$\boxed{\text{Prop.}}$ Let X be a TS, Y be Hausdorff (T_2) and $f, g: X \to Y$ cont. Then one has

(a) $\{x \in X : f(x) = g(x)\}$ is closed in X

(b) $\left.\begin{array}{l} D \subset X \text{ dense} \\ f|_D = g|_D \end{array}\right\} \Rightarrow f \equiv g.$

(c) The graph $\{(x,y) \in X \times Y : f(x) = y\}$ of f is closed in $X \times Y$.

Proof (a) The map $\begin{array}{l} X \to Y \times Y \\ x \mapsto (f(x), g(x)) \end{array}$ is cont. (why?)

and $A := \{x : f(x) = g(x)\}$ is the preimage of the diagonal $\Delta \subset Y \times Y$ under this map, so closed when Y is Hausdorff.

(b) $A \supset D$, A closed $\Rightarrow$ $A = \overline{A} \supset \overline{D} = X$, i.e., $A = X$ and $f \equiv g$.

by (a)

(c) Consider $\begin{array}{l} h: X \times Y \to Y \times Y \\ h(x,y) := (f(x), y) \end{array}$ - cont. (why?)

The graph of f is $h^{-1}(\Delta)$, so closed.

diagonal in $Y \times Y$, closed by (a) when Y is Hausdorff

$\boxed{\text{Def}}$ A TS is called

- regular if $T_1 + T_3$
- completely regular if $T_1 + T_{3a}$
- normal if $T_1 + T_4$.

Rem. normal $\Rightarrow$ regular (points are closed)
later: normal $\Rightarrow$ completely regular

So we have:

$$\begin{tikzcd} \text{normal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{completely regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Hausdorff} \arrow[d, equal] & & \\ T_4 & T_{3a} \arrow[r, Rightarrow] & T_3 & T_2 \arrow[r, Rightarrow] & T_1 \arrow[r, Rightarrow] & T_0 \end{tikzcd}$$

Vererbung in German

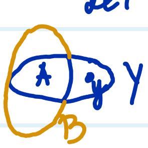
$\boxed{\text{Prop.}}$ (Inheritance of separation properties)

A subspace of a $T_0, T_1, T_2, T_3, T_{3a}$ spaces is again $T_0, T_1, T_2, T_3, T_{3a}$, respectively.
in part.,

$$\left.\begin{array}{l} X \text{ (compl.) regular} \\ Y \subset X \end{array}\right\} \Rightarrow Y \text{ (compl.) regular.}$$

Proof (for T_3, rest analog., exerc.)

Let X be T_3, $Y \subset X$, $A \subset Y$ closed in Y, i.e., $A = Y \cap B$ for some closed $B \subset X$,



and $y \in Y \setminus A$. Then $y \in X \setminus B$.

T_3 for X: $\exists\, \mathcal{U}, V$ open, disjoint with $y \in \mathcal{U}$, $B \subset V$.

Then $U \cap Y$, $V \cap Y$ are open in Y and disjoint with $y \in U \cap Y$ and $A \subset V \cap Y$ ∎

Rem. 1) Attention: $\left.\begin{array}{l} X \text{ normal} \\ Y \subset X \end{array}\right\} \not\Rightarrow Y$ normal (no proof)

2) Closed subspaces of a normal (resp., T_4) space is again normal (resp., T_4) – exerc

3) One can show:

$$\prod_\alpha X_\alpha \text{ is } T_0, T_1, T_2, T_3, T_{3a} \iff \forall \alpha, X_\alpha \text{ is } T_0, \dots, T_{3a}, \text{ respectively}$$

exerc: T_0, T_1, T_2.

But: $\forall \alpha\ X_\alpha$ is normal $\not\Rightarrow \prod_\alpha X_\alpha$ is normal (no proof)

4) in general, quotient spaces do not inherit sep. axioms

$\boxed{\text{Ex}}$ $X = [0,1]$ with natural top. – is $T_0, \dots, T_4$.

$x \sim y \overset{\text{def}}{\iff} (x = y \text{ or } x, y \in \mathbb{Q})$

exerc: $X/\sim$ is T_0 but not $T_1, \dots, T_4$.

$\boxed{\text{Ex}}$ (Order topology)

Let $(X, \leq)$ a totally ordered set. Def.

$(-\infty, a) := \{x \in X : x < a\}$ for $a \in X$.

$(a, \infty) := \{x \in X : a < x\}$

– why a subbasis of top.?

Let τ be a top. induced by such sets as a subbasis, called order topology.

($\boxed{\text{Ex}}$: $\mathbb{R}$ with natural top.)

Then (X, τ) is normal

Proof: exerc Hint for T_4: Let A, B be closed, disjoint.

Step 1 Show that

$A^* := \bigcup \{[a,b] : a, b \in A,\ [a,b] \cap B = \emptyset\}$

$B^* := \bigcup \{[c,d] : c, d \in B,\ [c,d] \cap A = \emptyset\}$

are disjoint

Step 2 Decompose A^* and B^* into convex components

$\forall a, b$ there, $[a,b]$ also there

and separate A^*, B^* componentwise by open sets

Step 3 Separate A and B

$\boxed{\text{Ex}}$ $\mathbb{R}$ with the following top. τ:

- $\forall x \neq 0$ let $\mathcal{U}(x)$ be as in natural top.
- $\mathcal{U}(0) =$ supersets of $(a,b) \setminus \{\frac{1}{n}, n \in \mathbb{N}\}$ with $a < 0 < b$

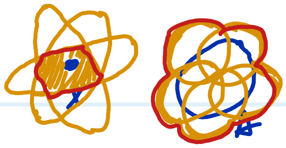
exerc:
- $(\mathcal{U}(x))_{x \in \mathbb{R}}$ is a topology
- $(\mathbb{R}, \tau)$ is Hausdorff but not regular, so not T_3

(one more example for $T_2 \not\Rightarrow T_3$)

$\boxed{\text{Prop.}}$ A compact Hausdorff space is T_4. In part., it is normal and regular.

Proof We saw before that comp. Hausdorff spaces are T_3:

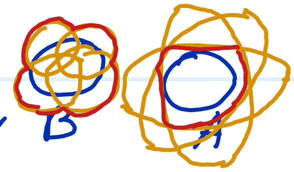
$\forall A \subset X$ closed ($\Rightarrow$ comp.) $\forall x \notin A\ \exists U_x \in \mathcal{U}(x)\ \exists V_x \supset A$ open



with $U_x \cap V_x = \emptyset$.

Recall: take an open cover of A by sets from the Hausdorff property, finite subcover $\rightsquigarrow$ V_x (= union), $U_x = \bigcap$ corresp. neighb. of x

For $x \in B$, (U_x) is an open cover of B as above separating x from A. X comp. $\Rightarrow$ $\exists$ finite subcover $U_{x_1}, \dots, U_{x_n}$. Thus take



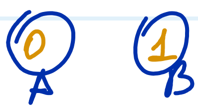
$$U := \bigcup_{j=1}^{n} U_{x_j} \supset B \text{ open}, \quad V := \bigcap_{j=1}^{n} V_{x_j} \supset A \text{ open},$$

disjoint. ■

Rem. The above proof also shows: comp. + $T_3 \Rightarrow T_4$.
without Hausdorff

IV.2 Continuous extensions of functions

Question 1) Let A, B be closed, disjoint. When can one separate A and B by a continuous fct, i.e., when $\exists f : X \to [0,1]$ cont. with $f|_A = 0$, $f|_B = 1$?



(Would be T_{4a})

Reformulation: when can we extend $\underset{A}{0}$ $\underset{B}{1}$ to a continuous fct on X (with values in $[0,1]$)?

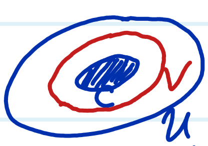
2) Let $f_A : \underset{\subset X}{A} \to \mathbb{R}$ cont. When $\exists f : X \to \mathbb{R}$ cont. with $f|_A = f_A$?

Rem. 1) $\Rightarrow T_4$ (why?), so we assume T_4.

$\boxed{\text{Thm}}$ (Lemma of Urysohn)
Let X be T_4 and $A, B \neq \emptyset$ disjoint, closed. Then $\exists f : X \to [0,1]$ cont. with $f|_A = 0$, $f|_B = 1$.

Proof Observation: X is $T_4 \Rightarrow \forall$ closed $C \subset X$ $\forall\, U \supset C$ open $\exists\, V$ open with

$$C \subset V \subset \overline{V} \subset U$$



Proof of observ.: C and $X \setminus U$ are closed, disj.

By T_4 $\exists\, V \supset C$, $\exists\, V_1 \supset X \setminus U$ disj, open. We have $C \subset V \subset U$ and check that $\overline{V} \subset U$. Assume $x \in \overline{V}$; if $x \notin U$ then $V_1 \in \mathcal{U}(x)$ with $V_1 \cap V = \emptyset$ $\lightning (x \notin \overline{V})$.
V_1 open, $x \in X \setminus U \subset V_1$
■ of observ.

Consider

$$D := \left\{ \frac{m}{2^n},\ n \in \mathbb{N},\ m \in \{0, \dots, 2^n\} \right\}$$

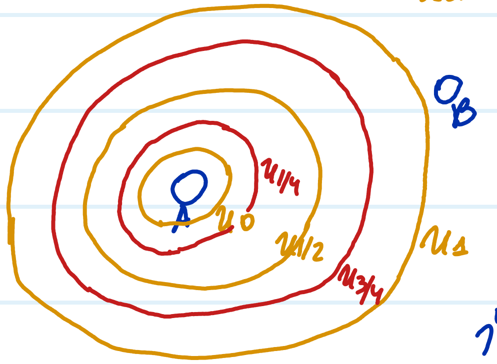
and form a sequence from it:

$$F := \left(0, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{8}, \dots, \tfrac{1}{2^n}, \tfrac{3}{2^n}, \dots, \tfrac{2^n - 1}{2^n}, \dots\right)$$

Using the observation several times: $\exists\, U_0, U_1$ open with

$$A \subset \mathcal{U}_0 \subset \overline{\mathcal{U}_0} \subset \mathcal{U}_1 \subset \overline{\mathcal{U}_1} \subset X \setminus B$$

closed   open   open



Def. $f|_{X\setminus \mathcal{U}_1} := 1$ and $f|_{\overline{\mathcal{U}_0}} := 0$

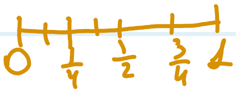
In part., $f|_B = 1$, $f|_A = 0$.

Def. $\mathcal{U}_{1/2}$ with $\overline{\mathcal{U}_0} \subset \mathcal{U}_{1/2} \subset \overline{\mathcal{U}_{1/2}} \subset \mathcal{U}_1$

$\mathcal{U}_{1/4}$ with $\overline{\mathcal{U}_0} \subset \mathcal{U}_{1/4} \subset \overline{\mathcal{U}_{1/4}} \subset \mathcal{U}_{1/2}$ etc.

inductively, given $\mathcal{U}_{d_1}, \dots, \mathcal{U}_{d_n}$, where $d_1, \dots, d_n$ are first elements of F, choose $\mathcal{U}_{d_{n+1}}$ s.t.

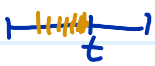
$$\overline{\mathcal{U}_d} \subset \mathcal{U}_{\tilde{d}} \quad \forall d < \tilde{d}, \quad d, \tilde{d} \in \{d_1, \dots, d_{n+1}\}$$



For $t \in [0,1]$ def.

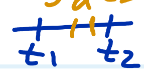
$$\mathcal{U}_t := \bigcup_{\substack{d \in D \\ d \le t}} \mathcal{U}_d$$

Check: $\mathcal{U}_t = \mathcal{U}_d$ for $t = d \in D$ (exer.)



- $\mathcal{U}_t \nearrow$ in t
- $\forall\, \mathcal{U}_t$ is open
- $t_1 < t_2 \Rightarrow \overline{\mathcal{U}_{t_1}} \subset \mathcal{U}_{t_2}$

$\exists\, d \in D$ in (t_1, t_2)



Def. now

$$f(x) := \inf\{t \in [0,1] : x \in \mathcal{U}_t\} \quad \text{for } x \in \mathcal{U}_1$$

– outside of $\mathcal{U}_1$ $f \equiv 1$

- $f : X \to [0,1]$ clear, $f|_A = 0$, $f|_B = 1$
- f is continuous:

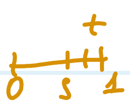
Recall: $[0,s)$, $(s,1]$ for $s \in (0,1)$ build a subbasis of top. on $[0,1]$, so it satisfies to show that

$f^{-1}([0,s))$, $f^{-1}((s,1])$ are open

$$f^{-1}([0,s)) = \{x \in X : f(x) < s\} = \{x : \exists\, t < s \text{ with } x \in \mathcal{U}_t\}$$

$$= \bigcup_{t<s} \mathcal{U}_t \ \text{– open}$$

$\overline{\mathcal{U}_{t_1}} \subset \mathcal{U}_{t_2}\ \forall t_1 < t_2$



$$f^{-1}([0,s]) = \bigcap_{t>s} f^{-1}([0,t)) = \bigcap_{t>s} \mathcal{U}_t \overset{\text{by monotonicity}}{=} \bigcap_{t>s} \overline{\mathcal{U}_t} \ \text{closed},$$

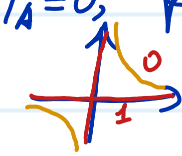
so its complement $f^{-1}((s,1])$ is open ■

$\boxed{\text{Cor}}$ X $\underbrace{\text{normal}}_{T_1 + T_4} \Rightarrow X$ $\underbrace{\text{completely regular}}_{T_1 + T_{3a}}$

Proof. $T_4 \overset{\text{Urysohn}}{\Rightarrow}$ sep. of 2 closed sets by cont. fcts

$T_1 \Rightarrow \{x\}$ is closed $\forall x \in X$, so T_{3a} follows ■

Exerc. Construct a cont. fct $f : \mathbb{R}^2 \to [0,1]$ with $f|_A = 0$, $f|_B = 1$, where $X = \mathbb{R}^2$, $A = \{(x,y) : y = \frac{1}{x}\}$, $B = \{(x,y) : x = 0 \text{ or } y = 0\}$



Rem. In the proof of Urysohn $f^{-1}(\{0\}) \supsetneq A$ and $f^{-1}(\{1\}) \supsetneq B$ in general. When can we achieve "="?

$\boxed{\text{Prop}}$ Let X be T_4 and $\emptyset \neq A \subset X$ closed. Then:

$\exists f: X \to \mathbb{R}$ continuous with $f^{-1}(\{0\}) = A$ $\iff$ A is a G_δ set, i.e., $A = \bigcap_{n=1}^{\infty} U_n$ for some open $U_1, U_2, \ldots$

Proof $\Rightarrow$ $A = \bigcap_{n=1}^{\infty} f^{-1}((-\frac{1}{n}, \frac{1}{n}))$

$\Leftarrow$ Let A be closed, $A = \bigcap_{n=1}^{\infty} U_n$, $\forall U_n$ open.

Urysohn: $\forall n\ \exists f_n: X \to [0,1]$ cont. with $f_n|_A = 0$, $f_n|_{X \setminus U_n} = 1$

(A) Consider $g := \sum_{n=0}^{\infty} \frac{f_n}{2^n}$

exerc. g is continuous as uniform limit of cont. fcts $g_N := \sum_{n=0}^{N} \frac{f_n}{2^n}$

Moreover, $g(x) = 0 \iff x \in A$:

$\Leftarrow$ clear

$\Rightarrow$ $g(x) = 0 \Rightarrow f_n(x) = 0\ \forall n \Rightarrow \underbrace{x \notin X \setminus U_n\ \forall n}_{x \in U_n\ \forall n,\ \text{so}\ x \in A.}$ ■

$\boxed{\text{Prop}}$ (Lemma of Urysohn for G_δ sets)

Let X be T_4, $A, B \subset X$ disj., closed, $\neq \emptyset$, G_δ. Then $\exists f: X \to [0,1]$ cont. with $A = f^{-1}(\{0\})$, $B = f^{-1}(\{1\})$

Proof: exerc. (play with the last prop., max., min).

Urysohn: $\exists$ cont. continuation of $\underset{A}{0}$ $\underset{B}{1}$

$\boxed{\text{Thm}}$ (Tietze-Urysohn)

X is T_4 $\iff$ $\forall A \subset X$ closed $\forall f: A \to \mathbb{R}$ cont. $\exists F: X \to \mathbb{R}$ cont. with $F_A = f$.

Proof $\Leftarrow$ Let A, B disj., closed, $\neq \emptyset$. Then $f: \underbrace{A \cup B}_{\text{closed}} \to \mathbb{R}$ $f|_A = 0$, $f|_B = 1$

$\underset{A}{0}$ $\underset{B}{1}$ - why cont.? By assumption $\exists F: X \to \mathbb{R}$ cont., $F|_A = 0$, $F|_B = 1$.

Take $U := \underbrace{f^{-1}((-\frac{1}{2}, \frac{1}{2}))}_{\supset A}$, $V = \underbrace{f^{-1}((\frac{1}{2}, \frac{3}{2}))}_{\supset B}$ - open, disj.

$\Rightarrow$ (for $f: A \to [-1,1]$)

Step 1 We show first: $\exists (g_n)$, $g_n: X \to [-1,1]$ cont. with

(a) $|g_n(x)| \leq 1 - (\frac{2}{3})^n \quad \forall x \in X$

(b) $|f(x) - g_n(x)| \leq (\frac{2}{3})^n \quad \forall x \in A$

(c) $|g_{n+1}(x) - g_n(x)| \leq \frac{1}{3} \cdot (\frac{2}{3})^n \quad \forall x \in X$

Def. $g_0 := 0$ - satisfies (a), (b).