℓ if $\ell \not| rDw_K$. We restrict here consideration to scalar valued theta functions. Our method in obtaining results on arithmeticity and ℓ-integrality will be rather rough: we consider usual standard bases of theta functions, whose integrality may be checked directly, and express the form b in these bases. The same method was used by Hickey [Hic2] to prove arithmeticity of the canonical scalar product.

Recall that after choosing and fixing a system of representatives $\mathcal{A}$ for Cl_K (which we assume to be stable under complex conjugation), we have

$$\mathcal{T}_{r,\rho} \simeq \bigoplus_{\mathfrak{a}\in\mathcal{A}} T^1_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho} \subseteq \bigoplus_{\mathfrak{a}\in\mathcal{A}} T_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho} =: \mathcal{T}'_{r,\rho}.$$

Here T^1 denotes the subspace of theta functions invariant under the action of the roots of unity. Of course, the space $\mathcal{T}'_{d,\rho}$ depends on the choice of $\mathcal{A}$.

Standard bases of theta functions We explain now the construction of convenient bases of the spaces $T_{d/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}$. These standard bases may be defined without assuming complex multiplication: for any lattice $L \subseteq \mathbb{C}$ let $H(x,y) = n\bar{x}y/a(L)$ for a positive integer n, which is a Riemann form, and let ψ an associated semicharacter. The space $T(H,\psi,L)$ of theta functions with respect to these choices is the space of all holomorphic functions ϑ on $\mathbb{C}$ satisfying

$$\vartheta(w+l) = \psi(l)e^{\pi H(l,w+l/2)}\vartheta(w)$$

for all $l \in L$. It has dimension n, as is well known. The canonical Heisenberg group operation on $T(H,\psi,L)$ is given by $(A_l\vartheta)(w) = e^{-\pi H(l,w+l/2)}\vartheta(w+l)$ for $l \in n^{-1}L$.

The following lemma provides a standard basis of $T(H,\psi,L)$. To state it, let theta functions with characteristics be defined as usual by

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(w,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\alpha)^2\tau + 2\pi i(n+\alpha)(w+\beta)}.$$

We also need the slightly modified functions

$$\phi_{\alpha\beta}(w,\tau) = e^{\pi w^2/2\mathrm{Im}(\tau)}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(w,\tau).$$

Lemma 3.17. *Let L, H and ψ be as above and (ω_1, ω_2) be a positively oriented basis of L, i. e. $\tau = \omega_2/\omega_1 \in \mathfrak{H}$. Then the functions*

$$g_{L;j}(w) = \phi_{\alpha_0+j/n,-n\beta_0}(nw/\omega_1, n\tau),$$

where j ranges over the residue classes mod n, are a basis of $T(H,\psi,L)$, if α_0 and β_0 are choosen such that

$$\psi(a\omega_1 + b\omega_2) = e^{\pi in(ab+2a\alpha_0+2b\beta_0)}.$$

The functions g_j are eigenfunctions of the (commuting) operators $A_{c\omega_1/n}$. More precisely,

$$A_{c\omega_1/n} g_j = e^{2\pi ic(\alpha_0+j/n)} g_j, \tag{43}$$

$$A_{c\omega_2/n} g_j = e^{2\pi ic\beta_0} g_{j+k}. \tag{44}$$

Finally, the g_j are orthogonal with respect to the standard scalar product, and we have $\langle g_j, g_j\rangle = |\omega_1|/(2na(L))^{1/2}$.

The proof is completely standard (see [I, Hic2]).

In the complex multiplication case it is not difficult to show the following arithmeticity and integrality properties of these bases.

Lemma 3.18. *Let L be a lattice with complex multiplication, take an oriented basis of L and construct a basis g of $T(H,\psi,L)$ as above. Setting $g'_j = \eta(n\tau)^{-1}g_j$, the functions g'_j are primitive integral theta functions (i. e. $\alpha g'_j$ is integral precisely when α is). Furthermore, every integral theta function in $T(H,\psi,L)$ may be expressed in the basis g' with algebraic coefficients of denominator dividing n.*

Proof. We introduce the classical Siegel functions [Lan, p. 262]. For $\tau \in \mathfrak{H}$, a, $b \in \mathbb{Q}$, they are defined by

$$g_{ab}(\tau) = -i\eta(\tau)^{-1}e^{\pi iaz}\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z,\tau), \quad z = a\tau + b.$$

Using this an elementary calculation yields

$$\begin{aligned}(A_{a\omega_1+b\omega_2}g_j)(0) &= e^{\pi i((\alpha_0+j/n-1/2)(n(b-\beta_0)+1/2)+a(n\beta_0+1/2)+1/2)}\\ &\quad \eta(n\tau)g_{a+\alpha_0+j/n-1/2,n(b-\beta_0)-1/2}(n\tau).\end{aligned}$$

It follows that the g'_j are primitive integral theta functions, since the Siegel functions g_{ab} take integral values at points in imaginary quadratic fields, and take units as values for suitable parameters a and b [Ra, p. 127].

To show the second assertion, let g be an integral theta function in $T(H,\psi,L)$. If $g = \sum_j \lambda_j g'_j$, we have from (43)

$$\lambda_j g'_j = n^{-1}\sum_{c \bmod n} e^{-2\pi ic(\alpha_0+j/n)}A_{c\omega_1/n}g,$$

and the assertion follows from the above.

Arithmeticity and integrality theorem Having assembled these tools, we are now able to state and prove the following proposition on the symmetric bilinear form b. We introduce its arithmetic version b_{ar} by

$$b_{\mathrm{ar}} = \Omega_0 b.$$

Proposition 3.19. *1. The bilinear form b_{ar} takes algebraic values at pairs of arithmetic theta functions in $\mathcal{T}_{r,\rho}(\bar{\mathbb{Q}})$.*

2. If $\ell \not| rD$, the bilinear form $i_\ell i_\infty^{-1}(b_{\mathrm{ar}})$ takes ℓ-integral values for ℓ-integral arguments (with respect to i_ℓ and i_∞).

3. Under the same assumptions, if in addition $\ell \not| w_K$, $i_\ell i_\infty^{-1}(b_{\mathrm{ar}})$ on $\mathcal{T}_{r,\rho}(\bar{\mathbb{Z}}_l)$ is non-degenerate modulo the maximal ideal, i. e. its determinant is a unit in $\bar{\mathbb{Z}}_\ell$.

Proof. Let μ be the weight corresponding to the one-dimensional representation ρ. Recall that $\vartheta \in \mathcal{T}_{r,\rho}$ is integral if and only if $\alpha(\mathfrak{a})^\mu \vartheta_{\mathfrak{a}}$ is integral for all $\mathfrak{a}$. Also

$$\langle \vartheta_1, \vartheta_2 \rangle = \sum_{\mathfrak{a} \in \mathcal{A}} \mathrm{N}(\mathfrak{a})^\mu \langle \vartheta_{1,\mathfrak{a}}, \vartheta_{2,\mathfrak{a}} \rangle.$$

We extend these definitions trivially to $\mathcal{T}'_{r,\rho}$.

To prove the first two assertions, we construct a basis h of $\mathcal{T}'_{r,\rho}$ over $\mathbb{C}$ from the explicit bases g_L defined above: choose for each $\mathfrak{a} \in \mathcal{A}$ an oriented basis, which defines an element $\tau_{\mathfrak{a}} \in \mathfrak{H}$, set $h_{\mathfrak{a},j} = \sqrt{rD}\gamma'(\mathfrak{a})^{-1}\eta(\tau_{\mathfrak{a}})^{-1} g_{\mathfrak{a},j}$ for $1 \le j \le rD$ as above, and extend it by zero to an element of $\mathcal{T}'_{r,\rho}$. Clearly, this gives a basis of $\mathcal{T}'_{r,\rho}$. Since $\sqrt{rD}\eta(rD\tau_{\mathfrak{a}})/\eta(\tau_{\mathfrak{a}})$ is an algebraic integer dividing $\sqrt{rD}$ (see [Lan, p. 164]), the functions $h_{\mathfrak{a},j}$ are integral, and even primitive except at primes dividing rD. They are mutually orthogonal, and by Lemma 3.2

$$\langle h_{\mathfrak{a},j}, h_{\mathfrak{a},j} \rangle = |\Omega(\mathfrak{a})|^{-1} \left(\frac{rD}{\sqrt{D}\mathrm{N}(\mathfrak{a})} \right)^{1/2} = u_{\mathfrak{a}} \Omega_0^{-1} (r\sqrt{D})^{1/2},$$

where $u_{\mathfrak{a}}$ is a unit.

From the above, we know that h and $h^\dagger$ are bases of the module of ℓ-integral theta functions over the ring of ℓ-integral algebraic numbers, and that $b_{\mathrm{ar}}(h^\dagger_{\mathfrak{a},j}, h_{\mathfrak{a},j})$ is ℓ-integral. Therefore the first two assertions follow.

As for the third assertion, it is clearly true for the larger space $\mathcal{T}'_{r,\rho}$. Now $\mathcal{T}_{r,\rho}$ is just the subspace of $\mathcal{T}'_{r,\rho}$ invariant under the action of $\mathfrak{o}_K^\times$ and this action fulfills

$$b_{\mathrm{ar}}(\varepsilon\vartheta, \vartheta') = b_{\mathrm{ar}}(\vartheta, \varepsilon\vartheta')$$

for $\varepsilon \in \mathfrak{o}_K^\times$. If $\ell \not| w_K$, we may decompose $\mathcal{T}'_{r,\rho}(\bar{\mathbb{Q}})$ ℓ-integrally into eigenspaces for the action of $\mathfrak{o}_K^\times$, and it is easily seen that b_{ar} vanishes if the arguments belong to different eigenspaces. From this the assertion follows.

3.5 Non-vanishing of the second factor

The aim of this section is to prove a modulo ℓ non-vanishing statement on the "second factor" $P(D^\lambda f, {\kappa^*}^{-1}\chi\varepsilon; \chi\varepsilon)$ in our formula for a primitive Fourier-Jacobi coefficient of $\mathcal{L}(f)$. The precise statement will be given in Theorem 3.21. We first give as a corollary of Section 2.3 an explicit description of the characters occuring in $\mathcal{V}^{\mathrm{prim}}_{r,C;\rho}$ for $(r, D) = 1$. Then we rewrite the "second factor" in an arithmetic way which allows us to apply the theory of geometric modular forms over rings.

Characters occuring in spaces of primitive theta functions For an integer $r > 0$ we know from Corollary 2.10 which Hecke characters κ^* correspond to characters κ occuring in the Shintani representation on $\mathcal{V}^{\mathrm{prim}}_{r,C;\rho}$. We consider here the case $(r, D) = 1$ and restrict our attention further to characters of exact conductor rD. Equivalently, we demand that $\mathfrak{d}_{\kappa^*} = \mathfrak{d}_\Theta = \mathfrak{o}_K$ for an associated eigenfunction Θ.

We write $\eta = (\kappa^*)^{-1}\chi\varepsilon$. If $\Theta \in \mathrm{im} d_{\nu\lambda}$, this is a Hecke character of weight $-m' = -(m+2\lambda)$ and conductor rD with $\eta|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$. The integer m', and the restrictions of the local components η_D and η_r to the unit groups $\mathfrak{o}_{K,D}^\times$ and $\mathfrak{o}_{K,r}^\times$ determine the character η up to twist by an ideal class character. The additional knowledge of the full local component η_D determines η up to twist by a character of $\mathrm{Cl}_K/\mathrm{Cl}_K^{\mathrm{inv}}$.

The restriction to the unit group of the component η_D may be parametrized by elements $\mu \in (\mathbb{Z}/D\mathbb{Z})^\times$ as follows:

$$\begin{aligned} \eta_D(1+d\delta) &= e^{2\pi i\mu d/D}, \quad 2 \not| D, \\ \eta_D(1+d(N+\sqrt{-N})) &= e^{2\pi i\mu d/D}, \quad D=4N,\ 2\not| N, \\ \eta_D(1+d\sqrt{-N}) &= e^{2\pi i\mu d/D}, \quad D=4N,\ 2|N. \end{aligned}$$

This is completely trivial for D odd and easily verified in the other cases.

To sum up, for fixed odd weight m' and arbitrary $r>0$, $(r,D)=1$, the classes under twist with the dual of Cl_K of Hecke characters η of weight $-m'$ and conductor rD with $\eta|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$ correspond bijectively to pairs of residue classes $\mu \in (\mathbb{Z}/D\mathbb{Z})^\times$ and primitive characters η_r^{unit} of $(\mathfrak{o}_K/r\mathfrak{o}_K)^\times/(\mathbb{Z}/r\mathbb{Z})^\times$ if $D>4$. In the two exceptional cases $D=3$ and $D=4$ we have to add a consistency condition $1 = \varepsilon^{m'}\eta_D(\varepsilon)\eta_r(\varepsilon)$ for a primitive third (resp. fourth) root of unity ε to obtain a bijection. The result of Corollary 2.10 takes now the following explicit form.

Lemma 3.20. *A character κ^* corresponding to η as above appears in $\mathcal{V}_{r,C;\rho}^{\mathrm{prim}}$ if and only if all of its twists by ideal class characters appear. This happens exactly when $\omega_q(\mu) = \omega_q(2r/c)$ (resp. $\omega_q(\mu) = \omega_q(r/c)$) for all primes $q|D$ if D is odd (resp. even). Here c denotes a prime-to-D representative of the class C.*

Proof. By Corollary 2.10 this boils down to the calculation of the epsilon factor $\varepsilon(\eta_q, \lambda_{K,q})$ for $q|D$. Let us shortly indicate the compuation for odd q, the case of even q being left to the reader. Since $a(\eta_q)=2$ and $n_{\lambda_{K,q}}=1$, we may take $c=q\delta$ and have to determine

$$S = \int_{\mathfrak{o}_{K,q}^\times} \eta_q^{-1}(u)\lambda_q(\mathrm{Tr}((q\delta)^{-1}u))du.$$

Assume $\eta_q(1+\delta x) = e^{2\pi i\mu_q x/q}$. Writing $u \equiv \xi(1+\delta x)\,(q)$ with $\xi \in (\mathbb{Z}/q\mathbb{Z})^\times$ and $x \in \mathbb{Z}/q\mathbb{Z}$ we obtain

$$S = \mathrm{vol}(q\mathfrak{o}_{K,q})\sum_{\xi,\,x}\omega_q(\xi)e^{2\pi i(-2\xi-\mu_q)x/q} = q\mathrm{vol}(q\mathfrak{o}_{K,q})\omega_q(-2\mu_q).$$

Therefore

$$\eta_q(\delta)\varepsilon(\eta_q,\lambda_{K,q}) = \omega_q(2(D/q)\mu_q),$$

and we are done.

Statement of the main result We now look at the "second factor" $P(\varphi,\eta;\sigma)$ more closely. The first step is to write it in a "geometric" way allowing reduction mod ℓ. For this purpose define the arithmetic variant of P by $P^{\mathrm{ar}}(\varphi,\eta;\sigma) = \Omega_0^{-m'}P(\varphi,\eta;\sigma)$.

Let r be a positive integer, and φ, η, σ as above. Choose a system of representatives $\mathfrak{c}$ for the ideal classes of K among ideals without rD-component and consider for each representative the elliptic curve $E_\mathfrak{c} \simeq \mathbb{C}/(2\pi i)\Omega(\mathfrak{c})\mathfrak{c}$ over $\bar{\mathbb{Z}}$ with invariant differential $\omega_\mathfrak{c} = dz$. The analytic parametrization of $E_\mathfrak{c}$ yields a natural isomorphism $\iota_{rD} : \mathfrak{c}/rD\mathfrak{c} \simeq E_\mathfrak{c}[rD]$.

Let η_f be the product of the p-components of η for primes $p|rD$. Then we have (just from the definitions)

$$\begin{aligned} P^{\mathrm{ar}}(\varphi,\eta;\sigma) &= \sum_{\mathfrak{c}} \left(\frac{\Omega(\mathfrak{c})}{\Omega_0}\right)^{m'} \mathrm{N}(\mathfrak{c})^{m'/2}\eta(\mathfrak{c})^{-1} \\ & \qquad \sum_{H'=\langle P\rangle < E_{\mathfrak{c}}[rD]} \varphi(E_{\mathfrak{c}}/H, \hat{\pi}^*(\omega_{\mathfrak{c}}), \pi(P)) \\ & \qquad \eta_f(\iota_{rD}^{-1}(P))^{-1}\mathrm{N}(\mathfrak{d}_P)^{1/2}\sigma(\mathfrak{d}_P). \end{aligned}$$

Here the second sum goes over all cyclic subgroups $H' = \langle P\rangle$ of order rD of $E_{\mathfrak{c}}[rD]$ such that $\mathfrak{o}_K H' = E_{\mathfrak{c}}[rD\mathfrak{d}_P^{-1}]$ with $\mathfrak{d}_P|\mathfrak{d}_\eta$. For each H' set $H = DH < E_{\mathfrak{c}}[r]$ and let $\pi : E_{\mathfrak{c}} \to E_{\mathfrak{c}}/H$ be the projection. (We identified a $\Gamma_1(D)$-structure with a point of order D, since the base is connected.)

As a corollary we see that for $\varphi = D^\nu f$ with $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; R)$ defined over a ring $\bar{\mathbb{Z}}[1/D] \subseteq R \subseteq \mathbb{C}$, we have the integrality property $P^{\mathrm{ar}}(\varphi,\eta;\sigma) \in R[1/r]$, as expected. Consequently for $R \subseteq \bar{\mathbb{Q}}$ it makes sense to consider $i_\ell i_\infty^{-1}(P^{\mathrm{ar}}(\varphi,\eta;\sigma))$. Let us now state the main result of this section on the non-vanishing mod ℓ of these values. We write $\mathfrak{L}$ for the prime ideal of $\bar{\mathbb{Z}}_\ell$.

Theorem 3.21. *Let ℓ be a prime split in K, and p be a split prime with $\ell \not| h_K p(p-1)$. Let μ and ν be integers with $m = \mu - \nu - 1$, λ an integer with $0 \le \lambda \le \nu$, and χ' an unramified Hecke character of K of weight $\nu + \mu$. If $\lambda > 0$, assume in addition $\ell \ge m + 2$. Let $f \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}_{(\ell)})$ be such that the reduction $\bar{f}$ of $i_\ell i_\infty^{-1}(f)$ modulo $\mathfrak{L}$ does not lie in the "trivial kernel" $\bar{\mathcal{K}}$.*

For each n consider the set $\mathcal{C}_n$ of all Hecke characters of the form $\eta = {\kappa^*}^{-1}\chi'$, where κ^* is a character of weight $2(\mu+\lambda) - 1$ and conductor Dp^n occuring in the Shintani representation on $\mathcal{T}^{\mathrm{prim}}_{p^n,\rho_{\nu\mu}}$. Then there are infinitely many characters $\eta \in \bigcup_n \mathcal{C}_n$ with

$$i_\ell i_\infty^{-1}(P^{\mathrm{ar}}(D^\lambda f, \eta)) \not\equiv 0\,(\mathfrak{L}).$$

In fact we can prove something a little stronger: for all large n there exists a character $\eta \in \mathcal{C}_n$ with the property above.

Before getting to the proof, we give the application to the non-vanishing of $\bar{\mathcal{L}}$, assuming a strong non-vanishing result on theta functions which will be proved in the next chapter.

Corollary 3.22. *Let $\mu \ge 6$ be an integer, and ε and χ Hecke characters of weight zero and μ, respectively. Let $\ell \not| 2h_K$ be a prime split in K and for all $n \ge 0$ let T_n be the set of all Shintani eigenspaces t in $\mathcal{T}^{\mathrm{prim}}_{p^n,\rho}$ corresponding to eigencharacters of maximal conductor Dp^n. Assume there exists a split prime p with $\ell \not| p(p-1)$ with the following property: there are only finitely many $t \in \bigcup_n T_n$ such that for all integral representatives $\vartheta \in t$ we have $i_\ell i_\infty^{-1}(\ell_{\chi\varepsilon^{-1}}(\vartheta)) \equiv 0\,(\mathfrak{L})$.*

Then for any modular form $f \in M^*_{\mu-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{F}}_\ell)\backslash\bar{\mathcal{K}}$ (assumed to be liftable to $M^*_{\mu-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}_\ell)$ if $\ell = 3$) we have $\bar{\mathcal{L}}_{0,\mu;\bar{\varepsilon},\bar{\chi}}(f) \ne 0$.

Proof. Lift f via $i_\infty i_\ell^{-1}$ to characteristic zero to obtain a modular form $f' \in M_{\mu-1}(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \bar{\mathbb{Z}}_{(\ell)})$. For the Fourier-Jacobi coefficients g_r^{ar} of the arithmetic lifting $\mathcal{L}^{\mathrm{ar}}_{0,\mu;\varepsilon,\chi}(f')$ we have

$$b_{\mathrm{ar}}(\vartheta, g_r^{\mathrm{ar}}) = \frac{r^{\mu-2}}{D}\ell_{\chi\varepsilon^{-1}}(\vartheta)P^{\mathrm{ar}}(f', {\kappa^*}^{-1}\chi\varepsilon; \chi\varepsilon),$$

where ϑ is a Shintani eigenfunction in $\mathcal{T}^{\mathrm{prim}}_{r,\rho}$ with eigencharacter κ.

Taking $r = p^n$ and choosing ϑ as an integral representative of a space in T_n, by assumption the "first factor" $i_\ell i_\infty^{-1}(\ell_{\chi\varepsilon^{-1}}(\vartheta))$ will be non-zero mod $\mathfrak{L}$ if n is large enough. On the other hand, by Theorem 3.21, the factor $i_\ell i_\infty^{-1}(P^{\mathrm{ar}}(f', \kappa^{*-1}\chi\varepsilon))$ will be non-zero infinitely often, if n grows. Because of the ℓ-integrality of b_{ar} on $\mathcal{T}_{p^n,\rho}$ we conclude that $g_{p^n}^{\mathrm{ar}}$ will be non-zero mod $\mathfrak{L}$ for some n, and the assertion follows.

The proof of Theorem 3.16 will be finished in the next chapter, where we will prove the non-vanishing assumption made above (see Theorem 4.1).

Proof of the non-vanishing theorem We now begin the proof of Theorem 3.21. Let us first consider the formula for $P^{\mathrm{ar}}(\varphi, \eta; \sigma)$ in the special case $(r, D) = 1$ and $\mathfrak{d}_\eta = \mathfrak{o}_K$. Here we have

$$P^{\mathrm{ar}}(\varphi, \eta) = \sum_{\mathfrak{c}} \left(\frac{\Omega(\mathfrak{c})}{\Omega_0}\right)^{m'} \mathrm{N}(\mathfrak{c})^{m'/2}\eta(\mathfrak{c})^{-1}P_{\mathfrak{c}}^{\mathrm{ar}}(\varphi, \eta)$$

with

$$P_{\mathfrak{c}}^{\mathrm{ar}}(\varphi, \eta) = \sum_{H'=\langle P\rangle < E_{\mathfrak{c}}[rD]} \varphi(E_{\mathfrak{c}}/H, \hat{\pi}^*(\omega_{\mathfrak{c}}), \pi(P))\eta_f(\iota_{rD}^{-1}(P))^{-1}.$$

Writing $\eta_f = \eta_r\eta_D$ it is not difficult to transform this expression further by separating the two components. Indeed, define a C^∞-modular form for $\Gamma(D)$ by

$$\varphi'(E, \omega, \alpha) = \sum_{X=\langle Q\rangle < E[D]} \varphi(E, \omega, Q)\eta_D^{-1}(\alpha^{-1}(Q))$$

for a level D structure $\alpha : (\mathfrak{o}_K/D\mathfrak{o}_K) \simeq E[D]$, where summation goes over all cyclic subgroups X of order D of $E[D]$ such that $\mathfrak{o}_K\alpha^{-1}(X) = \mathfrak{o}_K/D\mathfrak{o}_K$. Then

$$P_{\mathfrak{c}}^{\mathrm{ar}}(\varphi, \eta) = \omega_{K/\mathbb{Q}}(r) \sum_{H=\langle P\rangle < E_{\mathfrak{c}}[r]} \eta_r(\iota_r^{-1}(P))^{-1}\varphi'(E_{\mathfrak{c}}/H, \hat{\pi}^*(\omega_{\mathfrak{c}}), \pi(\alpha_{\mathfrak{c}})),$$

where ι_r is as above, $\alpha_{\mathfrak{c}}$ is the level D structure given by the natural isomorphism $\mathfrak{o}_K/D\mathfrak{o}_K \simeq \mathfrak{c}/D\mathfrak{c} \simeq E_{\mathfrak{c}}[D]$, and $\pi(\alpha_{\mathfrak{c}}) = \pi \circ \alpha_{\mathfrak{c}}$ is its projection down to $E_{\mathfrak{c}}/H$.

Assume now contrary to the assertion of Theorem 3.21, that for infinitely many n we have $i_\ell i_\infty^{-1}(P^{\mathrm{ar}}(D^\lambda f, \eta)) \equiv 0\,(\mathfrak{L})$ for all $\eta \in \mathcal{C}_n$. We may apply our formulas in the case $r = p^n$. Since

$$P_{\mathfrak{c}}^{\mathrm{ar}}(\varphi, \eta) = \frac{u(\mathfrak{c})}{h_K} \sum_{\tau \in \hat{\mathrm{Cl}}_K} P^{\mathrm{ar}}(\varphi, \eta\tau)\tau(\mathfrak{c})$$

with some unit $u(\mathfrak{c})$, our assumption implies $i_\ell i_\infty^{-1}(P_{\mathfrak{c}}^{\mathrm{ar}}(D^\lambda f, \eta)) \in \mathfrak{L}$ for all $\eta \in \mathcal{C}_n$ (since $\ell \not| h_K$). Fix now the (restricted) D-component of η by choosing some $\mu \in (\mathbb{Z}/D\mathbb{Z})^\times$ as above. This means we are fixing the modular form φ' associated to $D^\lambda f$ and η_D. Letting η_{p^n} range over all possible characters, we get by Fourier inversion on the group $(\mathfrak{o}_K/p^n\mathfrak{o}_K)^\times/(\mathbb{Z}/p^n\mathbb{Z})^\times \simeq (\mathbb{Z}/p^n\mathbb{Z})^\times$ (whose order is prime to ℓ by assumption) the fundamental periodicity property

$$i_\ell i_\infty^{-1}(\varphi'(E_{\mathfrak{c}}/H_1, \hat{\pi}_1^*(\omega_{\mathfrak{c}}), \pi_1(\alpha_{\mathfrak{c}}))) \equiv i_\ell i_\infty^{-1}(\varphi'(E_{\mathfrak{c}}/H_2, \hat{\pi}_2^*(\omega_{\mathfrak{c}}), \pi_2(\alpha_{\mathfrak{c}}))) \quad (\mathfrak{L})$$

for all pairs of cyclic subgroups $H_i < E_{\mathfrak{c}}[p^n]$ of order p^n with $\mathfrak{o}_K H_i = E_{\mathfrak{c}}[p^n]$ and $pH_1 = pH_2$. The restriction $\mathfrak{o}_K H_i = E_{\mathfrak{c}}[p^n]$ may be reformulated as $H_i \not\subseteq E_{\mathfrak{c}}[p^{n-1}\mathfrak{p}] \cup E_{\mathfrak{c}}[p^{n-1}\bar{\mathfrak{p}}]$. (Observe that this argument goes through unchanged in the case $w_K > 2$, since the components corresponding to characters η_f for which the "consistency condition" $1 = \varepsilon^{m'}\eta_f(\varepsilon)$ fails, are automatically zero.)

Equivalently, for all cyclic subgroups $H' < E_{\mathfrak{c}}[p^n]$ of order p^{n-1} not contained in $E_{\mathfrak{c}}[p^{n-1}\mathfrak{p}] \cup E_{\mathfrak{c}}[p^{n-1}\bar{\mathfrak{p}}]$ consider the elliptic curve $E' = E_{\mathfrak{c}}/H'$ with invariant differential ω' and level D structure α' obtained from $\omega_{\mathfrak{c}}$ and $\alpha_{\mathfrak{c}}$ by the canonical projection. These curves have the property that for all but possibly one of the $p+1$ subgroups $G < E'[p]$ of order p the values

$$i_\ell i_\infty^{-1}(\varphi'(E'/G, \hat{\pi}_G^*(\omega), \pi_G(\alpha')))$$

are congruent modulo $\mathfrak{L}$, where π_G is the projection $E' \to E'/G$.

Assume now $\nu = 0$. We will indicate later how to modify the argument in the case $\nu > 0$. The functor of isomorphism classes of triples (E, α, H) consisting of an elliptic curve with level D structure and a subgroup of order p is representable over $\bar{\mathbb{F}}_\ell$ by a smooth affine curve $\mathcal{M}$, a covering of degree $p+1$ of the level D moduli curve. Letting $\bar{\varphi}'$ denote the reduction mod $\mathfrak{L}$ of $i_\ell i_\infty^{-1}(\varphi')$, we define a subvariety $\mathcal{M}_0$ of the curve $\mathcal{M}$ by imposing on (E, α, H) the condition

$$\bar{\varphi}'(E/G_1, \hat{\pi}_1^*(\omega), \pi_1(\alpha)) = \bar{\varphi}'(E/G_2, \hat{\pi}_2^*(\omega), \pi_2(\alpha)) \tag{45}$$

for any non-trivial invariant differential ω and all order p subgroups G_i different from H. Our assumption implies that on each of the connected components of $\mathcal{M}$ corresponding to $\det\alpha = \zeta_D^d$ with $\omega_{K/\mathbb{Q}}(d) = 1$ there are infinitely many points of $\mathcal{M}_0$, namely the reductions mod $\mathfrak{L}$ of the curves E' (with level structure and p-subgroup) considered above. Here ζ_D is the primitive D-th root of unity in $\bar{\mathbb{F}}_\ell$ obtained from $i_\ell i_\infty^{-1}(e^{2\pi i/D})$ by reduction mod $\mathfrak{L}$. In fact, the curves E' have as ring of endomorphisms the order of conductor p^{n-1} in $\mathfrak{o}_K$, and since ℓ was assumed split in K, the endomorphism ring of the reduced curve $\bar{E}'$ is the same (cf. [Lan, p. 182, Thm. 12]). This shows that we get infinitely many non-isomorphic curves $\bar{E}'$, and since $\det\alpha = \zeta_D^d$ with $d \equiv \mathrm{N}(\mathfrak{c})^{-1}p^{n-1}\ (D)$ for the associated level structures, the determinant may be adjusted by varying $\mathfrak{c}$. Consequently, we see that each of the connected components above is contained in $\mathcal{M}_0$. This means that for all triples (E, α, H) over $\bar{\mathbb{F}}_\ell$ with $\det\alpha$ as above the equality (45) holds true for all pairs of subgroups G_i different from H. But this implies (by varying H) that (45) has to hold for all pairs of subgroups G_i of any E and any α under the determinant condition.

Let us consider the consequences for q-expansions as in [Ka1, p. 89-92]. If

$$\bar{\varphi}'(\mathrm{Tate}(q), \omega_{\mathrm{can}}, \alpha) = \sum_k a_k q^{k/D},$$

the values of $\bar{\varphi}'(\mathrm{Tate}(q)/H, \hat{\pi}^*(\omega_{\mathrm{can}}), \pi(\alpha))$ for the order p subgroups H of $\mathrm{Tate}(q)$ are as follows:

$$\bar{\varphi}'(\mathrm{Tate}(q)/H_i, \hat{\pi}^*(\omega_{\mathrm{can}}), \pi(\alpha)) = p^{-m}\sum_k a_k \zeta_p^{ik} q^{k/(Dp)},$$

for $H_i = \langle \zeta_p^{Di} q^{1/p}\rangle < \mathrm{Tate}(q)$, $0 \le i \le p-1$, and

$$\bar{\varphi}'(\mathrm{Tate}(q)/\mu_p, \hat{\pi}^*(\omega_{\mathrm{can}}), \pi(\alpha)) = \sum_k a_k q^{pk/D}.$$

Relation (45) implies first $a_k = 0$ if $p \nmid k$ (by using the subgroups H_i), and second $a_{k/p} = p^{-m}a_{pk}$ (by including μ_p), where $a_k = 0$ if k is not an integer. It follows quickly that $a_k = 0$ for all $k > 0$. But a modular form of weight m with constant q-expansion exists only if $(\ell-1)|m$, which is impossible since m is odd. Consequently, $\bar{\varphi}'$ vanishes for $\det\alpha = \zeta_D^d$ with $\omega_{K/\mathbb{Q}}(d) = 1$.

We will derive a contradiction by computing a q-expansion of $\bar{\varphi}'$ on each connected component. Of course, for this purpose we may compute q-expansions of φ' over the complex numbers. Given $d \in (Z/D\mathbb{Z})^\times$, let $E = \mathbb{C}/2\pi i(\mathbb{Z} + \tau\mathbb{Z})$, $\tau \in \mathfrak{H}$ with natural invariant differential ω and choose a level structure α of determinant $e^{2\pi i d/D}$ as follows:

$$\begin{aligned} \alpha(a + b\delta) &= (2\pi i/D)(b - 2da\tau), \quad 2 \not| D, \\ \alpha(a + b(N + \sqrt{-N})) &= (2\pi i/D)(b - da\tau), \quad D = 4N,\, 2 \not| N, \\ \alpha(a + b\sqrt{-N}) &= (2\pi i/D)(b - da\tau), \quad D = 4N,\, 2|N. \end{aligned}$$

Write $f|W'_D = \sum_k b_k q^k$, and let $\bar{b}_k$ be the reduction mod $\mathfrak{L}$ of $i_\ell i_\infty^{-1}(b_k)$. We easily get for the q-expansion coefficients c_k in $\varphi'(E, \omega, \alpha) = \sum_k c_k q^{k/D}$ the relation

$$c_k = \begin{cases} \omega(d')Db_k, & k \equiv -d'\mu\,(D), \\ 0, & \text{otherwise}, \end{cases}$$

where $d' = 2d$ in case D is odd, and $d' = d$ if D is even. For example, for D odd we compute

$$\begin{aligned} \varphi'(E, \omega, \alpha) &= \sum_{x \in \mathbb{Z}/D\mathbb{Z}} f(E, \omega, \frac{2\pi i}{D}(x + \tau))\eta_D^{-1}((-2d)^{-1} + x\delta) \\ &= \omega_{K/\mathbb{Q}}(-2d) \sum_{x \in \mathbb{Z}/D\mathbb{Z}} f| \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} e^{2\pi i(2d\mu)x/D} \\ &= \omega_{K/\mathbb{Q}}(2d) \sum_{x \in \mathbb{Z}/D\mathbb{Z}} (f|W'_D)((\tau + x)/D)e^{2\pi i(2d\mu)x/D} \\ &= \omega_{K/\mathbb{Q}}(2d)D \sum_{k \equiv -2d\mu\,(D)} b_k e^{2\pi i k\tau/D}. \end{aligned}$$

Letting now d run over all residue classes with $\omega_{K/\mathbb{Q}}(d) = 1$, we conclude $\bar{b}_k = 0$ for all k with $\omega_{K/\mathbb{Q}}(k) = -1$, and consequently $\bar{f} \in \bar{\mathcal{K}}$, a contradiction.

In the case $\lambda > 0$, we use Katz's comparison theorem [Ka2] to construct a modular form $\bar{\varphi}'$ whose values at ordinary CM elliptic curves coincide with the reduction of the values of φ'. To be precise, $\bar{\varphi}'$ is constructed from $\bar{\varphi} = \theta^\nu \bar{f}$ (with Serre's differential operator $\theta = qd/dq$) as φ' is from φ, and it is a modular form on the ordinary locus of the level D moduli scheme over $\bar{\mathbb{F}}_\ell$, obtained by omitting the points (finite in number) corresponding to supersingular elliptic curves. Therefore, using the same reasoning as above (but replacing $\mathcal{M}$ by the ordinary part $\mathcal{M}^{\text{ord}}$), we conclude that $\bar{\varphi}'$ vanishes on the connected components with $\det\alpha = \zeta_D^d$, $\omega_{K/\mathbb{Q}}(d) = 1$. But $\bar{\varphi}' = \theta^\nu(\bar{f}')$, where $\bar{f}'$ is just the modular form $\bar{\varphi}'$ in the case $\lambda = 0$. From [Ka3] we get (since $m \leq \ell - 2$) that $\bar{f}'$ itself vanishes on the corresponding components, and we may proceed as above.

4 A non-vanishing result for theta functions in characteristic ℓ

The purpose of this chapter is to prove the cruical non-vanishing result on theta functions needed to complete our strategy. Because of the connection to anti-cyclotomic L-functions furnished by the work of Yang [Y], the result may be of independent interest.

The main theorem We consider the following situation: fix a prime ℓ and embeddings $i_\ell : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_\ell$ and $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ as usual. For an integer $\mu \geq 0$ we have the spaces $\mathcal{V}_{r,C;\rho_{0\mu}}$ together with the Shintani representation $\mathcal{F}$ and the linear functionals ℓ_γ for unramified Hecke characters γ of weight μ.

We take a second prime $p \neq \ell$, assumed to be split in K, and look at Shintani eigenfunctions in the tower of spaces $\mathcal{V}_{r_0 p^m, Cp^m;\rho}$ for some fixed r_0 prime to p and arbitrary $m \geq 0$. If κ^* is the associated Hecke character of K, we fix the restriction β of κ^* (considered as a character of $\mathbb{A}_K^\times$) to the group $K_D^\times \mathfrak{o}_{K,r_0D}^\times$, and vary the p-component κ_p^* (more precisely, its restriction to $\mathfrak{o}_{K,p}^\times$). The choice of these local data (together with the datum at infinity given by the weight $2\mu - 1$) determines the character κ^* up to twist by a character of $\mathrm{Cl}_K/\mathrm{Cl}_K^{\mathrm{inv}}$. We have the consistency condition

$$\kappa_p^*(\delta) = i^{2\mu-1}(\kappa_{r_0D}^*(\delta))^{-1},$$

and in case $w_K > 2$ the additional condition $\kappa_p^*(\varepsilon) = \varepsilon^{2\mu-1}(\kappa_{r_0D}^*(\varepsilon))^{-1}$ for $\varepsilon \in \mathfrak{o}_K^\times/\{\pm 1\}$. Whether such a character κ^* actually occurs in $\mathcal{V}_{r_0p^m,Cp^m;\rho}$ is determined by the local epsilon conditions of Corollary 2.10 on $\beta|K_D^\times$, which are independent of m. The following theorem is our main result.

Theorem 4.1. *Let ℓ be a prime and $p \neq \ell$ be a prime split in K with $p \not| 2h_K$. Fix $\mu \geq 0$, an integer r_0 prime to p, a class $C = cN(K^\times)$, an unramified Hecke character γ of weight μ, and a character β of $K_D^\times \mathfrak{o}_{K,r_0D}^\times$, such that the epsilon conditions*

$$\varepsilon(\beta_q^{-1}, \lambda_{K,q})\beta_q^{-1}(\delta)\omega_q(r_0/c) = 1, \quad q|D,$$

hold true.

Then, if m is large enough, for every character κ^* with $\kappa^*|K_D^\times \mathfrak{o}_{K,r_0D}^\times = \beta$ appearing in $\mathcal{V}_{r_0p^m,Cp^m;\rho}^{\mathrm{prim}}$, there is an integral representative ϑ in the κ-eigenspace such that

$$i_\ell i_\infty^{-1}(w_K^{-1}\ell_\gamma(\vartheta)) \not\equiv 0 \,(\mathfrak{L}).$$

By Corollary 3.22 and Dirichlet's theorem on primes in arithmetic progressions, this theorem finishes in particular the proof of Theorem 3.16.

Our method used to obtain this result is based on ideas of Sinnott [Si1, Si2], who gave an algebraic proof of Washington's theorem [W1, W2] on the ℓ-adic behaviour of class numbers in cyclotomic $\mathbb{Z}_p$-extensions of abelian number fields. To state it, let F be an abelian number field, and $F_\infty = F\mathbb{Q}_\infty$ its cyclotomic $\mathbb{Z}_p$-extension with unique intermediate extensions F_n/F of degree p^n. Washington's theorem says that for a prime $\ell \neq p$ the ℓ-part of the class number h_n of F_n stays bounded for $n \to \infty$; indeed, the sequence $v_\ell(h_n)$ gets stationary. It is an ℓ-adic analogue of the theorem of Ferrero-Washington on the vanishing of the μ-invariant of F_∞/F, which implies by a well-known result of Iwasawa that the p-part of h_n grows linearly with n for $n \to \infty$. By the class number formula Washington's theorem is equivalent (see [W1]) to the following assertion on L-values: given an integer $n \geq 1$ and a Dirichlet character χ, for all but finitely many Dirichlet characters ψ of p-power conductor with $\chi\psi(-1) = (-1)^n$ we have

$$i_\ell i_\infty^{-1}(\frac{1}{2}L(1-n, \chi\psi)) \not\equiv 0 \,(\mathfrak{L}).$$

Sinnott's strategy is to use the fact that these L-values are closely connected to rational functions, which allows him to derive the non-vanishing from an algebraic independence result. The article [Si2] gives an excellent exposition.

Gillard [Gi] proved the analogous statement for $\mathbb{Z}_p$-extensions of imaginary quadratic fields in which exactly one of the primes lying above a split p is ramified. Here we are in effect considering anticyclotomic $\mathbb{Z}_p$-extensions. We hope that our method may also be applied to the determination of the μ-invariant of such an extension.

Explicit expression for $\ell_\gamma(\vartheta)$ For the proof of Theorem 4.1, we first have to give an expression for $\ell_\gamma(\vartheta)$ which allows us to separate the influence of the p-component. To begin, consider a space $\mathcal{V}_{r,C;\rho}$ of scalar-valued theta functions; we have the isomorphism

$$\mathcal{V}_{r,C;\rho} \simeq \bigoplus_{\mathfrak{c}\in\mathcal{C}} T^1_{r/\mathrm{N}(\mathfrak{a}),\mathfrak{a}\bar{\mathfrak{c}}/\mathfrak{c};\rho},$$

where $\mathfrak{a}$ is a fractional ideal of K with $\mathrm{N}(\mathfrak{a}) \in C$, and $\mathcal{C}$ a system of representatives for $\mathrm{Cl}_K/\mathrm{Cl}_K^{\mathrm{inv}}$ consisting of primitive integral ideals (i. e. integral ideals not divisible by n for any rational integer $n > 1$). If $\vartheta = (\vartheta_{\mathfrak{a}\bar{\mathfrak{c}}/\mathfrak{c}})_{\mathfrak{c}\in\mathcal{C}}$ is an eigenfunction with eigencharacter κ of the Shintani representation $\mathcal{F}$, we have

$$\vartheta_{\mathfrak{a}\bar{\mathfrak{c}}/\mathfrak{c}} = \kappa^*(\mathfrak{c})\mathcal{F}(\bar{\mathfrak{c}}/\mathfrak{c})\vartheta_{\mathfrak{a}}. \tag{46}$$

Consequently,

$$\ell_\gamma(\vartheta) = \gamma_{\mathrm{ar}}(\mathfrak{a})\varepsilon\left(\sum_{\mathfrak{c}\in\mathcal{C}}(\kappa^*\gamma^{-2})_{\mathrm{ar}}(\mathfrak{c})\mathcal{E}(\bar{\mathfrak{c}}/\mathfrak{c})\vartheta_{\mathfrak{a}}\right), \tag{47}$$

where we write $\chi_{\mathrm{ar}}(\mathfrak{a}) = \chi(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{w/2}$ for a Hecke character χ of weight w and denote by ε the map "evaluation at zero" on a space of theta functions.

Let us now specialize to the situation of Theorem 4.1. This means we fix in addition to C and ρ (or μ) the parameters ℓ, p, r_0, γ and β, and consider the spaces $\mathcal{V}_{r_0p^m,Cp^m;\rho}$. Since p is split, we have $p\mathfrak{o}_K = \mathfrak{p}\bar{\mathfrak{p}}$, and to the primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ above p correspond the two embeddings $\iota_{\mathfrak{p}}$ and $\iota_{\bar{\mathfrak{p}}}$ of $\mathfrak{o}_K$ into $\mathbb{Z}_p$. Together they induce an isomorphism $\mathfrak{o}_{K,p} \simeq \mathbb{Z}_p\times\mathbb{Z}_p$. We denote by (x, y) the element of $\mathfrak{o}_{K,p}$ corresponding to the pair (x, y) on the right hand side; the same applies to $\mathfrak{o}_K/p^n\mathfrak{o}_K$.

Take a fractional ideal $\mathfrak{a}$ (prime to p and ℓ) with $\mathrm{N}(\mathfrak{a}) \in C$. Choose a system of representatives $\mathcal{C}$ as above, and assume all the elements of $\mathcal{C}$ to be prime to p and ℓ. Our local datum β determines a character κ_{r_0} of $K^1(r_0)$ by

$$\kappa_{r_0}(z/\bar{z}) = (z/|z|)^{2\mu-1}\beta(z)^{-1}\prod_{q\,\mathrm{inert},\,q\nmid r_0D}(-1)^{v_q(z)},$$

and in turn an eigenspace in $T^{\mathrm{prim}}_{r_0/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}$ for the Shintani operators $\mathcal{F}(\eta)$ (cf. Section 2.3 for their definition), since the local conditions at primes dividing D are fulfilled. (The eigenspace does not necessaily lie in $T^1_{r_0/\mathrm{N}(\mathfrak{a}),\mathfrak{a};\rho}$, i. e. it does not need to come from restricting an element of $\mathcal{V}_{r_0,C;\rho}$.) From a representative $\vartheta_{\mathfrak{a}}$ of this space we construct elements $\vartheta_{\mathfrak{a}\bar{\mathfrak{p}}^m,\chi_0}$ of $T^{\mathrm{prim}}_{r_0/\mathrm{N}(\mathfrak{a}),\mathfrak{a}\bar{\mathfrak{p}}^m;\rho}$ by setting

$$\vartheta_{\mathfrak{a}\bar{\mathfrak{p}}^m,\chi_0} = \sum_{z\in(\mathbb{Z}/p^m\mathbb{Z})^\times}\chi_0(z)\psi(zl_0)A_{zl_0}\vartheta_{\mathfrak{a}}$$

for a Dirichlet character χ_0 (of the rational integers) of conductor p^m and a generator l_0 of $\mathfrak{a}\bar{\mathfrak{p}}^m\mathfrak{p}^{-m}/\mathfrak{a}\bar{\mathfrak{p}}^m$. This is certainly well-defined, and changing the parameter l_0 into λl_0 for $\lambda \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ changes $\vartheta_{\mathfrak{a}\bar{\mathfrak{p}}^m,\chi_0}$ only by a factor $\chi_0(\lambda)^{-1}$; therefore the associated vector space is uniquely determined by $\vartheta_{\mathfrak{a}}$ and χ_0. Since $\mathcal{E}(\eta)A_{\eta l} =$

$A_l\mathcal{E}(\eta)$ for $\eta \in K^1$ and $l \in \mathfrak{a}^* \cap \eta^{-1}\mathfrak{a}^*$ by [GlR, p. 72], it follows that $\vartheta_{\mathfrak{a}\bar{\mathfrak{p}}^m,\chi_0}$ is an eigenfunction of the operators $\mathcal{F}(\eta)$, $\eta \in K^1(r_0p)$, with eigencharacter $\kappa(\eta) = \kappa_{r_0}(\eta)\chi_0(\eta \bmod \mathfrak{p}^m)$.

In particular, $\vartheta_{\mathfrak{a}\bar{\mathfrak{p}}^m,\chi_0}$ is invariant under the action of the roots of unity if and only if $\kappa(\varepsilon) = 1$ for all $\varepsilon \in \mathfrak{o}_K^\times$. If this is the case, we can extend $\vartheta_{\mathfrak{a}\bar{\mathfrak{p}}^m,\chi_0}$ to an element of $\mathcal{V}_{r_0p^m,Cp^m;\rho}$ as in (46). Putting this together with (47) we have proved the following lemma.

Lemma 4.2. *The eigenspaces in $\mathcal{V}^{\mathrm{prim}}_{r_0p^m,Cp^m;\rho}$ of characters κ with $\kappa|K_D^\times\mathfrak{o}_{K,r_0}^\times = \beta$ may be described as follows. Let $\mathfrak{C}$ be the l. c. m. of all ideals $\mathfrak{c} \in \mathcal{C}$, $\mathfrak{A} = \mathfrak{a}\bar{\mathfrak{C}}$, and l_0 a generator of $\mathfrak{A}\bar{\mathfrak{p}}^m\mathfrak{p}^{-m}/\mathfrak{A}\bar{\mathfrak{p}}^m$. Moreover, let χ_0 be the character of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ given by $\chi_0(z) = \kappa_p^*((z,1))^{-1}$. Then $\vartheta = (\vartheta_{\mathfrak{a}\bar{\mathfrak{p}}^m\bar{\mathfrak{c}}/\mathfrak{c}})$ with*

$$\vartheta_{\mathfrak{a}\bar{\mathfrak{p}}^m\bar{\mathfrak{c}}/\mathfrak{c}} = \kappa^*(\mathfrak{c}) \sum_{z\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \chi_0(z)\psi(zl_0)A_{zl_0}(\mathcal{F}(\bar{\mathfrak{c}}/\mathfrak{c})\vartheta_{\mathfrak{a}})$$

is a generator of the κ-eigenspace, and we have

$$\ell_\gamma(\vartheta) = \gamma_{\mathrm{ar}}(\mathfrak{a}\bar{\mathfrak{p}}^m)\varepsilon\left(\sum_{\mathfrak{c}}(\kappa^*\gamma^{-2})_{\mathrm{ar}}(\mathfrak{c}) \sum_{z\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \chi_0(z)\psi(zl_0)A_{zl_0}(\mathcal{E}(\bar{\mathfrak{c}}/\mathfrak{c})\vartheta_{\mathfrak{a}})\right). \tag{48}$$

To achieve a complete separation of the p-component, we need a slight variation of this result. The character χ_0 is not enough for a complete parametrization of the possible eigenspaces (there are h_K' of them for each admissible χ_0). Such a parametrization may be constructed in the case $w_K = 2$ by setting $\kappa^* = \kappa_0^*\chi$, where κ_0^* is some fixed extension of $\beta|\mathfrak{o}_{K,r_0D}^\times$ to a Hecke character of K of weight $2\mu - 1$, and

$$\chi : \Gamma_m = I_K(p)/I_{\mathbb{Q}}(p)P_{K,p^m} \to \mathbb{C}^\times$$

is a finite order character extending the character $(z) \mapsto \chi_0(z/\bar{z} \bmod \mathfrak{p}^m)$ of the group of principal ideals prime to p. The local components $\chi(\mathfrak{q}) \in \{\pm 1\}$ for ramified prime ideals $\mathfrak{q}$ are fixed to let the D-component of κ^* come out correctly:

$$\chi(\mathfrak{q}) = \epsilon_{\mathfrak{q}} = (\beta_q {\kappa_{0,q}^*}^{-1})(\pi_q)$$

(where π_q denotes a prime element in $K_q^\times$). This implies the consistency condition $\chi(\delta\mathfrak{o}_K) = \chi_0(-1) = \prod_{\mathfrak{q}} \epsilon_{\mathfrak{q}}^{v_q(D)}$. We may now rephrase (48) as

$$\frac{1}{2}\ell_\gamma(\vartheta) = \gamma_{\mathrm{ar}}(\mathfrak{a}\bar{\mathfrak{p}}^m)\varepsilon\left(\sum_{\mathfrak{c}\in\mathcal{C},\, z\in(\mathbb{Z}/p^m\mathbb{Z})^\times/\{\pm1\}} \chi((z,1)\mathfrak{c})\psi(zl_0)A_{zl_0}(\vartheta_{\mathfrak{c}})\right), \tag{49}$$

if we introduce (changing our notation a bit) the theta functions

$$\vartheta_{\mathfrak{c}} = (\kappa_0^*\gamma^{-2})_{\mathrm{ar}}(\mathfrak{c})\mathcal{E}(\bar{\mathfrak{c}}/\mathfrak{c})\vartheta_{\mathfrak{a}}, \quad \mathfrak{c} \in \mathcal{C}.$$

Since the sections $\vartheta_{\mathfrak{c}}$ are even (resp. odd) according to whether $\chi((-1,1)) = \prod_{\mathfrak{q}} \epsilon_{\mathfrak{q}}^{v_q(D)} = 1$ or -1, summing over $z \in (\mathbb{Z}/p^m\mathbb{Z})^\times/\{\pm1\}$ instead of $z \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ yields a factor of $1/2$, as asserted.

In case $w_K > 2$ we have $h_K = 1$, and taking $\mathcal{C} = \{\mathfrak{o}_K\}$ we get

$$\frac{1}{w_K}\ell_\gamma(\vartheta) = \gamma_{\mathrm{ar}}(\mathfrak{a}\bar{\mathfrak{p}}^m)\varepsilon\left(\sum_{z\in(\mathbb{Z}/p^m\mathbb{Z})^\times/\iota_{\mathfrak{p}}(\mathfrak{o}_K^\times)} \chi_0(z)\psi(zl_0)A_{zl_0}(\vartheta_{\mathfrak{a}})\right).$$

For simplicity, we will assume $w_K = 2$ in the following, and leave it to the reader to figure out the minor modifications necessary for dealing with the two exceptional fields $K = \mathbb{Q}(\sqrt{-3})$ and $K = \mathbb{Q}(i)$.

Algebraic theta functions To deal with reduction mod ℓ and theta functions in characteristic ℓ, we have to use the elements of the theory of algebraic theta functions developed by Mumford [Mum1, Mum2]. We briefly state a few basic results in the form necessary for our purpose. Let E be an elliptic curve over an algebraically closed field $\bar{k}$, and L an ample line bundle on E. If L has degree dp^m with $p \nmid d$ and $m \geq 1$ for an odd prime p not dividing the characteristic, the subgroup

$$H(L) = \{x \in E(\bar{k}) \,|\, T_x^* L \simeq L\}$$

of $E(\bar{k})$ associated to L contains $E[p^m]$. We consider the subgroup $\mathcal{G}_p(L)$ of Mumford's Heisenberg group $\mathcal{G}(L)$ (see [Mum1, p. 289]) given as the set of pairs (x, φ) where $x \in E[p^m]$ and $\varphi : L \to T_x^* L$ is an isomorphism. This group fits into an exact sequence

$$1 \longrightarrow \bar{k}^* \longrightarrow \mathcal{G}_p(L) \longrightarrow E[p^m] \longrightarrow 0.$$

The commutator gives an alternating pairing $e_L : E[p^m] \times E[p^m] \to \mu_{p^m}$. The group $\mathcal{G}_p(L)$ acts naturally on the space of global sections $\Gamma(E, L)$ [Mum1, p. 295]. There is a smallest subgroup $\mathcal{G}_{p,f}(L)$ of $\mathcal{G}_p(L)$ such that the projection to $E[p^m]$ is still surjective; it is obtained by taking all elements of order dividing p^m and fits into an exact sequence

$$1 \longrightarrow \mu_{p^m} \longrightarrow \mathcal{G}_{p,f}(L) \longrightarrow E[p^m] \longrightarrow 0.$$

If the line bundle L is symmetric, i. e. $[-1]^* L \simeq L$, we may construct a canonical section of $\mathcal{G}_{p,f}$ by using the automorphism δ_{-1} of $\mathcal{G}_p(L)$ defined in [Mum1, p. 308]. It is of order two, its restriction to the center $\bar{k}^*$ is the identity, and its projection to $E[p^m]$ induces the map $[-1]$. For $x \in E[p^m]$ we now let $z = A'_x$ be the unique element in $\mathcal{G}_{p,f}$ with $\delta_{-1}(z) = z^{-1}$ projecting to x. We then have the addition law

$$A'_x A'_y = e_L(x, y)^{1/2} A'_{x+y},$$

where the square root is (uniquely) taken in μ_{p^m}.

As the reader may easily verify, over the complex numbers this construction gives (in essence) back the operators A_l. More precisely, let E be determined by a lattice L, and the line bundle by a pair (H, ψ), where ψ takes values in $\{\pm 1\}$ (because of the symmetry condition). Then we have $A'_x = \psi(x) A_x$ for $x \in p^{-m} L$, where ψ is canonically extended to $p^{-m} L$ by $\psi(x) := \psi(p^m x)$.

Application of an idea of Sinnott We now are able to reduce the result (49) modulo ℓ. First rephrase the characteristic zero situation algebraically. For each $m \geq 0$ we have on the elliptic curve $E_{\mathfrak{A}\bar{\mathfrak{p}}^m} = \mathbb{C}/\mathfrak{A}\bar{\mathfrak{p}}^m$ a symmetric line bundle $L_{\mathfrak{A}\bar{\mathfrak{p}}^m}$ whose space of global sections (over $\mathbb{C}$) is $T_{r_0/\mathrm{N}(\mathfrak{a}), \mathfrak{A}\bar{\mathfrak{p}}^m}$; the isogeny $\Phi_m : E_{\mathfrak{A}\bar{\mathfrak{p}}^m} \to E_{\mathfrak{A}}$ induces a map $\Phi_m^* : \Gamma(E_{\mathfrak{A}}, L_{\mathfrak{A}}) \to \Gamma(E_{\mathfrak{A}\bar{\mathfrak{p}}^m}, L_{\mathfrak{A}\bar{\mathfrak{p}}^m})$ which is just the natural inclusion of spaces of theta functions. We are considering the values at zero of the theta functions

$$\sum_{\mathfrak{c} \in \mathcal{C},\, z \in (\mathbb{Z}/p^m\mathbb{Z})^\times/\{\pm 1\}} \chi((z, 1)\mathfrak{c}) A'_{z l_0} \Phi_m^*(\vartheta_{\mathfrak{c}})$$

in $\Gamma(E_{\mathfrak{A}\bar{\mathfrak{p}}^m}, L_{\mathfrak{A}\bar{\mathfrak{p}}^m})$.

The objects $E_{\mathfrak{A}\bar{\mathfrak{p}}^m}$, $L_{\mathfrak{A}\bar{\mathfrak{p}}^m}$ and Φ_m are all defined over a fixed number field L, and indeed have good reduction at ℓ (for L big enough). Observe that (since $\mathfrak{a}$ and the elements of $\mathcal{C}$ were assumed prime to ℓ) the functions $\vartheta_{\mathfrak{c}}$ will be ℓ-integral whenever the original function $\vartheta_{\mathfrak{a}}$ is, and by (49) the number $(1/2)\ell(\vartheta)$ will be ℓ-integral, too. We may therefore reduce everything modulo ℓ via $i_\ell i_\infty^{-1}$. Choosing $\vartheta_{\mathfrak{a}}$ with non-zero reduction, all $\vartheta_{\mathfrak{c}}$ will reduce to non-zero sections of $\bar{L}_{\mathfrak{A}}$ over $\bar{E}_{\mathfrak{A}}$, since the Shintani operator $\mathcal{E}(\bar{\mathfrak{c}}/\mathfrak{c})$ is invertible even in characteristic ℓ. More precisely, the operators $\mathcal{E}(\bar{\mathfrak{c}}/\mathfrak{c})$ and $\mathcal{E}(\mathfrak{c}/\bar{\mathfrak{c}})$ both preserve integrality, and $\mathcal{E}(\mathfrak{c}/\bar{\mathfrak{c}})\mathcal{E}(\bar{\mathfrak{c}}/\mathfrak{c}) = \mathrm{N}(\mathfrak{c})$, which is prime to ℓ.

Let us now look at the situation modulo ℓ: let $k \subseteq \bar{\mathbb{F}}_\ell$ be a finite field of characteristic ℓ. We are given elliptic curves $E_m = \bar{E}_{\mathfrak{A}\bar{\mathfrak{p}}^m}$ over k (with complex multiplication by $\mathfrak{o}_K$) together with isogenies $\varphi_m : E_m \to E_0$ (over k) such that $\ker\varphi_m = E_m[\bar{\mathfrak{p}}^m]$. For $0 \le k \le m$ the isogeny φ_m may be factored as $\varphi_m = \varphi_k\psi_{mk}$ with an isogeny $\psi_{mk} : E_m \to E_k$ fulfilling $\ker\psi_{mk} = E_m[\bar{\mathfrak{p}}^{m-k}]$. In addition, we have a symmetric line bundle L_0 over E_0 (defined over k) and induced bundles $L_m = \varphi_m^* L_0$ over E_m of degree $p^m \deg L_0$. We give the bundles L_m rigidifications along the zero sections compatible with the φ_m^*; this induces (compatible) maps $\Gamma(E_m, L_m) \to \bar{\mathbb{F}}_\ell$ denoted by ε.

Furthermore we have non-zero sections $\vartheta_{\mathfrak{c}} \in \Gamma(E_0, L_0)$ for each $\mathfrak{c} \in \mathcal{C}$, and signs $\epsilon_{\mathfrak{q}}$ for all ramified prime ideals $\mathfrak{q}$. (We use here the same symbol $\vartheta_{\mathfrak{c}}$ for the characteristic ℓ theta function as for its characteristic zero counterpart; the same remark also applies to the character χ below. This should not yield to confusion.) Choose isomorphisms $i_m : \mathbb{Z}/p^m\mathbb{Z} \to E_m[\mathfrak{p}^m]$ and $j_m : \mathbb{Z}/p^m\mathbb{Z} \to E_m[\bar{\mathfrak{p}}^m]$; together they determine a primitive p^m-th root of unity ξ_m in $\bar{\mathbb{F}}_\ell$ by $e_{L_m}(i_m(x), j_m(y)) = \xi_m^{xy}$. For convenience we write $i(x,y) = i(x) + j(y)$. Note that for $\vartheta \in \Gamma(E_0, L_0)$ the section $\varphi_m^*(\vartheta) \in \Gamma(E_m, L_m)$ is invariant under the operators $A'_{j(y)}$ for $y \in \mathbb{Z}/p^m\mathbb{Z}$. (Because of the symmetry of L_m and L_0, it is easily seen that the "level subgroup" in the sense of Mumford associated to L_m, L_0 and φ_m consists out of the A'_X for $X \in E[\bar{\mathfrak{p}}^m]$.)

We assume (contrary to the assertion of our main theorem) that for infinitely many m there exists a character $\chi : \Gamma_m \to \bar{\mathbb{F}}_\ell^*$ of conductor p^m with $\chi(\mathfrak{q}) = \epsilon_{\mathfrak{q}}$ for all $\mathfrak{q}$ and $\varepsilon(\vartheta_\chi) = 0$, where $\vartheta_\chi \in \Gamma(E_m, L_m)$ is defined by

$$\vartheta_\chi = \sum_{\mathfrak{c}} \chi(\mathfrak{c}) \sum_{z \in (\mathbb{Z}/p^m\mathbb{Z})^\times/\{\pm 1\}} \chi((z,1)) A'_{i(z)} \varphi_m^*(\vartheta_{\mathfrak{c}}). \tag{50}$$

The group Γ_m is an extension of the class group Cl_K by $(\mathbb{Z}/p^m\mathbb{Z})^\times$, and the direct limit Γ_∞ of the Γ_m (with respect to the obvious direct system) is therefore an extension of Cl_K by $\mathbb{Z}_p^\times$. It is well known that $\mathbb{Z}_p^\times = \mu_{p-1} \times U$ with $U = 1 + p\mathbb{Z}_p \simeq \mathbb{Z}_p$; because $p \nmid h_K$ we may write $\Gamma_\infty = V \times U$ with a finite group V of order prime to p (indeed it is isomorphic to Γ_1). If $U_m = 1 + p^m\mathbb{Z}_p \subseteq U$, we identify $\mathbb{Z}_p^\times/U_m$ with $(\mathbb{Z}/p^m\mathbb{Z})^\times$.

We are now in a position to transfer Sinnott's ideas to our situation (cf. [Si2, p. 215]). Enlarge the finite field k, if necessary, such that it contains $\mu_{(p-1)h_K}$ and that furthermore the sections $\vartheta_{\mathfrak{c}}$ are defined over k. Let us agree to call a character $\chi : \Gamma_\infty \to \bar{\mathbb{F}}_\ell^*$ admissible, if $\chi(\mathfrak{q}) = \epsilon_{\mathfrak{q}}$ for all ramified prime ideals $\mathfrak{q}$, and assume that there exists an admissible χ of conductor p^m with $\varepsilon(\vartheta_\chi) = 0$. From this assumption we will derive relations for the $\vartheta_{\mathfrak{c}}$. We use the operation of the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_\ell/k)$ on L_m and on the spaces of global sections. The first observation is the following.

Lemma 4.3. *For any $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_\ell/k)$ and $\chi : \Gamma_m \to \bar{\mathbb{F}}_\ell^\times$ there is a constant $c(\sigma, \chi) \in \bar{\mathbb{F}}_\ell^\times$ with*

$$\vartheta_\chi^\sigma = c(\sigma, \chi)\vartheta_{\chi^\sigma}.$$

Proof. The action of σ on $E_m[\mathfrak{p}^m]$ is given by multiplication with some $\alpha(\sigma) \in (\mathbb{Z}/p^m\mathbb{Z})^\times$. Moreover, we have in general ${A'_x}^\sigma = A'_{x^\sigma}$ for $x \in E_m[p^m]$, since A'_x is the unique lifting z of x to $\mathcal{G}_p(L)$ which is of order dividing p^m, and fulfills $\delta^{-1}(z) = z^{-1}$. The assertion follows easily from these two facts.

Define n_0 by $p^{n_0} := \#(k \cap \mu_{p^\infty})$ and set $k_n := k(\mu_{p^{n_0+n}})$. Clearly we have for $\zeta \in \mu_{p^{n_0+n}}$:

$$\mathrm{Tr}_{k_n/k}(\zeta) = \begin{cases} p^n\zeta, & \zeta \in \mu_{p^{n_0}}, \\ 0, & \text{otherwise.} \end{cases} \tag{51}$$

The decomposition $\Gamma_\infty = V \times U$ gives a corresponding splitting of the character χ as a product $\chi = \omega\phi$, where ω and ϕ are characters of V and U, respectively; ϕ has by assumption conductor p^m. Assume now $m \geq 2n_0$ and set $n = m - n_0$. It is elementary that there exists a primitive p^{n_0}-th root of unity $\lambda \in \bar{\mathbb{F}}_\ell^\times$ such that $\phi(u) = \lambda^{(u-1)/p^n}$ for $u \equiv 1\,(p^n)$.

Splitting the class of every $\mathfrak{c} \in \mathcal{C}$ in $\Gamma_\infty = V \times U$ as $\langle \mathfrak{c} \rangle = v_\mathfrak{c} u_\mathfrak{c}$, we may rewrite the definition (50) of ϑ_χ as

$$\vartheta_\chi = \sum_{\mathfrak{c} \in \mathcal{C},\, \eta \in \mu_{p-1}/\{\pm 1\}} \omega(v_\mathfrak{c}\eta) \sum_{z \in U/U_m} \phi(z) A'_{i(\eta u_\mathfrak{c}^{-1} z)} \varphi_m^*(\vartheta_\mathfrak{c}).$$

Coming back to our assumption $\varepsilon(\vartheta_\chi) = 0$, by Lemma 4.3 it implies $\varepsilon(\vartheta_{\chi^\sigma}) = 0$ for $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_\ell/k)$. Since $\mu_{(p-1)h_K} \subseteq k^\times$, conjugation by σ leaves the V-part ω invariant. Consequently, we have for arbitrary $y \in U$ (using (51)):

$$\begin{aligned} 0 &= \varepsilon\left(p^{-n} \sum_{\sigma \in \mathrm{Gal}(k_n/k)} \phi^{-1}(y)^\sigma \vartheta_{\chi^\sigma} \right) \\ &= \varepsilon\left(\sum_{\mathfrak{c},\eta} \omega(v_\mathfrak{c}\eta) \sum_{z \in y(1+p^n\mathbb{Z})/p^m\mathbb{Z}} \phi(z/y) A'_{i(\eta u_\mathfrak{c}^{-1} z)} \varphi_m^*(\vartheta_\mathfrak{c}) \right). \end{aligned}$$

Since here $\phi(z/y) = \lambda^{(z/y-1)p^{-n}}$, we get

$$\varepsilon\left(\sum_{\mathfrak{c},\eta} \omega(v_\mathfrak{c}\eta) \sum_{u \in \mathbb{Z}/p^{n_0}\mathbb{Z}} \lambda^u A'_{i(\eta u_\mathfrak{c}^{-1} y(1+p^n u))} \varphi_m^*(\vartheta_\mathfrak{c}) \right) = 0. \tag{52}$$

We want to rewrite the inner sum in a more convenient form. Defining the projection

$$P_{n_0} = \sum_{u \in \mathbb{Z}/p^{n_0}\mathbb{Z}} A'_{i(p^n u)} = \sum_{X \in E_m[\mathfrak{p}^{n_0}]} A'_X$$

we observe the following identity of operators on $\Gamma(E_m, L_m)$:

$$\begin{aligned} A'_{i(\eta u_\mathfrak{c}^{-1} y, \eta^{-1} u_\mathfrak{c} v)} P_{n_0} &= \sum_{u \in \mathbb{Z}/p^{n_0}\mathbb{Z}} A'_{i(\eta u_\mathfrak{c}^{-1} y, \eta^{-1} u_\mathfrak{c} v)} A'_{i(p^n \eta u_\mathfrak{c}^{-1} y u)} \\ &= \xi_m^{-2^{-1} yv} \sum_{u \in \mathbb{Z}/p^{n_0}\mathbb{Z}} \xi_m^{-(p^n yv)u} A'_{i(\eta u_\mathfrak{c}^{-1} y(1+p^n u))} A'_{j(\eta^{-1} u_\mathfrak{c} v)}. \end{aligned}$$

If we choose $v \in \mathbb{Z}/p^m\mathbb{Z}$ with $\xi_m^{-p^n vy} = \lambda$, and apply this identity to $\varphi_m^*(\vartheta_\mathfrak{c})$, we get, because of the invariance of this theta function under $A'_{j(y)}$, just $\xi_m^{-2^{-1} yv}$ times the inner sum in (52). Therefore

$$\varepsilon\left(\sum_{\mathfrak{c},\eta} \omega(v_\mathfrak{c}\eta) A'_{i(\eta u_\mathfrak{c}^{-1} y, \eta^{-1} u_\mathfrak{c} v)} P_{n_0} \varphi_m^*(\vartheta_\mathfrak{c}) \right) = 0.$$

Since we have the factorization $\varphi_m = \varphi_{n_0}\psi_{m,n_0}$, and $A'_X\psi^*_{m,n_0}(\vartheta) = \psi^*_{m,n_0}(A'_{\psi(X)}\vartheta)$ for $\vartheta \in \Gamma(E_{n_0}, L_{n_0})$ and $X \in E_m[\mathfrak{p}^{n_0}]$ (cf. [Mum1, Prop. 2]), we conclude

$$P_{n_0}\varphi^*_m(\vartheta_\mathfrak{c}) = \psi^*_{m,n_0}(P_{n_0}\varphi^*_{n_0}(\vartheta_\mathfrak{c})),$$

denoting by P_{n_0} on $\Gamma(E_{n_0}, L_{n_0})$ the operator $\sum_{X\in E_{n_0}[\mathfrak{p}^{n_0}]} A'_X$, of course. Writing $\vartheta_{\mathfrak{c},n_0} = P_{n_0}\varphi^*_{n_0}(\vartheta_\mathfrak{c})$, we arrive at

$$\varepsilon\left(\sum_{\mathfrak{c},\,\eta}\omega(v_\mathfrak{c}\eta)A'_{i(\eta u_\mathfrak{c}^{-1}y,\eta^{-1}u_\mathfrak{c}v)}\psi^*_{m,n_0}(\vartheta_{\mathfrak{c},n_0})\right) = 0. \tag{53}$$

Note that $\vartheta_{\mathfrak{c},n_0}$ is non-zero; in fact, it is simply $\bar{\mathcal{E}}(\bar{\mathfrak{p}}^{n_0}/\mathfrak{p}^{n_0})\vartheta_\mathfrak{c}$ considered as a section of L_{n_0} on E_{n_0}, where $\bar{\mathcal{E}}$ is the characteristic ℓ version of the Shintani operator $\mathcal{E}$. More precisely, we may compute

$$\left(\sum_{Y\in E_{n_0}[\bar{\mathfrak{p}}^{n_0}]} A'_Y\right)(\vartheta_{\mathfrak{c},n_0}) = p^{n_0}\varphi^*_{n_0}(\vartheta_\mathfrak{c}),$$

and the non-vanishing follows.

We now want to deduce from these relations an algebraic identity saying that an algebraic variety contains a certain set of points. To this end, remember that $y \in U$ may be choosen arbitrarily, and that the condition $\xi_m^{-p^n vy} = \lambda$ relates only the classes of y and v mod p^{n_0}. Consider therefore $y' = y + w$ with $w \equiv 0\,(p^{n_0})$ and take for both y and y' the same value of v. We have

$$A'_{i(\eta u_\mathfrak{c}^{-1}y',\eta^{-1}u_\mathfrak{c}v)} = \xi_m^{2^{-1}vw}A'_{i(\eta u_\mathfrak{c}^{-1}y,\eta^{-1}u_\mathfrak{c}v)}A'_{i(\eta u_\mathfrak{c}^{-1}w)}.$$

Inserting this into (53) (with y replaced by y') yields

$$\varepsilon\left(\sum_{\mathfrak{c},\,\eta}\omega(v_\mathfrak{c}\eta)A'_{i(\eta u_\mathfrak{c}^{-1}y,\eta^{-1}u_\mathfrak{c}v)}A'_{i(\eta u_\mathfrak{c}^{-1}w)}\psi^*_{m,n_0}(\vartheta_{\mathfrak{c},n_0})\right) = 0.$$

Take a fixed integer $n_1 = n_0 + N > n_0$, assume $m \geq n_1 + n_0$, and set $w = p^{m-n_1}x$. Factorizing $\psi_{m,n_0} = \psi_{n_1,n_0}\psi_{m,n_1}$, we may write

$$\varepsilon\left(\sum_{\mathfrak{c},\,\eta}\omega(v_\mathfrak{c}\eta)A'_{i(\eta u_\mathfrak{c}^{-1}y,\eta^{-1}u_\mathfrak{c}v)}\psi^*_{m,n_1}(A'_{[(\eta u_\mathfrak{c}^{-1},\eta^{-1}u_\mathfrak{c})]X}\psi^*_{n_1,n_0}(\vartheta_{\mathfrak{c},n_0}))\right) = 0, \tag{54}$$

where $X = \psi_{m,n_1}(i_m(p^{m-n_1}x)) \in E_{n_1}[\mathfrak{p}^{n_1}]$. Here we use the natural operation of $\mathfrak{o}_{K,p}$ on $E_{n_1}[p^\infty] \simeq K_p/\mathfrak{o}_{K,p}$. Because of the invariance of $\vartheta_{\mathfrak{c},n_0}$ under A'_Y, $Y \in E_{n_0}[\mathfrak{p}^{n_0}]$, the value of X is only important modulo $E_{n_1}[\mathfrak{p}^{n_0}]$.

Setting $\theta_{\mathfrak{c},X} = A'_X\psi^*_{n_1,n_0}(\vartheta_{\mathfrak{c},n_0})$ for $X \in E_{n_1}[\mathfrak{p}^{n_1}]$, we have morphisms $\Phi_\mathfrak{c} : E := E_{n_1} \to \mathbb{P}^{p^N-1}$ given by the global sections $\theta_{\mathfrak{c},i_{n_1}(x)}$, $x \in \mathbb{Z}/p^N\mathbb{Z}$, of L_{n_1}. Every element $\alpha \in (\mathbb{Z}/p^N\mathbb{Z})^\times$ defines an automorphism c_α of $\mathbb{P}^{p^N-1}$ by $(v_x)_x \mapsto (v_{\alpha x})_x$. Set $P_m := \psi_{m,n_1}(i_m(y,v)) \in E[\mathfrak{p}^m\bar{\mathfrak{p}}^{n_1}]$ for $y \in U$ and corresponding v, and look at the points $c_{\eta u_\mathfrak{c}^{-1}}(\Phi_\mathfrak{c}([(\eta u_\mathfrak{c}^{-1},\eta^{-1}u_\mathfrak{c})]P_m))$. For $x \in E(\bar{\mathbb{F}}_\ell)$ define $L_{n_1}(x)$ as the tensor product with $\bar{\mathbb{F}}_\ell$ of the stalk of L_{n_1} at x (cf. [Mum1, p. 299]). Then the map $\Gamma(E, L_{n_1}) \to \bar{\mathbb{F}}_\ell$ given by $\vartheta \mapsto \varepsilon(A'_X\psi^*_{m,n_1}\vartheta)$ induces an identification of $L_{n_1}(\psi_{m,n_1}(X))$ with $\bar{\mathbb{F}}_\ell$. The relations (54) imply now the existence of a non-trivial linear dependency between the $c_{\eta u_\mathfrak{c}^{-1}}(\Phi_\mathfrak{c}([(\eta u_\mathfrak{c}^{-1},\eta^{-1}u_\mathfrak{c})]P_m))$: they have to lie in a projective space of dimension $(p-1)h'_K/2 - 2$.

Set $q = (p-1)/2$, enumerate the elements of $\mu_{p-1}/\{\pm 1\}\times\mathcal{C}$ as $(\eta_\nu, \mathfrak{c}_\nu)$ and define $\alpha_\nu = \eta_\nu u_{\mathfrak{c}_\nu}^{-1}$ for $0 \le \nu \le qh'_K - 1$. Write $(\alpha, \alpha^{-1})P = ([(\alpha_\nu, \alpha_\nu^{-1})]P)_\nu \in E^{qh'_K}[p^\infty]$ for $P \in E[p^\infty]$. Summarizing, we have obtained the following intermediate result.

Let $\mathcal{D} \subseteq E^{qh'_K}$ be the subvariety defined by the relation

$$\bigwedge_\nu c_{\alpha_\nu}(\Phi_{\mathfrak{c}_\nu}(P_\nu)) = 0.$$

Then for each $m \ge n_1 + n_0$ such that there exists an admissible character $\chi : \Gamma_\infty \to \bar{\mathbb{F}}_\ell$ of conductor p^m with $\varepsilon(\vartheta_\chi) = 0$, we have $(\alpha, \alpha^{-1})P_m \in \mathcal{D}$ for all

$$P_m = \psi_{m,n_1}(i_m(y, v)) \in E[\mathfrak{p}^m \bar{\mathfrak{p}}^{n_1}]$$

with arbitrary $y \in U$ and $v \in \mathbb{Z}/p^m\mathbb{Z}$ such that $\xi_m^{-p^n vy} = \lambda_m$ for a primitive p^{n_0}-th root of unity $\lambda_m \in \bar{\mathbb{F}}_\ell^\times$ depending on χ.

A geometric "independence" result Assuming contrary to Theorem 4.1 the existence of infinitely many admissible χ with $\varepsilon(\vartheta_\chi) = 0$, we proceed to derive a contradiction. Let us first consider the Zariski closure of the infinite set of all points $(\alpha, \alpha^{-1})P_m$ obtained in this way[7].

Lemma 4.4. *Let $r \ge 1$ and $\beta_\nu = (\alpha_\nu, \alpha'_\nu) \in \mathfrak{o}_{K,p}$ for $0 \le \nu \le r-1$ be given, and $\beta : E[p^\infty] \to E^r[p^\infty]$ the map $P \mapsto ([\beta_\nu]P)_\nu$. Let*

$$\mathcal{R} = \{x \in \mathfrak{o}_K^r \mid \sum_{\nu=0}^{r-1} \iota_{\mathfrak{p}}(x_\nu)\alpha_\nu = 0\}$$

be the $\mathfrak{o}_K$-module of relations between the α_ν, and

$$\mathcal{A} = \{P = (P_\nu) \in E^r \mid \sum_{\nu=0}^{r-1} [x_\nu] P_\nu = 0 \,\forall x \in \mathcal{R}\}$$

the abelian subvariety of E^r defined by these relations. Assume we have a chain of sets $E[p^\infty] \supseteq \mathcal{M} = \mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \ldots$, where $\mathcal{M}_i + E[\mathfrak{p}^i] = \mathcal{M}_i$. Then the Zariski closure B of $\beta(\mathcal{M})$ in E^r contains a translate $\mathcal{A} + X$ (for some $X \in E^r(\bar{\mathbb{F}}_\ell)$).

Proof. Let B_i be the Zariski closure of $\beta(\mathcal{M}_i)$ in E^r. This gives a chain $B = B_1 \supseteq B_2 \supseteq \ldots$ Clearly B_i is stable under translation by $\beta(E[\mathfrak{p}^i])$. Since the chain of the B_i gets stationary at some point, the closed subset $B_\infty = \bigcap B_i$ is non-empty, and we see that it is stable under translation by $\beta(E[\mathfrak{p}^\infty])$. Consider the algebraic subgroup T of E^r consisting out of all X with $B_\infty + X = B_\infty$. Since $\beta(E[\mathfrak{p}^\infty]) \subseteq T$, and we will see shortly that $\mathcal{A}$ is the minimal algebraic subgroup of E^r containing $\beta(E[\mathfrak{p}^\infty])$, we have $\mathcal{A} \subseteq T$, which implies that B_∞ (and a fortiori B) contains a translate of $\mathcal{A}$.

Let now A be an algebraic subgroup of E^r containing $\beta(E[\mathfrak{p}^\infty])$. If A_0 is the connected component of zero (an abelian variety), we have $\beta(E[\mathfrak{p}^\infty]) \subseteq A_0$. If not

[7] The following lemma is inspired by [B]. I would like to thank Don Blasius for bringing this paper to my attention. A similar lemma is contained in [Gi, p. 351, Prop. 1.2], but the proof given there is insufficient. Using our lemma we are able to fill the gap in Gillard's proof. Although I do not know how to prove Gillard's Proposition 1.2 in general, he applies it only in the case where (using Gillard's notation) the ideal Λ is prime to its conjugate Λ'. Assuming $\Lambda = \mathfrak{p}$ is a prime ideal different from its conjugate, if for an infinite subset $W \subseteq E[\mathfrak{p}^\infty]$ the points $P(w)$, $w \in W$, are contained in a proper subvariety $S \subset E^s$, we may enlarge the base field k such that S is defined over k, and consider the set W' of all Galois conjugates of elements of W. Since the image of Galois in $\mathrm{Aut}E[\mathfrak{p}^\infty] \simeq \mathbb{Z}_p^\times$ has to contain $1 + p^N\mathbb{Z}_p$ for some N, we see that the set of points in W' of order $\ge n$ is invariant under translation by $E[\mathfrak{p}^{n-N}]$; consequently, we may apply our lemma and get a contradiction. The case of general Λ (prime to Λ') can be dealt with accordingly.

$\mathcal{A} \subseteq A$, there exists a non-trivial homomorphism $\varphi : \mathcal{A}/(\mathcal{A} \cap A) \to E$. By Poincaré's complete reducibility theorem [Mum2, p. 173], we can extend $\varphi \circ [N]$ to E^r for some integer $N > 0$, and obtain a homomorphism $\varphi' : E^r \to E$ mapping $\beta(E[\mathfrak{p}^\infty])$ to zero. Since φ' has to be of the form $P \mapsto \sum_\nu \xi_\nu(P_\nu)$ with $\xi_\nu \in \mathrm{End} E$, necessarily $\mathcal{A} \subseteq \ker \varphi'$, contradicting the assumption that φ is non-trivial on $\mathcal{A}$. (Observe that this holds true even if E is supersingular and $\mathrm{End} E$ is strictly bigger than $\mathfrak{o}_K$.) Therefore $\mathcal{A} \subseteq A$, and the lemma is proved.

It is clear how to apply this lemma to our situation: the set of all points $\psi_{m,n_1}(i_m(y,v))$ for a fixed m is certainly invariant under $E[\mathfrak{p}^{m-n_0}]$, and we may take as $\mathcal{M}_i$ the set of all such points for $m \geq n_0 + i$. We conclude that a translate $\mathcal{A} + X$, $X \in E^{qh'_K}(\bar{\mathbb{F}}_\ell)$, is contained in the subvariety $\mathcal{D}$.

To derive a contradiction, we use the fact that translations by elements of $E[\mathfrak{p}^{n_1}\bar{\mathfrak{p}}^N]$ operate on $\Phi_\mathfrak{c}(E)$ via projective automorphisms. In fact, for $y \in \mathbb{Z}/p^N\mathbb{Z}$ we have

$$\begin{aligned} A'_{j(p^{n_0}y)}\theta_{\mathfrak{c},i(x)} &= A'_{j(p^{n_0}y)}A'_{i(x)}\psi^*_{n_1,n_0}(\vartheta_{\mathfrak{c},n_0}) \\ &= e_{L_{n_1}}(i(x), j(p^{n_0}y))^{-1}A'_{i(x)}A'_{j(p^{n_0}y)}\psi^*_{n_1,n_0}(\vartheta_{\mathfrak{c},n_0}) \\ &= \zeta_{p^N}^{xy}\theta_{\mathfrak{c},i(x)} \end{aligned}$$

with the primitive p^N-th root of unity $\zeta_{p^N} = \xi_{n_1}^{-p^{n_0}}$, and consequently

$$\Phi_\mathfrak{c}(X + j(p^{n_0}y)) = \tau_y(\Phi_\mathfrak{c}(X)),$$

where τ_y is the automorphism $(v_x)_x \mapsto (\zeta_{p^N}^{xy}v_x)_x$ of $\mathbb{P}^{p^N-1}$.

From $\mathcal{A}+X \subseteq \mathcal{D}$ trivially $P + \mathcal{A}[\bar{\mathfrak{p}}^N] \subseteq \mathcal{D}$ for every $P \in \mathcal{A}+X$. Parametrize the elements Y of $\mathcal{A}[\bar{\mathfrak{p}}^N]$ by writing $Y = (j(p^{n_0}x_\nu))_\nu$ with $x \in (\mathbb{Z}/p^N\mathbb{Z})^{qh'_K}$ satisfying $\rho^{\mathrm{tr}}x = 0$ for all $\rho \in \iota_{\bar{\mathfrak{p}}}(\mathcal{R})/p^N\iota_{\bar{\mathfrak{p}}}(\mathcal{R})$. If for each ν the vector v_ν is a representative for $c_{\alpha_\nu}(\Phi_{\mathfrak{c}_\nu}(P_\nu))$ in $V = \bar{\mathbb{F}}_\ell^{qh'_K}$, the fact $P + Y \in \mathcal{D}$ translates into

$$\bigwedge_\nu \tau_{\alpha_\nu x_\nu}(v_\nu) = \bigwedge_\nu c_{\alpha_\nu}(\tau_{x_\nu}(c_{\alpha_\nu}^{-1}(v_\nu))) = 0, \tag{55}$$

using the commutation rule between the τ_y and c_α: $c_\alpha\tau_y = \tau_{\alpha y}c_\alpha$ for $y \in \mathbb{Z}/p^N\mathbb{Z}$ and $\alpha \in (\mathbb{Z}/p^N\mathbb{Z})^\times$.

We will get a contradiction by forming suitable linear combinations of these relations, which will force the vanishing of some coordinate of a v_ν, provided N was choosen large enough. Let $a : V^{\otimes qh'_K} \to \bigwedge^{qh'_K} V$ be the canonical projection, and expand the vectors v_ν as $v_\nu = \sum_{i \in \mathbb{Z}/p^N\mathbb{Z}} v_{\nu i}e_i$ in terms of the standard basis (e_i). We have then

$$v_0 \otimes \ldots \otimes v_{qh'_K-1} = \sum_{i \in (\mathbb{Z}/p^N\mathbb{Z})^{qh'_K}} \prod_\mu v_{\mu i_\mu} \bigotimes_\mu e_{i_\mu}$$

and

$$\bigotimes_\nu \tau_{\alpha_\nu x_\nu} v_\nu = \sum_{i \in (\mathbb{Z}/p^N\mathbb{Z})^{qh'_K}} \zeta_{p^N}^{\sum_\mu \alpha_\mu x_\mu i_\mu} \prod_\mu v_{\mu i_\mu} \bigotimes_\mu e_{i_\mu}.$$

Applying a Fourier transform, we get

$$\sum_x \zeta_{p^N}^{-\lambda^{\mathrm{tr}}x} \bigotimes_\nu \tau_{\alpha_\nu x_\nu} v_\nu = (\#\mathcal{A}[\bar{\mathfrak{p}}^N]) \sum_{i,\, \alpha i - \lambda \in \iota_{\bar{\mathfrak{p}}}(\mathcal{R})/p^N\iota_{\bar{\mathfrak{p}}}(\mathcal{R})} \prod_\mu v_{\mu i_\mu} \bigotimes_\mu e_{i_\mu}$$

for all $\lambda \in (\mathbb{Z}/p^N\mathbb{Z})^{qh'_K}$. From (55) we know that application of a to this equation yields zero. Since $\#\mathcal{A}[\bar{\mathfrak{p}}^N]$ is a power of p, we conclude

$$\sum_{i,\, \alpha i - \lambda \in \iota_{\bar{\mathfrak{p}}}(\mathcal{R})/p^N\iota_{\bar{\mathfrak{p}}}(\mathcal{R})} \prod_\mu v_{\mu i_\mu} \bigwedge_\mu e_{i_\mu} = 0.$$

If we can find a summation index i with $i_\nu \neq i_\mu$ $(\nu \neq \mu)$ such that no non-trivial permutation $\sigma(i)$, $\sigma \in \mathfrak{S}_{qh'_K}\backslash\{\mathrm{id}\}$, occurs in the sum for the same value of λ, a multiple of the multivector $\bigwedge_\mu e_{i_\mu}$ appears only once. Therefore

$$\prod_\mu v_{\mu i_\mu} = 0,$$

i. e. one of the coordinates $v_{\mu i_\mu}$ has to vanish. But it is easily seen that the subvariety of $\mathcal{A} + X$ cut out by the condition that one of the coordinates of the $\Phi_{\mathfrak{c}_\nu}(P_\nu)$ should vanish, has codimension one, and so choosing a point P outside of this exceptional set yields a contradiction. It remains to check the existence of an index i with the required property; this is provided by the following two lemmas, which finish the proof of the main theorem.

Lemma 4.5. *The module of relations $\mathcal{R} \subseteq \mathfrak{o}_K^{qh'_K}$ does not contain any vectors (except zero) which have less than three non-zero entries.*

Proof. It is clear that no element of $\mathcal{R}$ can have exactly one non-zero coordinate. Assume there exists a vector in $\mathcal{R}$ with two non-zero entries. This implies $(\eta/\eta')(u_{\mathfrak{c}'}/u_{\mathfrak{c}}) \in K$ where either $\mathfrak{c} \neq \mathfrak{c}'$ or $\eta \neq \eta'$. In case $\mathfrak{c} = \mathfrak{c}'$ we get immediately a contradiction. If $\mathfrak{c} \neq \mathfrak{c}'$, let γ and γ' be generators of the principal ideals $\mathfrak{c}^{h_K}$ and $\mathfrak{c}'^{h_K}$. Then $u_{\mathfrak{c}}^{h_K} = (\gamma/\bar{\gamma})\zeta$ for some $\zeta \in \mu_{p-1}$, and the same for $u_{\mathfrak{c}'}$. From our assumption, $\gamma'\gamma^{-1}/\overline{\gamma'\gamma^{-1}}$ is an element of μ_{p-1} times the h_K-th power of an element of K, and therefore the product of a unit of K and a h_K-th power. By Hilbert 90, there is some $\alpha \in K^\times$ such that $\gamma'\gamma^{-1}\alpha^{h_K}$ generates an ideal of K invariant under complex conjugation. But this means that $\mathfrak{c}'\mathfrak{c}^{-1}\alpha$ has to be invariant under complex conjugation, which contradicts the fact that $\mathfrak{c}$ and $\mathfrak{c}'$ represent different classes in $\mathrm{Cl}_K/\mathrm{Cl}_K^{\mathrm{inv}}$.

Lemma 4.6. *For N large enough, there exists an element $i \in (\mathbb{Z}/p^N\mathbb{Z})^{qh'_K}$ such that $i_\nu \neq i_\mu$ $(\nu \neq \mu)$ and*

$$\alpha(i - \sigma(i)) \notin \iota_{\bar{\mathfrak{p}}}(\mathcal{R})/p^N\iota_{\bar{\mathfrak{p}}}(\mathcal{R})$$

for every $\sigma \in \mathfrak{S}_{qh'_K}\backslash\{\mathrm{id}\}$.

Proof. We use a simple counting argument. The number of $i \in (\mathbb{Z}/p^N\mathbb{Z})^{qh'_K}$ such that $i_\nu \neq i_\mu$ $(\nu \neq \mu)$ is simply

$$p^N(p^N - 1)\cdots(p^N - qh'_K + 1),$$

i. e. grows like $p^{Nqh'_K}$. We bound the number of i, for which there exists some $\sigma \in \mathfrak{S}_{qh'_K}\backslash\{\mathrm{id}\}$ with

$$\alpha(i - \sigma(i)) \in \iota_{\bar{\mathfrak{p}}}(\mathcal{R})/p^N\iota_{\bar{\mathfrak{p}}}(\mathcal{R}),$$

by considering each σ separately. We have the linear map $f_\sigma : (\mathbb{Z}/p^N\mathbb{Z})^{qh'_K} \to (\mathbb{Z}/p^N\mathbb{Z})^{qh'_K}$ defined by $i \mapsto \alpha(i-\sigma(i))$ and want to count the number n_σ of elements in $f_\sigma^{-1}(\iota_{\bar{\mathfrak{p}}}(\mathcal{R})/p^N\iota_{\bar{\mathfrak{p}}}(\mathcal{R}))$. If b_σ is the number of orbits of σ on $\{0, \ldots, qh'_K - 1\}$, the kernel of f_σ has p^{Nb_σ} elements, and the image consists out of all $x \in (\mathbb{Z}/p^N\mathbb{Z})^{qh'_K}$ with

$$\sum_{\nu \in B} \alpha_\nu^{-1} x_\nu = 0$$

for all orbits B. Standard results on the number of solutions of a system of linear congruences imply that n_σ is equal to $p^{N(b_\sigma + r_\sigma) + c_\sigma}$ for N large enough, where c_σ is some integer independent of N, and r_σ is the rank of the $\mathbb{Z}_p$-module $\iota_{\bar{\mathfrak{p}}}(\mathcal{R}) \cap I_\sigma$,

$$I_\sigma = \{x \in \mathbb{Z}_p^{qh'_K} \mid \sum_{\nu \in B} \alpha_\nu^{-1} x_\nu = 0 \,\forall B\}.$$

Since the rank of I_σ is $qh'_K - b_\sigma$, we have $r_\sigma \le qh'_K - b_\sigma$, and equality can occur only if $I_\sigma \subseteq \iota_{\bar{\mathfrak{p}}}(\mathcal{R})$. But I_σ is generated by vectors with only two non-zero components, and none of these generators can be contained in $\iota_{\bar{\mathfrak{p}}}(\mathcal{R})$. Therefore $r_\sigma < qh'_K - b_\sigma$ for every $\sigma \neq \mathrm{id}$, the number of excluded multiindices i is bounded by a constant times $p^{N(qh'_K-1)}$, and we see that for N large enough there will be a multiindex satisfying the assertion. The lemma is proved.

Application to anticyclotomic L-functions Because of Proposition 2.11 our non-vanishing theorem has immediate consequences for the values of anticyclotomic L-functions. In fact, let ϑ be a Shintani eigenfunction in the space $\mathcal{V}_{r,C;\rho_{0\mu}}$. The character $\kappa^*\gamma^{-2}$ has weight -1, and setting $L_{\mathrm{ar}}(\xi, 1/2) = (\sqrt{D}/2\pi\Omega_0)L(\xi, 1/2) \in \bar{\mathbb{Q}}$ for anticyclotomic ξ of weight -1, we may rewrite the result of Proposition 2.11 as

$$\frac{1}{2}L_{\mathrm{ar}}(\kappa^*\gamma^{-2}, 1/2) = h_K \prod_{q|r}(1-\omega_{K/\mathbb{Q}}(q)q^{-1})b_{\mathrm{ar}}(\vartheta,\vartheta)^{-1}\left(\frac{\ell_\gamma(\vartheta)}{w_K}\right)^2.$$

Observe here that the anti-linear map $\vartheta \mapsto \vartheta^\dagger$ preserves the eigenspaces of the Shintani operators, and that therefore $\vartheta^\dagger$ is a constant multiple of ϑ for all eigenfunctions ϑ. It is now easy to deduce the following result under the restriction $\ell \not| h_K r D \prod_{q|rp}(q - \omega_{K/\mathbb{Q}}(q))$, but by a more careful consideration of the form b_{ar} (or the scalar product) we get a stronger statement.

Corollary 4.7. *Let ℓ be a prime, $p \neq \ell$ be a prime split in K, $p \not| 2h_K$, and ξ_0 a Hecke character of K of weight -1 and conductor $rD\mathfrak{d}^{-1}$ for some product $\mathfrak{d}$ of ramified prime ideals, which fulfills $\xi_0|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$. Assume in addition*

$$\ell \not| r_{\mathrm{in}} D \prod_{q,\, \omega_{K/\mathbb{Q}}(q)=-1, v_q(r)=1} (q+1),$$

where r_{in} denotes the product of the inert prime factors of r. Then there are only finitely many twists ξ of ξ_0 with finite order characters unramified outside p such that $\xi|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$, the global root number $\varepsilon(\xi, 1/2) = 1$ and

$$i_\ell i_\infty^{-1}\left(\frac{1}{2^{\nu(D)}}L_{\mathrm{ar}}(\xi, 1/2)\right) \in \mathfrak{L}.$$

It should be possible to lift the restriction on the weight by considering ℓ-adic L-functions, at least for split ℓ. That some restriction on ℓ of the type above is necessary is indicated by examples of Gillard [Gi, Section 6].

5 Eisenstein series

In this chapter we apply our results to the determination of the congruence primes of (scalar valued) Eisenstein series on $GU(3)$. We first review general results on arithmetic moduli stacks and schemes, compactifications and q-expansions for $GU(3)$ due mainly to Larsen [Lar1, Lar2]. The result on congruences is then easily deduced from our non-vanishing theorem with the help of a geometric lemma (Lemma 5.2 below). Our analysis is analogous to Ribet's in his famous paper [Ri].

5.1 Review of the arithmetic moduli problem for $GU(3)$

Shimura varieties for $GU(3)$ We briefly review the situation over the complex numbers, which is described by the theory of Shimura varieties of PEL type. See [Go] for more details.

Let K_∞ be a maximal compact subgroup of (the derived group of) $G(\mathbb{R})$ and K be an open compact subgroup of $G(\mathbb{A}_f)$. Then we may form the quotient

$$S_K(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/Z(\mathbb{R})K_\infty K \simeq G(\mathbb{Q})\backslash(\mathfrak{D} \times G(\mathbb{A}_f))/K,$$

which is in fact the set of complex points of a quasi-projective algebraic variety $S_K/\mathbb{C}$ of dimension two, the Shimura variety (surface) associated to G and K. The character $\nu = \det/\mu : G \to T$, where T is the torus $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$, gives a map

$$\nu : S_K(\mathbb{C}) \to T(\mathbb{A})/T(\mathbb{Q})\nu(Z(\mathbb{R})K)$$

from $S_K(\mathbb{C})$ to a finite group (a generalized ideal class group), whose fibers are the connected components of $S_K(\mathbb{C})$. Especially, for $K = G(L)_f$ the Shimura surface S of level one has h_K connected components. The principal congruence subgroup K_N of $G(L)_f$ is the normal subgroup obtained as the kernel of the map "reduction mod N" on $G(L)_f$, i. e. the subgroup operating trivially on $\hat{L}/N\hat{L}$. We will mainly consider the Shimura varieties $S_N = S_{K_N}$.

The Shimura varieties obtained this way allow an interpretation as moduli spaces for polarized abelian varieties with additional endomorphism and level structures. To sketch this interpretation, let $V = K^3$ and $L = \mathfrak{o}_K^3$ as in the introduction, and $\langle\cdot,\cdot\rangle$ the skew-hermitian space given by the matrix R. Given a point $\mathfrak{z} \in \mathfrak{D}$, we obtain a splitting $V_\mathbb{C} = V_+ \oplus V_-$ of $V_\mathbb{C} = V \otimes_K \mathbb{C}$ as a sum of an one-dimensional space V_+ and a two-dimensional space V_-, on which the hermitian form $(\cdot,\cdot) = -\delta^{-1}\langle\cdot,\cdot\rangle$ is positive, resp. negative definite. Defining a new complex structure j on $V_\mathbb{C}$ by $j(z)(v_+ + v_-) = \bar{z}v_+ + zv_-$, the alternating $\mathbb{R}$-linear form $E = \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}\langle\cdot,\cdot\rangle$ has, as is easily checked, the properties $E(j(i)x, j(i)y) = E(x,y)$ and $E(j(i)x,x) > 0$ for $x \in V_\mathbb{C}\backslash\{0\}$. If therefore $\Lambda \subseteq V$ is an $\mathfrak{o}_K$-lattice such that the form E is integral on Λ, we obtain a polarized abelian threefold $V_\mathbb{C}/\Lambda$ (with complex structure j and polarization given by E) together with an $\mathfrak{o}_K$-operation (given simply by multiplication of elements of $V_\mathbb{C}$ with scalars - with respect to the usual complex structure, of course), such that the Rosati involution induces the non-trivial automorphism of $\mathfrak{o}_K$.

We now associate to a pair $(\mathfrak{z}, g_f) \in \mathfrak{D} \times G(\mathbb{A}_f)$ the lattice $\Lambda = g_f L$ and change the skew-hermitian form by the factor $|\mu(g_f)|_\mathbb{A}$. Then our construction gives a polarized abelian threefold A with additional $\mathfrak{o}_K$-structure. Multiplication of g_f by an element of $G(L)_f$ on the right does not change anything, and multiplication of $(\mathfrak{z}, g_f)$ by an element of $G(\mathbb{Q})$ from the left induces an isomorphism of the two structures. Consequently, we have an interpretation (over $\mathbb{C}$) of the Shimura variety of level one as a moduli space. To get a corresponding interpretation of the level N Shimura variety, we have to add a level structure, i. e. an isomorphism $\alpha : L/NL \to A[N]$. See [Go] for proofs; instead of giving more details, we directly turn to the arithmetic case.

Arithmetic moduli spaces Since $S_K(\mathbb{C})$ parametrizes abelian varieties with additional structure over $\mathbb{C}$, it is natural to give it an arithmetic structure by showing the representability of the corresponding moduli functor over a number ring. We pose for an integer N with $(N,D) = 1$ the following moduli problem over the ring $\mathcal{R}_N = \mathfrak{o}_K[1/DN]$: let S be a scheme over $\mathcal{R}_N$. We consider quadruples (A, ι, ϕ, α), where the first datum is an abelian scheme $\pi : A \to S$ and the second one an endomorphism structure $\iota : \mathfrak{o}_K \hookrightarrow \mathrm{End}_S(A)$. The locally free sheaf $\omega_{A/S} = \pi_*\Omega^1_{A/S}$ of rank three acquires a natural structure as an $\mathcal{O}_S \otimes \mathfrak{o}_K \simeq \mathcal{O}_S \oplus \mathcal{O}_S$-module and therefore splits as $\omega_{A/S} = \omega^-_{A/S} \oplus \omega^+_{A/S}$, where $\omega^-_{A/S}$ is the identity component and $\omega^+_{A/S}$ the non-identity component. We require $\omega^-_{A/S}$ and $\omega^+_{A/S}$ to be of rank two and one, respectively. Furthermore, as third piece we have a polarization $\phi : A \to \hat{A}$ of type $(1, D, D)$ such that the Rosati involution induces the non-trivial automorphism of $\mathfrak{o}_K$. Considering for each $p|D$ the relative étale cohomology $V_p = R^1_{\text{et}}\pi_*\mathbb{Z}_p$, a smooth sheaf of rank six on S, we want the canonical alternating pairing on V_p induced by ϕ to be equivalent to $\text{Tr}\langle\cdot,\cdot\rangle$ on $L \otimes \mathbb{Z}_p$. (Here we have to choose identifications $\mathbb{Z}_p(1) \simeq \mathbb{Z}_p$, since the pairing takes at first values in $\mathbb{Z}_p(-1)$.) The last piece is a level N-structure $\alpha : L/NL \to A[N]$ required to carry $\text{Tr}\langle\cdot,\cdot\rangle$ on the left hand side into the pairing $e_N(x, \phi(y))$ on the right hand side, identifying again $\mu_N \simeq \mathbb{Z}/N\mathbb{Z}$ in some (fixed) way.

By the work of Larsen [Lar1, Lar2, Lar3] the functor of isomorphism classes of quadruples (A, ι, ϕ, α) is representable by a smooth two-dimensional moduli stack $\mathcal{M}_N/\text{Spec}(\mathcal{R}_N)$. In fact, by Serre's lemma [Mum2, p. 207] for $N \geq 3$ the quadruples have no automorphisms, and $\mathcal{M}_N$ is an algebraic space, even a scheme.

We can now define geometric automorphic forms for $GU(3)$ as global sections of certain (automorphic) vector bundles over $\mathcal{M}_N$. The stack $\mathcal{M}_N$ with its universal abelian threefold $\mathcal{A}$ carries the locally free sheaves $\omega^+_{\mathcal{A}/\mathcal{M}_N}$ and $\omega^-_{\mathcal{A}/\mathcal{M}_N}$ of rank one and two, respectively. For any $\mathcal{R}_N$-algebra R an element of $A(\rho_{\nu\mu}, L, N; R)$ is now a global section over $\mathcal{M}_N$ of the sheaf $\mathcal{V}_{\nu\mu} \otimes R$, where

$$\mathcal{V}_{\nu\mu} = (\omega^+_{\mathcal{A}/\mathcal{M}_N})^{\otimes\mu} \otimes \text{Sym}^*_\nu(\omega^-_{\mathcal{A}/\mathcal{M}_N})$$

(Sym^*_ν denotes the dual of the symmetric ν-th power). Alternatively, we can give an equivalent description in terms of "test objects", as it was done in Chapter 3 for elliptic modular forms: we interpret an automorphic form $f \in A(\rho_{\nu\mu}, L, N; R)$ as a functorial rule associating to any quadruple (A, ι, ϕ, α) (as above) defined over an R-algebra R', together with bases ω_+ and $(\omega_{-,1}, \omega_{-,2})$ of the R'-modules $\Omega^{1,+}_{A/R'}$ and $\Omega^{1,-}_{A/R'}$, an element of $\text{Sym}^*_\nu((R')^2)$. Of course, we require f to be homogeneous of degree $-\mu$ in ω_+, and to transform naturally if we change the basis ω_-. Automorphic forms with central character may be defined using the operation $A \mapsto A \otimes \mathfrak{c}$ for ideals $\mathfrak{c}$ of K as in [Ka4, p. 207, 1.0.5].

The "dictionary" to the complex analytic case is as follows ($N = 1$ for simplicity): for an (analytic) automorphic form $f \in A(\rho_{\nu\mu}, L)$ we have the functions f_{g_f} on $\mathfrak{D}$ for all $g_f \in G(\mathbb{A}_f)$ defined in Section 2.1, and a corresponding function $f'(\mathfrak{z}, g_f) = |\mu(g_f)|_{\mathbb{A}}^{(\nu-\mu)/2} f_{g_f}(\mathfrak{z})$ on $\mathfrak{D} \times G(\mathbb{A}_f)/G(L)_f$, holomorphic in the first variable, and satisfying

$$f'(\gamma(\mathfrak{z}), \gamma g_f) = j_1(\gamma, \mathfrak{z})^\mu \kappa(\gamma, \mathfrak{z})(f'(\mathfrak{z}, g_f)), \quad \gamma \in G(\mathbb{Q}).$$

To a pair $(\mathfrak{z}, g_f)$ we associated above a polarized abelian threefold A over $\mathbb{C}$ with additional structure. We can further define canonical bases of $\Omega^{1,+}_{A/\mathbb{C}}$ and $\Omega^{1,-}_{A/\mathbb{C}}$ by taking the differentials of the j-linear maps $V \to \mathbb{C}$, $v \mapsto (2\pi i)|\mu(g_f)|_{\mathbb{A}}\langle v, P_+(\mathfrak{z})\rangle$ and $v \mapsto (2\pi i)|\mu(g_f)|_{\mathbb{A}}\langle P_-(\mathfrak{z}), v\rangle$. Given a geometric automorphic form f_{geom} over $\mathbb{C}$, we may now evaluate it on the triple (A, ι, ϕ) and these bases of $\Omega^{1,+}$ and $\Omega^{1,-}$ to get a function $f'(\mathfrak{z}, g_f)$ on $\mathfrak{D} \times G(\mathbb{A}_f)/G(L)_f$, and therefore an analytic automorphic form $f \in A(\rho_{\nu\mu}, L)$.

We recall the Kodaira-Spencer isomorphism for $\mathcal{M}_N$. By [FaC, p. 81, Prop. 9.2] the canonical Kodaira-Spencer homomorphism

$$\rho_{\mathcal{A}} : \omega_{\mathcal{A}} \otimes \omega_{\hat{\mathcal{A}}} \to \Omega^1_{\mathcal{M}_N/\mathcal{R}_N}$$

gives a symmetric homomorphism

$$\rho_{\mathcal{A}} : \omega_{\mathcal{A}} \otimes \omega_{\mathcal{A}} \to \Omega^1_{\mathcal{M}_N/\mathcal{R}_N},$$

if we identify $\omega_{\mathcal{A}}$ and $\omega_{\hat{\mathcal{A}}}$ by ϕ. Furthermore, by [Lar1, Lar3] it is compatible with the endomorphism structure:

$$\rho_{\mathcal{A}}(\iota(\bar{\alpha})^*\omega_1 \otimes \omega_2) = \rho_{\mathcal{A}}(\omega_1 \otimes \iota(\alpha)^*\omega_2).$$

Writing

$$\mathrm{Sym}^2(\omega_{\mathcal{A}}) = (\omega_{\mathcal{A}}^+)^{\otimes 2} \oplus \mathrm{Sym}^2(\omega_{\mathcal{A}}^-) \oplus (\omega_{\mathcal{A}}^+ \otimes \omega_{\mathcal{A}}^-),$$

we see that $\rho_{\mathcal{A}}$ induces in fact an isomorphism

$$\rho_{\mathcal{A}} : \omega_{\mathcal{A}}^+ \otimes \omega_{\mathcal{A}}^- \to \Omega^1_{\mathcal{M}_N/\mathcal{R}_N},$$

which in turn gives an isomorphism of $\Omega^2_{\mathcal{M}_N/\mathcal{R}_N}$ and $(\omega^+_{\mathcal{A}/\mathcal{M}_N})^{\otimes 3}$ modulo tensoring with a torsion sheaf, since $\bigwedge^2 \omega^-_{\mathcal{A}/\mathcal{M}_N} \simeq \omega^+_{\mathcal{A}/\mathcal{M}_N}$ modulo torsion. For $N \geq 3$ the moduli problem is rigid and no torsion sheaf occurs.

Compactification and q-expansions We now sketch the compactification of the stack $\mathcal{M}_N$ following Larsen [Lar1, Lar2]. The construction is based on the theory of degeneration of abelian varieties [C, FaC]. We obtain the compactification by attaching a boundary consisting out of finitely many elliptic curves parametrizing split semi-abelian threefolds with $\mathfrak{o}_K$-action, that is extensions

$$0 \to T \to G \to E \to 0$$

of an elliptic curve E with complex multiplication by $\mathfrak{o}_K$ by a two-dimensional split torus T (with an action of $\mathfrak{o}_K$); the $\mathfrak{o}_K$-actions on the two pieces lift to an action on G. There are h_K^2 isomorphism classes of pairs (E, T), since there are h_K many curves E and h_K many possibilities for an $\mathfrak{o}_K$-module structure on the character group $X(T)$; they correspond to the "cusps" of the level one moduli stack. Fixing a pair (E, T) the extensions G are parametrized by the CM elliptic curve $E' = \mathrm{Hom}_{\mathfrak{o}_K}(X(T), \hat{E})$. These boundary components are defined over $\mathfrak{o}_H[1/ND]$, where H is the Hilbert class field of K. — We denote the resulting compactified moduli stack by $\bar{\mathcal{M}}_N$; it is smooth and proper over $\mathcal{R}_N$ and carries a semi-abelian scheme $\mathcal{G}$ with $\mathfrak{o}_K$-action whose restriction to $\mathcal{M}_N$ is the universal abelian threefold $\mathcal{A}$.

The sheaves $\omega^+_{\mathcal{A}/\mathcal{M}_N}$ and $\omega^-_{\mathcal{A}/\mathcal{M}_N}$ extend to $\omega^+_{\mathcal{G}/\bar{\mathcal{M}}_N}$ and $\omega^-_{\mathcal{G}/\bar{\mathcal{M}}_N}$, and we may also extend global sections, i. e. automorphic forms (by the Koecher principle). By [FaC, p. 86, Cor. 9.8] the Kodaira-Spencer isomorphism extends to an isomorphism of $\omega_{\mathcal{G}}^+ \otimes \omega_{\mathcal{G}}^-$ with $\Omega^1_{\bar{\mathcal{M}}_N/\mathcal{R}_N}[d\log C]$, the sheaf of differentials with logarithmic poles at the boundary C. Consequently, $(\omega^+_{\mathcal{G}/\bar{\mathcal{M}}_N})^{\otimes 3}$ is isomorphic to $\Omega^2_{\bar{\mathcal{M}}_N/\mathcal{R}_N}[d\log C]$ modulo a torsion sheaf.

For a pair (E, T) as above, the theory of degenerations allows us to construct quotients of the semi-abelian variety G over E', which are generically abelian, by "dividing through a period group". As the Tate curve (semi-abelian scheme) over $\mathbb{Z}[[q]]$ is the "universal degenerating elliptic curve", we get "universal degenerating abelian threefolds with $\mathfrak{o}_K$-action" over formal schemes related to E'. From the polarization and endomorphism data a certain line bundle L of degree D on E' is canonically constructed, and we obtain a semi-abelian scheme $\mathrm{Tate}_{E,T}$ with $\mathfrak{o}_K$-action over the completion of the variety $L^{\otimes -1}$ at the zero section (more precisely, $\mathrm{Tate}_{E,T}$ is a relative scheme over this formal scheme, cf. [C]). $\mathrm{Tate}_{E,T}$ is abelian away from the zero section of $L^{\otimes -1}$, where it degenerates to the universal semi-abelian variety G over E' described above. The bundle $\omega^+_{\mathcal{G}/\bar{\mathcal{M}}_N}$ getting trivial at the boundary, we can choose an invariant differential in $\Omega_G^{1,+}$ and evaluate a global section f of its μ-th power (or a scalar valued automorphic form) on our semi-abelian scheme $\mathrm{Tate}_{E,T}$ to obtain an element of the completion of the ring $\Gamma(L^{\otimes -1}, \mathcal{O}_{L^{\otimes -1}})$, which is the homogeneous coordinate ring of the elliptic curve E' and the ample line bundle L. This gives an element of the completed homogeneous coordinate ring, which is the algebraic Fourier-Jacobi expansion of f. We skip the case of vector valued automorphic forms since it will not be needed in the following.

To compare with the analytic situation (in the case $N=1$), let E be a model of the elliptic curve $\mathbb{C}/\mathfrak{b}$ over $\mathfrak{o}_H[1/D]$, and $\mathfrak{a}$ be a fractional ideal such that there exists an $\mathfrak{o}_K$-isomorphism $X(T) \simeq \delta^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}^{-1}$. Every $\lambda \in X(T)$ gives an invariant differential $\lambda^*(dq/q)$ on T, and by projection an element of $\Omega_T^{1,+}$. This gives us the possibility of constructing an element $\omega_T \in \Omega_T^{1,+}$ as $\bar{x}^{-1}$ times the differential associated to a non-zero $x \in \delta^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}^{-1}$; obviously it is independent of x. Now ω_T lifts uniquely to $\omega_G \in \Omega_G^{1,+}$, and with this choice we get q-expansion coefficients $g_{r,\mathfrak{a},\mathfrak{b}}^{\mathrm{geom}}$ corresponding to the analytic ones by $g_{r,\mathfrak{a},\mathfrak{b}}^{\mathrm{geom}} = g_{r,\mathfrak{a},\mathfrak{b}}\mathrm{N}(\mathfrak{b})^{\mu/2}$.

We now have the "q-expansion principle" as in the elliptic modular case (cf. [C, FaC]): the q-expansion homomorphism is injective, and if the q-expansion of an automorphic form f over R (R being a $\mathfrak{o}_H[1/ND]$-algebra) is already defined over a subalgebra S, the modular form f is defined over S. This means that we can identify the geometric concept with the "naive" concept of an arithmetic automorphic form defined via q-expansions.

Let us finally remark that Hecke operators $T_{\mathfrak{p}}^{\mathrm{ar}} = \mathrm{N}(\mathfrak{p})^{(\nu+\mu)/2}T_{\mathfrak{p}}$ on forms of level one are defined over the field of definition K of the level one moduli variety, and that they preserve integrality. In particular, the associated eigenvalues $\lambda_{\mathfrak{p}}^{\mathrm{ar}}$ are algebraic integers (cf. [Fi]).

5.2 Congruences between Eisenstein series and cusp forms

Recall that for ν, μ with $m = \mu - \nu - 1 \geq 5$ we have the Eisenstein series $\mathcal{E}_{\nu,\mu;\varepsilon,\chi} = \mathcal{L}_{\nu,\mu;\varepsilon,\chi}(E_{m,\omega_{K/\mathbb{Q}}})$ in $A(\rho_{\nu\mu}, L)$ associated to pairs of Hecke characters (ε, χ) of weight zero and $\nu+\mu$, respectively. Here $E_{m,\omega_{K/\mathbb{Q}}}$ is the unnormalized standard Eisenstein series of weight m and character $\omega_{K/\mathbb{Q}}$ as in Corollary 2.13. The form $\mathcal{E}_{\nu,\mu;\varepsilon,\chi}$ has central character χ, and a look at the constant terms shows that the collection of these h_K^2 many forms is linearly independent. Consequently, in the weights considered here, these forms span the space of Eisenstein series. In Corollary 2.13, we computed their (primitive) Fourier-Jacobi coefficients. Their Hecke eigenvalues were determined in Proposition 2.2. The eigenvalues $\lambda_{\mathfrak{p}}^{\mathrm{ar}}$ of $T_{\mathfrak{p}}^{\mathrm{ar}}$ on $\mathcal{E}_{\nu,\mu;\varepsilon,\chi}$ are:

$$\lambda_{\mathfrak{p}}^{\mathrm{ar}} = \begin{cases} \varepsilon(\bar{\mathfrak{p}})(p^{\nu+1} + p^{\mu-1}) + (\chi\varepsilon^{-2})_{\mathrm{ar}}(\bar{\mathfrak{p}}), & \mathfrak{p} \neq p\mathfrak{o}_K, \\ p^{2(\nu+1)} + p^{2(\mu-1)} + p^{\mu+\nu-1}, & \mathfrak{p} = p\mathfrak{o}_K. \end{cases}$$

We may normalize $E_{m,\omega_{K/\mathbb{Q}}}$ by setting

$$E_{m,\omega_{K/\mathbb{Q}}}^{\mathrm{ar}} = -\frac{(m-1)!}{2\delta}\left(\frac{2\pi i}{D}\right)^{-m} E_{m,\omega_{K/\mathbb{Q}}},$$

and the normalized form has q-expansion

$$E_{m,\omega_{K/\mathbb{Q}}}^{\mathrm{ar}} = L(1-m, \omega_{K/\mathbb{Q}})/2 + \sum_{k=1}^{\infty} \sigma_{m,\omega_{K/\mathbb{Q}}}(k)q^k.$$

In particular, $E_{m,\omega_{K/\mathbb{Q}}}^{\mathrm{ar}} \in M_m(\Gamma_0(D), \omega_{K/\mathbb{Q}}; \mathbb{Z}[1/D])$. By the integrality theorem for the lifting $\mathcal{L}$ (Theorem 3.5) the forms $\mathcal{E}_{\nu,\mu;\varepsilon,\chi}^{\mathrm{ar}} = \mathcal{L}_{\nu,\mu;\varepsilon,\chi}^{\mathrm{ar}}(E_{m,\omega_{K/\mathbb{Q}}}^{\mathrm{ar}})$ are defined over $\bar{\mathbb{Q}}$ and all their Fourier-Jacobi coefficients except for the constant term are integral away from D. Write the constant term as

$$g_{0,\mathfrak{a}}(\psi) = \psi(0,1)(\varepsilon\chi^{-1})(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{(\nu-\mu)/2}C_{\nu,\mu;\varepsilon,\chi}$$

with

$$C_{\nu,\mu;\varepsilon,\chi} = w_K \delta^{-\nu} \frac{(\mu-1)!}{4\Omega_0^{\mu+\nu}(2\pi i)^{\mu}} L(\chi\varepsilon^{-3}, (\mu-\nu)/2) L(1-m, \omega_{K/\mathbb{Q}}) \in \bar{\mathbb{Q}}.$$

The following theorem shows that (in the scalar valued case $\nu = 0$, and for $\ell \not| 6h_K$) the ℓ-adic behaviour of $C_{\nu,\mu;\varepsilon,\chi}$ determines whether or not there exists a congruence modulo ℓ between $\mathcal{E}^{\mathrm{ar}}_{\nu,\mu;\varepsilon,\chi}$ and a cusp form. Choose embeddings $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_\ell : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_\ell$, as always, and let $\mathfrak{L}$ be the prime ideal of $\bar{\mathbb{Z}}_\ell$.

Theorem 5.1. *Let $\ell \not| 6h_K$ be a split prime in K, $\mu \geq 6$ be an integer with $\mu \not\equiv 2(\ell-1)$, χ and ε Hecke characters of weight μ and zero, respectively. If now $i_\ell i_\infty^{-1}(C_{0,\mu;\varepsilon,\chi}) \in \mathfrak{L}$, we have the following:*

1. *There is a cusp form $f \in A_0(\rho_{0\mu}, L, i_\ell i_\infty^{-1}(\chi_{\mathrm{ar}}); \bar{\mathbb{Z}}_\ell)$ such that*

$$f \equiv i_\ell i_\infty^{-1}(\mathcal{E}^{\mathrm{ar}}_{0,\mu;\varepsilon,\chi}) \not\equiv 0\,(\mathfrak{L}).$$

2. *There is a Hecke eigenform $f \in A_0(\rho_{0\mu}, L, i_\ell i_\infty^{-1}(\chi_{\mathrm{ar}}); \bar{\mathbb{Z}}_\ell)$ such that*

$$\lambda^{\mathrm{ar}}_{\mathfrak{p}}(f) \equiv i_\ell i_\infty^{-1}(\lambda^{\mathrm{ar}}_{\mathfrak{p}}(\mathcal{E}_{0,\mu;\varepsilon,\chi}))\,(\mathfrak{L})$$

for all $\mathfrak{p}$ prime to ℓ.

Let us remark that we are unable to prove the existence of a Hecke eigenform congruent to the Eisenstein series, although this assertion is probably true under our conditions; we only obtain a congruence of eigenvalues. It is almost trivial that the first assertion is in fact equivalent to $i_\ell i_\infty^{-1}(C_{0,\mu;\varepsilon,\chi}) \in \mathfrak{L}$. On the other hand, if the second assertion holds true, I do not know if it is possible to conclude $i_\ell i_\infty^{-1}(C_{0,\mu;\varepsilon,\chi}) \in \mathfrak{L}$, since it is not clear whether there exists a modular form congruent to the Eisenstein series. As mentioned in the introduction, explicit computations indicate that in the case of inert ℓ (excluded here) the eigenvalues of a Hecke eigenform may be congruent to Eisenstein eigenvalues without the form itself begin congruent to an Eisenstein series.

The restriction $\mu \not\equiv 2(\ell-1)$ is necessary for rather simple reasons. Namely, for $m \equiv 1\,(\ell-1)$ we have (for a prime ideal $\mathfrak{l}$ over ℓ in a suitable number field) the congruence $E^{\mathrm{ar}}_{m,\omega_{K/\mathbb{Q}}} \equiv \vartheta_\chi\,(\mathfrak{l})$ of the $\Gamma_0(D)$ Eisenstein series to a theta series ϑ_χ associated to a Hecke character χ of K of weight $m-1$. This can be checked by considering the non-constant terms in the q-expansion and gives a congruence of the constant terms since $(\ell-1) \not| m$. Therefore, for such m we have $L(1-m,\omega_{K/\mathbb{Q}})/2 \equiv 0\,(\ell)$, but since $E^{\mathrm{ar}}_{m,\omega_{K/\mathbb{Q}}}$ is congruent to a theta series, by Theorem 3.15 the entire Eisenstein series $\mathcal{E}^{\mathrm{ar}}_{\nu,\mu;\varepsilon,\chi}$ vanishes modulo ℓ, and in general we can not expect to get a congruence in this case. The divisibility of $L(1-m,\omega_{K/\mathbb{Q}})/2$ by ℓ can be explained by the existence of a trivial zero at $s=0$ of the Kubota-Leopoldt ℓ-adic L-function associated to the character $\omega_{K/\mathbb{Q}}\omega_\ell$, ω_ℓ being the ℓ-adic Teichmüller character. The trivial zero arises from the vanishing of the Euler factor at ℓ, which appears in the defining interpolation property of $L_\ell(s,\omega_{K/\mathbb{Q}}\omega_\ell)$.

The proof of Theorem 5.1 is based on our main result Theorem 3.16 together with the following cruical lemma.

Lemma 5.2. *Let $\ell \not| 6D$ be a prime, and $\mu \geq 4$ be an integer. Then there exist modular forms $f \in A(\rho_{0,\mu}, L; \bar{\mathbb{Z}}_\ell)$ with arbitrarily prescribed constant terms in $\bar{\mathbb{Z}}_\ell$ at the h_K^2 cusps.*

If in addition $\ell \not| h_K$, for every pair (χ,ε) of a $\bar{\mathbb{Q}}_\ell$-valued unramified Hecke character χ with $\chi(\lambda\mathfrak{o}_K) = \lambda^\mu$ for all $\lambda \in K^\times$, and an ideal class character ε, there exists $f \in A(\rho_{0,\mu}, L, \chi; \bar{\mathbb{Z}}_\ell)$ with constant terms $g^{\mathrm{geom}}_{0,\mathfrak{a},\mathfrak{b}} = \chi(\mathfrak{b})(\varepsilon\chi^{-1})(\mathfrak{a})$ for all fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$.

Postponing the proof of this Lemma for a moment, let us show how it immediately implies Theorem 5.1. Since $\mathcal{E}^{\text{ar}}_{0,\mu;\varepsilon,\chi} = \mathcal{L}^{\text{ar}}_{0,\mu;\varepsilon,\chi}(E^{\text{ar}}_{\mu-1,\omega_{K/\mathbb{Q}}})$, we may apply our non-vanishing theorem. Namely, $E^{\text{ar}}_{\mu-1,\omega_{K/\mathbb{Q}}}|W'_D$ has q-expansion $\sum_{k=0}^{\infty} b_k q^k$ such that

$$b_p = -\delta^{-1}(\omega_{K/\mathbb{Q}}(p) + p^{\mu-2})$$

for all primes p. Since $\mu - 2 \not\equiv 0\,(\ell - 1)$, by Dirichlet's theorem there exists a prime p with $\omega_{K/\mathbb{Q}}(p) = -1$ and $p^{\mu-2} \not\equiv 1\,(\ell)$, which implies $b_p \not\equiv 0\,(\ell)$. Therefore, by Theorem 3.16 we have: $i_\ell i_\infty^{-1}(\mathcal{E}^{\text{ar}}_{0,\mu;\varepsilon,\chi}) \not\equiv 0\,(\mathfrak{L})$.

But by Lemma 5.2 there exists a modular form $g \in A(\rho, L, i_\ell i_\infty^{-1}(\chi_{\text{ar}}); \bar{\mathbb{Z}}_\ell)$ with constant term $g_{0,\mathfrak{a},\mathfrak{b}} = i_\ell i_\infty^{-1}(\chi_{\text{ar}}(\mathfrak{b})(\varepsilon\chi_{\text{ar}}^{-1}(\mathfrak{a})))$. Consequently,

$$f = i_\ell i_\infty^{-1}(\mathcal{E}^{\text{ar}}_{0,\mu;\varepsilon,\chi}) - C_{0,\mu;\varepsilon,\chi}\, g$$

is a cusp form in $A_0(\rho, L, i_\ell i_\infty^{-1}(\chi); \bar{\mathbb{Z}}_\ell)$, and

$$f \equiv i_\ell i_\infty^{-1}(\mathcal{E}^{\text{ar}}_{0,\mu;\varepsilon,\chi}) \not\equiv 0\,(\mathfrak{L}).$$

The second assertion follows almost immediately from the Deligne-Serre lemma. The forms f and $i_\ell i_\infty^{-1}(\mathcal{E}^{\text{ar}}_{0,\mu;\varepsilon,\chi})$ are already defined over the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_\ell$. Since the reduction $\bar{f}$ of f is a Hecke eigenform with eigenvalues $i_\ell i_\infty^{-1}(\lambda^{\text{ar}}_{\mathfrak{p}}(\mathcal{E}_{0,\mu;\varepsilon,\chi}))$ modulo $\mathfrak{L} \cap \mathcal{O}$, by Deligne-Serre [DS, p. 522, Lemme 6.11] there exists a discrete valuation ring $\mathcal{O}'$ finite over $\mathcal{O}$, and a Hecke eigenform $f' \in A_0(\rho_{0\mu}, L, i_\ell i_\infty^{-1}(\chi); \mathcal{O}')$ with the same eigenvalues as $\bar{f}$ modulo $\mathfrak{L} \cap \mathcal{O}'$, which is what we want.

It remains to prove Lemma 5.2. For this we have to use the geometric results recalled above. Let us first prove the first assertion in characteristic ℓ. Let $N \geq 3$ to rigidify the moduli problem and consider the characteristic ℓ moduli scheme $S = \mathcal{M}_N \otimes \bar{\mathbb{F}}_\ell$ with auxilary level N-structure, its compactification $\bar{S} = \bar{\mathcal{M}}_N \otimes \bar{\mathbb{F}}_\ell$, and let C be the compactification divisor (consisting out of finitely many elliptic curves). We have the exact sequence of invertible sheaves on $\bar{S}$ (denoting tensoring with $\bar{\mathbb{F}}_\ell$ by a tilde):

$$0 \longrightarrow \tilde{\mathcal{V}}_{0,\mu}(-C) \longrightarrow \tilde{\mathcal{V}}_{0,\mu} \longrightarrow \tilde{\mathcal{V}}_{0,\mu;C} \longrightarrow 0.$$

Here $\tilde{\mathcal{V}}_{0,\mu;C}$ denotes the restriction of $\tilde{\mathcal{V}}_{0,\mu}$ to the boundary of the compactification; it is isomorphic to $\mathcal{O}_C$, since $\omega^+_{\mathcal{G}}$ gets trivial at the boundary. The natural map $\Gamma(\bar{S}, \tilde{\mathcal{V}}_{0,\mu}) \to \Gamma(\bar{S}, \tilde{\mathcal{V}}_{0,\mu;C})$ associates to an element of $A(\rho, L; \bar{\mathbb{F}}_\ell)$ the vector of its values at the cusps. From the cohomology exact sequence, we see that this map will be surjective if $H^1(\bar{S}, \tilde{\mathcal{V}}(-C)) = 0$. Let $\tilde{\mathcal{V}}_{0,1} = \mathcal{O}_{\bar{S}}(D)$ with some divisor D. Then $\tilde{\mathcal{V}}(-C) = L \otimes \omega_{\bar{S}}$ with $L = \mathcal{O}_{\bar{S}}(\mu D - K_{\bar{S}} - C)$. But by the Kodaira-Spencer isomorphism $3D$ is linearly equivalent to $K_{\bar{S}} + C$, where $K_{\bar{S}}$ is the canonical divisor of the surface $\bar{S}$, and therefore $L = \mathcal{O}_{\bar{S}}((\mu - 3)D)$. By Serre duality we have to show $H^1(\bar{S}, L^{-1}) = 0$.

Now from [MB], if $\pi : \mathcal{G} \to \bar{S}$ is the "universal" semi-abelian variety above $\bar{S}$, the restriction of the Hodge bundle $\tilde{H} = \bigwedge^3 \pi_* \Omega_{\mathcal{G}/\bar{S}}$ to S is ample. But $\tilde{H} = \tilde{\mathcal{V}}_{0,2}$ up to torsion, and therefore $\tilde{\mathcal{V}}_{0,1}$ and also L are ample on S. Therefore the invertible sheaf L is numerically positive, and we conclude $H^1(\bar{S}, L^{-1}) = 0$ from the characteristic ℓ version of the Kodaira vanishing theorem obtained by Deligne-Illusie-Raynaud [DI, p. 257, Cor. 2.8], because the smooth projective surface $\bar{S}$ lifts (obviously) to characteristic zero. Having obtained surjectivity for level N modular forms, we get surjectivity for level one by taking invariants under the factor group, if its order is not divisible by ℓ. The reader may easily verify that because of $\ell \geq 5$ it is always possible to find such an N.

We still have to prove the assertion for $\bar{\mathbb{Z}}_\ell$. For this it is enough to know surjectivity of base change from $\bar{\mathbb{Z}}_\ell$ to $\bar{\mathbb{F}}_\ell$. By the same argument as above, it is enough to consider the level N case. We have to show $H^1(\bar{\mathcal{M}}_N, \mathcal{V}_{0,\mu} \otimes \bar{\mathbb{Z}}_\ell) = 0$ (cf. [Ka1, p. 85]). But for this it is enough to know $H^1(\bar{S}, \tilde{\mathcal{V}}_{0,\mu}) = 0$ which follows from the ampleness of $\tilde{\mathcal{V}}_{0,1}$ on S as above. — The second assertion follows very simply by projecting onto the eigenspaces of the center.

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Abstract

The main subject of this thesis is the study of some finer arithmetic properties of a certain theta lift from $GU(2,\mathbb{Q})$ (or equivalently $\mathrm{GL}_2(\mathbb{Q})$) to $GU(3,\mathbb{Q})$, where $GU(n)$ denotes the quasi-split unitary similitude group in n variables with respect to a fixed imaginary quadratic field extension $K/\mathbb{Q}$. The lifting considered here was first studied by Kudla [Ku1, Ku2]: it takes holomorphic elliptic modular forms of level D (the negative of the discriminant of K) and character $\omega_{K/\mathbb{Q}}$ (the quadratic Dirichlet character associated to the extension $K/\mathbb{Q}$) to holomorphic, in general vector valued, modular forms of level one on $GU(3)$.

After integrality (away from the discriminant) of the suitably normalized lifting is demonstrated, it makes sense to reduce it mod ℓ for a prime ℓ, unramified in K. The main result is a precise determination of the kernel of the reduction mod ℓ in the case where ℓ splits in K, and the lifting goes to scalar modular forms, under the weak technical restriction $\ell \nmid 2h_K$. Since Eisenstein series go to Eisenstein series under the lifting, as an application a criterion on congruences between Eisenstein series and cusp forms is obtained.

These results depend mainly on a careful study of the Fourier-Jacobi expansion of the lifting. A closed expression for the Fourier-Jacobi expansion is derived from Kudla's work, and its coefficients are then decomposed into primitive components as defined by Shintani [Shin]. The resulting formula may be of some independent interest, but it also allows to prove the cruical non-vanishing (modulo ℓ) of the lifting away from the expected kernel. As a second main ingredient a characteristic ℓ non-vanishing result on theta functions is proved, which is an analogue of a theorem of Washington [W2, Si2] on non-divisibility of Bernoulli numbers (special values of Dirichlet L-functions). In fact, by the work of Yang [Y] our result implies corresponding non-divisibility statements for special values of anticyclotomic L-functions of the field K.

Zusammenfassung

Hauptthema dieser Arbeit sind arithmetische Eigenschaften einer Thetakorrespondenz (theta lifting) zwischen den Gruppen $GU(2,\mathbb{Q})$ (oder $\mathrm{GL}_2(\mathbb{Q})$) und $GU(3,\mathbb{Q})$, wobei $GU(n)$ die quasizerfallende Gruppe unitärer Ähnlichkeiten in n Variablen bezüglich eines festen imaginärquadratischen Zahlkörpers K bezeichnet. Die hier betrachtete Korrespondenz wurde zuerst von Kudla behandelt [Ku1, Ku2]: sie ordnet holomorphen elliptischen Modulformen der Stufe $D = -\mathrm{disc}(K)$ zum Charakter $\omega_{K/\mathbb{Q}}$ (dem quadratischen Dirichlet-Charakter, der zu der Erweiterung $K/\mathbb{Q}$ gehört), holomorphe, im allgemeinen Fall vektorwertige Modulformen der Stufe eins auf $GU(3)$ zu.

Nachdem die Ganzzahligkeit dieser geeignet normalisierten Abbildung (außerhalb der Diskriminante) bewiesen ist, kann ihre Reduktion modulo einer unverzweigten Primzahl ℓ betrachtet werden. Als Hauptergebnis wird der Kern der modulo ℓ reduzierten Abbildung in dem Fall bestimmt, daß ℓ in K zerfällt, die Thetakorrespondenz skalarwertige Modulformen als Werte annimmt, und ℓ nicht $2h_K$ teilt. Da Eisenstein-Reihen Bilder von Eisenstein-Reihen unter der Thetakorrespondenz sind, wird als Anwendung ein Kriterium für Kongruenzen zwischen Eisenstein-Reihen und Spitzenformen auf $GU(3)$ erhalten.

Diese Ergebnisse basieren hauptsächlich auf einer genauen Untersuchung der Fourier-Jacobi-Entwicklung einer Form im Bild der Thetakorrespondenz. Eine geschlossene Formel für die Fourier-Jacobi-Entwicklung wird aus Kudlas Ergebnissen hergeleitet, und daran anschließend werden die Fourier-Jacobi-Koeffizienten in primitive Komponenten nach Shintani [Shin] zerlegt. Die resultierende Formel mag auch an sich von Interesse sein; sie dient hier jedenfalls zum Beweis der zentralen Nichtverschwindungsaussage (modulo ℓ). Zum Beweis des Hauptsatzes wird zusätzlich ein Nichtverschwindungssatz für Thetafunktionen in Charakteristik ℓ bewiesen, der zu einem Satz von Washington [W2, Si2] über Bernoullische Zahlen (spezielle Werte von Dirichletschen L-Reihen) analog ist. In Verbindung mit einer Arbeit von Yang [Y] impliziert dieses Ergebnis einen entsprechenden Nichtteilbarkeitssatz für spezielle Werte antizyklotomischer L-Funktionen des Körpers K.