Polynomially bounded C_0-semigroups

Tatjana Eisner

Abstract

We characterize generators of polynomially bounded C_0-semigroups in terms of an integrability condition for the second power of the resolvent on vertical lines. This generalizes results by Gomilko, Shi and Feng on bounded semigroups and by Malejki on polynomially bounded groups.*†

1 Introduction

The Hille-Yosida estimates for the norm of the resolvent $R(z, A)$ (see [E-N, Theorem II.3.8]) characterize generators A of C_0-semigroups $(T(t))_{t\geq 0}$ on Banach spaces and yield exponential estimates of the form

$$\|T(t)\| \leq Me^{t\omega}, \quad t \geq 0.$$

However, except in the case $M = 1$, one needs estimates for all powers of $R(z, A)$, a task rarely possible in concrete situations. In addition, these results do not characterize polynomial growth for $\|T(t)\|$. It is the purpose of this paper to deal with these problems.

The key to our results is an integrability condition for the square $R(a + i\cdot, A)^2$ of the resolvent along imaginary lines. This condition and the corresponding estimates (see (6) below) imply that A generates a C_0-semigroup which grows only polynomially. In the case of Hilbert spaces such estimates are necessary (Theorem 2.6).

For bounded C_0-semigroups such conditions already appeared in a paper by Gomilko [Gom] with an alternative proof given by Shi and Feng [Sh-F]. On the other side, Malejki [Mal] characterized polynomially bounded C_0-groups. This paper is a generalization of these results.

We now fix the notation to be used in this paper. We denote a linear operator on a Banach space X by $(A, D(A))$ and its spectrum by $\sigma(A)$. The following two constants related to the spectrum of A are used.

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Definition 1.1 *We call*

$$s(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\}$$

the spectral bound of A *and*

$$s_0(A) := \inf\{a \in \mathbb{R} : R(\lambda, A) \textit{ is uniformly bounded on } \{\lambda : Re\lambda \geq a\}\}$$

the pseudo spectral bound of A.

If A is the generator of a C_0-semigroup X, we denote the semigroup by $(T(t))_{t\geq 0}$. For more detailed information about theory of C_0-semigroups we refer to [E-N].

2 Results

We first state a simple property following from the resolvent identity.

Lemma 2.1 *Let* A *be densely defined on a Banach space* X *and* $s_0(A) < \infty$. *Then for every* $a > s_0(A)$ *and every* $x \in X$

$$R(z, A)x \to 0, \quad |z| \to \infty, Rez \geq a.$$

Proofs for the results in this section will be given in Section 3.

The following property is the basis of our approach:

$$< R(a + i\cdot, A)^2 x, y > \in L^1(\mathbb{R}) \text{ for all } x \in X,\ y \in X^*, \tag{1}$$

where $a > s_0(A)$. Indeed, this property allows us to construct the inverse Laplace transform of the resolvent of the operator A which actually is a semigroup. Note that this semigroup need not to be strongly continuous.

Lemma 2.2 *Let* A *be a densely defined linear operator on a Banach space* X *satisfying* $s_0(A) < \infty$. *Assume that for some* $a > s_0(A)$ *the condition (1) holds. Then this condition also holds for all* $a > s_0(A)$ *and the bounded linear operators defined by* $T(0) = Id$ *and*

$$\begin{aligned} T(t)x &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(a+is)t} R(a+is, A)x ds && (2)\\ &= \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{(a+is)t} R(a+is, A)^2 x ds && (3)\end{aligned}$$

are independent of $a > s_0(A)$. *In addition, the family* $(T(t))_{t\geq 0}$ *is a semigroup which is strongly continuous on* $(0, \infty)$ *and satisfies*

$$\lim_{t\to 0+} T(t)x = x \textit{ for all } x \in D(A^2). \tag{4}$$

Finally, we have

$$R(z, A)x = \int_0^{\infty} e^{-zt} T(t)x ds \textit{ for all } x \in D(A), \quad Rez > s_0(A). \tag{5}$$

Note that condition (1) holds for generators of analytic semigroups by the inequality $||R(z,A)|| \leq \frac{M}{|z-\omega|}$. By Plancherel's theorem and the Schwarz inequality this condition always holds for generators on a Hilbert space.

As a first application of Lemma 2.2 we will give an alternative proof of a result of Kaashoek and Verduyn Lunel ([Kaa-Lu]) generalizing Gearhart's stability theorem (see [E-N, Theorem V.1.11]) to semigroups on Banach spaces, provided condition (1) holds. Here and later we denote by $\omega_0(T)$ the growth bound of the semigroup $(T(t))_{t\geq 0}$ (see [E-N, Definition I.5.6]).

Theorem 2.3 *([Kaa-Lu]) Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. If condition* (1) *for the resolvent of A holds for some $a > s_0(A)$, then $s_0(A) = \omega_0(T)$.*

Our main result is the following generalization of Malejki's characterization of generators of polynomially bounded C_0-groups (see [Mal]).

Theorem 2.4 (generation of polynomially bounded semigroups) *Let X be a Banach space and A be a densely defined operator on X with $s(A) \leq 0$ and $d \geq 0$. If the condition*

$$\int_{-\infty}^{\infty} |\left\langle R(a+is,A)^2 x, y\right\rangle| ds \leq \frac{M}{a}(1+a^{-d})||x||||y||,\ \forall x \in X,\ \forall y \in X^* \qquad (6)$$

holds for all $a > 0$, then A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ which does not grow faster than t^d, i.e.,

$$||T(t)|| \leq K(1+t^d) \qquad (7)$$

for some constant K and all $t > 0$.

Note that in the case given in Theorem 2.4 the semigroup is exponentially stable if and only if the resolvent of A exists and is uniformly bounded on $i\mathbb{R}$. This follows from Theorem 2.3 and the proof of Theorem 2.4.

It can be seen from the proof of Theorem 2.4 that condition (6) for large a is responsible for the strong continuity of the semigroup, while this condition for small a is responsible for its polynomial growth. Therefore the following corollary from Theorem 2.4 holds.

Proposition 2.5 *Let A be the generator of a C_0-semigroup on a Banach space X with $s_0(A) \leq 0$. If condition* (6) *holds for all $a \in (0, a_0)$ for some $a_0 > 0$, then the semigroup satisfies growth estimate* (7)*.*

As in the paper of Malejki the converse implication in Theorem 2.4 holds on Hilbert spaces.

Theorem 2.6 *Let A generate a C_0-semigroup on a Hilbert space X satisfying growth estimate* (7)*. Then estimate* (6) *for the resolvent of A holds with $d_1 := 2d$ for all $a > 0$.*

Note that for $d = 0$ in Theorem 2.4 and Theorem 2.6 we obtain the generation theorem of Gomilko for bounded semigroups (see [Gom]). The method used in the proof of Lemma 2.2 and Theorem 2.4 is based on the alternative proof of Gomilko's result given by Shi and Feng (see [Sh-F]) and on the papers of Kaiser and Weis [Kai-We] and Batty [Ba].

3 Proofs

Proof of Lemma 2.1. Let $a > s_0(A)$. Then there exists a constant $M > 0$ such that $\|R(z, A)\| \leq M$ for all $z \in \mathbb{C}$ with $Rez \geq a$. Let now $x \in D(A)$ and z with $Rez \geq a$. Then

$$\|R(z, A)x\| = \frac{1}{|z|}\|x + R(z, A)Ax\| \leq \frac{1}{|z|}(\|x\| + M\|Ax\|),$$

and therefore we have

$$R(z, A)x \to 0, \quad |z| \to \infty, \ Rez \geq a$$

for all $x \in D(A)$. Since $D(A)$ is dense in X and the resolvent of A is uniformly bounded on $\{z : \ Rez \geq a\}$, this is true for all $x \in X$. ∎

Proof of Lemma 2.2. Let us first prove that condition (1) holds for all $a > s_0(A)$ if it is true for some $a_0 > s_0(A)$. Let $x \in X$, $y \in X^*$ and $a > s_0(A)$. Then the resolvent identity implies

$$R(a + is, A)^2 x = [Id + (a_0 - a)R(a + is, A)]^2 R(a_0 + is, A)^2 x$$

and therefore

$$\| < R(a + i\cdot, A)^2 x, y > \|_1 \leq [1 + L|a_0 - a|]^2 \ \| < R(a_0 + i\cdot, A)^2 x, y > \|_1,$$

where $L = L(a) := \sup_{s\in\mathbb{R}} \|R(a + is, A)\|$ (which is finite because $a > s_0(A)$). So (1) is true for all $a > s_0(A)$. Let us define now $T(0) := Id$ and

$$T(t)x := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(a+is)t} R(a + is, A)xds \tag{8}$$

(the inverse Laplace transform of the resolvent) for all $x \in X$, $t > 0$ and some $a > 0$. We prove that the integral on the right hand side of (8) converges for all $a > 0$ and all $x \in X$ and does not depend on $a > 0$. Let us fix $t > 0$. Since $\frac{d}{dz}(R(z, A)) = -R(z, A)^2$, we have for any $r > 0$

$$\begin{aligned} it \int_{-r}^{r} e^{(a+is)t} R(a + is, A)xds \ &= \ e^{(a+ir)t} R(a + ir, A)x - e^{(a-ir)t} R(a - ir, A)x \\ &+ \ i \int_{-r}^{r} e^{(a+is)t} R(a + is, A)^2 xds, \end{aligned}$$

and by Lemma 2.1 the first two summands converge to zero if $r \to +\infty$. Therefore

$$t \int_{-\infty}^{\infty} e^{(a+is)t} R(a+is,A)xds = \int_{-\infty}^{\infty} e^{(a+is)t} R(a+is,A)^2 xds, \tag{9}$$

and by condition (1) the integral on the right hand side converges. Indeed, for all $r, R \in \mathbb{R}$ and all $x \in X$ we have, by the uniform boundedness principle, that

$$\begin{aligned} \| \int_r^R e^{ist} R(a+is,A)^2 xds \| &= \sup_{y \in B^*} \int_r^R < e^{ist} R(a+is,A)^2 x, y > ds \\ &\leq \sup_{y \in B^*} \| < R(a+i\cdot,A)^2 x, y > \|_1 \leq L_1(a) \|x\|, \end{aligned}$$

where $B^* = \{y \in X^* : \ \|y\| = 1\}$, holds for some constant $L_1(a)$ independend on x. This implies the convergence of the integral on the right hand side of (9).

Therefore the integral on the right hand side of (8) converges and

$$T(t)x = \frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{(a+is)t} R(a+is,A)^2 xds \tag{10}$$

for every $x \in X$ and $t > 0$. We show next that $T(t)$ does not depend on $a > 0$. Indeed, by Cauchy's theorem we obtain for all $a, b > s_0(A)$

$$\begin{aligned} &\int_{-r}^{r} e^{(a+is)t} R(a+is,A)^2 xds \quad - \quad \int_{-r}^{r} e^{(b+is)t} R(b+is,A)^2 xds \\ = - &\int_a^b e^{\tau+ir} R(\tau+ir,A)^2 xd\tau \quad + \quad \int_a^b e^{\tau-ir} R(\tau-ir,A)^2 xd\tau. \end{aligned}$$

By Lemma 2.1 the right hand side converges to zero if $r \to +\infty$. So we have proved that $T(t)$ does not depend on $a > 0$ and formula (10) holds.
Again by (10) we obtain

$$| < T(t)x, y > | \leq \frac{e^{at}}{2\pi t} \| < R(a+i\cdot,A)^2 x, y > \|_1 \tag{11}$$

and by the uniform boundedness principle, each $T(t)$ is a bounded linear operator satisfying

$$\|T(t)\| \leq \frac{Ce^{at}}{t}, \quad t > 0, \tag{12}$$

for some constant C depending on $a > s_0(A)$.

By [Kai-We, Lemma 4.2] we obtain that $T(t+s)x = T(t)T(s)x$ for all $x \in D(A^4)$. Since $D(A^4)$ is dense, the semigroup law holds for all $x \in X$. Let us prove that (5) holds for all $x \in D(A)$. Take $x \in D(A)$, z with $Re(z) > s_0(A)$ and $a \in (s_0(A), Rez)$. Then by Fubini's theorem and Cauchy's integral theorem for bounded functions on a right half-plane we have

$$\begin{aligned}
&\int_0^\infty e^{-zt}T(t)xdt \\
&= \frac{1}{2\pi}\int_0^\infty e^{-zt}\int_{-\infty}^\infty e^{(a+is)t}R(a+is,A)xdsdt \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \left\{\int_0^\infty e^{(a+is-z)t}dt\right\}\frac{R(a+is,A)Ax+x}{a+is}ds \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{R(a+is,A)Ax+x}{(a+is)(z-a-is)}ds = \frac{R(z,A)Ax+x}{z} = R(z,A)x.
\end{aligned}$$

So equality (5) is proved.

Finally we show strong continuity of our semigroup on $(0,\infty)$. Since by (12) the semigroup is uniformly bounded on all compact intervals from $(0,\infty)$, it is enough to show that (4) holds for all $x \in D(A^2)$. Take such $x \in D(A^2)$ and any $a > 0$. By [Kai-We, Lemma 4.1 and 4.2] we have

$$T(t)x - x = \frac{1}{2\pi}\int_{-\infty}^\infty e^{(a+is)t}\frac{R(a+is,A)Ax}{a+is}ds$$

and $||R(a+is,A)Ax|| \leq \frac{c||A^2x||}{1+|a+is|}$ for some constant c. Therefore, by Lebesgue's theorem,

$$\lim_{t\to 0+}(T(t)x - x) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{R(a+is,A)Ax}{a+is}ds \tag{13}$$

and the integral on the right hand side converges absolutely.

We now show that

$$\int_{-\infty}^\infty \frac{R(a+is,A)Ax}{a+is}ds = 0. \tag{14}$$

By Cauchy's theorem and Lemma 2.1 we have

$$\begin{aligned}
\left\|\int_{-r}^{r}\frac{R(a+is,A)Ax}{a+is}ds\right\| &= \left\|\int_{-\pi/2}^{\pi/2}\frac{ire^{i\phi}}{a+re^{i\phi}}R(a+re^{i\phi},A)Axd\phi\right\| \\
&\leq \int_{-\pi/2}^{\pi/2}\|R(a+re^{i\phi},A)Ax\|d\phi \to 0, \quad r\to\infty.
\end{aligned}$$

So equality (14) is proved and (13) implies (4) and the strong continuity of our semigroup on $(0,\infty)$. ■

Proof of Theorem 2.3. It is sufficient to prove that $s_0(A) < 0$ implies $\omega_0(T) < 0$. Indeed, if $s_0(A) < \omega_0(T)$, then we can rescale our semigroup so that

$s_0(A) < 0 < \omega_0(T)$. Note that property (1) is still true for the rescaled semigroup. So assume $s_0(A) < 0$. By Lemma 2.2 our semigroup can be represented as

$$T(t)x = \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{(a+is)t}R(a+is,A)^2 x ds$$

for all $x \in X$, $t > 0$ and $a > s_0(A)$. Taking here $a = 0$ we obtain

$$| < T(t)x, y > | \leq \frac{1}{2\pi t}\int_{-\infty}^{\infty} | < R(is,A)^2 x, y > |ds$$

for all $x \in X$ and $y \in X^*$. By the principle of uniformly boundedness there exists a constant K independent on x and y such that

$$\|T(t)\| \leq \frac{K}{t}$$

holds for all $t > 0$. Therefore we have

$$e^{t\omega_0(T)} = r(T(t)) \leq \|T(t)\| \to 0, \quad t \to \infty,$$

and $\omega_0(T) < 0$ holds. ■

Proof of Theorem 2.4. Step 1. Let us first prove that by condition (6) we have $s_0(A) \leq 0$. Since $\frac{d}{dz}R(z,A) = -R^2(z,A)$ we have for all $a > 0$, $x \in X$ and $y \in X^*$,

$$< R(a+is,A)x, y > = < R(a,A)x, y > - i\int_0^s < R(a+i\tau,A)^2 x, y > d\tau. \quad (15)$$

By the absolute convergence of the integral on the right hand side we obtain that $< R(a+is,A)x, y > \to 0$ if $s \to \infty$. From (15) and condition (6) it follows that

$$\|R(a+is,A)\| \leq \frac{M}{a}(1+a^{-d}),$$

hence $s_0(A) \leq 0$ holds.

Step 2. By Lemma 2.2 the operators given by (2) form a semigroup. Let us estimate the norm of $T(t)$. From representation (3) and condition (6) we have

$$\begin{aligned} | < T(t)x, y > | &\leq \frac{e^{at}}{2\pi t}\int_{-\infty}^{\infty} | < R(a+is,A)^2 x, y > |ds \\ &\leq \frac{Me^{at}}{2\pi ta}(1+a^{-d})\|x\|\|y\|. \end{aligned}$$

Taking $a := t^{-1}$ we obtain for $C := \frac{Me}{2\pi}$ the desired estimate

$$\|T(t)\| \leq C(1+t^d). \tag{16}$$

The strong continuity of $(T(t))_{t\geq 0}$ follows from estimate (16) and Lemma 2.2. ■

4 Remarks

Note that although the representation (2) of the semigroup as the inverse Laplace transform always holds in UMD-spaces (see [ABHN, Theorem 3.12.2]), property (1) studied in this paper is not automatically true for generators on UMD-spaces. Indeed, by Theorem 2.3 property (1) implies Gearhart's stability theorem for the semigroup. On the other hand side the example considered in [We] shows that Gearhart's stability theorem does not hold in L_p-spaces for all $p \in (1,\infty)$, $p \neq 2$.

A direct application of the characterization of generators of polynomially bounded semigroups given in Theorem 2.4 is made in the paper by J. Goldstein and M. Wacker (see [Go-Wa]). However, condition (6) in Theorem 2.4 is not so easy to check. In addition, the converse implication is true for Hilbert spaces but not for Banach spaces. Even condition (1) is not always satisfied which follows by Theorem 2.3. So it remains an open problem to find criteria for the property that a densely defined operator on a Banach space satisfying $s(A) \leq 0$ is the generator of a polynomially bounded semigroup .

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Tatjana Eisner
Arbeitsbereich Funktionalanalysis
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D-72076 Tübingen
Germany
e-mail: talo@fa.uni-tuebingen.de