Recall: for $f: A \to [-1,1]$ continuous we first construct $g_n : X \to [-1,1]$, $n \in \mathbb{N}$ with

(a) $|g_n(x)| \leq 1 - \left(\frac{2}{3}\right)^n \quad \forall x \in X$

(b) $|f(x) - g_n(x)| \leq \left(\frac{2}{3}\right)^n \quad \forall x \in A$

(c) $|g_{n+1}(x) - g_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n \quad \forall x \in X.$

and def $g_0 := 0$ (satisf (a) and (b)).

Assume $g_1, \dots, g_n$ are given with (a)-(c). Def.

$$B_{n+1} := \left\{ x \in A : f(x) - g_n(x) \geq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n \right\}$$

$$C_{n+1} := \left\{ \quad - \text{''} - \quad \leq -\frac{1}{3} \left(\frac{2}{3}\right)^n \right\}$$

- disjoint, closed (why?) if both are $\neq \emptyset$, by Urysohn's lemma

$\exists r_n : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^n, \frac{1}{3}\left(\frac{2}{3}\right)^n\right]$ continuous with

$$r_n|_{B_{n+1}} = \frac{1}{3}\left(\frac{2}{3}\right)^n, \quad r_n|_{C_{n+1}} = -\frac{1}{3}\left(\frac{2}{3}\right)^n.$$

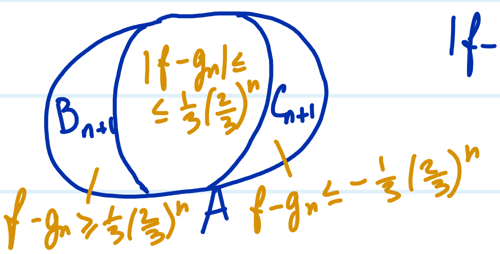
Now, if both $C_{n+1} = B_{n+1} = \emptyset$, take $r_n := 0$ and if one is $= \emptyset$, take $r_n := \frac{1}{3}\left(\frac{2}{3}\right)^n$ if $C_{n+1} = \emptyset$ or $r_n := -\frac{1}{3}\left(\frac{2}{3}\right)^n$ if $B_{n+1} = \emptyset$, resp.

Def. $g_{n+1} := g_n + r_n$ - cont. on X.

(a) $|g_{n+1}| \leq |g_n| + |r_n| \leq 1 - \left(\frac{2}{3}\right)^n + \frac{1}{3}\left(\frac{2}{3}\right)^n = 1 - \left(\frac{2}{3}\right)^{n+1}$

(b) On $A \setminus (B_{n+1} \cup C_{n+1})$ we have

$$|f - g_{n+1}| \leq |f - g_n| + |r_n| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n + \frac{1}{3}\left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^{n+1}$$



On B_{n+1}:

$$f - g_{n+1} = \underbrace{f - g_n}_{\frac{1}{3}\left(\frac{2}{3}\right)^n \leq \cdot \leq \left(\frac{2}{3}\right)^n} - \frac{1}{3}\left(\frac{2}{3}\right)^n \in \left[0, \left(\frac{2}{3}\right)^{n+1}\right]$$

Analogously, on C_{n+1}

$$f - g_{n+1} = \underbrace{f - g}_{-\left(\frac{2}{3}\right)^n \leq \cdot \leq -\frac{1}{3}\left(\frac{2}{3}\right)^n} + \frac{1}{3}\left(\frac{2}{3}\right)^n \in \left[-\left(\frac{2}{3}\right)^{n+1}, 0\right).$$

So (b) holds

(c) $|g_{n+1} - g_n| = |r_n| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$ ✓

<u>Step 2: construction of F</u>

By (c) $\forall n, k \in \mathbb{N}$

$$|g_{n+k} - g_n| \leq |g_{n+k} - g_{n+k-1}| + \dots + |g_{n+1} - g_n| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n \underbrace{\sum_{j=0}^{k-1} \left(\frac{2}{3}\right)^j}_{\leq \frac{1}{1-\frac{2}{3}} = 3}$$

$$\leq \left(\frac{2}{3}\right)^n \xrightarrow[n \to \infty]{} 0,$$

i.e., (g_n) is uniformly Cauchy. Since $\mathbb{R}$ is complete,

$\exists F(x) := \lim_{n\to\infty} g_n(x) \quad \forall x \in X.$

- $|F(x)| \le 1 \quad \forall x$ by (a)
- $F|_A = f$ by (b)
- F is continuous as uniform limit of cont. fcts. $\blacksquare$

Rem. 1) We have shown more, namely $\exists$ cont. extension of f with the same bound as f

2) The proof of the general case consists of two more steps:

Step 1 Show that $\forall$ cont. $f: A \to (-1,1)$
$\exists F: X \to (-1,1)$ cont. with $F|_A = f$
idea: multiply the constructed continuation $\tilde{F}: X \to \underline{[-1,1]}$ from the above thm with

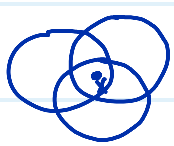
$$g|_A = 1, \quad g|_{\tilde{F}^{-1}\{-1,1\}} = 0$$

($\exists$ by Urysohn)

Step 2 For $f: A \to \underline{\mathbb{R}}$ use $(-1,1) \sim \mathbb{R}$ and step 1 exerc.

IV.3 Partitions of unity

$\boxed{\text{Def}}$. Let $\mathcal{A} := (A_\alpha)_{\alpha\in I}$ be a family of subsets of X. It is called



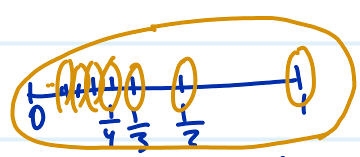
- locally finite if $\forall x \in X \ \exists U \in \mathfrak{U}(x)$ s.t. $U \cap A_\alpha \neq \emptyset$ for at most finitely many α
- point finite if $\forall x \in X \quad x \in A_\alpha$ for —"—
- cover of X if $\bigcup_{\alpha\in I} A_\alpha = X$
- open cover if cover and $\forall A_\alpha$ is open.

Rem. Locally finite $\Rightarrow$ point finite

$\boxed{\text{Ex}}$ ($\not\Leftarrow$ in general)

$X := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with natural top.

$\mathcal{A} := \{\{\frac{1}{n}\} : n \in \mathbb{N}\} \cup X$

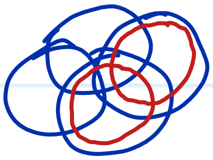


Then $\mathcal{A}$ is point finite (with at most 2 sets $\forall x$) but not locally finite ($x=0$ does not have the desired neighb.)

$\boxed{\text{Prop.}}$ Let X be T_4, $(U_\alpha)_{\alpha\in I}$ be a point finite open cover of X. Then $\exists$ open cover $(V_\alpha)_{\alpha\in I}$ of X with

$$\overline{V_\alpha} \subset U_\alpha$$

Proof (sketch) Consider



$\mathcal{M} := \{ (V_k)_{k\in K} : K \subset I,\ \overline{V_k} \subset U_k \ \forall k \in K$ and $(V_k)_{k\in K} \cup (U_k)_{k\in I\setminus K}$ is open cover of $X \}$

Then $K = \emptyset$ is ok, so $\mathcal{M} \neq \emptyset$.

Def $\leq$ on $\mathcal{M}$ by $\underset{= (V_k)_{k\in K}}{C} \leq \underset{= (V'_k)_{k\in K'}}{C'} \overset{\text{def}}{\iff} K \subset K',\ V_k = V'_k \ \forall k \in K$

This is a partial order (why?) and $\forall$ totally ordered subset of $\mathcal{M}$ has an upper bound (why? exerc.)
take $K = \bigcup K_i$ — use point finite

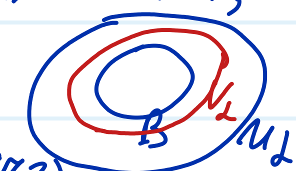
Lemma of Zorn: $\exists$ max. element $C^* = (V_k)_{k\in K^*}$. We want: $K^* = I$.
Assume not. $\exists \alpha \in I \setminus K^*$.
Consider

$$B := X \setminus \Big(\bigcup_{k\in K^*} V_k \cup \bigcup_{\substack{l\in I\setminus K^* \\ l\neq\alpha}} U_l \Big)$$

- B is closed
- $B \subset U_\alpha$ (since $\bigcup_{k\in K^*} V_k \cup \bigcup_{l\in I\setminus K^*} U_l$ is a cover)

Since X is T_4, see observation in the proof of Urysohn's lemma, $\exists\ V_\alpha$ open with

$$B \subset V_\alpha \subset \overline{V_\alpha} \subset U_\alpha$$



So $C' := \{ (V_k)_{k\in K^*}, V_\alpha \}$ satisfies $C' > C^*$ (why in $\mathcal{M}$?)
↯ (C^* is not maximal). So $K^* = I$ and $(V_k)_{k\in K^*}$ is a desired cover. ■

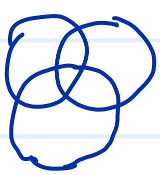
$\boxed{\text{Def + Prop}}$ (Existence of partitions of unity)

Let X be T_4 and $(U_\alpha)_{\alpha\in I}$ be a locally finite open cover of X.
Then $\exists$ a partition of unity corresp. to $(U_\alpha)_{\alpha\in I}$, i.e., a family $(f_\alpha)_{\alpha\in I}$ of continuous fcts on X with

(a) $f_\alpha \geq 0 \quad \forall \alpha$

(b) $\forall \alpha \in I \quad \underset{= \overline{\{x : f_\alpha \neq 0\}}}{\operatorname{supp} f_\alpha} \subset U_\alpha$ and (here: in particular)

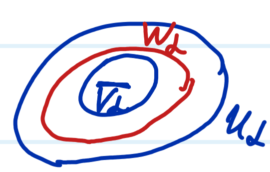
$(\operatorname{supp} f_\alpha)$ form a locally finite system



(c) $\sum_{\alpha\in I} f_\alpha(x) = 1 \quad \forall x \in X$.
finite sum because loc finite $\Rightarrow$ point finite

Proof Prop. above: $\exists (V_\alpha)_{\alpha\in I}$ open cover with $\overline{V_\alpha} \subset U_\alpha$
Since X is T_4, $\forall \alpha\ \exists W_\alpha$ open with

$$\overline{V_\alpha} \subset W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$$



Lemma of Urysohn: $\exists\, g_\alpha : X \to [0,1]$ cont. with

$$g_\alpha|_{\overline{V_\alpha}} = 1, \quad g_\alpha|_{X \setminus W_\alpha} = 0$$

- $\operatorname{supp} g_\alpha \subset \overline{W_\alpha} \subset U_\alpha$
- $g(x) := \sum_{\alpha \in I} g_\alpha(x)$ is well defined (finite sum) and cont. fct.
  use loc. finite
- (V_α) cover $\Rightarrow g(x) \geq 1 \quad \forall x \in X$

Def. $\quad f_\alpha := \dfrac{g_\alpha}{g}$ cont. Then

$$\operatorname{supp} f_\alpha \subset U_\alpha, \quad \sum_{\alpha \in I} f_\alpha = 1. \qquad \blacksquare$$

Rem. One can show:
For a Hausdorff TS X $\underline{\forall}$ open cover $\exists$ partition of unity corresp. to it $\Longleftrightarrow$ X is paracompact, i.e., (Hausdorff and) $\forall$ open cover $\mathcal{U}$ $\exists$ open cover $\mathcal{V}$ with

1) $\mathcal{V}$ is locally finite
2) $\mathcal{V}$ is a refinement of $\mathcal{U}$, i.e.,
$\forall V \in \mathcal{V}\ \exists U \in \mathcal{U}$ with $V \subset U$.

$\Rightarrow$ Take $V_f := \{x : f(x) \neq 0\}$ for $f \in$ partition of unity to $\mathcal{U}$
open, locally finite, cover, refinement

$\Leftarrow$ without (idea: show first: paracomp $\Rightarrow T_4$, then as before)

Rem. Partitions of unity are used in many areas and allow to work locally

V Locally compact spaces and compactifications

V.1. Locally comp. spaces

Def X is called locally compact if $\forall x \in X$ $\exists$ compact superset of a neighb. U of x.

Rem. 1) X loc. compact $\Leftarrow$ $\forall x \in X\ \exists U \in \mathcal{U}(x) \cdot \overline{U}$ compact
if X Hausdorff

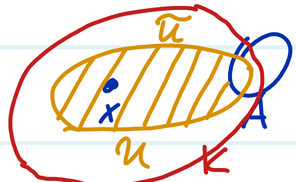
For $\Rightarrow$: $\exists U \in \mathcal{U}(x)\ \exists K \supset U$ comp. Then $\overline{U} \subset \overline{K} = K$, so $\overline{U}$ comp
Hausd. (K comp)

2) X comp. $\Rightarrow$ X locally comp. (Take $K=X$)

$\not\Leftarrow$ $\boxed{\text{ex}}$ $X=\mathbb{R}$ not comp., but $\forall x$ $[a,b]\supset(a,b)\ni x$

$\boxed{\text{Prop.}}$ $\forall$ locally comp. Hausdorff space is regular.

T_3+T_1



Proof Since $T_2 \Rightarrow T_1$, we have to show T_3.

Let $x\in X$, $A\subset X$ closed with $x\notin A$

Let $U\in \mathcal{U}(x)$ and $K\supset U$ comp.

- K is closed (X Hausd.)
- K is T_3 (since comp. Hausdorff spaces are $T_4 \Rightarrow T_3$)

Consider $\tilde{U} := U\cap(X\setminus A)\in\mathcal{U}(x)$, $\tilde{U}\subset K$

Since K is T_3, one can separate x and $\underbrace{K\setminus\tilde{U}}_{\text{closed}}$ by open sets

in part., $\exists V\in\mathcal{U}(x)$ with

$$V\subset\overline{V}\subset\tilde{U}\subset K\cap(X\setminus A)$$

Then V and $X\setminus\overline{V}$ separate x an A ($\overline{V}\subset X\setminus A \Leftrightarrow A\subset X\setminus\overline{V}$) why ∎

$\boxed{\text{Cor.}}$ Let X be loc comp. and Hausdorff. Then

$$\forall x\in X\ \forall U\in\mathcal{U}(x)\ \exists V\in\mathcal{U}(x) \text{ with}$$

$$V\subset\overline{V}\subset U,\ \overline{V} \text{ comp.}$$

Proof Let $x\in X$, $U\in\mathcal{U}(x)$. Since X is T_3, one can separate x and $X\setminus U$. in part., $\exists V\in\mathcal{U}(x)$ with $V\subset\overline{V}\subset U$

Let $K\supset U_0$ be comp. for some $U_0\in\mathcal{U}(x)$. Then

$$V\cap U_0\in\mathcal{U}(x),\ V\cap U_0\subset U,$$

and

$$V\cap U_0\subset\overline{V\cap U_0}\subset\overline{U_0}\subset K \text{ - comp.}$$

(K closed since X Hausd.)

So $\overline{V\cap U_0}$ is comp. (closed subset of a comp. set) and

$$\overline{V\cap U_0}\subset\overline{V}\subset U$$ ∎

Rem. in part., $\forall x$ has neighb' basis with compact closures.

$\boxed{\text{Ex}}$ (Hausdorff property cannot be dropped)

$X=\mathbb{R}\cup\{\infty\}$. Closed sets: X, $\emptyset$, $A\subset X$ if $\infty\notin A$ and A is at most countable

exerc: topology, X compact, but $\forall x\in\mathbb{R}$, $U=\mathbb{R}$ is a neighb. of x not containing a compact subset of a neighb. of x

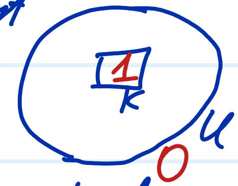
$\mathcal{U}(\infty)=\{X\}$

We now show more than regularity

$\boxed{\text{Thm}}$ (Urysohn's lemma for locally compact spaces)

Let X be locally comp. and Hausdorff, $K \subset X$ comp. and $\mathcal{U} \supset K$ open. Then $\exists$ continuous $f: X \to [0,1]$ with compact support s.t.

$$f|_K = 1, \quad \operatorname{supp} f \subset \mathcal{U}$$

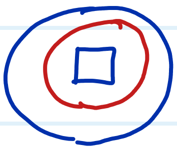


in part., $f|_{X \setminus \mathcal{U}} = 0$

Rem. So in such spaces one can separate comp. sets from disjoint closed sets by cont. fcts. Since $\forall \{x\}$ is comp., X is in part. completely regular. why?

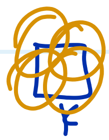
Proof Observation$^{T_{3a}+T_1}$ $K \subset \mathcal{U}$, K comp., $\mathcal{U}$ open $\Rightarrow \exists V$ open with

$$K \subset V \subset \overline{V} \subset \mathcal{U}$$



Proof of observ. Corollary above: $\forall x \in K \; \exists V_x \in \mathcal{U}(x)$ with

$$x \in V_x \subset \overline{V_x} \subset \mathcal{U}, \quad \overline{V_x} \text{ comp.}$$



Since K is comp. and $(V_x)_{x \in K}$ is an open cover of K, $\exists$ finite subcover $V_{x_1}, \dots, V_{x_n}$. Then

$$K \subset V_{x_1} \cup \dots \cup V_{x_n} \subset \overline{V_{x_1} \cup \dots \cup V_{x_n}} \overset{\text{why}}{=} \underbrace{\overline{V_{x_1}} \cup \dots \cup \overline{V_{x_n}}}_{\text{comp.-why?}} \subset \mathcal{U},$$

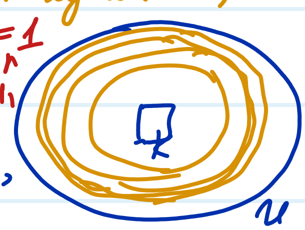
so $V_{x_1} \cup \dots \cup V_{x_n}$ is the desired neighb of K ■ of observ.

As in the proof of classical Urysohn construct $(\mathcal{U}_t)_{t \in [0,1]}$ open $\forall t$ with

$$K \subset \mathcal{U}_0 \subset \overline{\mathcal{U}_0} \subset \mathcal{U}_r \subset \overline{\mathcal{U}_r} \subset \mathcal{U}_s \subset \overline{\mathcal{U}_s} \subset \mathcal{U}_1 \subset \overline{\mathcal{U}_1} \subset \mathcal{U}$$

$\forall r < s$ in $(0,1)$ s.t. $\forall \overline{\mathcal{U}_r}$ is compact (first for dyadic t)

Def. now $\tilde{f}(x) := \inf\{t \in [0,1] : x \in \mathcal{U}_t\}$ on $\mathcal{U}_1$, $\tilde{f} = 1$ on $X \setminus \mathcal{U}_1$



As in the classical Urysohn, $\tilde{f}: X \to [0,1]$ is cont., $\tilde{f}|_K = 0$, $\tilde{f}|_{X \setminus \mathcal{U}_1} = 1$. Now,

$$f := 1 - \tilde{f}$$

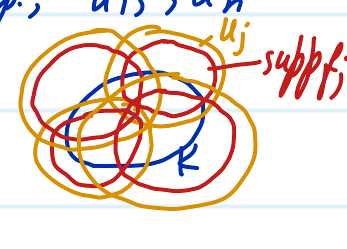
cont, $f|_K = 1$, $f|_{X \setminus \mathcal{U}_1} = 0$ (so also on $X \setminus \mathcal{U}$).

Since $\overline{\mathcal{U}_1} \subset \mathcal{U}$ and $\overline{\mathcal{U}_1}$ is compact, $\underbrace{\operatorname{supp} f}_{\text{closed}} \subset \underbrace{\overline{\mathcal{U}_1}}_{\text{comp.}} \subset \mathcal{U}$, so $\operatorname{supp} f$ is comp. ■

$\boxed{\text{Prop.}}$ (Partition of unity for locally comp spaces)

Let X be locally comp., Hausdorff, $K \subset X$ comp., $\mathcal{U}_1, \dots, \mathcal{U}_n$ an open cover of K. Then $\exists f_1, \dots, f_n : X \to [0,1]$ continuous with compact support s.t. $\operatorname{supp} f_j \subset \mathcal{U}_j \; \forall j$

$$\sum_{j=1}^{n} f_j(x) = 1 \quad \forall x \in K$$



<u>Proof</u> Let $x \in K$, take j: $x \in \mathcal{U}_j$. The corollary above from regularity:
$\exists V_x \in \mathcal{U}(x)$ with

$$x \in V_x \subset \overline{V_x} \subset \mathcal{U}_j, \quad \overline{V_x} \text{ comp.}$$

Since K is comp. and $(V_x)_{x\in K}$ is an open cover, $\exists x_1, \dots, x_m$ s.t.

$$K \subset V_{x_1} \cup \dots \cup V_{x_m}.$$

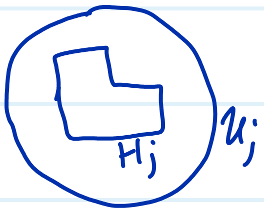
Let

$$H_j := \bigcup_{x_k \in \mathcal{U}_j} \overline{V_{x_k}} \text{ - comp. (why?)}, \; H_j \subset \mathcal{U}_j$$

finite union comp

By Urysohn above for loc. comp spaces $\exists$ cont $g_1, \dots, g_n : X \to [0,1]$ with comp. support s.t.

$$g_j|_{H_j} = 1, \quad \operatorname{supp} g_j \subset \mathcal{U}_j \quad \forall j.$$



Def.

$$f_1 := g_1$$
$$f_2 := (1-g_1)\cdot g_2$$
$$f_3 := (1-g_1)\cdot(1-g_2)g_3$$
$$\vdots$$
$$f_n := (1-g_1)\cdot\ldots\cdot(1-g_{n-1})g_n$$

- $\forall f_j$ is cont.
- $\underbrace{\operatorname{supp} f_j}_{\text{closed}} \subset \underbrace{\operatorname{supp} g_j}_{\text{comp.}} \subset \mathcal{U}_j$ comp. $\forall j$
- $f_1 + \dots + f_j = 1 - (1-g_1)\cdot(1-g_2)\cdot\ldots\cdot(1-g_j)$

<u>Proof</u>: induction on j.

$j=1$ $\quad f_1 = 1-(1-g_1) = g_1$ ✓

$j \to j+1$ $\quad f_1 + \dots + f_{j+1} \overset{\text{ind. hyp.}}{=} 1-(1-g_1)\cdot\ldots\cdot(1-g_j) + (1-g_1)\cdot\ldots\cdot(1-g_j)g_{j+1}$

$$= 1-(1-g_1)\cdot\ldots\cdot(1-g_j)(1-g_{j+1}) \quad ✓ \; \blacksquare$$

Since $\forall x \in K \; \exists V_{x_k}$ with $x \in V_{x_k}$ (subcover) and $g_j|_{H_j} = 1$ for j with $V_{x_k} \subset \mathcal{U}_j$ $(x \in V_{x_k} \subset H_j \subset \mathcal{U}_j)$, we have

$$f_1 + \dots + f_n = 1 \text{ on } K \quad \text{(because } \forall x \; \exists j: g_j(x) = 1) \; \blacksquare$$

<u>Question</u> when does local compactness transfer?

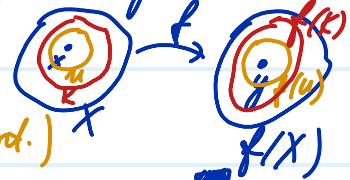
1) X loc. comp., $A \subset X$ closed $\Rightarrow$ (A, τ_A) loc. comp. (exer.)
subspace top.
2) False for arb. sets.

$\boxed{\text{Ex}}$ $X = \mathbb{R}$ (Hausdorff!), $A := \mathbb{Q}$ is <u>not</u> loc. compact (why?)
for ex., $\forall [a,b] \cap \mathbb{Q}$ is not comp.! (a, b, irrat.)

$\boxed{\text{Lemma}}$ Let X be loc. comp., Y TS, $f: X \to Y$ cont.

and open ($f(U)$ open $\forall U$ open). Then $f(X)$ is loc. comp.

Proof Let $y \in f(X)$ and take $x \in X : f(x) = y$. By assumption on X
$\exists\, U \in \mathcal{U}(x)\ \exists\, K \supset U$ comp. So



$$f(x) \in \underbrace{f(U)}_{\text{open}} \subset \underbrace{f(K)}_{\text{comp.}} \quad (K \text{ comp., } f \text{ cont.})$$

So $f(X)$ is loc. comp. ■

$\boxed{\text{Thm}}$ (Tychonoff for locally comp. spaces)

Let $((X_\alpha, \tau_\alpha))_{\alpha \in I}$ be a family of TS and $X := \prod_\alpha X_\alpha$. Then

$$X \text{ loc. comp.} \iff \begin{cases} \forall\, X_\alpha \text{ loc. comp.} \\ \text{at most finitely many } X_\alpha \text{ are } \underline{\text{not}} \text{ comp.} \end{cases}$$

Proof

($\Rightarrow$) Since $\forall\, \pi_\alpha : X \to X_\alpha,\ (x_\alpha) \mapsto x_\alpha$ is cont., surj and open (why? (image of a cylinder set is open))
proj. onto α th coord.

By lemma above, $\forall\, X_\alpha = \pi_\alpha(X)$ is loc. comp.

Let $x \in X$. Then $\exists$ cylinder set (basis) $U \in \mathcal{U}(x)\ \exists\, K \supset U$ comp.

Then for all but finitely many α

$$\pi_\alpha(U) = X_\alpha = \underbrace{\pi_\alpha(K)}_{\text{comp.!}} \text{ comp.} \quad (K \text{ comp, } \pi_\alpha \text{ cont.})$$

($\Leftarrow$) Let $x = (x_\alpha) \in X$ and let $\alpha_1, \dots, \alpha_n$ be such that X_α comp. $\forall \alpha \notin \{\alpha_1, \dots, \alpha_n\}$

Since $X_{\alpha_1}, \dots, X_{\alpha_n}$ are loc. comp.,

$\exists\, U_1 \in \mathcal{U}(x_{\alpha_1}), \dots, U_n \in \mathcal{U}(x_{\alpha_n})$ with comp. supersets $K_1, \dots, K_n$, resp.
($\subset X_{\alpha_1}$, $\subset X_{\alpha_n}$; $\subset X_{\alpha_1}$, $\subset X_{\alpha_n}$)

Consider the corr. cylinder set U with restrictions $U_1, \dots, U_n$ in the coord $\alpha_1, \dots, \alpha_n$ and the cylinder set K with restrictions $K_1, \dots, K_n$ in these coord. Then

$$x \in U \subset K, \quad K = \prod_\alpha K_\alpha \text{ comp. by Tychonoff.}$$

(K_α either X_α or K_j; if $\alpha \notin \{\alpha_1, \dots, \alpha_n\}$) ■

So X is loc. compact

$\boxed{\text{Thm}}$ (Baire for locally comp. spaces)

Let X be a locally comp. Hausdorff space and $(G_n)_{n=1}^\infty$ a sequence of open dense sets of X. Then

$$G := \bigcap_{n=1}^\infty G_n$$

is dense in X.

Rem. Spaces where $\forall$ countable intersection of open dense sets is dense are called Baire spaces. FA1: also complete metric spaces are Baire (weaker form)

Proof of Baire Let $\mathcal{U} \neq \emptyset$ open. To show: $G \cap \mathcal{U} \neq \emptyset$.

Def. $(\underbrace{\mathcal{U}_n}_{\text{open}})_{n=0}^{\infty}$ inductively as follows:

- $\mathcal{U}_0 := \mathcal{U}$
- Let $\underbrace{\mathcal{U}_0, \dots, \mathcal{U}_n}_{\text{open}}$ be def. Then $G_{n+1} \cap \mathcal{U}_n \neq \emptyset$ and open

  (G_{n+1} is dense) (G_{n+1} is open)

  Take $x \in G_{n+1} \cap \mathcal{U}_n$. By the corollary of regularity

  $\exists\, \mathcal{U}_{n+1} \in \mathfrak{U}(x)$ with
  $$x \in \mathcal{U}_{n+1} \subset \underbrace{\overline{\mathcal{U}_{n+1}}}_{\text{comp.}} \subset G_{n+1} \cap \mathcal{U}_n$$

Since $\underbrace{\overline{\mathcal{U}_n}}_{\text{comp.}} \searrow$, $\forall$ finitely many such sets have non-empty intersection

so, by $\overline{\mathcal{U}_n} \subset \overline{\mathcal{U}_1}$ comp. $\forall n$,
$$\bigcap_{n=1}^{\infty} \overline{\mathcal{U}_n} \neq \emptyset$$

(charact. of compactness with closed sets). Since $\overline{\mathcal{U}_n} \subset G_n$ and $\overline{\mathcal{U}_1} \subset G_1 \cap \mathcal{U}_0 = G_1 \cap \mathcal{U}$, we have
$$\emptyset \neq \bigcap_{n=1}^{\infty} \overline{\mathcal{U}_n} \subset \mathcal{U} \cap \Big(\underbrace{\bigcap_{n=1}^{\infty} G_n}_{G}\Big) = \mathcal{U} \cap G.$$

So G is dense. ∎

V.2. Compactifications

What if X is not compact?

Def. of X Let X, Y be TS. Then Y is called a **compactification** if Y is compact and
$$\exists\, i : X \to Y \text{ cont, open as } i : X \to \underbrace{i(X)}_{\text{with subspace top.}}, \text{ inj with}$$
$$\overline{i(X)} = Y$$

i.e., Y is comp. and X is homeom. to a dense subset of Y

Def. Let (X, τ) be a TS which is not comp. Def
$$X^* := X \cup \{\infty\}$$
$$\tau^* := \tau \cup \{X^* \setminus A,\ A \subset X \text{ closed and comp. in } (X,\tau)\}$$

The space (X^*, τ^*) is called **Alexandroff (one-point) compactification of (X, τ)**

(exerc.) τ^* is a topology

Prop. 1) (X^*, τ^*) is a compactif. of (X, τ)

2) (X^*, τ^*) is Hausdorff $\Leftrightarrow$ (X, τ) is loc comp., Hausdorff

Proof 1) • X^* comp.: Let $(\mathcal{U}_\alpha)_{\alpha \in I}$ be an open cover and let α_0 be s.t. $\infty \in \mathcal{U}_{\alpha_0}$. Then $X^* \setminus \mathcal{U}_{\alpha_0}$ is compact in X, and $(\mathcal{U}_\alpha \cap X)_{\alpha \neq \alpha_0, \alpha \in I}$ is an open

since sets $\subset X$ open in $X^* \Rightarrow$ in X

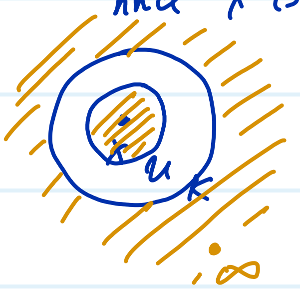
cover of it in X. So $\exists$ open subcover $(U_{\alpha_j}\cap X)_{j=1}^{n}$ of $X^*\setminus U_{\alpha_0}$ in X ($\Rightarrow$ in X^* by $\tau\subset\tau^*$), so $U_{\alpha_0}, U_{\alpha_1},\dots,U_{\alpha_n}$ is an open subcover of X^*.

- X homeom. to a dense subset of X^*

Consider $id: X\to X^*$

- id is inj
- id is open (since $\tau\subset\tau^*$)
- id is contin: For $U\in\tau^*\cap\tau$, $(id)^{-1}(U)=U$ open in X.
  For $U\in\tau^*\setminus\tau$ $\exists A$ closed $\underset{\text{in } X}{}$ $U=X^*\setminus A$, i.e.,
  $(id^{-1})(U)=X\setminus A$ open
- $id(X)=X$ is dense in X^*.
  Let $\emptyset\neq U\subset X^*$ open. To show: $U\cap X\neq\emptyset$.
  if $U\subset X$, then $U\cap X=U\neq\emptyset$.
  if $\infty\in U$, then $U=X^*\setminus A$, $A\subset X$ comp. in X.
  since X is not comp., $A\neq X$, i.e., $U\cap X\neq\emptyset$

2) $\Leftarrow$ To show: $\forall x\in X$ can be separated from ∞ by open sets (for $x,y\in X$ clear)
Let $x\in X$. Since X is loc. comp., $\exists U\in\mathcal{U}(x)$, $K\subset X$ comp. in X with $K\supset U$.
Since X is Hausdorff, K is closed, so $X^*\setminus K$ is open in X^*,
so neighb. of ∞ disjoint from U.



$\Rightarrow$ Since X^* is Hausdorff, $\{\infty\}$ is closed and hence $X\subset X^*$ is open in X^*. We know:

$$\exists V \text{ open in } X^* \quad x\in V\subset \underbrace{\overline{V}}_{\text{comp. in } X^*}\subset \underbrace{X}_{\text{open in } X^*}$$

Since $V\subset X$, $V\in\tau$.

We show: $\overline{V}$ is comp. in X (then X is loc comp.)
Let (U_α) be an open cover of $\overline{V}$ in X. Then it is open in X^* (by $\tau\subset\tau^*$). Since X^* is comp. and $\overline{V}$ is closed in X^*, $\overline{V}$ comp. in X^*
$\Rightarrow\exists$ finite subcover. So $\overline{V}$ is comp. in X and X is loc comp.

It remains to show: X Hausdorff. Take $x,y\in X$. Then $\exists U,V\in\tau^*$, $U\cap V=\emptyset$, $x\in U$, $y\in V$. Since $\{\infty\}$ is closed in X^*, X is open in X^*, so $X\cap U$, $X\cap V$ open in X^*, so $\underbrace{X\cap U, X\cap V}_{\text{desired neighb. of } x \text{ and } y}\in\tau$ ∎

<u>Rem.</u> We know, X^* comp. $\Rightarrow\forall$ ultrafilter in X^* conv. So $\forall$ divergent ultrafilter $\mathcal{F}$ on X conv. to ∞ (extend $\mathcal{F}$ to an ultraf. on X^*)

<u>Question</u> Find a Hausdorff compactif. with more freedom for ultrafilters which do not conv. in X.

$\boxed{\text{Def}}$ A Hausdorff compactif. βX of a TS X is called <u>Stone–Čech</u>

compactif. of X if $\forall$ comp. Hausdorff space Y $\forall f: X \to Y$ contin. $\exists! \, \bar{f}: \beta X \to Y$ cont. s.t. the diagram commutes
also denoted by βf

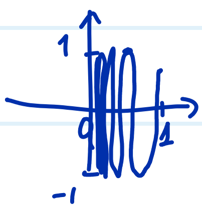
$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & \beta X \arrow[dl, dashed, "\bar{f}"] \\ Y & \end{tikzcd}$$

- universal property of βX

Rem. 1) βX is a „maximal" Hausdorff comp. of X ($\forall$ other such compact. is a quotient space of βX)

2) Alexandroff compact $\neq \beta X$ in general:

$\boxed{\text{Ex}}$ $X := (0,1]$, $f: X \to [-1,1] =: Y$ with $f(x) := \sin \frac{1}{x}$
euclid. top



Then f cannot be extended continuously to the one-point compactif. why

$X^* = [0,1]$
eucl. top.

Exer. X comp Hausd. $\Rightarrow \beta X = X$.

Question: When does βX exist?

$\boxed{\text{Prop.}}$ For a TS X,

$$\exists \, \beta X \iff X \text{ is completely regular}$$
$T_{3a} + T_1$

in this case βX is unique up to a homeom.

Proof (idea) Uniqueness follows from def. exer.

Existence $\Rightarrow$ βX comp. $\Rightarrow$ normal (Urysohn)

$\Rightarrow$ completely regular $\Rightarrow$ $i(X)$ as a subspace is completely regular
βX Hausdorff, so $\forall$ point is closed
exer.

$\Rightarrow$ X is compl. regular
why? (X homeom. to $i(X)$)

$\Leftarrow$ Def. $C(X) := \{$all continuous fcts $f: X \to [0,1]\}$ and
eucl top.

$$\tilde{X} := [0,1]^{C(X)} = \prod_{f \in C(X)} [0,1]$$ - Tychonoff cube

Tychonoff's thm: $\tilde{X}$ is comp. and Hausdorff for the product top.
$[0,1]$ is so

Consider $\forall x \in X$

$$h_x : C(X) \to [0,1]$$
$$f \mapsto f(x).$$
evaluation at x

Then $h_x \in \tilde{X}$ $\forall x$.

Consider
$$i : X \to [0,1]^{C(X)}$$
$$x \mapsto h_x$$

and def. $\beta X := \overline{i(X)} \subset \tilde{X}$

Check:

- βX is comp., Hausdorff (since $\tilde{X}$ is so)
- i is inj.: $x_1 \neq x_2 \underset{T_{3a}}{\Longrightarrow} \exists f \in C(X)$ with $f(x_1)=0$, $f(x_2)=1$, $\{x_2\}$ closed by T_1
  i.e., $h_{x_1} \neq h_{x_2}$
- i is open, continuous - sorry, no time (exer?)

To show: universal property

Let Y be comp. Hausdorff, $f: X \to Y$ cont. Observe: $\beta Y = Y$

Def. $P: [0,1]^{C(X)} \to [0,1]^{C(Y)}$ by

$$(x_\lambda)_{\lambda \in C(X)} \mapsto (x_{\gamma \circ f})_{\gamma \in C(Y)}$$

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & \beta(X) \subset [0,1]^{C(X)} \arrow[dl, dashed, "\tilde{f}"'] \arrow[d, "P"] \\ Y \arrow[r, "i_Y"] & {[0,1]^{C(Y)}} \end{tikzcd}$$

and check: $\tilde{f} := i_Y^{-1} \circ P|_{\beta(X)}$ is well-defined and as desired $\beta X \to Y$.

(no time ☹) ∎

<u>Rem.</u> Since locally comp. Hausdorff spaces are completely regular, they have βX. In part., $\forall$ discrete space has βX

(Exer.) 1)* $|\beta \mathbb{N}| = |[0,1]^{[0,1]}| > \mathfrak{c}$ ......)

($\mathbb{N}$ with discrete top.) continuum

$[0,1]^{[0,1]}$ - many elements

2) X is open in $\beta X \Leftrightarrow X$ loc. comp.

$\boxed{\text{Ex}}$ ($\beta \mathbb{N}$) seminar (approach of Stone):

$$\beta \mathbb{N} \cong \{ \mathcal{F} \text{ ultrafilters on } \mathbb{N} \}$$

with the Stone topology with basis $\{\mathcal{F} \in \mathrm{Ult}(\mathbb{N}) : U \in \mathcal{F}\}$ for $U \subset \mathbb{N}$

with $i: x \mapsto \mathcal{F}_x$ fixed ultrafilter (supersets of x)

Univers. property:

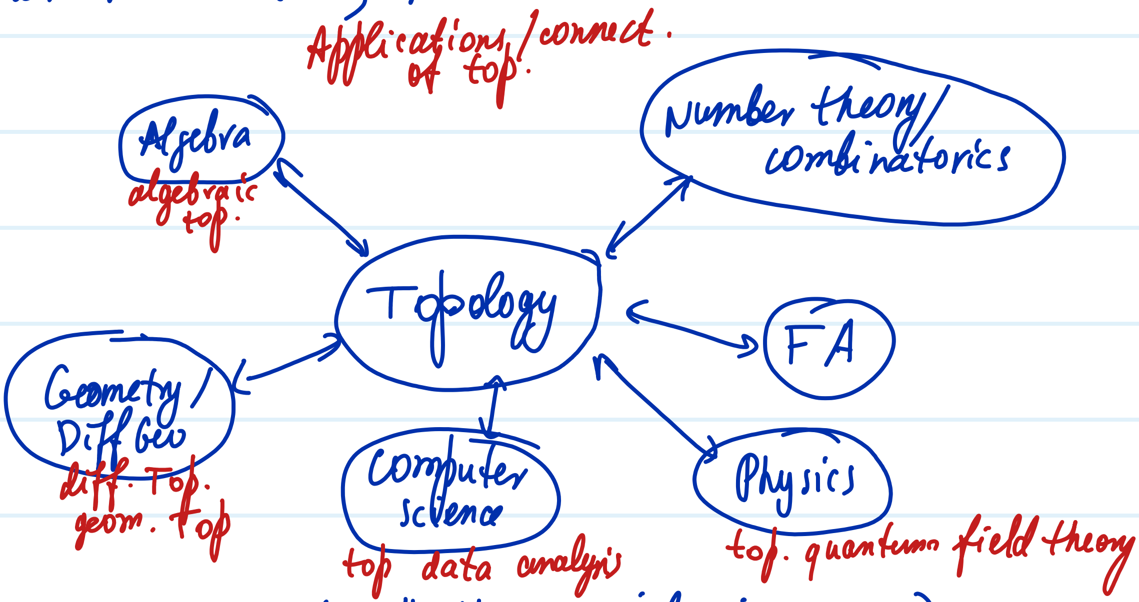
$$\begin{tikzcd} \mathbb{N} \arrow[r, "i"] \arrow[d, "f"'] & \mathrm{Ult}(\mathbb{N}) \arrow[dl, dashed] \\ Y & \end{tikzcd}$$

$\mathcal{F} \mapsto \underbrace{\lim f(\mathcal{F})}_{\in \mathrm{Ult}(Y)}$

Y comp., Hausd.

- $\beta \mathbb{N}$ is totally disconnected ($\forall$ connect. comp. is one-point)
- $\boxed{\ell^\infty(\mathbb{N}) \simeq C(\beta \mathbb{N})}$
- $\beta \mathbb{N}$ inherits the sgr structure of $\mathbb{N}$ (left top. noncomm. sgr - see seminar)

In the seminar: connection of $\beta\mathbb{N}$ to combinatorics (van der Waerden thm, Hindman thm)



many more (graph theory, social sciences, ...)