22.4.26

II. Convergence

II.1 Nets

Def Let $(x_n) \subset X$ be a sequence, $x \in X$. We say that (x_n) converges to x (and write $x_n \xrightarrow[n\to\infty]{} x$ or $\lim_{n\to\infty} x_n = x$) if

$$\forall U \in \mathcal{U}(x) \quad \exists n_0 : \forall n \geq n_0 \quad x_n \in U.$$

Ex $\mathbb{R}^{\mathbb{R}} = \prod_{r \in \mathbb{R}} \mathbb{R}$ with product topology. Then (exerc.)

$$f_n \to f \iff f_n(s) \to f(s) \quad \forall s \in \mathbb{R} \quad \text{(pointwise conv.)}$$

since a neighb. basis of f is (exerc.)

$$U_{\varepsilon, s_1, \dots, s_k} := \{ g : \underbrace{|f(s_j) - g(s_j)| < \varepsilon}_{g(s_j) \in \underbrace{(f(s_j)-\varepsilon, f(s_j)+\varepsilon)}_{\text{open interval}}} \ \forall j \in \{1, \dots, k\} \}$$

Exerc. Go through all examples of TS and think what $x_n \to x$ means there.

Ex Indiscrete top.: all seq. converge (to any point)
Discrete top.: only constant seq. from some n_0 conv.

Recall For X, Y metric spaces we have:

(a) For $A \subset X$ and $x \in X$

$$x \in \overline{A} \iff \exists (x_n) \subset A \text{ with } x_n \to x$$

(b) $f : X \to Y$ is continuous at $x \in X \iff \forall (x_n) \subset X$

$$x_n \to x \implies f(x_n) \to f(x) \quad \text{(sequential continuity)}$$

Prop. Let (X, τ) be a TS s.t. $\forall x \in X$ has countable neighb. basis. Then (a) and (b) hold, i.e., closure and continuity can be characterised via sequences.

Proof Let $x \in X$ and (U_n) a count. neighb. basis for x.

WLOG (without loss of generality) $U_1 \supset U_2 \supset U_3 \dots$ (otherwise $V_1 := U_1$, $V_2 := U_2 \cap U_1 \dots$)

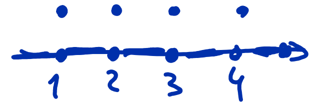
(a) $x \in \overline{A} \iff \forall U \in \mathcal{U}(x) \ \exists y \in A \cap U$
$\iff \forall n \ \exists x_n \in A \cap U_n$ (neighb. basis) $\iff \exists (x_n) \subset A : x_n \to x$ ($U_n \searrow$)

(b) exerc.

Rem. If $\mathcal{U}(x)$ does not have a countable basis, then it might happen that only eventually const. seq. converge (too many neighb.)
(then $\forall f$ is seq. continuous)

Ex $X = \mathbb{R}$, $\tau := \{ A : \mathbb{R} \setminus A \text{ is at most countable} \} \cup \{\emptyset\}$

$f := \mathbb{1}_{\mathbb{N}}$ is not cont. because



$$f^{-1}(\underbrace{\mathbb{R}\setminus\{0\}}_{\text{open}}) = \mathbb{N} \quad \text{not open!}$$

Now, $x_n \to x$ only if $x_n = x$ $\forall n \geq n_0$ some, so f is sequentially continuous.

$\mathcal{U} := \mathbb{R} \setminus \{x_n : x_n \neq x\}$

So one needs convergence of different („larger") objects (for different, „larger" index set)

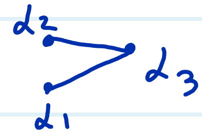
$\boxed{\text{Def}}$ Let $I \neq \emptyset$ be a set with $\preceq$ relation on I. Then I (or the pair $(I, \preceq)$) is called **directed** if:

gerichtet in German

(I1) $d \preceq d$ $\quad \forall d \in I$ (reflexivity)

(I2) $\left.\begin{array}{l} d_1 \preceq d_2 \\ d_2 \preceq d_3 \end{array}\right\} \Rightarrow d_1 \preceq d_3$ (transitivity)

(I3) $\forall d_1, d_2 \in I$ $\exists d_3 \in I$ with $d_1 \preceq d_3$, $d_2 \preceq d_3$



We def. $d_1 \prec d_2$ if $d_1 \preceq d_2$ and $d_2 \not\preceq d_1$.

$\boxed{\text{Ex}}$ 1) I totally ordered $\Rightarrow$ directed ($d_3 = d_1$ or d_2)

So $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{R}, \leq)$, $(\mathbb{Q}, \leq)$, $((0,1), \leq)$

also $\geq$

2) A tree I

with $d_1 \preceq d_2$ if d_1 and d_2 are connected by a „vertical" path and d_2 is „above" d_1

3) Let (X, τ) be a TS and $x \in X$. Then

$$(\mathcal{U}(x), \supset)$$

is a directed set. (I1), (I2) clear, for (I3) $\forall U, V \in \mathcal{U}(x)$ we have $U \cap V \subset U$, $U \cap V \subset V$

$\supseteq$

4) $P := \{$ partitions $a = x_0 < x_1 < \ldots < x_n = b$ of $[a,b]\}$

$a < b$, $a, b \in \mathbb{R}$ with „$<$" (for (I3) take the union (refinement) of two part.)

$\boxed{\text{Def}}$ 1) A **net** (or **Moore-Schmidt sequence**) in a set $X_{\neq \emptyset}$ is a map

Netz in German

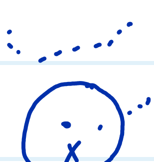
$$f : I \to X,$$

where I is an (arbitrary) directed set.

We write $(x_d)_{d \in I}$ instead of $d \mapsto x_d$

2) A net $(x_d)_{d \in I} \subset X$, (X, τ) a TS, is called **convergent to $x \in X$** if

$$\forall U \in \mathcal{U}(x)\ \exists d_0 \in I : \forall d \succeq d_0 \quad x_d \in U.$$



Write: $x_d \xrightarrow[d \in I]{} x$ or $\lim_{d \in I} x_d = x$

$\boxed{\text{Ex}}$ 1) $\forall$ sequence is a net ($I := \mathbb{N}$ directed) and conv. is the same

2) $(x_r)_{r\in\mathbb{R}} \subset \mathbb{R}$ a net conv. to x:

$$\forall \varepsilon>0 \ \exists R_0 : \ \forall r \in [R_0,\infty) \ |x_r - x| < \varepsilon$$

3) Let $\mathcal{P} := \{$partitions of $[a,b]\}$ with $\subset$

Let $f: [a,b] \to \mathbb{R}$ bounded. Then $(\overline{S}_p)_{p\in\mathcal{P}} \subset \mathbb{R}$ for

$$\overline{S}_p := \sum_{j=1}^{n} (x_j - x_{j-1}) \sup\{f(x) : x \in [x_{j-1}, x_j]\}$$

$p = \{a = x_0, x_1, \dots, x_n = b\}$

is a net in $\mathbb{R}$. Also $(\underline{S}_p)_{p\in\mathcal{P}} \subset \mathbb{R}$ for

$$\underline{S}_p := \ \text{—''—} \quad \inf \ \text{—''—}$$

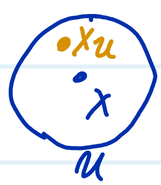
is a net in $\mathbb{R}$.

exerc. f is Riemann integr. $\iff (\overline{S}_p)_{p\in\mathcal{P}}$ and $(\underline{S}_p)_{p\in\mathcal{P}}$ converge to the same $c \in \mathbb{R}$. In this case

$$c = \int_a^b f$$

4) Let (X,τ) be a TS, $x \in X$. Choose $\forall U \in \mathcal{U}(x)$ $x_U \in U$.

Then $(x_U)_{U\in\mathcal{U}(x)}$ is a net.



$\boxed{\text{Thm.}}$ Let X, Y be TS and $x \in X$.

(a) For $A \subset X$

$$x \in \overline{A} \iff \exists \text{ net } (x_\alpha)_{\alpha\in I} \subset A \text{ with } x_\alpha \to x$$

(b) A fct $f: X \to Y$ is continuous at $x \iff \forall$ net $(x_\alpha)_{\alpha\in I}$

$$x_\alpha \to x \implies f(x_\alpha) \to f(x)$$

Proof (a) ($\Leftarrow$) Let $x_\alpha \to x$, then: $\forall U \in \mathcal{U}(x)$ $\exists \alpha$ with $x_\alpha \in U \cap A$,

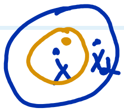
so $x \in \overline{A}$

($\Rightarrow$) $\forall U \in \mathcal{U}(x)$ let $x_\alpha \in A \cap U$. We have

$\alpha := U$

- $(x_\alpha)_{\alpha\in\mathcal{U}(x)}$ is a net in A
- Let $U \in \mathcal{U}(x)$ and $\alpha_0 := U$. Then

$\forall \alpha \succcurlyeq \alpha_0$ (i.e., $U_\alpha \subset U$) $x_\alpha \in U_\alpha \subset U$, so $x_\alpha \xrightarrow[\alpha\in\mathcal{U}(x)]{} x$ ■



(b) exerc.

Recall $(x_n) \subset X$ a seq., $m: \mathbb{N} \to \mathbb{N}$ $\nearrow$, then $(x_{m(n)})_{n=1}^{\infty}$ is a subseq.

$\boxed{\text{Def}}$ Let $(x_\alpha)_{\alpha\in I} \subset X$ be a net, $(K, \preccurlyeq)$ a directed set and

$m: K \to I$ monotone $\nearrow$: $\forall \alpha_0 \in I \ \exists k_0 \in K : \forall k \succcurlyeq k_0 \ \ m(k) \succcurlyeq \alpha_0$

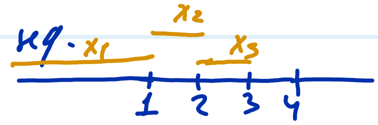
$k_1 \preccurlyeq k_2 \implies m(k_1) \preccurlyeq m(k_2)$

Then the net $(x_{m(k)})_{k\in K}$ is called a <u>subnet</u> of (x_α)

Rem. A ~~subnet~~ exerc. subnet of a sequence is not a sequence in general.

$\boxed{\text{Ex}}$ Let $x: \mathbb{N} \to X$ be a seq. in X.
$m: \mathbb{R} \to \mathbb{N}$ with $m(r) := \min\{n \in \mathbb{N} : r \le n\} = \lceil r \rceil$

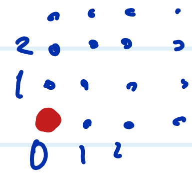
Then $(x_{m(r)})_{r \in \mathbb{R}}$ is a subnet of (x_n) but a seq.



Exerc.: $\left.\begin{array}{l} x_\lambda \to x \\ (x_{m(k)})_{k \in K} \text{ a subnet of } (x_\lambda) \end{array}\right\} \Longrightarrow x_{m(k)} \to x$

$\boxed{\text{Ex}}$ Consider $X := \mathbb{N}_0 \times \mathbb{N}_0$ with τ, where

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$



(1) $\tau|_{(\mathbb{N}_0 \times \mathbb{N}_0) \setminus \{(0,0)\}}$ = discrete top. on $(\mathbb{N}_0 \times \mathbb{N}_0) \setminus \{(0,0)\}$

$\forall$ set not containing $(0,0)$ is open

(2) $U \subset \mathbb{N}_0 \times \mathbb{N}_0$ is a neighb. of $(0,0)$ if $(0,0) \in U$ and for all but finitely many $m \in \mathbb{N}_0$
$\{n \in \mathbb{N}_0 : (n,m) \notin U\}$
is finite.

(3) $\nexists\, (x_n)_1^\infty \subset X \setminus \{(0,0)\}$ with $x_n \to (0,0)$

Assume $\exists\, \underset{\ne (0,0)}{x_n} \to (0,0)$, then $\forall m$ $\exists$ at most finitely many elements in $m \times \mathbb{N}_0$ (take $U := \{(\tilde{m}, \tilde{n}) : \tilde{m} \ne m\}$)

So $U := (\mathbb{N}_0 \times \mathbb{N}_0) \setminus (x_n)$ is open and $\ni (0,0)$ ↯

(4) The seq. $(x_n) \subset X \setminus \{(0,0)\}$ taking all values of $X \setminus \{(0,0)\}$ has $(0,0)$ as a $\underbrace{\text{limit point}}_{\forall U \in \mathcal{U}((0,0))\ \exists\ \infty\text{-many elements of } (x_n)}$ (why?),
but $\nexists$ conv. subsequence by (3) (subnet yes!)

II.2. Filters

Alternative notion of convergence, introduced by Nicolas Bourbaki (pseudonym for a group founded 1934 by Weil and Cartan)

$\boxed{\text{Def}}$ Let $X \ne \emptyset$ be a set. A system $\mathcal{F}$ of subsets of X is called filter if

(F1) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$ equiv. $\mathcal{F} \ne \emptyset$

(F2) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$ (finite intersection property)

(F3) $F \in \mathcal{F}$, $F' \supset F \Rightarrow F' \in \mathcal{F}$ (supersets belong to $\mathcal{F}$)

A filter $\mathcal{F}$ is called free if $\bigcap_{F \in \mathcal{F}} F = \emptyset$, otherwise fixed. (Obermengen i. German)

$\boxed{\text{Ex}}$ 0) Trivial filter $\{X\}$

1) Let $\emptyset \ne A \subset X$, then
$$\mathcal{F}_A := \{F \subset X \text{ with } A \subset F\}$$
is a fixed filter with $\bigcap_{F \in \mathcal{F}} F = A$

2) Let (X,τ) be a TS and $x \in X$. Then

$$\mathcal{F}_x := \{ V \subset X : \exists\, U \in \mathcal{U}(x) \text{ with } U \subset V \}$$

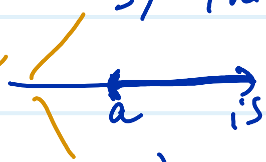
— supersets of neighb. of x.

is a fixed filter ($x \in \bigcap_{F \in \mathcal{F}} F$), called the <u>neighbourhood filter</u> of x

3) The filter

$$\mathcal{F} := \{ V \subset \mathbb{R} : (a,\infty) \subset V \text{ for some } a \in \mathbb{R} \}$$

free



is sometimes called <u>Fréchet filter on $\mathbb{R}$</u>

4) For an infinite set X, the <u>cofinite filter</u> is

$$\mathcal{F} := \{ A : X \setminus A \text{ is finite or } \emptyset \}$$

$\boxed{\text{Def}}$ A subset $\mathcal{B}$ of $\mathcal{F}$ is called <u>basis</u> for $\mathcal{F}$ if

$$\forall F \in \mathcal{F} \ \exists B \in \mathcal{B} \text{ with } B \subset F.$$

Rem. 1) $\mathcal{B}$ is a filter basis $\iff \forall B_1, B_2 \in \mathcal{B} \ \exists B_3 \in \mathcal{B}$ with $B_3 \subset B_1 \cap B_2$ (exerc.)

2) A filter is defined uniquely by its basis, (basis not unique!) namely $\mathcal{F} := \{ F \supset B,\ B \in \mathcal{B} \}$

$\boxed{\text{Ex}}$ 1) $\{A\}$ is a filter basis for $\mathcal{F}_A$

2) $\mathcal{B} := \{ (a,\infty),\ a \in \mathbb{R} \}$ is a basis for the Fréchet filter (also $\mathbb{N}, \mathbb{Q}, \ldots$)

$\boxed{\text{Def}}$ 1) Let $\mathcal{F}_1, \mathcal{F}_2$ filter on X. $\mathcal{F}_1$ is called <u>finer</u> than $\mathcal{F}_2$ (and $\mathcal{F}_2$ <u>coarser</u> than $\mathcal{F}_1$) if $\mathcal{F}_1 \supset \mathcal{F}_2$

2) A filter $\mathcal{F}$ on X is called <u>ultrafilter</u> if it is maximal w.r.t. (with respect to) inclusion, i.e., $\nexists\, \tilde{\mathcal{F}} \neq \mathcal{F}$ with $\tilde{\mathcal{F}} \supset \mathcal{F}$

$\boxed{\text{Ex}}$ $\mathcal{F}_A$ is an ultrafilter $\iff A = \{x\}$ for some x

$\boxed{\text{Prop}}$ <u>(Properties of ultrafilters)</u>

(a) $\forall$ filter $\mathcal{F}$ is contained in an ultrafilter

(b) $\mathcal{F}$ ultrafilter $\iff \forall A \subset X \quad A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$

(c) A filter $\mathcal{F}$ is fixed ultrafilter $\iff \mathcal{F} = \mathcal{F}_{\{x\}}$ for some $x \in X$

Proof (a) Consider $\Phi := \{ \text{all filters on } X \supset \mathcal{F} \}$ $\{\tilde{\mathcal{F}} \ni x\}$ with „$\subset$" ($\neq \emptyset$ by $\mathcal{F} \in \Phi$)
Let (Φ_α) be a totally ordered subset of Φ, then $\bigcup \Phi_\alpha$ is also a filter $\supset \mathcal{F}$ (why?), so it is in Φ.
Lemma of Zorn: $\exists$ maximal element which is an ultrafilter then $\supset \mathcal{F}$.

(b) ($\Rightarrow$) since $A\cap(X\setminus A)=\emptyset$, $\nexists F_1,F_2\in\mathcal{F}$: $F_1\subset A$, $F_2\subset X\setminus A$

So $\underbrace{(\forall F\in\mathcal{F}\ \ F\cap A\neq\emptyset)}_{\text{WLOG}}$ or $(\forall F\in\mathcal{F}\ \ F\cap(X\setminus A)\neq\emptyset)$

Consider $\mathcal{B}:=\{F\cap A,\ F\in\mathcal{F}\}$

which is a basis for a filter $\tilde{\mathcal{F}}$ with $\tilde{\mathcal{F}}\supset\mathcal{F}$

But $\mathcal{F}$ is ultrafilter, so $\tilde{\mathcal{F}}=\mathcal{F}$ and so $A=A\cap X\in\mathcal{F}$

($\Leftarrow$) Assume $\forall A\subset X$ $A\in\mathcal{F}$ or $X\setminus A\in\mathcal{F}$ for a filter $\mathcal{F}$

Assume $\mathcal{F}$ not maximal, so $\exists\,\tilde{\mathcal{F}}\supsetneq\mathcal{F}$, so $\exists\,G\in\tilde{\mathcal{F}}\setminus\mathcal{F}$

By assumption, $\underbrace{G\in\mathcal{F}}_{\lightning\ G\in\tilde{\mathcal{F}}\setminus\mathcal{F}}$ or $\underbrace{X\setminus G\in\mathcal{F}}_{\lightning \text{ since } G\in\tilde{\mathcal{F}} \text{ and then } G\cap(X\setminus G)\in\tilde{\mathcal{F}}}$, $\underbrace{G\cap(X\setminus G)}_{\emptyset}$.

(c) ($\Leftarrow$) clear

($\Rightarrow$) $\mathcal{F}$ fixed $\Rightarrow$ $A:=\bigcap_{F\in\mathcal{F}}F\neq\emptyset$

Assume $|A|\geq 2$ and consider $A_1\subsetneq A$ non-empty.

By (b) $\underbrace{A_1\in\mathcal{F}}_{\lightning \text{ to def. of } A}$ or $\underbrace{X\setminus A_1\in\mathcal{F}}_{\lightning \text{ to def. of } A}$,

So $|A|=1$. ■

Def Let (X,τ) be a TS and $\mathcal{F}$ a filter on X. A point $x\in X$ is called

- limit of $\mathcal{F}$ if $\mathcal{F}\supset\mathcal{U}(x)$. One says: $\mathcal{F}$ converges to x, write $\mathcal{F}\to x$.
- cluster point of $\mathcal{F}$ if $\forall\,U\in\mathcal{U}(x)$ $\forall F\in\mathcal{F}$ $F\cap U\neq\emptyset$

  accumulation point

Rem. a) limit is not unique in general ($\mathcal{F}:=\{X\}$)

exerc. $\forall\,\mathcal{F}$ on X has at most 1 limit $\Leftrightarrow$ X is Hausdorff (the same for nets)

b) The set of all cluster points of $\mathcal{F}=\bigcap_{F\in\mathcal{F}}\overline{F}$ exerc.

c) A point x is a limit $\Rightarrow$ cluster point; $\not\Leftarrow$ exerc.

Ex (a) Let $(x_n)\subset X$ be a seq. The corresponding filter is def. by

$$\mathcal{F}:=\{F\supset\underbrace{\{x_n,x_{n+1},\dots\}}_{\text{tail of }(x_n)} \text{ for some } n\}$$

Such filter is called elementary filter (Elementarfilter in German.)

Exerc. 1) $\mathcal{F}$ is a filter

2) $\mathcal{F}\to x$ $\Leftrightarrow$ $x_n\to x$

3) x is a cluster point of $\mathcal{F}$ $\Leftrightarrow$ x accumulation point of (x_n)

(b) The Fréchet filter on $\mathbb{R}$ has no cluster points ($\Rightarrow$ no limits)
$\longleftrightarrow$ exerc.

(c) Let (X,τ) be a TS, $A \subset X$, $A \neq \emptyset$, $\mathcal{F}_A := \{\text{sets} \supset A\}$
Then $\overline{A} = \{\text{cluster points of } \mathcal{F}\}$ exerc.

(d) Def. $\mathcal{F}$ on $\mathbb{R}$ by
$$\mathcal{F} := \{\text{sets} \supset (0,\tfrac{1}{n}) \text{ for some } n\}$$
(i.e., $\left((0,\tfrac{1}{n})\right)_{n=1}^{\infty}$ is a basis of $\mathcal{F}$)
Then $\mathcal{F} \to 0$ $\quad (\forall U \in \mathcal{U}(0)\ \exists n : (0,\tfrac{1}{n}) \subset U)$
for euclidean top. on $\mathbb{R}$

Prop. Let (X,τ) be TS, $\mathcal{F}$ filter on X, $x \in X$. Then
x is a cluster point of $\mathcal{F}$ $\Leftrightarrow$ $\exists \mathcal{G} \supset \mathcal{F}$, $\mathcal{G}$ filter with $\mathcal{G} \to x$.

Proof ($\Rightarrow$) Def. $\mathcal{G}$ by the basis
$$\mathcal{U}(x) \cap \mathcal{F} = \{U \cap F,\ U \in \mathcal{U}(x), F \in \mathcal{F}\}$$
(filter basis since $\neq \emptyset$ by assumption and $(U_1 \cap F_1) \cap (U_2 \cap F_2) = \underbrace{(U_1 \cap U_2)}_{\in \mathcal{U}(x)} \cap \underbrace{(F_1 \cap F_2)}_{\in \mathcal{F}}$)
Then:
- $\mathcal{G} \supset \mathcal{F}$ (superset property)
- $\mathcal{G} \to x$ because $\forall U \in \mathcal{U}(x)$ $\quad U = U \cap \underbrace{X}_{\in \mathcal{F}} \in \mathcal{G}$

($\Leftarrow$) Let $\mathcal{G} \to x$ with $\mathcal{G} \supset \mathcal{F}$. Then $\mathcal{U}(x) \subset \mathcal{G}$,
so $U \cap F \neq \emptyset$ $\forall U \in \mathcal{U}(x)$ $\forall F \in \mathcal{F}$ since $\mathcal{G}$ is a filter ■

Rem. Finer filters are analogue to subnets/subseq.

Def Let X, Y be a TS, $f : X \to Y$ and $\mathcal{F}$ a filter on X

Then the filter on Y with basis $\{f(F) : F \in \mathcal{F}\}$ is called the image filter of $\mathcal{F}$ under f. We write $f(\mathcal{F})$.
Bildfilter i.G.

Rem. Filter basis by $f(F_1) \cap f(F_2) \supset f(F_1 \cap F_2)$ $\quad \forall F_1, F_2 \subset X$

Prop. Let X, Y be a TS and $A \subset X$.
(a) $x \in \overline{A} \Leftrightarrow \exists$ filter $\mathcal{F}$ on X with $\mathcal{F} \to x$ and $A \in \mathcal{F}$
(b) A map $f : X \to Y$ is continuous (at x) $\Leftrightarrow$ $\forall$ filter $\mathcal{F}$ on X $\cancel{\forall x \in X}$
$$(\mathcal{F} \to x \Rightarrow f(\mathcal{F}) \to f(x))$$

Proof (a) ($\Rightarrow$) Let $x \in \overline{A}$. Then
$$\mathcal{B} := \{\underbrace{A \cap U}_{\neq \emptyset \text{ by assumption}},\ U \in \mathcal{U}(x)\}$$
is a basis of a filter $\mathcal{F}$.
Clear: $A \in \mathcal{F}$ and $\mathcal{U}(x) \subset \mathcal{F}$ by the superset property of filters.
($\Leftarrow$) Let $\mathcal{F} \ni A$, $\mathcal{F} \to x$. Then x is a cluster point of $\mathcal{F}$, i.e.,
$$x \in \bigcap_{F \in \mathcal{F}} \overline{F} \subset \overline{A} \quad \text{because } A \in \mathcal{F}$$

(b) ($\Rightarrow$) Let $f : X \to Y$ be cont. at x, $\mathcal{F} \to x$

Let $V \in \mathcal{U}(f(x))$. Then $f^{-1}(V) \in \mathcal{U}(x)$ by continuity of f,
so $f^{-1}(V) \in \mathcal{F}$ and $\underbrace{V \in f(\mathcal{F})}_{f(\underbrace{f^{-1}(V)}_{\in \mathcal{F}})}$

($\Leftarrow$) Assume $\forall \mathcal{F}$ $\mathcal{F} \to x \Rightarrow f(\mathcal{F}) \to f(x)$.
Consider $\mathcal{F} := \{\text{sets} \supset \underbrace{\text{neighb. of } x}_{\text{basis}}\}$. Then $\mathcal{F} \to x$.
By assumption $f(\mathcal{F}) \to f(x)$, so $\mathcal{U}(f(x)) \subset f(\mathcal{F})$,
i.e., $\forall V \in \mathcal{U}(f(x))$ $\exists U \in \mathcal{U}(x)$ with $\underbrace{f(U)}_{\text{form a basis for } f(\mathcal{F})} \subset V$ ∎

Rem. Let $f: X \to Y$, $\mathcal{F}$ is a filter on Y. Then $f^{-1}(F)$, $F \in \mathcal{F}$ generate a filter (called preimage filter) on X
$\Leftrightarrow f^{-1}(F) \neq \emptyset \quad \forall F \in \mathcal{F}$

$X \xrightarrow{f} Y$

II.3. Nets via filters

Nets $\rightsquigarrow$ filters Let $(x_\alpha)_{\alpha \in I}$ be a net in X. Then the „tails"

$$E_{\alpha_0} := \{x_\alpha : \alpha \succcurlyeq \alpha_0\}, \quad \alpha_0 \in I$$

generate a filter $\mathcal{F}$ as a basis (why?). One has:

$$x_\alpha \to x \Leftrightarrow \mathcal{F} \to x \quad \text{exer.}$$

Filters $\rightsquigarrow$ nets Let $\mathcal{F}$ be a filter on X. Def.

$$I := \{(F, y) : F \in \mathcal{F}, y \in F\}$$

with the direction given by

$$(F_1, y_1) \preccurlyeq (F_2, y_2) \Leftrightarrow F_2 \subset F_1$$

(why is I directed?)
Define the net $(x_\alpha)_{\alpha \in I}$ by

$$x_{(F,y)} := y$$

Claim: $x_\alpha \to x \Leftrightarrow \mathcal{F} \to x$.

Proof $x_\alpha \to x \Leftrightarrow \forall U \in \mathcal{U}(x)\ \exists \alpha_0 = (F_0, y_0) \in I$:
$\underbrace{\forall \alpha = \underbrace{(F, y) \succcurlyeq (F_0, y_0)}_{\forall F \subset F_0\ \forall y \in F} \quad x_\alpha = y \in U}_{\forall F \subset F_0\ F \subset U \Leftrightarrow F_0 \subset U}$

$\Leftrightarrow \forall U \in \mathcal{U}(x)\ \exists F_0 \in \mathcal{F} : F_0 \subset U$

$\Leftrightarrow \mathcal{U}(x) \subset \mathcal{F} \Leftrightarrow \mathcal{F} \to x.$ ∎

Rem. The analogue of ultrafilters are so-called universal nets, where $(x_\alpha)_{\alpha \in I}$ is called universal if
$\forall A \subset X \ \exists \alpha_0 \in I$:
$(\forall \alpha \succcurlyeq \alpha_0\ x_\alpha \in A)$ or $(\forall \alpha \succcurlyeq \alpha_0\ x_\alpha \notin A)$

III Compactness

III.1. Compactness, countable comp. and sequential comp.

Def 1) A TS (X,τ) is called compact if $\forall$ open cover of X has a finite subcover, i.e.,

$$\bigcup_{\alpha\in I} U_\alpha = X,\ \forall U_\alpha \text{ open} \Rightarrow \exists n\ \exists \alpha_1,\dots,\alpha_n \in I : \bigcup_{j=1}^n U_{\alpha_j} = X.$$

2) A subset $A\subset X$ is called compact if A is compact for the subspace topology, i.e.,

$$A \subset \bigcup_{\alpha\in I} U_\alpha\ \forall U_\alpha \in \tau \Rightarrow \exists n\ \exists \alpha_1,\dots,\alpha_n \in I : A \subset \bigcup_{j=1}^n U_{\alpha_j}$$

Ex X finite or τ finite $\Rightarrow$ (X,τ) comp.

Rem. Often one takes the Hausdorff property into the def. of compactness (for ex., Bourbaki)

Ex Let (X,τ), τ cofinite top. Then X is comp. (why?) but not Hausdorff! (infinite set)

Prop. (Characterisation of compactness)

TFAE:

(i) X is compact

(ii) $\forall$ family $(B_\alpha)_{\alpha\in I}$ of closed sets with $\bigcap_{\alpha\in I} B_\alpha = \emptyset$ contains a finite subfamily $(B_{\alpha_j})_{j=1}^n$ with $\bigcap_{j=1}^n B_{\alpha_j} = \emptyset$

(iii) $\forall$ filter on X has a cluster point

(iv) $\forall$ ultrafilter on X converges in X.

Proof (i) $\Rightarrow$ (ii): go to the complement (exer.)

(ii) $\Rightarrow$ (iii) Recall: x is a cluster point of $\mathcal{F}$ if $F\cap U \neq \emptyset\ \forall F\in\mathcal{F}\ \forall U\in \mathcal{U}(x)$, i.e., $x\in \overline{F}\ \forall F\in\mathcal{F}$.

Assume $\exists$ filter $\mathcal{F}$ without cluster point, i.e.,

$$\bigcap_{F\in\mathcal{F}} \overline{F} = \emptyset$$

By (ii), $\exists n\ \exists F_1,\dots,F_n \in \mathcal{F}$: $\bigcap_{j=1}^n \underbrace{\overline{F}_j}_{\in\mathcal{F}} = \emptyset$ $\lightning$ $(\emptyset\notin\mathcal{F})$

(iii) $\Rightarrow$ (iv) Let $\mathcal{F}$ be an ultrafilter. We know: x is a cluster point of $\mathcal{F}$ implies $\exists\ \mathcal{G}\supset\mathcal{F}$ $\mathcal{G}\to x$. But $\mathcal{F}$ is maximal, so $\mathcal{F}=\mathcal{G}\to x$.

(iv) $\Rightarrow$ (i) Assume $\exists\ \mathcal{U} = (U_\alpha)_{\alpha\in I}$ an open cover without a finite subcover. Def. $\forall L\subset I$ finite

$$A_L := X \setminus \Big(\bigcup_{\alpha\in L} U_\alpha\Big) \neq \emptyset$$

By construction, $A_{L\cup L'} = A_L \cap A_{L'} \neq \emptyset$ $\forall L, L'$ finite

so $\exists$ filter $\mathcal{F}$ with finite basis $(A_L)_{L\subset I \text{ finite}}$.

Let $\mathcal{G}$ be an ultrafilter $\supset \mathcal{F}$. By (iv) $\exists\, x \in X$ with $\mathcal{G} \to x$, i.e., $\mathcal{U}(x) \subset \mathcal{G}$.

Since (U_α) is an open cover of X, $x \in U_\alpha$ for some $\alpha \in I$. Then $U_\alpha \in \mathcal{G}$. But $X \setminus U_\alpha = A_{\{\alpha\}} \in \mathcal{F} \subset \mathcal{G}$ ↯ $(\emptyset \notin \mathcal{G})$. ■

Cor. 1 Let X be comp. Then $\forall$ sequence $(x_n) \subset X$ / $\forall$ net $(x_\alpha)_{\alpha \in I}$ has a cluster point.

cluster point x if $\forall U \in \mathcal{U}(x)$ $\forall \alpha \in I$ $\exists \alpha' \geq \alpha$ with $x_{\alpha'} \in U$

Proof: Consider the filter $\mathcal{F}$ generated by (x_n) or $(x_\alpha)_{\alpha \in I}$, i.e.,

$\mathcal{F}$ = all supersets of $\{x_n, x_{n+1}, \dots\}$ or $\{x_\alpha, \alpha \geq \alpha_0\}$ respect.

and use (iii) from Prop.: $\exists\, x \in X$ cluster point for $\mathcal{F}$ ■

Rem. X comp. $\Leftrightarrow$ $\forall$ net has cluster point (exer.)

Ex ($\forall$ seq. has a cluster point $\not\Rightarrow$ X comp.)

Consider $\{0,1\}$ with discrete top. and $X := \{0,1\}^{\mathbb{R}}$ with the product top.

Consider
$$A := \left\{ x \in X : \exists \text{ count. (at most?) } I \subset \mathbb{R} \text{ with } \pi_\alpha(x) = \begin{cases} 0, & \alpha \in I \\ 1, & \alpha \notin I \end{cases} \right\}$$

projection onto α-th coord.

Then:
- $\forall$ seq. in A has a cluster point in A
- A not comp.

} (exer.)

Cor 2 Let X be a TS and $A \subset X$. TFAE:

(i) A comp.

(ii) $\forall$ ultrafilter $\mathcal{F}$ with $A \in \mathcal{F}$ conv. to an element of A

(iii) $\forall$ net $(x_\alpha)_{\alpha \in I} \subset A$ has cluster point in A.

Proof (exer.)

Ex (order topology)

Let $(X, \leq)$ be a totally ordered set with so-called order topology generated by open intervals (for $a < b$ take $(a,b) := \{c : a < c < b\}$). Assume that $\forall A \subset X$ has a supremum and an infimum in X. Then X is comp. (exer.)
(Hint: imitate the proof that $[0,1]$ is comp.)

Prop. (a) Closed subsets of compact spaces are comp.

(b) X Hausdorff, $A \subset X$ comp. $\Rightarrow$ A closed.

Proof (a) Take (B_α), $\forall B_\alpha$ closed, with $\bigcap (\underbrace{B_\alpha \cap A}_{\text{closed}}) = \emptyset$ since A closed

X comp. $\Rightarrow$ $\exists$ finite family $B_{\alpha_1}, \dots, B_{\alpha_k}$: $\bigcap_{j=1}^{k} (B_{\alpha_j} \cap A) = \emptyset$

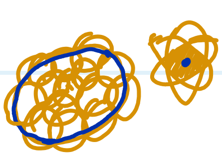
(use Prop. (i)$\Leftrightarrow$(ii))

(b) Observation: X Hausdorff, $A \subset X$ comp., $x \notin A$
$\Rightarrow$ one can separate x and A by open sets

Proof of observ. Choose $\forall a \in A$ $U_a \in \mathcal{U}(a)$, $V_a \in \mathcal{U}(x)$ with $U_a \cap V_a = \emptyset$ (Hausdorff)

$A \subset \bigcup_a U_a$, A comp. $\Rightarrow$ $\exists\, U_{a_1}, \dots, U_{a_n}$ a finite subcover of A. Def. $U := U_{a_1} \cup \dots \cup U_{a_n} \supset A$, $U \cap V = \emptyset$

open $-$ $V := V_{a_1} \cap \dots \cap V_{a_n} \ni x$



■ observ.

By the observation, $X \setminus A$ is open, i.e., A closed. Prop: ■

$\boxed{\text{Thm}}$ (Alexander's subbasis thm)
Let X be a TS with subbasis $\mathcal{S}$ of top. Then

X comp. $\Longleftrightarrow$ $\forall$ cover of X with elements from $\mathcal{S}$ has a finite subcover.

Proof ($\Rightarrow$) clear

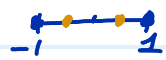
($\Leftarrow$) Assume X not comp. Then $\exists \mathcal{F}$ ultrafilter without a limit, i.e., $\forall x \in X \; \exists U_x \in \mathcal{S}$ with $U_x \notin \mathcal{F}$ and $x \in U_x$.
Since $(U_x)_{x\in X}$ is a cover of X from $\mathcal{S}$, $\exists n, x_1, \dots, x_n$ with
$X = U_{x_1} \cup \dots \cup U_{x_n}$.
Since $\mathcal{F}$ is an ultrafilter and $U_{x_j} \notin \mathcal{F}$ $\forall j$, we know that

$$X \setminus U_{x_j} \in \mathcal{F} \quad \forall j \quad \xRightarrow{\mathcal{F}\text{ filter}} \quad \underbrace{\bigcap_{j=1}^{n} (X \setminus U_{x_j})}_{= X \setminus \bigcup_{j=1}^{n} U_{x_j} = \emptyset} \in \mathcal{F} \quad ↯ \; (\emptyset \notin \mathcal{F}) \quad ■$$

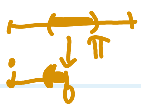
$\boxed{\text{Ex}}$ $[a,b]$ with natural (= order) topology is comp:
take subbasis of intervals $[a,c)$, $(c,b]$ for $c \in [a,b]$
Check: $\forall$ cover by such intervals has a subcover of 1 or 2 elements (exer.)

$\boxed{\text{Ex}}$ ($A \subset X$ comp $\not\Rightarrow$ A closed)
Let $X := [-1,1]$ and consider the equiv. relation

$$x \sim y \; :\Longleftrightarrow \begin{cases} x = \pm y & \text{for } x \neq \pm 1 \\ x = y & \text{for } x = \pm 1 \end{cases}$$



Then $\tilde{X} := X/\sim$ satisfies:
with 'quotient top.' ($A \subset \tilde{X}$ is open $\Leftrightarrow$ $\pi^{-1}(A) \subset X$ is open, $\pi: x \mapsto [x]$)



$\tilde{X} \setminus [1]$ is a comp. neighb. of $[-1]$, but it is not closed.
$[1]$ is not open because $\{1\}$ is not open in $[-1,1]$
(anal. for $\tilde{X} \setminus [-1]$ —"— $[1]$ —"—)

To show comp., let $\mathcal{U}$ be an open cover of $\tilde{X} \setminus [1]$, then $\pi^{-1}(\mathcal{U})$ is an open cover of $[-1,1)$ and hence also of $[-1,0]$.
$[-1,0]$ comp. $\Rightarrow$ $\exists$ finite subcover of $\pi^{-1}(\mathcal{U})$, say $\pi^{-1}(U_1), \dots, \pi^{-1}(U_k)$. Then $U_1, \dots, U_k$ cover $\tilde{X} \setminus [1]$.
exer.: which points from $\tilde{X} \setminus [1]$ cannot be sep. by open sets

$\boxed{\text{Prop.}}$ (Continuous images of comp. spaces are comp.)
X, Y TS. Then $\left.\begin{array}{l} X \text{ comp.} \\ f: X \to Y \text{ continuous} \end{array}\right\} \Rightarrow \underbrace{f(X)}_{\subset Y}$ is comp.

in part., continuous fcts $: X \to \mathbb{R}$ for comp. X attain their maximum and minimum.

Proof Let $(U_\alpha)_{\alpha \in I}$ be an open cover of $f(X)$. Since $(f^{-1}(U_\alpha))_{\alpha\in I}$ is an open cover of X and X is comp., $\exists$ finite subcover $(f^{-1}(U_{\alpha_j}))_{j=1}^{n}$, so $(U_{\alpha_j})_{j=1}^{n}$ is a finite subcover of $f(X)$.
"in part.": $f(X) \subset \mathbb{R}$ is comp. $\Rightarrow$ closed (Hausdorff) and bdd (take an open cover $\{(-N,N)\}_{N=1}^{\infty}$)
Bdd $\Rightarrow$ $\exists \sup f(X)$, since $f(X)$ is closed, $\sup f(X) \in f(X)$
(the same with inf) ■

$\boxed{\text{Cor}}$ Let X be comp., Y Hausdorff and $f: X \to Y$ continuous and bij. Then f is a homeomorphism, i.e., $f^{-1}: Y \to X$ is also cont.

Proof We show: preimages under f^{-1} of closed sets are closed.
Let $A \subset X$ closed. Then $(f^{-1})^{-1}(A) = f(A)$, and

$f(A)$ is comp. (f cont., A closed, X comp. $\Rightarrow$ A comp.)
Y Hausdorff $\Rightarrow$ $\underbrace{f(A)}_{\text{comp.}} \subset Y$ is closed. ■

$\boxed{\text{Thm}}$ (Tychonoff)
Let $(X_\alpha)_{\alpha \in I}$ be a family of TS. Then
arbitrary

$$X := \prod_{\alpha \in I} X_\alpha \text{ is comp.} \iff \forall X_\alpha \text{ comp.}$$

Recall: product top. is generated by cylinder sets of the form OO A_1 OO A_2 ...

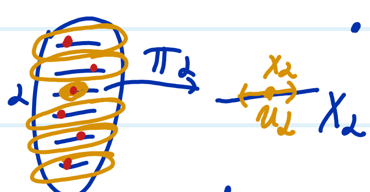
Proof ($\Rightarrow$) Let X be comp. Since the projections $\pi_\alpha : X \to X_\alpha$ are cont. (why?), $\pi_\alpha(X) = X_\alpha$ is comp.

($\Leftarrow$) Let $\mathcal{F}$ be an ultrafilter on X. Consider for $\alpha \in I$ the image filter $\pi_\alpha \mathcal{F} := \{\text{supersets of } \pi_\alpha(F), F \in \mathcal{F}\}$

- $\pi_\alpha(\mathcal{F})$ is ultrafilter on X_α
(if it is not maximal $\Rightarrow$ the preimage filter $\mathcal{F}$ also not maximal ↯)
- X_α comp. $\Rightarrow$ $\exists x_\alpha \in X_\alpha$ with $\pi_\alpha(\mathcal{F}) \to x_\alpha$.

We show: $\mathcal{F} \to x := (x_\alpha)_{\alpha \in I}$.
it suffices to show: $\forall\, \mathcal{U} \in \mathfrak{U}(x)$ from a nbhd bas's of product top. (sets of form $\pi_\alpha^{-1}(\mathcal{U}_\alpha)$)
$\mathcal{U} \in \mathcal{F}$.
Take $\mathcal{U} := \pi_\alpha^{-1}(\mathcal{U}_\alpha)$ for some α, $\mathcal{U}_\alpha$ open in X_α with $x \in \mathcal{U}$, i.e., $x_\alpha \in \mathcal{U}_\alpha$. Then:

- $\mathcal{U} \in \mathfrak{U}(x)$
- $\mathcal{U}_\alpha \in \pi_\alpha(\mathcal{F})$ since $\pi_\alpha(\mathcal{F}) \to x_\alpha$, i.e.,
$\exists\, F \in \mathcal{F}$ with $\mathcal{U}_\alpha = \pi_\alpha(F)$
$\Rightarrow \mathcal{U} = \pi_\alpha^{-1}(\mathcal{U}_\alpha) = \pi_\alpha^{-1}(\pi_\alpha(F)) \supset F$,
i.e., $\mathcal{U} \in \mathcal{F}$.



So $\mathcal{F} \to x$, i.e., X is comp. ■