Topology

SS 2026

Homepage:

www.math.uni-leipzig.de/~eisner/Topology-SS2026.html

$\tau o\pi o\varsigma$ = place
$\lambda o\gamma o\varsigma$ = word
Born: 1914, Felix Hausdorff („top. space")
modern definition: Kazimierz Kuratowski / Heinrich Tietze 1922
Before: Euler, Cantor, ...

I introduction

I.1 Basic notions

$\boxed{\text{Def.}}$ Let X be a set and τ a family of subsets of X. The pair (X,τ) is called top. space and τ topology on X if:

top. Raum Topologie

(a) $\emptyset \in \tau$, $X \in \tau$

(b) arbitrary unions of sets from τ belong to τ:

$$U_\alpha \in \tau,\ \alpha \in I \Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \tau$$

(c) finite intersections –"–

$$U_1, \dots, U_n \in \tau \Rightarrow \bigcap_{k=1}^{n} U_k \in \tau$$

Thereby $A \subset X$ is called

- offen: open if $A \in \tau$
- abgeschlossen: closed if $X \setminus A \in \tau$
- Umgebung: neighbourhood of $x \in X$ if $x \in A$ and A is open (in some books: $\exists U$ open. $x \in U \subset A$)

Rem. (b) $\Leftrightarrow$ arbitrary intersections of closed sets are closed
(c) $\Leftrightarrow$ finite unions –"–

$\boxed{\text{Ex}}$ 1) $\forall$ set X has (at least) 2 topologies:

$\{\emptyset, X\}$ – indiscrete (or trivial) topology indiskrete

2^X – discrete top. ($\forall$ set is open) diskrete Top.

2) $X = \{\cdot, *, \circ\}$

$\tau := \{\emptyset, \{*\}, \{\cdot, *\}, \{\cdot, *, \circ\}\}$

Exerc. Find all top. on X

3) $X = \mathbb{R}$, $\tau := \{(-\infty, a), a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$

or –″– $\cup \{(-\infty, a], a \in \mathbb{R}\}$ exerc.

4) let X be infinite.

$\tau := \{A \subset X : X \setminus A \text{ finite}\} \cup \{\emptyset, X\}$

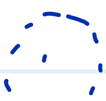
is a top. (exerc.), called cofinite top. cofinite Top.

5) $\mathbb{R}$ with $\tau :=$ arb. unions of open intervals (a, b)

– the natural top. on $\mathbb{R}$ natürliche Top

Analogously: let (X, d) be a metric space. Then

$\tau :=$ arb. union of open balls of the form

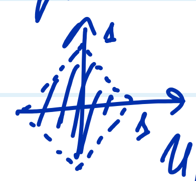


$U_r(a) := \{x \in X : d(x, a) < r\}$

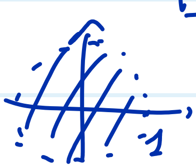
is a top (exerc.) One says: τ is induced by the metric d induziert

Rem. Different metrics can induce the same top.:

$\boxed{\text{Ex}}$ $\mathbb{R}^2$, $d_1(x, y) := |x_1 - y_1| + |x_2 - y_2|$

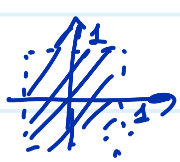


$U_1(0)$

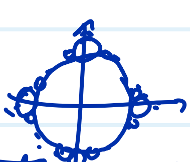


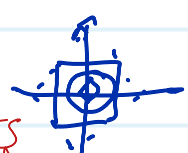
$d_2(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$



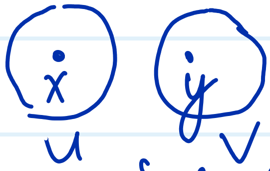
induce the same (Euclidian) topology since unions of open balls are the same.





$\boxed{\text{Def}}$ A top. space (TS) (X, τ) is called metrizable if τ is induced by a metric on X. metrisierbar

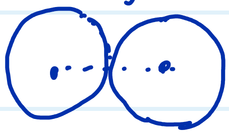
Rem. (X,τ) metrisable $\Rightarrow$ (X,τ) is a Hausdorff space,



i.e., $\forall x, y \in X$, $x \neq y$ $\exists U, V$ open with $x \in U$, $y \in V$, $U \cap V = \emptyset$

(one can separate points by open sets)

Indeed, $d := d(x,y)$ and def. $U := U_{d/2}(x)$, $V := U_{d/2}(y)$



in part., $\mathbb{R}$ with $\tau := \{(-\infty, a), a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is not metrisable

Def Let (X,τ) be TS. A family $\mathcal{B} \subset \tau$ is called

Basis der Top. • basis of the top. if

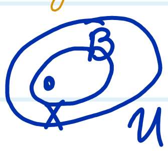
$\forall U \in \tau$ $\forall x \in U$ $\exists B \in \mathcal{B}$ with $x \in B \subset U$

(exerc. $\Leftrightarrow$ $\forall U \in \tau$ is a union of sets from $\mathcal{B}$)

Subbasis der Top. • subbasis of the top. if the set of all finite intersections of the sets from $\mathcal{B}$ is a basis of the top.

Umgebungsbasis • neighbourhood basis of $x \in X$ if

$\forall$ neighb. U of x $\exists$ neighb $B \in \mathcal{B}$ of x with $B \subset U$



Rem. 1) τ is uniquely determined by its basis $\mathcal{B}$

(U open $\Leftrightarrow$ $U = \bigcup$ sets from $\mathcal{B}$)

Analog.: subbasis S

(U open $\Leftrightarrow$ $U = \bigcup$ (finite inters. of sets of S))
arbitr.

2) Basis, subbasis, neighb. basis are not unique (see the example of $\mathbb{R}^2$ above)

Ex 1) $\mathbb{R}$, $\tau :=$ natural topology

$\mathcal{B} := \{(a,b), a, b \in \mathbb{R}\}$ – basis

$S := \{(-\infty, a), (b, \infty), a, b \in \mathbb{R}\}$ – subbasis

But also – " – $a, b \in \mathbb{Q}$ exerc.

2) X arb., $\tau =$ discrete (2^X)

$\mathcal{B} := \{\{x\}, x \in X\}$ basis of τ

Def Let (X,τ) be TS. One says: (X,τ) satisfies

• the first axiom of countability if 1. Abzählbarkeitsaxiom

Q

$\forall x \in X$ $\exists$ countable neighb. basis of x.

- the **second axiom of countability** if $\exists$ countable basis of the top. 2. –"–

Rem. 2^{nd} axiom of count. $\Rightarrow$ 1^{st} –"– exerc.
$\not\Leftarrow$
(hint: for $\not\Leftarrow$: $\mathbb{R}$ with discrete top.)

Rem. (X,d) metric $\Rightarrow$ satisf 1^{st} axiom of count.
(take $(U_{1/n}(x))$ as a neighb. basis for x)

$\dot{x}$

Exer.: (X,d) metric satisf. 2^{nd} axiom of count.
$\Leftrightarrow$ (X,d) is **separable**, i.e., $\exists$ countable dense subset $D \subset X$. separabel

Here, $D \subset X$ is **dense** if $\forall$ open ball U $\exists x \in D$ with $x \in U$. dicht

$\boxed{\text{Ex}}$ $\mathbb{R}$ with natural top., induced by
$d(x,y) := |x-y|$
sep.: $\mathbb{Q} \subset \mathbb{R}$ dense, count.
$\{U_r(a),\ r,a \in \mathbb{Q}\}$ count. basis of τ

Further def.

$\boxed{\text{Def}}$ Let (X,τ) be TS and $A \subset X$. A point $x \in X$ is called

innerer Punkt

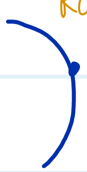
- **inner point** of A if $\exists U$ open. $x \in U \subset A$

A

The set of all inner points: $\mathring{A}$

Randpunkt

- **boundary point** of A if $\forall U$ open with $x \in U$ $U \cap A \neq \emptyset$, $U \cap A^c \neq \emptyset$,



where $A^c := X \setminus A$

Notation: $x \in \partial A$

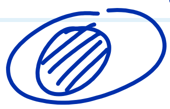
$A \subset X$ is called

- **dense** (in X) if $\forall U$ open $U \cap A \neq \emptyset$

nirgends dicht

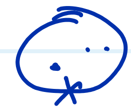
- **nowhere dense** if $\forall U$ open $\exists V$ open: $V \subset U$ and $V \cap A = \emptyset$

U



($\exists$ „holes" everywhere)

The set $\overline{A} := \{x \in X : \forall \underbrace{U \text{ neighb of } x}_{U \in \mathcal{U}(x)} \ U \cap A \neq \emptyset\}$

Abschluss is called the closure (or closed hull) of A. 

Prop. Let (X,τ) be TS, $A \subset X$.

(a) $\overline{A}$ is closed and $\boxed{\overline{A} = \bigcap_{\substack{F \text{ closed} \\ A \subset F}} F}$,

i.e., $\overline{A}$ is the smallest closed set containing A.

In part., A closed $\Leftrightarrow A = \overline{A}$

(b) $\mathring{A}$ is open and $\boxed{\mathring{A} = \bigcup_{\substack{U \text{ open} \\ U \subset A}} U}$, i.e.,

$\mathring{A}$ is the largest open set contained in A.

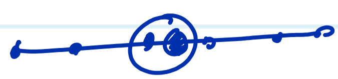
In part., A open $\Leftrightarrow A = \mathring{A}$

(c) $\partial A = \overline{A} \setminus \mathring{A}$.

Proof (exer.)

Rem
$$\left.\begin{array}{l} A \text{ dense} \Leftrightarrow \overline{A} = X. \\ A \text{ nowhere dense} \Leftrightarrow (\overline{A})^{\circ} = \emptyset \end{array}\right\} \text{(exer.)}$$

Ex 1) $X = \mathbb{R}$, $\mathbb{Q}$ dense,
$\mathbb{Z}$ nowhere dense



2) C Cantor set: $C \subset \mathbb{R} = X$ (always natural top. if nothing said)

$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

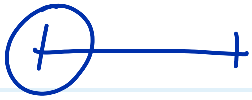
$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$

(delete the middle third)

$C := \bigcap_{m=1}^{\infty} C_m$ - Cantor ternary set

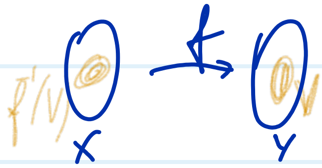
Then:

- C closed
- C is uncount.
  ($x \in C \Leftrightarrow x = 0,a_1a_2a_3\ldots$ with $a_j \in \{0,2\}$)
  in ternary repres. (base 3)
- C is nowhere dense:
  $\forall$ open U (interval) $\exists V \subset U$ open with



$V \subset [1,\infty)$, $V \subset (-\infty,-1)$ or $I_j \subset V$, where I_j is one of the omitted intervals (why?)

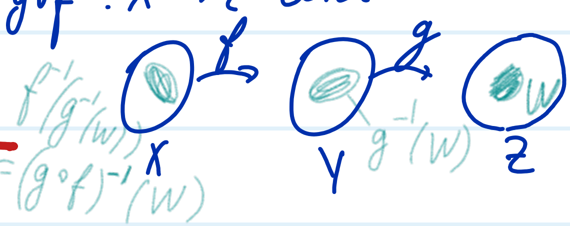
$\boxed{\text{Def}}$ Let (X,τ), (Y,ν) be TS. A map $f: X \to Y$ is called



- continuous if preimages of open sets are open:
  $\forall V \in \nu \quad f^{-1}(V) \in \tau$
- open if images of open sets are open:
  $\forall U \in \tau \quad f(U) \in \nu$
- homeomorphism if f is bij and f, f^{-1} are continuous

Rem. 1) $f: X \to Y$ is
- cont. $\iff$ preimages of closed sets are closed
  $\iff$ preimages of elements of a basis are open
- homeom. $\iff$ f bij, continuous and open.

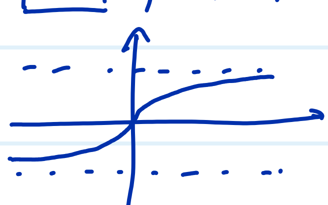
2) $f: X \to Y$, $g: Y \to Z$ cont. $\Rightarrow$ $g \circ f: X \to Z$ cont



$\boxed{\text{Def}}$ (X,τ), (Y,ν) are called homeomorphic if $\exists f: X \to Y$ homeomorphism.

Notation: $X \sim Y$ (exerc) $\sim$ is equiv. relation

$\boxed{\text{Ex}}$ 1) $(-1,1) \sim \mathbb{R}$ (both with natural top. induced by $|x-y|$)

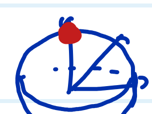


Need f: open intervals $\leftrightarrow$ open intervals

$f(x) := \dfrac{x}{1+|x|}$, $f: \mathbb{R} \to (-1,1)$ (exerc. homeom)

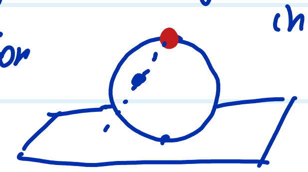
Also: $(0,1) \sim (a,b)$ $\forall a < b$ (exerc.)

2) Let $S^2 := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$



and $X := S^2 \setminus \{(0,0,1)\}$ with Euclidean topology induced by the metric in $\mathbb{R}^3$

(Exerc.) $X \sim \mathbb{R}^2$ via the stereographic projection

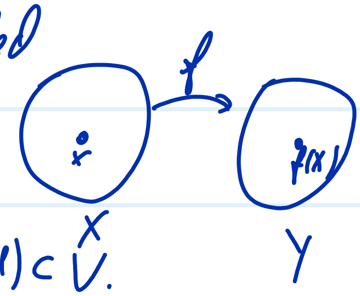


3) Later we will see: $\mathbb{R} \not\sim \mathbb{R}^2$, $(0,1) \not\sim [0,1]$

Def Let $f: X \to Y$ and $x \in X$. Then f is called continuous at x if preimages of neighb. of $f(x)$ contain a neighb. of x, i.e.,

$$\forall V \in \mathcal{U}(f(x)) \;\; \exists \mathcal{U} \in \mathcal{U}(x) : f(\mathcal{U}) \subset V.$$

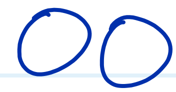
all neighb.



Exerc. f is cont. on $X \iff f$ cont. at every $x \in X$.

Def A TS (X, τ) is called connected if $\forall \mathcal{U}, V \in \tau$

$$\left.\begin{array}{l} \mathcal{U} \cup V = X \\ \mathcal{U} \cap V = \emptyset \end{array}\right\} \Rightarrow \mathcal{U} = \emptyset \text{ or } V = \emptyset$$

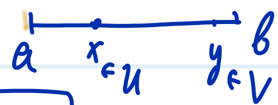


Ex 1) $\underbrace{(0,1)}_{\mathcal{U}} \cup \underbrace{(2,3)}_{V}$ not connect.

2) (a, b) is connected exerc.

if not, $\exists x \in \mathcal{U}, y \in V$, wlog $x < y$ — without loss of generality (OBdA)

consider $s := \sup\{s \in \mathcal{U} : s < y\}$



Thm

$$\left.\begin{array}{l} f: X \to Y \text{ cont} \\ X \text{ connected} \end{array}\right\} \Rightarrow f(X) \text{ is connected.}$$

Proof Assume $f(X) = \mathcal{U} \cup V$, $\mathcal{U}, V$ open in Y, disj. Then

$$X = \underbrace{f^{-1}(\mathcal{U}) \cup f^{-1}(V)}_{\text{open, disj.}}$$

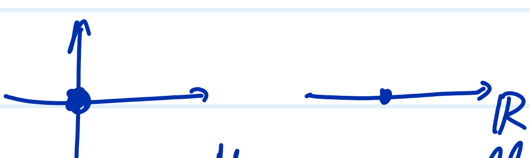
X connected $\Rightarrow f^{-1}(\mathcal{U}) = \emptyset$ or $f^{-1}(V) = \emptyset$, so $\mathcal{U} = \emptyset$ or $V = \emptyset$ since $\mathcal{U}, V \subset f(X)$ ∎

Ex 1) $\mathbb{R}$ is connected $(\mathbb{R} \sim (a, b))$

2) $\mathbb{R} \not\sim \mathbb{R}^2$ Assume $\exists f: \mathbb{R}^2 \to \mathbb{R}$ continuous, bij.

Consider $X := \mathbb{R}^2 \setminus \{0\}$ – connected (why?)

$Y := \mathbb{R} \setminus \{f(0)\}$ – not connected!



↯ to thm.

More generally, $\mathbb{R}^n \not\sim \mathbb{R}^m$ for $n \neq m$ (without proof)

Def Let X be a set and τ_1, τ_2 top. on X.

τ_1 is called finer (feiner in G) (or stronger) then τ_2 and τ_2 coarser (gröber in German) (or weaker) than τ_1 if $\tau_2 \subset \tau_1$

Rem. Discrete top. is the finest, indiscrete top. is the coarsest top.

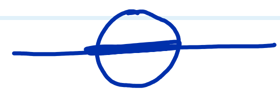
I.2. Basic constructions

Def Let (X,τ) be a TS and $A\subset X$. The <u>induced top.</u> (or <u>subspace top.</u> or <u>relative top.</u>) on A is

$$\tau_A := \tau\cap A = \{U\cap A : U\in\tau\}.$$

Exerc. τ_A is a top. on A

Ex $\mathbb{R}\subset\mathbb{R}^2$ with euclidean top



One has: $\underbrace{\tau_{\mathbb{R}}}_{\text{subspace top.}}$ is the euclidean top. on $\mathbb{R}$

But: $A\subset\mathbb{R}$ are never open in $\mathbb{R}^2$

Rem The inclusion map $\underbrace{j: A\to X}_{j(x)=x}$ is cont. w.r.t. (with respect to) τ_A and τ since $j^{-1}(U)= A\cap U$. Moreover, τ_A is the coarsest top. s.t. $j: A\to X$ is cont.

Def Let $I\neq\emptyset$ be an index set and $(X_j,\tau_j), j\in I$ TS. Def. $X:=\prod_{j\in I} X_j = \{(x_j)_{j\in I}, x_j\in X_j\}$ and $p_j: X\to X_j$ the proj. onto jth coordinate. The <u>product topology</u> on X is def. via the basis

$$\mathcal{B} := \Big\{\underbrace{\bigcap_{j\in F} p_j^{-1}(U_j)}_{\text{cylinder sets}} : F\subset I \text{ finite},\ U_j\in\tau_j\Big\}.$$

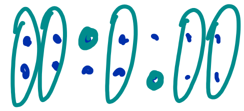
The space (X,τ) is called <u>product space</u>.

Rem. So: $\mathcal{S} := \{p_j^{-1}(U_j),\ j\in I,\ U_j\in\tau_j\}$ is a subbasis of the product top.

Ex 1) $\mathbb{R}^n=\mathbb{R}\times\dots\times\mathbb{R}$. The product top. on $\mathbb{R}^n$ is the natural top. on $\mathbb{R}^n$ (products of intervals build a basis)

2) if I is infinite, then $\mathcal{B}$ consists of sets where only finitely many coordinates are restricted, that's why „cylinders"

3) Let $X := \prod_{n=1}^{\infty} \underbrace{\{0,2\}}_{X_n}$, where $\{0,2\}$ has discrete top. $(\{\emptyset,\{0\},\{2\},\{0,2\}\})$



Exerc. $(X,\tau) \sim (C, \tau_C)$ via
Cantor set, subspace top. (from $\mathbb{R}$)

$$(t_1, t_2, \dots) \longmapsto \frac{t_1}{3} + \frac{t_2}{9} + \dots = \sum_{n=1}^{\infty} \frac{t_n}{3^n}$$ ternary expansion (base 3)

(Hint: show: f: cylinders $\longleftrightarrow$ $C \cap$ finite union of intervals of the form $\left(\frac{a}{3^n}, \frac{b}{3^n}\right)$,

for ex., $f\left(\{(t_j) \cdot t_2 = 0\}\right) = C \cap \left([0, \tfrac{1}{9}] \cup [\tfrac{2}{3}, \tfrac{7}{9}]\right)$)

Prop. Let (X_j, τ_j) be TS, $j \in I$, and (X,τ) be the product space.

a) The projections $p_j : X \to X_j$ are open and cont.

b) τ is the coarsest top. s.t. $\forall\, p_j$ is cont.

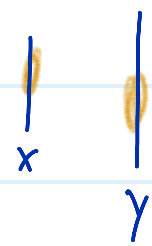
Proof a) Let $U \subset X$ be a cylinder, then $p_j(U) = X_j$ or $p_j(U) = U_j$ open, so p_j is open (why?) if jth coord not restricted / if jth coord is restr. to U_j.

continuity: $p_j^{-1}(U_j)$ (U_j open in X_j) is cylinder by def., so open.

b) $p_j^{-1}(U_j)$ must be open if p_j is cont. and they generate τ, so τ is coarsest top. ∎

Def Let (X,τ), (Y,ν) be TS. The sum of X and Y (or disjoint union) is

$$X + Y := (X \times \{0\}) \cup (Y \times \{1\})$$



with $U \subset X+Y$ open $:\Longleftrightarrow \begin{cases} U \cap p_2^{-1}(0) \text{ open in } X \\ U \cap p_2^{-1}(1) \text{ open in } Y \end{cases}$

$\Longleftrightarrow U = U_1 + U_2$ for $U_1 \subset X$ open, $U_2 \subset Y$ open

Ex $\mathbb{R} + \mathbb{R} = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \subset \mathbb{R}^2$

The sum topology = euclidean top. (subspace top. from $\mathbb{R}^2$)

Def Let X be a set, (X_j, τ_j), $j \in I$, TS — initialtop. in German

and $f_j : X \to X_j$, $j \in I$. The initial top. τ on X is the coarsest top. on X s.t. $\forall f_j$ is cont., i.e., it has subbasis

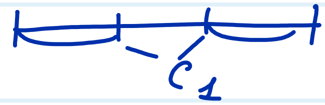
$$\mathcal{S} := \{ f_j^{-1}(U_j),\ j \in I,\ U_j \in \tau_j \}$$

$\boxed{\text{Ex}}$ 1) The product top. on $X = \prod_j X_j$ is initial top. for projections p_j.

2) The subspace top. on $A \subset X$ is initial top. for the inclusion.

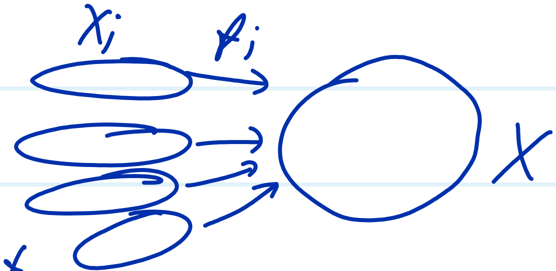
3) The natural (subspace top.) on the Cantor set C is the initial top. for the inclusions $j_n : C \to C_n$, $n \in \mathbb{N}$

exerc.



The dual property:

$\boxed{\text{Def}}$ Let X be a set, (X_j, τ_j) TS, $j \in I$, and $f_j : X_j \to X$ $\forall j$.



The final top. on X is the

Finaltop. in German

finest top. for which all f_j are cont.,

i.e., $\quad U$ open $\Leftrightarrow f_j^{-1}(U) \in \tau_j \ \forall j$.

The final top. is $\bigcap_{j \in I} \{ U \subset X : f_j^{-1}(U) \in \tau_j \}$

$\boxed{\text{Def}}$ Let (X, τ) be TS with an equiv. relation $\sim$ and $p : X \to X/\sim$ the canonical quotient map $\quad x \mapsto [x]_{\text{equiv. class.}}$

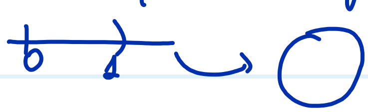
The quotient topology on $X/\sim$ is the final top. for p, i.e.,

$$U \subset X/\sim \text{ open} \Leftrightarrow p^{-1}(U) \subset X \text{ open}$$

$\boxed{\text{Ex}}$ $\mathbb{R}$ with $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$

unit circle in $\mathbb{C}$

exerc. The quotient space $\mathbb{R}/\sim$ is homeom. to $\mathbb{T}$



II Convergence

II.1. Nets