Ergodic Theory

SS 2026

Homepage: www.math.uni-leipzig.de/~eisner/Ergodic-Theory-SS2026
(+ Moodle)

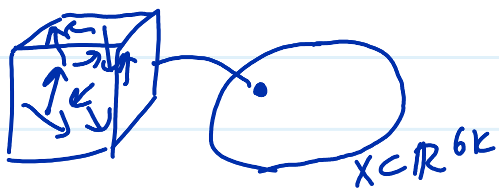
0. What is ET?

Birth: ~1880 Boltzmann

Origin: statistical mechanics

Given: room in $\mathbb{R}^3$, ideal gas, k particles.

$3k$ coordinates + $3k$ velocities = $6k$ (degree of freedom)



So state of the system = point in $\mathbb{R}^{6k}$

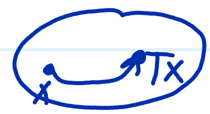
Not all states are possible, so

let $X :=$ {all possible states of the system} $\subseteq \mathbb{R}^{6k}$

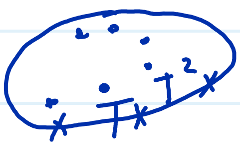
~ the state space

The system changes its state in one time unit (say second) according to a physical law $\rightsquigarrow$ $T : X \to X$

~ transformation



The orbit $\{x, Tx, T^2x, \dots\}$ describes the development of the system in time.

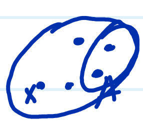


Question: How does the system behave for large times, so $T^n x$ for large n? (asymptotic behaviour)

Difficulties:
- x not always completely known
- even if known, $T^n x$ is difficult to compute.

Idea (Bolzmann): Statistical approach: $x \in A$ or $x \notin A$ is easier to check (for $A \subset X$)

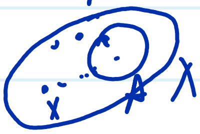
What is the probability that $T^n x \in A$?
(How often does x visit A?)



Boltzmann's ergodic hypothesis:
The time the system spends in $A \subset X$ is (in mean and asymptotically) equal to $\mathrm{Vol}(A)$:

„time mean = space mean"

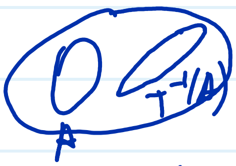
Mathematically: $\dfrac{\#\{n\in\{1,\dots,N\}: T^n x\in A\}}{N} \xrightarrow[N\to\infty]{} \mathrm{Vol}(A)$



Or: $\lim\limits_{N\to\infty} \frac{1}{N}\sum\limits_{n=1}^{N} \mathbb{1}_A(T^n x) = \int \mathbb{1}_A \, d\mu \quad (\mu = \mathrm{Vol})$ rescaled!

<u>Question:</u> For which T, f, x do we have
$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(T^n x) = \int f\,d\mu \ ?$$
(f - observable, result of measurements)

<u>Remark</u> Physics (Liouville's thm) $\Rightarrow$ T preserves the volume.
$$\mathrm{Vol}(T^{-1}(A)) = \mathrm{Vol}(A) \quad \forall A\subset\mathbb{R}^{6k} \text{ meas.}$$



So we have a system (X, Vol, T), T Vol-preserving
(X: state space)

<u>Ergodic Theory</u>: theory of measure-preserving systems/transformations studying the behaviour of orbits of points and sets.
Beautiful theory with connections to many other areas (stochastics, group theory, number theory, combinatorics, functional analysis, ...)

<u>Plan of the course:</u> Part I: classical theory
<u>Part II</u>: ergodic theoretic proof of Szemerédi's thm:
$A\subset\mathbb{N}$ „large" $\Rightarrow$ $A\supset$ arithm. progressions $a, a+n, a+2n, \dots, a+kn$ of arbitrary length.

<u>Rem.</u> Origin of „ergodic" not clear:
- ergodos = difficult
- ergon = work, odos = way

1. Measure theoretic dyn. systems

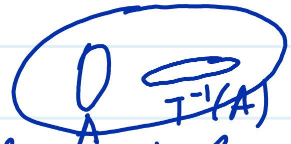
1.1 Definitions and examples

Let (X,Σ,μ) be a prob. space. We assume:
- μ is complete: $\mu(A)=0$, $B\subset A \Rightarrow B\in\Sigma$ (and hence $\mu(B)=0$)
- Σ is generated by a countable family (+ null sets)

$\boxed{\text{Ex}}$ X separable metric space, μ (completion of) a Borel prob.

$\boxed{\text{Def. 1.1}}$ Let (X,μ,T) be a prob. space (often shortened to (X,μ)) and let $T: X \to X$ be measurable and μ-pres., i.e.,

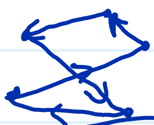
$$\mu(T^{-1}(A)) = \mu(A) \quad \forall A \in \Sigma.$$



Then (X,μ,T) is called a measure theoretic dynamical system (or measure preserving system), short: MDS.

$\boxed{\text{Ex 1.2}}$ (Finite systems)

X finite set, μ rescaled counting measure, $T: X \to X$ bij.

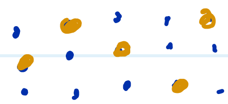


Then (X,μ,T) is an MDS.

$\boxed{\text{Ex 1.3}}$ (Bernoulli shifts)

1) Two-sided shifts:

let $k \in \mathbb{N}$, $X := \{0,\dots,k-1\}^{\mathbb{Z}}$

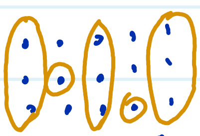


$0,\dots,k-1$ – letters

$X = \left\{ \begin{array}{l} \text{words infinite} \\ \text{in both directions} \end{array} \right\}$

$T \leftarrow$ (left shift): $T((x_j)) := (x_{j+1})$

$\Sigma :=$ product σ-algebra generated by cylinders



$$U_{n,j_1,\dots,j_n,a_1,\dots,a_n} := \{x : x_{j_1} = a_1, \dots, x_{j_n} = a_n\}$$

Σ is T-invariant, since cylinder $\longleftrightarrow$ cylinder

Let p be a prob. measure on $\{0,\dots,k-1\}$, $p(\{j\}) =: p_j$.

Def.

$$\mu(U_{n,\dots}) := p_{a_1} \cdots p_{a_n}$$

and extend it to Σ:

(Recall: $\mathcal{H}$ is a semi-ring if
- $\emptyset \in \mathcal{H}$
- $A, B \in \mathcal{H} \Rightarrow A \cap B \in \mathcal{H}$, $A \setminus B = \bigcup_{j=1}^{n} C_j$ for some n and disjoint $C_1,\dots,C_n \in \mathcal{H}$.

Here: $\mathcal{H} = \{\text{cylinders}\} \cup \{\emptyset\}$

Extension thm of Carathéodory:

$\forall$ (σ-finite) pre-measure on a semi-ring $\mathcal{H}$ extends uniquely to a measure on the σ-algebra generated by $\mathcal{H}$)

μ is called product measure on Σ induced by p.

(We take completion and denote it by μ again)

One has
$$\mu(T^{-1}(A)) = \mu(A) = \mu(TA) \quad \forall A \in \Sigma$$

check! since true for cylinders

The MDS (X, Σ, μ, T) is called two-sided (or invertible) Bernoulli shift.

Rem. This is a special case of a general product system $\prod_{n=-\infty}^{\infty} Y$ for a prob. space (Y, Σ_Y, p). Cylinders here:

$$\mathcal{U}_{n, j_1, \dots, j_n, B_1, \dots, B_n} := \left\{ x : \begin{matrix} x_{j_1} \in B_1 \\ x_{j_n} \in B_n \end{matrix} \right\}$$

$\dots \, 0\,0\,0\,0\,0\,0 \dots$

for measur. sets $B_1, \dots, B_n \subset Y$, $n \in \mathbb{N}$ and

$$\mu(\mathcal{U}_{n, \dots}) := p(B_1) \cdot \ldots \cdot p(B_n).$$

2) One-sided shifts

$X := \{0, \dots, k-1\}^{\mathbb{N}}$, T : $T((x_j)) := (x_{j+1})$, words infinite in one direction

cylinders as before, μ also. We have

$$\mu(\mathcal{U}_{n, \dots}) = \mu(T^{-1}(\mathcal{U}_{n, \dots}))$$

cylinder shifted to the right, first coordinate free.

So (X, μ, T) is an MDS,
one-sided Bernoulli shift.

$\boxed{\text{Ex. 1.4}}$ (Group rotations)

Let G be a compact metrizable group — so that Σ is countably generated

G compact space + group s.t. $\cdot : G \times G \to G$, $(\cdot)^{-1} : G \to G$ are continuous.

Σ its Borel σ-algebra. Then $\exists!$ prob. measure μ on Σ which is left rotation invariant, i.e.,

$$\mu(a \cdot A) = \mu(A) \quad \begin{matrix} \forall A \in \Sigma \\ \forall a \in G \end{matrix} \quad \text{(without proof)}$$

This μ is called Haar measure on G.

Properties of the Haar measure

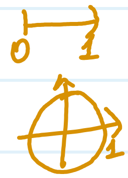
exercise sheet

- μ is automatically right invariant, i.e.,
$$\mu(A \cdot a) = \mu(A) \quad \forall A \in \Sigma \ \forall a \in G.$$
- μ is invariant under $g \mapsto g^{-1}$, i.e.,
$$\mu(A^{-1}) = \mu(A) \quad \forall A \in \Sigma$$
$A^{-1} = \{g^{-1} : g \in A\}$
- $\operatorname{supp} \mu = G$, i.e., $\forall \mathcal{U} \subset G$ open $\mu(\mathcal{U}) > 0$.

Often one knows μ:

$\boxed{Ex}$: 1) G finite, $\mu(A) := \frac{|A|}{|G|}$ — shift inv. (Haar) measure
with discrete metric/topology
$\{0,\dots,k\}$, $+ \bmod k$

2) $[0,1)$ with $(a,b) \mapsto a+b \bmod 1$
$\mu = Leb$ — the unique translation invariant measure



On $\mathbb{T}$ unit circle: $\mu(\mathrm{Arc}(\varphi,\psi)) = \frac{\varphi-\psi}{2\pi}$
— rescaled Lebesgue measure

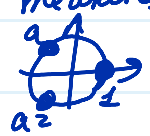
The MDS (G,μ,a) is called rotation system.
rotation by a: $x \mapsto ax$

Attention: For a fixed a there can be other a-invariant measures.

Exercise — $\boxed{Ex}$ $(\mathbb{T}, a)$

Case 1 a irrational, i.e., $a^n \neq 1$ $\forall n \neq 0$.
Then the Haar measure is a unique a-inv. measure

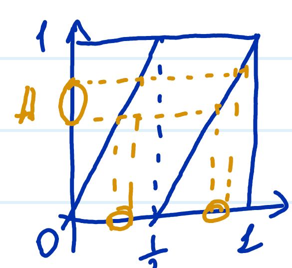
Case 2 a rat. — $\exists$ many a-inv. measures
exer.: describe them all



$\boxed{Ex. 1.5}$ (Baker's map)
$X = [0,1]^2$ with $\mu = Leb^2$, $T: X \to X$ def. by
$$T(x,y) := \begin{cases} (2x, \frac{y}{2}), & 0 \le x < \frac{1}{2} \\ (2x-1, \frac{y+1}{2}), & \frac{1}{2} \le x \le 1. \end{cases}$$

So (X,μ,T) is an MDS

$\boxed{Ex. 1.6}$ (Doubling map)
$X := [0,1]$ with Leb., $T: X \to X$
$$Tx := 2x \bmod 1 = \begin{cases} 2x, & x \in [0,\frac{1}{2}) \\ 2x-1, & x \in [\frac{1}{2}, 1] \end{cases}$$



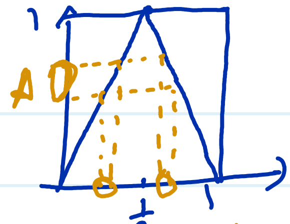
$\mu(T^{-1}[a,b]) = \cancel{2} \cdot \frac{b-a}{\cancel{2}} = \mu([a,b])$

(X,μ,T) is an MDS (exercise)

$\boxed{Ex. 1.7}$ (Tent map)

$X = [0,1]$ with Leb., $T: X \to X$

$$Tx := \begin{cases} 2x, & x \in [0, \frac{1}{2}) \\ 2-2x, & x \in [\frac{1}{2}, 1] \end{cases}$$



(X, μ, T) is an MDS.

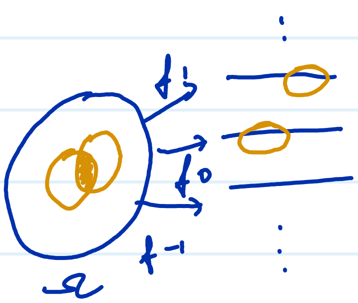
again $\mu(T^{-1}[a,b]) = 2 \cdot \frac{b-a}{2} = \mu([a,b])$

Ex. 1.8 (Stationary stochastic processes)

Let (Ω, p) be a prob. space, $\dots, f_{-1}, f_0, f_1, \dots : \Omega \to \mathbb{R}$ measurable, fcts, i.e., random variables.

Assume that the process is stationary, i.e.,

$\forall m \in \mathbb{N} \ \forall n_1, \dots, n_m \in \mathbb{Z} \ \forall B_1, \dots, B_m \subset \mathbb{R}$ Borel $\forall k \in \mathbb{Z}$



$$p\left(\omega \in \Omega : \begin{matrix} f_{n_1}(\omega) \in B_1 \\ \vdots \\ f_{n_m}(\omega) \in B_m \end{matrix}\right) = p\left\{\omega \in \Omega : \begin{matrix} f_{n_1+k}(\omega) \in B_1 \\ \vdots \\ f_{n_m+k}(\omega) \in B_m \end{matrix}\right\},$$

i.e.,

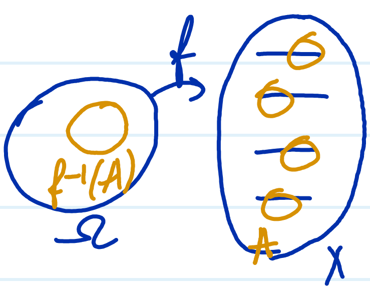
$$(*) \quad \boxed{p\left(\bigcap_{j=1}^m f_j^{-1}(B_j)\right) = p\left(\bigcap_{j=1}^m f_{n_j+k}^{-1}(B_j)\right)}.$$

(p ($\cap$ of preimages) is shift invariant, so no matter, where we have time „0")

Let $X := \mathbb{R}^{\mathbb{Z}} = \{(\dots, x_{-1}, x_0, x_1, \dots),\ x_j \in \mathbb{R}\}$ - infinite $\mathbb{R}$-words

$f: \Omega \to X$ with $f(\omega) := (\dots, f_{-1}(\omega), f_0(\omega), f_1(\omega), \dots)$

For $A \subset X$ Borel def.



$$\mu(A) := p(f^{-1}(A))$$

and consider

$(\mathbb{R}^{\mathbb{Z}}, \mu, T_L)$ left shift $\leftarrow$:

$T((x_j)) := (x_{j+1})$.

Since f^{-1}(cylinder) has the form

$$\bigcap_{j=1}^m f_{n_j}^{-1}(B_j),$$

by $(*)$ we have

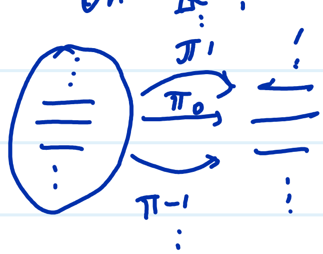
$$\mu(\text{cylinder}) = p(f^{-1}(\text{cylinder})) = p(f^{-1}(\underbrace{T^{-1}(\text{cylinder})}_{\text{shifted cylinder}}))$$

$$= \mu(T^{-1}(\text{cylinder}))$$

So T is μ-preserving and $\boxed{(\mathbb{R}^{\mathbb{Z}}, \mu, \leftarrow) \text{ is an MDS}}$

$\mathbb{R}^{\mathbb{Z}}, \leftarrow$ the same for all processes, only μ varies!

Conversely, let $(\mathbb{R}^{\mathbb{Z}}, \mu, \leftarrow)$ be an MDS, i.e., μ be $\leftarrow$-invariant on $\mathbb{R}^{\mathbb{Z}}$. We construct the corresponding $(\Omega, P, (f_j))$.



Def. $\Omega := \mathbb{R}^{\mathbb{Z}}$, $f_j := \pi_j : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ - projection onto the j-th coord.

$P := \mu$ on Ω.

We have:

- $\forall \pi_j$ is measurable ($\pi_j(\underset{\subset \mathbb{R}}{B})$ is a cylinder)
- The process is stationary:

$$\mu\left(\bigcap_{j=1}^{m} \pi_{n_j}^{-1}(B_j)\right) \underset{\mu \text{ shift inv.}}{=} \mu\left(\bigcap_{j=1}^{m} \pi_{n_j+k}^{-1}(B_j)\right)$$

- $\mu(A) = P(\pi^{-1}(A))$ for $\pi := (\dots, \pi_{-1}, \pi_0, \pi_1, \dots)$ since $\pi = id$ and $P = \mu$,

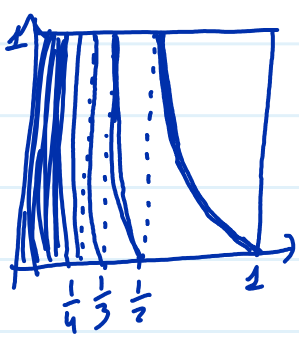
so this (Ω, P, π) corresponds to $(\mathbb{R}^{\mathbb{Z}}, \mu, \leftarrow)$.

So: $\boxed{\text{stationary stochastic processes} \longleftrightarrow \text{shift invariant prob. measures on } \mathbb{R}^{\mathbb{Z}}}$

Rem. The same construction for $f_0, f_1, \dots$ and $\mathbb{R}^{\mathbb{N}_0}$ $\quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}$
(time not invertible)

Ex. 1.8 (Gauss transformation)

Let $X := [0,1]$, $Tx := \frac{1}{x} - \lfloor \frac{1}{x} \rfloor = \{\frac{1}{x}\}$ for $x \neq 0$, $T0 := 0$.
fractional part



For $x \in (\frac{1}{n+1}, \frac{1}{n}]$ we have $Tx = \frac{1}{x} - n$

Therefore

(∗∗) $T^{-1}(\{y\}) = \{\frac{1}{y+n}, n \in \mathbb{N}\}$ $\forall y \in (0,1]$

Let μ be defined by

$$\mu(A) := \frac{1}{\log 2} \int_A \frac{dx}{x+1}$$

Then μ is a prob. measure: $\mu([0,1]) = \frac{1}{\log 2}(\log 2 - \log 1) = 1$.

Claim: T is μ-preserving

Proof It suffices to show $\mu(T^{-1}[0,b]) = \mu([0,b])$ $\forall b \in (0,1]$ (why?)

Let $b \in (0,1]$. Then

$$\log 2 \cdot \mu(T^{-1}[0,b]) \underset{(**)}{=} \sum_{n=1}^{\infty} \int_{\frac{1}{b+n}}^{\frac{1}{n}} \frac{dx}{1+x} = \sum_{n=1}^{\infty} \underbrace{\left(\log\left(1+\tfrac{1}{n}\right) - \log\left(1+\tfrac{1}{b+n}\right)\right)}_{\log \frac{1+\frac{1}{n}}{1+\frac{1}{b+n}}}$$

$$\boxed{\frac{1+\frac{1}{n}}{1+\frac{1}{b+n}} = \frac{n+1}{n} \cdot \frac{b+n}{b+n+1} = \frac{1+\frac{b}{n}}{1+\frac{b}{n+1}}}$$

$$= \sum_{n=1}^{\infty} \left(\log\left(1+\tfrac{b}{n}\right) - \log\left(1+\tfrac{b}{n+1}\right)\right) = \int_0^b \frac{dx}{1+x} = \log 2 \cdot \mu([0,b]).$$

So (X,μ,T) is an MDS.

$\boxed{\text{Def}}$ (continuous fraction expansion) Kettenbruchentwicklung in German

Let $x \in [0,1]\setminus\mathbb{Q}$. Write $x = \frac{1}{a_1 + r_1}$, $a_1 \in \mathbb{N}$, $r_1 \in (0,1)$ irrat.

Repeat: $r_1 = \frac{1}{a_2 + r_2}$, ..., so

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$ (never ends since x irrat.)

The sequence $(a_n)_{n=1}^{\infty}$ is called (simple) continued fraction expansion of x. (Bombelli 1579, Lagrange, Euler, ...)

One writes $x = [0; a_1, a_2, \dots]$ ($a_i \in \mathbb{N}$)

$\boxed{\text{Ex}}$ $\cfrac{1}{1+\cfrac{1}{1+\dots}} = \varphi$ – the golden ratio (indeed, $\frac{1}{1+\varphi} = \varphi$, so $\varphi^2 + \varphi - 1 = 0$, $\varphi = \frac{-1+\sqrt{5}}{2}$)

so $\varphi = [0; 1,1,1,\dots]$

$$\pi - 3 = \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \dots}}}$$

14.4.26

Properties of (a_n)

Def. $\frac{p_n}{q_n} := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots \frac{1}{a_n}}}$ (stop after n iteration)

Then $\frac{p_n}{q_n} \in \mathbb{Q}$ and one has (no proof)

- $\frac{p_2}{q_2} < \dots < \frac{p_{2n}}{q_{2n}} < \dots < x < \dots < \frac{p_{2m+1}}{q_{2m+1}} < \dots < \frac{p_1}{q_1}$

  and $\frac{p_n}{q_n} \to x$ (so $\frac{p_{2n}}{q_{2n}} \nearrow x$, $\frac{p_{2n+1}}{q_{2n+1}} \searrow x$)

- $\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$

- $\left(\frac{p_n}{q_n}\right)$ is the best rational approximation of x with denominator $\leq q_n$, i.e.,

  $\forall$ other $\frac{p}{q}$ (reduced) with $q \in \{1,\dots,q_n\}$

$$|x-\frac{p_n}{q_n}| < |x-\frac{p}{q}|$$

Ex: $\frac{22}{7}$ for π

- $\forall (a_n) \subset \mathbb{N}$ $\exists \lim \cfrac{1}{a_1+\cfrac{1}{a_2+\dots}} \in [0,1]\setminus\mathbb{Q}$,

 i.e., irrat. number in $[0,1] \longleftrightarrow [0; a_1, a_2, \dots]$

$\boxed{\text{Prop}}$ Let $x \in [0,1]\setminus\mathbb{Q}$. Then $\forall n$:

$$\boxed{a_n = \left\lfloor \frac{1}{T^{n-1}x} \right\rfloor}$$

Proof Induction on n. We show: $a_n = \left\lfloor \frac{1}{T^{n-1}x} \right\rfloor$, $r_n = T^n x$

$n=1$: $x = \frac{1}{a_1+r_1}$, $r_1 \in (0,1)$

So $\frac{1}{x} = a_1 + r_1$, i.e., $a_1 = \lfloor \frac{1}{x} \rfloor$, $r_1 = \{\frac{1}{x}\} = Tx$

$n \to n+1$: $r_n = \frac{1}{a_{n+1}+r_{n+1}}$ (def. of a_{n+1}, r_{n+1})

$\frac{1}{r_n} = a_{n+1} + r_{n+1}$, so $a_{n+1} = \lfloor \frac{1}{r_n} \rfloor \underset{\text{induction hyp.}}{=} \lfloor \frac{1}{T^n x} \rfloor$ and $r_{n+1} = \{\frac{1}{r_n}\} \underset{\text{induct. hyp.}}{=} \{\frac{1}{T^n x}\}$

$= T(T^n x) = T^{n+1}x$ ∎

$\boxed{\text{Lemma 1.8}}$ Let (X,μ) be a prob. space and $T: X \to X$ measurable. Then T is μ-preserving $\Longleftrightarrow$

$$(*) \quad \boxed{\int_X f\,d\mu = \int_X f\circ T\,d\mu} \quad \forall f \in \mathcal{L}^\infty(X,\mu)$$

In this case, $(*)$ holds $\forall f \in \mathcal{L}^1(X,\mu)$.

Proof ($\Leftarrow$) Let $A \subset X$ meas., $f := \mathbb{1}_A$. Then

$$\mu(A) = \int_X \mathbb{1}_A\,d\mu \underset{(*)}{=} \int_X \underbrace{\mathbb{1}_A(Tx)}_{\substack{=\begin{cases}1 & \text{if } x\in T^{-1}(A)\\ 0 & \text{otherw.}\end{cases} \\ = \mathbb{1}_{T^{-1}(A)}(x)}}\,d\mu(x) = \mu(T^{-1}(A))$$

($\Rightarrow$) (+ last assertion)

As in $\Leftarrow$, $(*)$ holds for $\forall f = \mathbb{1}_A$, so $\forall f$ simple (lin. comb. of char. fcts).

Let $f \in \mathcal{L}^1$. WLOG (without loss of generality) $f \geq 0$. Then $\exists f_n \nearrow f$ (simple fcts) a.e. and so

$$\int_X f(Tx)\,d\mu \underset{\text{monotone conv. thm}}{=} \lim_{n\to\infty} \int f_n(Tx)\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$$ ∎

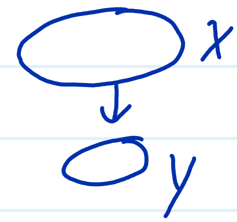
$\boxed{\text{Remark}}$ Here we used $\boxed{L^\infty(X,\mu) \subset L^p(X,\mu) \subset L^1(X,\mu) \quad \forall p \in (1,\infty)}$

1.2 Basic constructions (and more examples)

$\boxed{\text{Def. 1.9}}$ (Factors, extensions and isomorphisms)

Let (X,μ,T) and (Y,ν,S) be MDS.

(1) (Y,ν,S) is called factor of (X,μ,T) (and (X,μ,T) is extension of (Y,ν,S)) if $\exists \varphi: X \to Y$ measure preserving $(\mu(\varphi^{-1}B)=\nu(B)\ \forall B \in \Sigma_Y)$



s.t.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "S"'] & Y \end{tikzcd}$$

$$\varphi(Tx) = S(\varphi x) \quad \text{for a.e. } x \in X.$$

Such φ is called factor map.

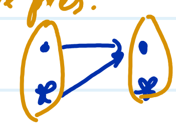
(2) (X,μ,T) and (Y,ν,S) are called isomorphic if φ above can be chosen to be invertible, i.e., $\exists \psi: Y \to X$ measur.:

$$\varphi \circ \psi = id_Y \quad \text{and} \quad \psi \circ \varphi = id_X \quad \text{a.e.}$$

Rem. if in (1) $\varphi(X)$ is measurable, then $\nu(\varphi(X)) = \mu(X) = 1$, i.e., φ is „almost surjective".
φ measure pres. preimage under φ

But $\varphi(X)$ not always measurable:

$\boxed{\text{Ex}}$ $X = \{\bullet, *\} = Y$, T: $\bullet \leftrightarrows *$, $S = id$, φ:



$\Sigma = \{\emptyset, X\}$

Here, φ is a factor map, but $\varphi(X)$ not meas.

Rem. ψ above (in (2)) is autom. measure pres. (why?)

Rem (1) is equiv. to

(1*) $\exists X' \subset X, \mu(X')=1, TX' \subset X'$ $\exists \varphi: X' \to Y$ meas. pres.:

$$\varphi(Tx) = S(\varphi x) \quad \forall x \in X'$$

and (2) is equiv. to

(2*) $\exists X' \subset X, \mu(X')=1, TX' \subset X'$ $\exists \varphi: X' \to Y'$ meas., bij, with measur. φ^{-1}:
$\exists Y' \subset Y$ $\nu(Y')=1$, $SY' \subset Y'$

$$\varphi(Tx) = S(\varphi x) \quad \forall x \in X'.$$

(here, $\varphi(X') = Y'$ measurable)

$\boxed{\text{Ex 1.10}}$ (Doubling map is isomorphic to Bernoulli shift)

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

Let $X = \{0,1\}^{\mathbb{N}_0}$, $\mu = \prod_{n=0}^{\infty} \underbrace{\left(\tfrac{1}{2}, \tfrac{1}{2}\right)}_{p \text{ on } \{0,1\}}$ – product measure induced by cylinders

$T: X \to X$ left shift $\leftarrow$ $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$

(X,μ,T) is one-sided Bernoulli shift from Ex. 1.3 (with $\mathbb{N}_0$)

Let $Y = [0,1]$ with $\nu = Leb$

$\varphi: \{0,1\}^{\mathbb{N}} \to [0,1]$ with $\varphi(x_0, x_1, x_2, \dots) := \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}$

$$\begin{tikzcd} \{0,1\}^{\mathbb{N}_0} \arrow[r, "\leftarrow"] \arrow[d, "\varphi"'] & \{0,1\}^{\mathbb{N}_0} \arrow[d, "\varphi"] \\ {[0,1]} \arrow[r, "S"'] & {[0,1]} \end{tikzcd}$$

Find S: (X,μ,T) is isom. to (Y,ν,S)

$$S\left(\sum_{n=0}^{\infty}\frac{x_n}{2^{n+1}}\right) = S(\varphi(x)) \underset{\substack{\uparrow\\ \text{commuting}\\ \text{diagram}}}{=} \varphi(Tx) = \varphi(x_1,x_2,\dots) = \sum_{n=0}^{\infty}\frac{x_{n+1}}{2^{n+1}} = \sum_{n=1}^{\infty}\frac{x_n}{2^n} = 2\sum_{n=1}^{\infty}\frac{x_n}{2^{n+1}}.$$

So $Sy = 2y \mod 1$ – doubling map

(Exerc.) Check: isom. (what are X', Y' from (2^*) ?)

$\boxed{\text{Def 1.11}}$ (Products)

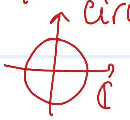
Let (X,μ,T), (Y,ν,S) be MDS. The product system is $(X\times Y, \mu\times\nu, T\times S)$, where:

- $(T\times S)(x,y) = (Tx, Sy)$
- $\Sigma_{X\times Y}$ is generated by $\underset{\Sigma_X}{\overset{}{A}}\times\underset{\Sigma_Y}{B}$ ($A\in\Sigma_X$, $B\in\Sigma_Y$)
- $(\mu\times\nu)(A\times B) := \mu(A)\cdot\nu(B)$ for $\begin{matrix}A\in\Sigma_X\\ B\in\Sigma_Y\end{matrix}$

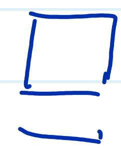
Analogously: $(X_1\times\dots\times X_n, \mu_1\times\dots\times\mu_n, T_1\times\dots\times T_n)$

$\boxed{\text{Ex}}$ $\mathbb{T}^d$, $T(x_1,\dots,x_d) := (\underset{\mathbb{T}}{a_1}x_1,\dots,\underset{\mathbb{T}}{a_d}x_d)$, μ Leb.

$\mathbb{T}$ = unit circle

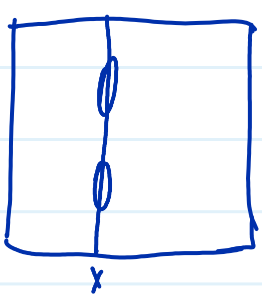


Such products are extensions of each (X_j,μ_j,T_j) via $\pi_j : (x_1,\dots,x_n)\mapsto x_j$ – jth projection – why?] $\square$



$\boxed{\text{Def 1.14}}$ (Skew products)

Let (X,μ,T) be an MDS and (Y,ν) prob. space,



$\varphi : X\times Y\to Y$ measur. s.t.

for a.e. $x\in X$ $\varphi(x,\cdot): Y\to Y$ is ν-pres.

(i.e., $\nu(A) = \nu(\{y\in Y : \varphi(x,y)\in A\})$, so φ is ν-pres. in each fiber

set of the form $\{(x,y), y\in Y\}$

Def. $S : X\times Y\to X\times Y$

$S(x,y) := (Tx, \varphi(x,y))$ – skew product along φ.

Claim: S is $\mu\times\nu$ pres.

Proof
$$(\mu\times\nu)(S^{-1}(\underbrace{A\times B}_{\text{generate the }\sigma\text{-alg.}})) = \int_{X\times Y}\underbrace{\mathbb{1}_{S^{-1}(A\times B)}(x,y)}_{\mathbb{1}_{A\times B}(\underbrace{S(x,y)}_{(Tx,\varphi(x,y))})}\,d(\mu\times\nu)(x,y)$$

$$=\int_{X\times Y}\underbrace{\mathbb{1}_{A\times B}(Tx,\varphi(x,y))}_{\mathbb{1}_A(Tx)\cdot\mathbb{1}_B(\varphi(x,y))}\,d(\mu\times\nu)(x,y) = [\text{Fubini}]$$

$$=\int_X\Big[\underbrace{\int_Y \mathbb{1}_B(\varphi(x,y))\,d\nu(y)}_{\nu(\{y:\varphi(x,y)\in B\}) = \nu(B)\text{ by assump. on }\varphi}\Big]\,\mathbb{1}_A(Tx)\,d\mu(x)$$

$= \nu(B) \int_X \mathbb{1}_A(Tx)\, d\mu(x) = \mu(A)\nu(B) = (\mu\times\nu)(A\times B)$.

$\underbrace{\int_X \mathbb{1}_A(Tx)\,d\mu(x)}_{\mu(T^{-1}(A)) = \mu(A)}$

Such sets generate the σ-alg., so we are fine. (why?) ∎

So $(X\times Y, \mu\times\nu, S)$ is an MDS

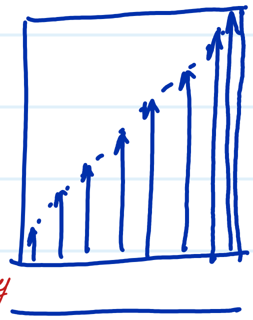
if $Y = G$ comp. (metrisable) group with ν = Haar and

$\varphi(x,g) = \varphi(x)\cdot g$ for some $\varphi: X \to G$,

then $(X\times G, S)$ is called a group extension of (X,μ,T)

$\boxed{\text{Ex}}$ $[0,1]^2$, $S(x,y) = (\underbrace{\alpha + x}, \underbrace{x+y}_{\text{mod } 1})$, α fixed.

Here: $Tx = x+\alpha \mod 1$



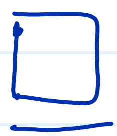
Or multiplicatively: $\mathbb{T}^2$, $S(z,w) := (\lambda z, z\cdot w)$

Exerc.: $([0,1], \text{Leb}, \alpha + \cdot \mod 1)$ is isom. to $(\mathbb{T}, \text{Haar}, e^{2\pi i\alpha})$ rotation by

Rem. (X,μ,T) is a factor of the skew product

with factor map $\pi: X\times Y \to X$, $\pi(x,y) := x$ – proj. onto X

$$\begin{CD} X\times Y @>S>> X\times Y \\ @V\pi VV @VV\pi V \\ X @>T>> X \end{CD}$$



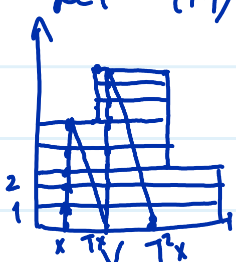
- π is measure pres.:
  $(\mu\times\nu)(\pi^{-1}(A)) = (\mu\times\nu)(A\times Y) = \mu(A)$
- $\underbrace{\pi(S(x,y))} = \pi(Tx, \varphi(x,y)) = Tx = \underbrace{T(\pi(x,y))}$ $\forall x,y$.

$\boxed{\text{Ex + Def. 1.13}}$ (Transformation under a fct)

Let (X,μ,T) be an MDS, $f: X\to\mathbb{N}$ simple, i.e., $f = \sum_{j=1}^m n_j \mathbb{1}_{A_j}$,

A_j meas., disjoint, $n_j \in \mathbb{N}$

normally $\bigcup A_j = X$ up to null sets



Def. $D := \{(x,n),\ x\in X,\ n\in\{0,\dots,f(x)\}\} \subset X\times\mathbb{N}_0$

with rescaled product measure

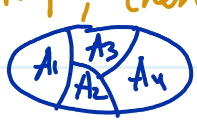
$\nu_1(A\times\{n\}) := \mu(A)$ $\forall A\subset A_j$ $\forall n\in\{0,\dots,n_j\}$

$\nu := \dfrac{\nu_1}{\nu_1(D)}$

Def. $S: D\to D$ as follows:

$$S(x,n) := \begin{cases} (x, n+1) & \text{if } (x,n+1)\in D \\ (Tx, 0) & \text{otherwise} \end{cases}$$

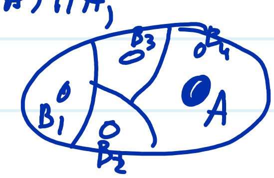
(go up to the „roof", then down to $(Tx,0)$)

Different view  – delay of n_j if in A_j

We show: S is ν-pres.

Enough to show $A\subset A_k$ (why?). Def. $B_j := T^{-1}(A)\cap A_j$

Then $T^{-1}(A) = \bigcup_{j=1}^m B_j$, B_j disjoint up to null sets



For $n\in\{1,\dots,n_k\}$ we have $S^{-1}(A\times\{n\}) = A\times\{n-1\}$ - the same measure

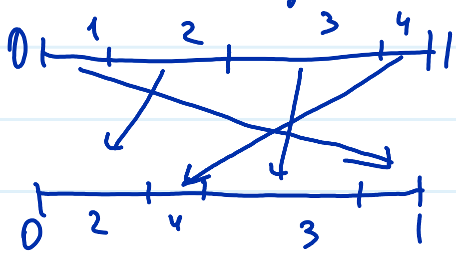
For $n=0$:

$$\nu(\underbrace{S^{-1}(A\times\{0\})}_{\bigcup_{j=1}^{m} B_j\times\{n_j\}}) = \sum_{j=1}^{m} \underbrace{\nu(B_j\times\{n_j\})}_{\frac{\mu(B_j)}{\nu_1(D)}} = \frac{1}{\nu_1(D)}\cdot \underbrace{\sum_{j=1}^{m}\mu(B_j)}_{\underbrace{\mu(T^{-1}(A))}_{=\mu(A)}} = \frac{\mu(A)}{\nu_1(D)}$$

$$= \nu(A\times\{0\})$$

So: (D,ν,S) is an MDS.

Rem. There are many more classical transformations, such as interval exchange transformations and many more:



k intervals, shift them around :)

1.3 invariant measures for topological dynamical systems

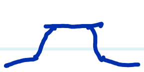
Let X be a compact metric space, $T: X\to X$ continuous. The pair (X,T) is called a topological dynamical system (TDS)

Examples we saw before: finite systems (with discrete metric), group rotations

Question: $\exists$? Borel (completed) prob. measure μ on X s.t. (X,μ,T) is an MDS, so μ is T-invariant ($\Leftarrow$ T μ-pres.)

Rem. 1) $\forall$ Borel measure μ on X is regular, i.e.,

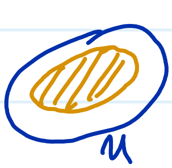
$\forall A$ meas. $\mu(A) = \sup\{\mu(K), K\subset A \text{ comp.}\}$
$= \inf\{\mu(U), U\supset A \text{ open}\}$



2) $\forall U\subset X$ open $\exists (f_n)\subset C(X)$, $0\le f_n \nearrow 1_U$

Proof: let (K_n) comp. sets, $K_n\subset U$, $K_n\nearrow$, $\bigcup K_n = U$ ($\exists$ by 1))

For example, $K_n := \{x\in A: d(x,\partial U)\ge \frac{1}{n}\}$ - closed, so comp. (X comp.)



Then $\exists f_n$ contin. with $f_n = \begin{cases} 1 & \text{on } K_n \\ 0 & \text{on } X\setminus U\end{cases}$ on U, $\{f_n(x) = \min\{1, n\,dist(x,\partial U)\}\}$

$$f_n(x) := \begin{cases} n\cdot dist(x,\partial U) & \text{on } U\setminus K_n \\ 1 & \text{on } K_n \\ 0 & \text{on } X\setminus U \end{cases}$$

Exerc.: f_n is cont. and
$0 \leq f_n \leq 1_U$, $f_n \nearrow 1_U$ clear ■

$\boxed{\text{Lemma 1.14}}$ For a TDS (X,T) and a Borel prob. meas. μ on X,
T is μ-pres. $\Leftrightarrow$
$$\int_X f \, d\mu = \int_X f \circ T \, d\mu \quad \forall f \in C(X)$$

Proof ($\Rightarrow$) follows from lemma 1.8 (f cont. $\Rightarrow f \in L^\infty(X,\mu)$)
($\Leftarrow$) We need to show this for $f = 1_A$, A meas.
it suffices to assume that A is open (why?)
Let $(f_n) \subset C(X)$ be as in Rem. 2) : $0 \leq f_n \leq 1_A$, $f_n \nearrow 1_A$
Then
$$\underbrace{\int f_n \, d\mu \overset{\text{assumption}}{=} \int f_n(Tx) \, d\mu(x)}_{\text{monotone conv. thm}}$$
$$\int f \, d\mu \qquad \int f(Tx) \, d\mu(x)$$

Recall (FA)
$\boxed{\text{Thm}}$ (Riesz) : $C(X)' \simeq \{$finite $\mathbb{C}$-valued Borel measures on $X\}$
via $\langle f, \mu \rangle = \int f \, d\mu$
functional measure
The topology/convergence on measures is the weak* top./conv.
$$\mu_n \to \mu \Leftrightarrow \int f \, d\mu_n \to \int f \, d\mu \quad \forall f \in C(X)$$

$\boxed{\text{Ex}}$ The point evaluation $f \mapsto f(a)$, $a \in X$ fixed,
corresponds to the Dirac meas. $\delta_a(A) := \begin{cases} 1, & a \in A \\ 0, & \text{otherwise} \end{cases}$

$\boxed{\text{Thm}}$ (Banach-Alaoglu) The closed unit ball in $C(X)'$ is weak*-compact. In particular, $\forall$ sequence of Borel prob. measures on X have a weak*-convergent subsequence.

μ Borel prob. measure $\Rightarrow |\int f \, d\mu| \leq \|f\|_\infty$ and "=" for $f = 1$,
so $\|\mu\|_{C(X)'} = 1$ - in the closed unit ball of $C(X)'$.

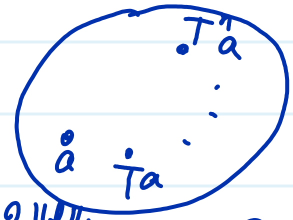
$\boxed{\text{Thm 1.15}}$ (Krylov-Bogolyubov)
$\forall$ comp. (metric) space X $\forall$ cont. $T : X \to X$ $\exists$ T-inv. Borel prob. measure μ on X.

So : TDS $\rightsquigarrow$ MDS

Rem. μ is in general not unique : $T = id$ - $\forall \mu$ is T-inv.

Proof Take $a \in X$ and consider

$$\mu_n := \frac{\delta_a + \delta_{Ta} + \dots + \delta_{T^n a}}{n+1} \quad \text{- prob. measure.}$$



Then

$$\left|\int_X (f(Tx) - f(x))\, d\mu_n(x)\right| = \left|\frac{\cancel{f(Ta)} \cancel{+ \dots +} f(T^{n+1}a)}{n+1} - \frac{f(a) \cancel{+ \dots +} \cancel{f(T^n a)}}{n+1}\right| = \left|\frac{f(T^{n+1}a) - f(a)}{n+1}\right| \le \frac{2\|f\|_\infty}{n+1} \xrightarrow[n\to\infty]{} 0$$

Banach-Alaoglu: $\exists\, \mu_{n_k} \to \mu$ (weak* conv. subseq.)

Then μ is also Borel $\underbrace{\text{prob measure}}$

$\mu \geq 0$ because $\int f\, d\mu \geq 0 \ \forall f \geq 0$

$\underset{\int 1\, d\mu}{\mu(X)} = \lim \mu_{n_k}(X) = 1$

and

$$\int f(Tx)\, d\mu - \int f\, d\mu = 0,$$

so μ is T-inv. By Lemma 1.14.

Rem. If such μ is unique, then (X, T) is called uniquely ergodic