$\boxed{\text{Cor.}}$ (Heine-Borel)

$A \subset \mathbb{R}^n$ is comp. $\Leftrightarrow$ A is closed and bdd.

Proof ($\Rightarrow$) • bdd: take the open cover $\{U_n(0)\}_{n=1}^{\infty}$ open balls with radius n

• closed since compact and $\mathbb{R}^n$ is Hausdorff.

($\Leftarrow$) A bdd $\Rightarrow$ $A \subset \underbrace{[a,b]^n}_{\text{comp. by Tychonoff}}$ for some $a<b$

A closed $\Rightarrow$ also comp. ∎

Exerc. $X := \prod_\lambda X_\lambda$ is Hausdorff $\Leftrightarrow$ $\forall X_\lambda$ is Hausdorff

(so Tychonoff holds also if one takes Hausdorff property to the def. of compactness)

III.2. Countable compactness and sequential comp.

$\boxed{\text{Def}}$ A top. space X is called

- countably compact if $\forall$ countable open cover has a finite subcover
- sequentially comp. if $\forall$ sequence in X has a convergent subsequence.

$\boxed{\text{Prop.}}$ A TS X is countably compact $\Leftrightarrow$ $\forall$ seq. in X has an $\underbrace{\text{accumulation point}}_{\forall U \in \mathcal{U}(x)\ \exists \infty\text{-many } x_n \in U}$

Proof ($\Rightarrow$) Let X be count. compact and $(x_n) \subset X$. Consider

$$U_k := X \setminus \underbrace{\overline{\{x_n : n \geq k\}}}_{=:A_k} \quad \text{- open}$$

Assume that (x_n) has no accum. point, i.e., $\bigcap_{k=1}^{\infty} A_k = \emptyset$

This means that $\bigcup_k U_k = X$ - open cover (count.!)

By assumption on X $\exists$ finite subcover, i.e., $\exists m$: $\bigcap_{k=1}^{m} A_k = \emptyset$ (we use $A_k \downarrow$)

↯ $\left(x_m \in \bigcap_{k=1}^{m} A_k \right)$

($\Leftarrow$) Let (U_n) be a count. open cover. Assume $\nexists$ finite subcover.

Take $x_n \in X \setminus \bigcup_{j=1}^{n} U_j$. By assumption $\exists x$ accum. point of (x_n).

Since (U_n) is a cover, $\exists j_0$: $x \in U_{j_0}$. By constr., $x_j \notin U_{j_0}$ $\forall j \geq j_0$,

so $\nexists$ ∞-many elements in U_{j_0} ↯ (not an accum. point) ∎

$\boxed{\text{Cor}}$ X count. compact, $A \subset X$ closed $\Rightarrow$ A count. comp.

(the same for sequential comp., too)

Rem. We see that both comp. and sequent. comp. imply count. comp. No further directions hold, i.e.,

$$\begin{tikzcd} \text{comp.} \arrow[rr, shift left, "\not\Rightarrow \ (2)"] \arrow[dr, shift left, Rightarrow] & & \text{sequential comp.} \arrow[ll, shift left, "\not\Leftarrow \ (3)"] \arrow[dl, shift left, Rightarrow] \\ & \text{countable comp.} \arrow[ul, shift left, "\not\Rightarrow \ (1)"] \arrow[ur, shift left, "\not\Rightarrow \ (4)"'] & \end{tikzcd}$$

$\boxed{\text{Ex}}$ (1: count. comp. $\not\Rightarrow$ comp.)

Consider $[0,1]^{\mathbb{R}}$ with the product top. and def.

$$X := \left\{ \begin{array}{l} (x_t)_{t\in\mathbb{R}} \subset [0,1]^{\mathbb{R}} \text{ s.t. } x_t \neq 0 \text{ for} \\ \text{at most count. many } t \in \mathbb{R} \end{array} \right\}$$

Then:

exerc. $\left\{ \begin{array}{l} \bullet\ X \text{ is not closed in } [0,1]^{\mathbb{R}}, \text{ so not comp.} \\ \bullet\ X \text{ is count. compact} \end{array} \right.$

even dense!

(Tychonoff: $[0,1]^{\mathbb{R}}$ comp. and Hausdorff)

separation by cylinder sets

Hint: show first that $\forall$ countable subset of $X \subset$ set homeom. to (at most) count. product of $[0,1]$.

Rem. This example also shows: $\left. \begin{array}{l} X \text{ Hausdorff} \\ A \subset X \text{ count. comp.} \end{array} \right\} \not\Rightarrow A$ closed

$\boxed{\text{Ex}}$ (2: comp. + Hausdorff $\not\Rightarrow$ seq. comp.)

$X := \{0,1\}^{P(\mathbb{N})}$ — all subsets of $\mathbb{N}$

- X is comp. (Tychonoff) and Hausdorff (since $\{0,1\}$ is so) with discrete top.
- X is not seq. comp.:

Consider $(x_n)_{n=1}^{\infty} \subset X$ def. by:

$$(x_n)_M := \begin{cases} 0 & \text{if } n \notin M \\ 1 & \text{if } n \in M \text{ and } \#\{m \in M : m < n\} \text{ is even} \\ 0 & \text{— " — odd} \end{cases}$$

for $M \subset \mathbb{N}$.

Then (x_n) has no conv. subseq.: Assume $\exists (n_k)\nearrow$ s.t. (x_{n_k}) conv. Consider $M := \{n_1, n_2, \dots\}$. Then

$$(x_{n_k})_M = \begin{cases} 1 & \text{if } 2 \mid n \\ 0 & \text{if not} \end{cases} \quad \text{— does not conv.!}$$

Recall: convergence in product spaces = coordinatewise conv.

$\boxed{\text{Ex}}$ (3: seq. comp. $\not\Rightarrow$ comp.)

idea: use example from (1)

$$X = \left\{ \begin{array}{l} (x_t)_{t\in\mathbb{R}} \subset [0,1]^{\mathbb{R}} : x_t \neq 0 \text{ for} \\ \text{at most count. many } t \in \mathbb{R} \end{array} \right\}$$

We know: X is not compact. To show: seq. comp.

Let $(x^{(n)})$ be a seq. in X and $D \subset \mathbb{R}$ count. s.t.
$x^{(n)}_t = 0 \ \forall t \notin D \ \forall n$.
Write $D := \{d_1, d_2, \dots\}$. Diagonal argument:
take a subseq. where d_1-coord. conv., then a subseq. of it where d_2-coord. conv. etc. exerc.

Rem. For ④ use the counterex. to ② (if ④ false, ② also false)
(this example: comp. $\Rightarrow$ count. comp.)

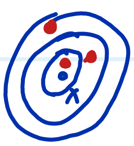
Good news:

Prop if X satisfies $\underbrace{\text{1. countability axiom}}_{\forall x \text{ has a count. neighb. basis}}$, then

$$X \text{ seq. comp} \Leftrightarrow X \text{ count. comp.}$$

Proof (of $\Leftarrow$, other direction see above)
Let X be count. comp. and sat. 1. count. axiom.
Take $(x_n) \subset X$. By assumption $\exists x$ accum. point of (x_n).
Let $(U_n)_{n=1}^{\infty}$ be count. neighb basis of x. Then

$$V_n := \bigcap_{j=1}^{n} U_j$$



also form a (count.) neighb basis of x with $V_1 \supset V_2 \supset V_3 \dots$
($\forall U$ neighb. of x $\exists n$: $U_n \subset U$, then $V_n \subset U_n \subset U$)
Def. a subseq. (x_{n_k}) of (x_n) inductively:

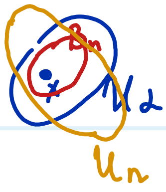
- Take $x_{n_1} \in V_1$
- take $n_{k+1} > n_k$ with $x_{n_k} \in V_k$
  possible since accum. point

We show: $x_{n_k} \to x$.
Let $U \in \mathcal{U}(x)$. Since (V_n) is a neighb. basis of x, $\exists k_0$: $V_{k_0} \subset U$
Then $\forall k \geq k_0$ $\quad x_{n_k} \in V_k \subset V_{k_0} \subset U$

Prop. if X satisf. $\underbrace{\text{2. count. axiom}}_{\exists \text{ count. basis of top.}}$, then

$$X \text{ comp.} \Leftrightarrow X \text{ count. comp.}$$

Proof ($\Rightarrow$) always
($\Leftarrow$) **Proof 1**: Use Alexander subbasis thm – exerc.
Proof 2 We show:
2. count. axiom $\Rightarrow$ $\underbrace{\forall \text{ open cover has an (at most) count. subcover}}_{\text{such spaces are called Lindelöf}}$
(then clear)
Let $(U_\alpha)_{\alpha \in I}$ be an open cover.
For every B_n from a countable basis (B_n) of top.,
if $\exists U_\alpha \supset B_n$, call this U_α U_n.



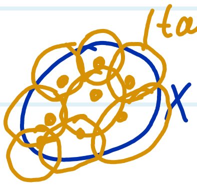
We claim: (U_n) is an (at most) count. subcover.
Let $x \in X$. Since (U_α) is a cover, $\exists \alpha$ with $x \in U_\alpha$.
Since (B_n) is a basis of top., $\exists n$: $x \in B_n \subset U_\alpha$,
so $x \in U_n$ for this n. So it is a cover! ■

$$\boxed{\begin{tikzcd}
\text{comp.} \arrow[rr, phantom, "\not\Leftrightarrow"] \arrow[dr, Rightarrow, shift left] & & \text{seq. comp.} \arrow[dl, Rightarrow, shift right] \\
& \text{count. comp} \arrow[ul, Rightarrow, shift left, "\text{if 2. count. axiom holds}"] \arrow[ur, Rightarrow, shift right, "\text{if 1. count. axiom holds}"'] &
\end{tikzcd}}$$

$\boxed{\text{Thm}}$ For metric spaces,

$$\text{comp.} \Leftrightarrow \text{count. comp.} \Leftrightarrow \text{seq. comp.}$$

<u>Proof</u> Since metric spaces sat. 1. count. axiom, (consider $(U_{1/n}(x))_{n=1}^{\infty}$ as count. neighb. basis of x)
we only need to show:
$$\text{count. comp.} \Rightarrow \text{comp.}$$
We show: X metric, count. comp. $\Rightarrow$ X satisf. 2. count. axiom
We first show: X is <u>separable</u> ($\exists$ at most count. dense set)
Let $n \in \mathbb{N}$. Since $\forall$ seq. in X has an accumulation point (count. comp.),
$\nexists$ infinite set $E \subset X$ with $d(x,y) \geq \frac{1}{n}$ $\forall x \neq y$, $x, y \in E$ (why?)
So $\exists$ finite set $K_n \subset X$ with the property
$$\forall x \in X \ \exists y \in K_n : d(x,y) < \tfrac{1}{n}$$ – balls with centers in K_n and radius $1/n$ cover X.
(take y_1 arb., y_2 with $d(y_1, y_2) \geq \frac{1}{n}$ (if $\exists$) etc, only finitely many are possible)



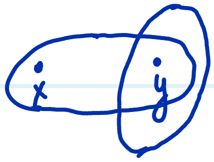
Then
$$K := \bigcup_{n=1}^{\infty} K_n$$
is an at most count. <u>dense</u> set in X ($\forall x \ \forall n \ \exists y \in K$ with $d(x,y) < \frac{1}{n}$)
Recall: $\forall$ sep. metric space satisf. 2. count. axiom.
$$\mathcal{B} := \{ U_{1/n}(y),\ n \in \mathbb{N}, y \in K \}$$
is a count. basis of topology (why?).
So X is compact $\left(\text{comp.} \xLeftarrow{\text{if 2. count. axiom}} \text{count. comp.} \quad \text{seq. comp.} \right)$ ■

<u>Rem.</u> 1) Since 2. count. axiom $\Rightarrow$ 1. count. axiom, all three comp. notions are equiv. for spaces with 2. count. axiom
2) <u>Countable</u> (at most) products of seq. compact spaces are seq. comp. (exerc)
But even a product of 2 count. compact spaces does not need to be count. comp. (no proof)

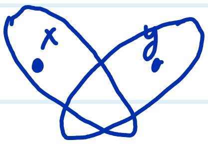
IV Separation axioms and continuous extensions of functions

IV.1 Separation axioms

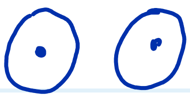
Def A TS (X,τ) is called



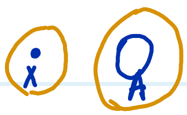
- T_0 space if ∀ 2 different point (at least) one has a neighb. without the other point, i.e., $\forall x,y\in X$ with $x\neq y$ $(\exists U\in\mathcal{U}(x)$ with $y\notin U)$ or $(\exists V\in\mathcal{U}(y)$ with $x\notin V)$



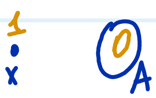
- T_1 space if ∀ 2 diff. points each one —''—, i.e., $\forall x,y\in X$ with $x\neq y$ $\exists U\in\mathcal{U}(x)\ V\in\mathcal{U}(y): y\notin U,\ x\notin V$.



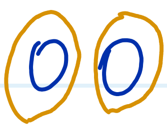
- T_2 space (or Hausdorff space) if ∀ 2 point have disjoint neighb., $\forall x,y\in X$ with $x\neq y$ $\exists U\in\mathcal{U}(x)\ \exists V\in\mathcal{U}(y): U\cap V=\emptyset$



- T_3 space if ∀ closed set $A\subset X$ and $\forall x\notin A$ have disjoint neighb., i.e., $\forall A\subset X$ closed $\forall x\notin A$ $\exists U\supset A$ open $\exists V\in\mathcal{U}(x): U\cap V=\emptyset$



- T_{3a} space if $\forall A\subset X$ closed $\forall x\notin A$ $\exists f:X\to[0,1]$ continuous with $f(x)=1$ and $f|_A\equiv 0$ (i.e., cont. fcts separate points from open sets)



- T_4 space if ∀ 2 disjoint closed sets A, B have disjoint neighb., i.e., $\forall A,B\subset X$ closed with $A\cap B=\emptyset$ $\exists U\supset A$ open $\exists V\supset B$ open : $U\cap V=\emptyset$

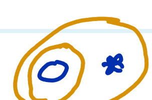
Rem. These properties are called axioms or separation axioms.

Connection between the sep. axioms

(a) $T_2 \underset{\not\Leftarrow}{\Rightarrow} T_1 \underset{\not\Leftarrow}{\Rightarrow} T_0$

Proof: $T_2\Rightarrow T_1\Rightarrow T_0$ clear

$T_0\not\Rightarrow T_1$ $X=\{0,*\}$, $\tau:=\{\emptyset,X,\{0\}\}$



$T_1\not\Rightarrow T_2$ ~~exerc. find X with 4 points~~

other ex.: $(\mathbb{N}$, cofinite top.) ∎

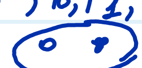
(b) $T_{3a}\Rightarrow T_3$.

Proof Let $f:X\to[0,1]$ be cont., $f(x)=1$, $f|_A=0$ $(x\notin A$, A closed)

Def $U:=f^{-1}([0,\frac{1}{2}))$, $V:=f^{-1}((\frac{1}{2},1])$ open, disjoint, $A\subset U$, $x\in V$ ∎

(c) $T_3\not\Rightarrow T_1$, $T_3\not\Rightarrow T_2$, $T_3\not\Rightarrow T_0$

Ex: $X:=\{0,*\}$, $\tau=\{\emptyset,X\}$, T_3 holds, T_0,T_1,T_2 do not hold



(d) $T_4\not\Rightarrow T_3$

Ex $X=\{1,2,3\}$ $\tau:=\{\emptyset,\{1\},\{1,2\},\{1,2,3\}\}$

- X is not T_3 : $x = \{1\}$, $A = \{3\}$ – closed
- X is T_4: closed sets are $\emptyset, \{2,3\}, \{3\}, \{1,2,3\}$,
  so A, B closed, disjoint $\Rightarrow A = \emptyset$ or $B = \emptyset$

(e) X metric $\Rightarrow T_0, T_1, T_2, T_3, T_{3a}, T_4$.

<u>Proof</u> T_2 clear ⊙ ⊙ take balls with radius $\frac{d(x,y)}{3}$. ($\Rightarrow T_0, T_1$)

T_{3a} ($\Rightarrow T_3$) and T_4 : construct cont. fct $f: X \to [0,1]$ with $f|_A = 0$, $f|_B = 1$ using metric exerc.

(For T_{3a}: A $\cdot x_0$: use $f(x) := \frac{d(x,A)}{d(x_0,A)}$) "■" :)

<u>Rem.</u> Be careful: points are not always closed!

$\boxed{\text{Prop.}}$ For a TS X TFAE:

(i) X is T_1
(ii) $\forall$ one-point set is closed
(iii) $\forall A \subset X$ is intersection of all its <u>neighb.</u> (open set $\supset A$)

<u>Proof</u> (i) $\Rightarrow$ (ii) $|X| = 1$ clear

For $|X| \geq 2$: Let $x \in X$ and $y \neq x$.

By T_1 $\exists\, U_y \in \mathcal{U}(y)$: $x \notin U_y$. So

$$X \setminus \{x\} = \bigcup_{y \neq x} U_y \text{ – open}$$

(ii) $\Rightarrow$ (iii) Let $A \subset X$ and take $x \notin A$.

By (ii) $X \setminus \{x\}$ is open and $\supset A$, so

$$A = \bigcap \{X \setminus \{x\} : x \notin A\}$$

and $\forall$ other open neighb. of A is in this $\bigcap$ ($\bigcap$ cannot get smaller than A)

(iii) $\Rightarrow$ (i) By assumption, $\forall x \in X$

$$\{x\} = \bigcap_{U \in \mathcal{U}(x)} U .$$

Take $y \neq x$. Since $y \notin \bigcap_{U \in \mathcal{U}(x)} U$, $\exists\, U \in \mathcal{U}(x)$ with $y \notin U$

The same argument gives $\exists\, V \in \mathcal{U}(y)$ with $x \notin V$. ■

<u>Rem.</u> 1) T_0 and T_1 are called weak sep. axioms

2) it follows from this prop. that

$$T_4 + T_1 \Rightarrow T_3$$
$$T_4 + T_1 \Rightarrow T_2$$
$$T_3 + T_1 \Rightarrow T_2$$

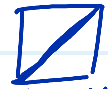
$\boxed{\text{Prop.}}$ (Characterisation of Hausdorff spaces)

For a TS X TFAE:

(i) X is T_2, i.e., Hausdorff
(ii) $\forall$ convergent filter/net (also sequences) has exactly 1 limit point. – limits are unique

(iii) $\forall x \in X$ $\quad \{x\} = \bigcap_{\mathcal{U} \in \mathcal{V}(x)} \overline{\mathcal{U}}$

(iv) The diagonal $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$.



Proof (i) $\Rightarrow$ (ii) We do here filters (nets – exerc.)

Let $\mathcal{F}$ be a conv. filter, say, $\mathcal{F} \to x$, i.e., $\mathcal{V}(x) \subset \mathcal{F}$.

Assume $\mathcal{F} \to y$ with $y \neq x$, i.e., $\mathcal{V}(y) \subset \mathcal{F}$.

Take $\mathcal{U}, V$ as in T_2, then $\underbrace{\mathcal{U} \cap V}_{=\emptyset} \in \mathcal{F}$ ↯

(ii) $\Rightarrow$ (iii) For filters (nets: exerc.)

$\subset$ clear

$\supset$ Let $y \in \bigcap_{\mathcal{U} \in \mathcal{V}(x)} \overline{\mathcal{U}}$. Then y is an accumul. point of ($\Leftrightarrow \forall F \in \mathcal{F}\ y \in \overline{F}$)

the filter $\mathcal{F}_{\mathcal{V}(x)}$ = supersets of $\mathcal{V}(x)$

We know (see Section II.2): $\exists\ \tilde{\mathcal{F}} \supset \mathcal{F}$ with $\tilde{\mathcal{F}} \to y$.

But $\tilde{\mathcal{F}} \to x$ by $\mathcal{V}(x) \subset \mathcal{F} \subset \tilde{\mathcal{F}}$. By (ii) $x = y$.

(iii) $\Rightarrow$ (i) Let $x \neq y$. By (iii) $y \notin \bigcap_{\mathcal{U} \in \mathcal{V}(x)} \overline{\mathcal{U}}$, so $\exists\ \mathcal{U} \in \mathcal{V}(x)$: $y \notin \overline{\mathcal{U}}$.

Then $V := X \setminus \overline{\mathcal{U}}$ is a neighb. of y disjoint from $\mathcal{U}$. ■

(i) $\Leftrightarrow$ (iv) exerc.

$\boxed{\text{Ex}}$ (Uniqueness of limits of seq. $\not\Rightarrow T_2$)

Ex. at the end of sect. II.2:

$(\mathbb{R}, \tau)$ with $\tau := \{A \subset \mathbb{R} : \mathbb{R} \setminus A$ is at most count.$\} \cup \{\emptyset\}$

"X"

- X is not T_2: $\forall\ U, V$ open, $\neq \emptyset$ $\quad U \cap V \neq \emptyset$
- Limits of seq. are unique:

  $x_n \to x \Leftrightarrow x_n = x \quad \forall n \geq n_0$ for some n_0

  $\Leftarrow$ clear

  $\Rightarrow$ take $\mathcal{U} := \mathbb{R} \setminus \{x_n \neq x,\ n \in \mathbb{N}\}$