Banach spaces with the Daugavet property

(joint papers with Vladimir Kadets, Nigel Kalton, Miguel Martín, Javier Merí and others)

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The Daugavet equation

Proposition (I. Daugavet 1963)

Each compact linear operator $T\colon C[0,1] \to C[0,1]$ satisfies

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

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$C[0,1]$, $L_1[0,1]$, $L_\infty[0,1]$, $A(\mathbb{D})$, H^∞, $\mathrm{Lip}(K)$ ($K \subset \mathbb{R}^d$ convex), type II von Neumann algebras and their preduals, ...

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More generally: $C(K)$ for a compact Hausdorff space K without isolated points; $L_1(\mu)$ and $L_\infty(\mu)$ for a non-atomic measure μ.

Counterexamples

c_0, ℓ_1, ℓ_∞, $L_p(\mu)$ for $1 < p < \infty$, $\mathrm{Lip}(K)$ ($K \subset \mathbb{R}^d$ compact and not convex), type I von Neumann algebras and their preduals, ...

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Definition

A Banach space X has the Daugavet property if

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for all operators $T\colon X \to X$ of the form $T(x) = x_0^*(x)\,x_0$.

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- X has the Daugavet property.
- For all $\|x_0\| = 1$, $\varepsilon > 0$ and all slices S of the unit ball B_X there exists some $z \in S$ such that
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- For all $\|x_0\| = 1$, $\varepsilon > 0$ and all slices S of the unit ball B_X there exists some $z \in S$ such that
$$\|z - x_0\| \geq 2 - \varepsilon.$$
- For all $\|x_0\| = 1$ and $\varepsilon > 0$, the convex hull of $\{z \in B_X \colon \|z - x_0\| \geq 2 - \varepsilon\}$ is dense in B_X.

Weak compactness

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Proposition

If X has the Daugavet property, then $\|\mathrm{Id} + T\| = 1 + \|T\|$ for all weakly compact operators T.

T is weakly compact if the closure of $T(B_X)$ is weakly compact, i.e., compact for the weak topology.

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If X has the Daugavet property, then $\|\mathrm{Id} + T\| = 1 + \|T\|$ for all strong Radon-Nikodym operators T.

T is a strong Radon-Nikodym operator if the closure of $T(B_X)$ has the Radon-Nikodym property.

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Theorem

If X has the Daugavet property, then $\|\mathrm{Id} + T\| = 1 + \|T\|$ for all ℓ_1-singular operators T.

T is called ℓ_1-singular if *no* restriction of T to any copy of ℓ_1 is an (into-) isomorphism, i.e., bounded below.

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A separable Banach space with the Daugavet property fails to have an unconditional basis.
Even more, a separable Banach space with the Daugavet property does not even embed into a space with an unconditional basis.

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Theorem (here used as a Definition)

Let X be a Banach space with the Daugavet property. A closed subspace Y is called rich if every closed subspace between Y and X has the Daugavet property.

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Theorem (Kadets, Popov)

If a separable Banach space contains a complemented copy of $C[0,1]$, then it is isomorphic to a rich subspace of $C[0,1]$ and can hence be renormed to have the Daugavet property.

Theorem

If X has the Daugavet property, then

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

for all weakly compact operators $T\colon X \to X$; in fact this is so for all “strong Radon-Nikodym operators” (i.e., $\overline{T(B_X)}$ has the RNP).

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Common roof:

- narrow operators;
- SCD operators.

SCD sets

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Definition (Avilés, Kadets, Martín, Merí, Shepelska 2010)

A bounded subset A of a Banach space is called slicely countably determined if there is a sequence of slices S_n of A with the following property: If $B \subset A$ intersects all the S_n, then $A \subset \overline{\operatorname{conv}} B$.

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Theorem

If X has the Daugavet property and $T\colon X \to X$ is such that $T(B_X)$ is an SCD-set, then

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Note that a continuous linear operator $T\colon X \to Y$ is

- a bounded map on the closed unit ball, and the norm is the sup norm;
- a Lipschitz map, and the norm is the Lipschitz norm.

Lipschitz maps

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Lip(X) stands for the Banach space of all Lipschitz maps from X to X that map 0 to 0, endowed with the Lipschitz norm, i.e.,

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is an SCD-set (e.g., relatively weakly compact), then

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Elements of the proof: Lipschitz slices

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Introduce Lipschitz slices for Lipschitz functionals $f\colon X \to \mathbb{R}$:

$$\Sigma(f,\varepsilon) = \left\{ \frac{x-y}{\|x-y\|} \colon \frac{f(x)-f(y)}{\|x-y\|} > (1-\varepsilon)\|f\|_{\mathrm{Lip}} \right\}$$

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Key lemma

If $A \subset S_X$ and $A \cap \Sigma(f,\varepsilon) = \emptyset$, then $\overline{\mathrm{conv}}(A) \cap \Sigma(f,\varepsilon) = \emptyset$.

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If X has the Daugavet property, then for all $\|x_0\| = 1$, $\varepsilon > 0$ and all Lip-slices Σ of the unit sphere S_X there exists some $z \in \Sigma$ such that

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The numerical range of an operator

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Definition (O. Toeplitz 1918; G. Lumer / F.L. Bauer 1961/62)

- Let X be a Hilbert space and $T\colon X \to X$ a linear operator. The numerical range of T is

$$V(T) = \{\langle Tx, x\rangle \colon \|x\| = 1\}.$$

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- Let X be a Hilbert space and $T\colon X \to X$ a linear operator. The numerical range of T is

$$V(T) = \{\langle Tx, x\rangle\colon \|x\| = 1\}.$$

- Let X be a Banach space and $T\colon X \to X$ a linear operator. The numerical range of T is

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The number $v(T) = \sup\{|\lambda|: \lambda \in V(T)\}$, that is

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Note:
Daugavet property $\not\Rightarrow$ $n(X) = 1$ (e.g. $X = C([0,1], \mathbb{R}^2)$);
$n(X) = 1 \not\Rightarrow$ Daugavet property (e.g. $X = c_0$).

Lush Banach spaces

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A (real) Banach space X is called lush if for all $\|x_0\| = 1$, $\|y_0\| = 1$ and $\varepsilon > 0$ there exists an ε-slice S containing x_0 such that $\operatorname{dist}(y_0, \operatorname{conv}(S \cup -S)) \le \varepsilon$.

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Proposition

Every lush space has numerical index 1.

Theorem

If X is lush, then the “Lipschitz numerical index” is 1, i.e.,

$$\max_{\pm} \|\mathrm{Id} \pm T\|_{\mathrm{Lip}} = 1 + \|T\|_{\mathrm{Lip}}$$

for all Lipschitz maps $T\colon X \to X$.

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- $Y^\perp \cong (X/Y)^*$ and $X/Y \cong \{(x,y,z) \in \ell_\infty^3 \colon x + y + z = 0\}$.
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Theorem

There is a real Banach space with $n(X) = 1$, but $n(X^*) = 0$.

Epilogue

"I have noticed," said Mr. K., "that we put many people off our teachings because we have an answer to everything. Could we not, in the interests of propaganda, draw up a list of the questions that appear to us completely unsolved?"

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All contributions are welcome!