27.10.26 **Part 2: X Hilbert.** To show: $\lambda - T$ invert. $\Leftrightarrow$ $\overline{\lambda} - T^* = (\lambda - T)^*$ invert.

WLOG $\lambda = 0$ (otherwise $S := \lambda - T$), i.e., to show: T invert. $\Leftrightarrow$ T^* invert.

($\Rightarrow$) if $\exists T^{-1}$, then $(T^{-1})^* = (T^*)^{-1}$ by

$$T^* (T^{-1})^* = (T^{-1} T)^* = I \quad \text{and analog. } (T^{-1})^* T^* = I$$

($\Leftarrow$) follows from $\Rightarrow$ for T^* (recall: $T^{**} = T$) ∎

Rem. 2.11 in the proof of Part 1 we saw:

if T is **bounded from below**, i.e., $\exists c > 0$:

$$\|Tx\| \geq c\|x\| \quad \forall x,$$

then $\operatorname{Ran} T$ is closed (and T is inj). So such operators are invertible

$\Leftrightarrow$ T^{-1} is injective ($\operatorname{Ran} T$ is dense) ($\Leftarrow$ T^* is inj) if X is Hilbert

Ex 2.12 ℓ^p, $p \in [1, \infty)$, $\sigma(T_\ell) =$

By $T_r = T_\ell'$ on ℓ^q, $\frac{1}{q} + \frac{1}{p} = 1$ (left shift) we see that $\sigma(T_r) =$

(But $P\sigma(T_r) = \emptyset$ - exer.)

3. Classification of spectrum; spectrum of isometries and s.a. operators

Def. 3.1 For $T \in \mathcal{L}(X)$, the **approximate point spectrum** is

$$A\sigma(T) := \{\lambda \in \mathbb{C} : \exists (x_n)_{n=1}^{\infty} \subset X \text{ with } \|x_n\| = 1 \ \forall n \text{ and } \|\lambda x_n - T x_n\| \xrightarrow[n\to\infty]{} 0\}$$

(approximate eigenvector; approximate eigenvalue)

and the **residual spectrum**

$$R\sigma(T) := \{\lambda \in \mathbb{C} : \operatorname{Ran}(\lambda - T) \text{ is not dense in } X\}$$

Prop. 3.2 (Characterisation of $A\sigma$)

For $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$, TFAE:

(i) $\lambda \notin A\sigma(T)$

(ii) $\lambda - T$ is inj and $\operatorname{Ran}(\lambda - T)$ is closed

(iii) T is bdd from below: $\exists c > 0 : \|(\lambda - T)x\| \geq c\|x\| \ \forall x$.

in particular, $A\sigma(T) = \{\lambda \in \mathbb{C} : \operatorname{Ran}(\lambda - T) \text{ is not closed}\} \cup P\sigma(T)$

Proof WLOG $\lambda = 0$ (otherwise $S := \lambda - T$)

(i)⇒(iii) Assume (iii) fails: $\forall n \ \exists x_n \neq 0 : \|Tx_n\| < \frac{\|x_n\|}{n}$

Def. $y_n := \frac{x_n}{\|x_n\|}$. Then $\|y_n\| = 1 \ \forall n$ and

$$\|Ty_n\| = \frac{\|Tx_n\|}{\|x_n\|} < \frac{1}{n} \longrightarrow 0$$

and $0 \in A\sigma(T)$.

(iii)⇒(ii) see Remark 2.11

(ii)⇒(i) Let $Y := \mathrm{Ran}\, T$ - Banach space (was closed)

Consider $T : X \to Y$ bij and continuous

Continuous inverse thm. $\exists S : Y \to X$ contin, lin with $ST = I_X$, $TS = I_Y$

So $\forall x$ with $\|x\| = 1$

$$1 = \|x\| = \|STx\| \leq \|S\| \, \|Tx\|,$$

i.e., $\|Tx\| \geq \frac{1}{\|S\|} \Rightarrow 0 \notin A\sigma(T)$. ∎

Cor 3.3 One has $\boxed{\sigma(T) = A\sigma(T) \cup R\sigma(T)}$.

($\cup P\sigma(T)$)

Proof $\lambda - T$ not bij $\Leftrightarrow$ not inj or $\mathrm{Ran}(\lambda - T) \neq X$

$\mathrm{Ran}(\lambda - T)$ not closed, or not dense

$A\sigma$ $R\sigma$ ∎

Rem. $A\sigma(T)$ and $R\sigma(T)$ are not disjoint in general.

Ex • $T = 0$ $\sigma(T) = P\sigma(T) = A\sigma(T) = R\sigma(T) = \{0\}$

• Analog., $\sigma(\lambda I) = \dots = R\sigma(\lambda I) = \{\lambda\}$

• Also $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $0 = \sigma(T) = P\sigma(T) = A\sigma(T) = R\sigma(T)$

Generally for matrices! range 1-dim

Thm 3.4 Let $T \in \mathcal{L}(X)$ and denote by $\partial\sigma(T)$ the topological boundary of $\sigma(T)$ in $\mathbb{C}$. Then

$$\boxed{\partial\sigma(T) \subset A\sigma(T)}$$

in part., $A\sigma(T) \neq \emptyset$

Proof Let $\lambda \in \partial\sigma(T)$ and take $(\lambda_n) \subset \rho(T)$ with $\lambda_n \to \lambda$.

By Cor 1.11:

$$\|R(\lambda_n, T)\| \geq \frac{1}{\mathrm{dist}(\lambda_n, \sigma(T))} \xrightarrow[n\to\infty]{} \infty.$$

For $n \in \mathbb{N}$ take x_n with $\|x_n\| = 1$ s.t.

$$\|R(\lambda_n, T)x_n\| \geq \|R(\lambda_n, T)\| - \frac{1}{n} \xrightarrow[n\to\infty]{} \infty.$$

$= \sup_{\|x\|=1} \|R(\lambda_n, T)x\|$

Def. now $y_n := \frac{R(\lambda_n, T)x_n}{\|R(\lambda_n, T)x_n\|}$. Then $\|y_n\| = 1 \ \forall n$ and

$$(\lambda - T)y_n = (\lambda_n - T)y_n + (\lambda - \lambda_n)y_n = \frac{x_n}{\|R(\lambda_n, T)x_n\|} + (\lambda - \lambda_n)y_n$$

So

$$\|(\lambda - T)y_n\| = \frac{1}{\|R(\lambda_n, T)x_n\|} + |\lambda - \lambda_n| \xrightarrow{n\to\infty} 0$$

So (y_n) is approx eigenvector to λ ■

Examples.

Prop. 3.5 (Spectrum of isometries)

Let $T \in \mathcal{L}(X)$ be an isometry, i.e., $\|Tx\| = \|x\|$ $\forall x \in X$. Then:

(a) $A\sigma(T) \subset \mathbb{T}$ (unit circle)

(b) if T is invertible, then $\sigma(T) \subset \mathbb{T}$

(c) if T is not invert., then $\sigma(T) = \{\lambda : |\lambda| \le 1\}$ in this case in part. $R\sigma(T) \supset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

(d) in (b), $\forall K \subset \mathbb{T}$ closd, $\neq \emptyset$ can be attained as $\sigma(T)$.

Proof (a) Let $\lambda \in A\sigma(T)$ then $\exists (x_n) \subset X$: $\|x_n\| = 1$:

$$\big|\underbrace{|\lambda|\|x_n\|}_{=\|x_n\|} - \|Tx_n\|\big| \le \|\lambda x_n - Tx_n\| \longrightarrow 0,$$

$\|\lambda| - 1|$

i.e., $|\lambda| = 1$.

(b) T invert. $\Rightarrow 0 \in \rho(T)$. By (a) and $\partial\sigma(T) \subset A\sigma(T)$ we must have

$$\{\lambda : |\lambda| < 1\} \subset \rho(T)$$

(why?) By $\|T\| = 1$ we know $\sigma(T) \subset \{\lambda : |\lambda| \le 1\}$, so $\subset \mathbb{T}$.

(c) if $0 \in \sigma(T)$, by $\partial\sigma(T) \subset \mathbb{T}$ (by (a)) we must have

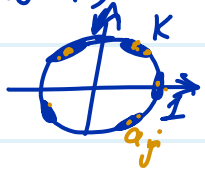
$$\{\lambda : |\lambda| < 1\} \subset \sigma(T).$$

Since $\sigma(T)$ is closed and by $\|T\| = 1$, $\sigma(T) =$

"in this case" follows from (a) and $\sigma(T) = A\sigma(T) \cup R\sigma(T)$.

(d) Consider multipl. on ℓ^2: take $(a_j) \subset K$ dense and M_a given by

$$M_a(t_1, t_2, \dots) := (a_1 t_1, a_2 t_2, \dots)$$



- isometry since $a_j \in \mathbb{T}$ $\forall j$ and $\sigma(M_a) = \overline{\{a_j, j \in \mathbb{N}\}} = K$ ■

Rem. if X is Hilbert, then

$$T \text{ isometry} \Leftrightarrow \langle Tx, Ty\rangle = \langle x, y\rangle \ \forall x, y \Leftrightarrow \langle T^*Tx, y\rangle = \langle x, y\rangle \ \forall x, y$$

(polarisation identity)

$$\Leftrightarrow T^*T = I$$

So T is unitary $\Leftrightarrow T$ is invertible isometry, and in this case $\sigma(T) \subset \mathbb{T}$ by (b) in Prop. 3.5

Ex (Right shift) $X = \ell^\infty, \ell^p, c_0, c, \dots$, $T \to$.

$$T(t_1, t_2, \dots) := (0, t_1, t_2, \dots)$$

- $\|Tx\| = \|x\|$ $\forall x$ clear, T not invert. clear (not surj) $\Rightarrow \sigma(T) =$

Moreover, $A\sigma(T) = \Pi$ ($\supset$ boundary, $\subset$ isometry: Prop. 3.5 (a))

- $P\sigma(T) = \emptyset$. $Tx = \lambda x \Rightarrow 0 = \lambda t_1$, $t_1 = \lambda t_2$ (case 1: $\lambda = 0 \Rightarrow \forall t_j = 0$; case 2: $t_1 = 0$, $\lambda \neq 0 \Rightarrow t_2 = 0 \Rightarrow \dots$) $\Rightarrow x = 0$.

Ex. of an approximate eigenvector:

$\lambda = 1$, $X = \ell^\infty$. Find (x_n) $\|x_n\| = 1$ $\forall n$, $\|x_n - Tx_n\| \to 0$

Take $x_n = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1, 1, 1, \dots\right)$

$Tx_n = \left(0, \frac{1}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n}, 1, 1, \dots\right)$

$x_n - Tx_n = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}, 0, \dots\right)$ $\|\cdot\|_\infty = \frac{1}{n} \to 0$.

exer. $X = \ell^p, c_0, c, \dots$ $\lambda = 1$ (and $\lambda \neq 1$)

Ex $X = \ell^\infty, \ell^p, c_0, c, \dots$

$T(t_1, t_2, \dots) := (0, t_1, 0, t_2, \dots)$ - the same $\sigma(T)$, $A\sigma(T)$, $R\sigma(T)$, $P\sigma(T)$.

exer

Rem. Moreover, one has the following (sheet 3):

(a) $A\sigma(T)$ is always closed

(b) $P\sigma(T') = R\sigma(T)$ and, if X Hilbert, $P\sigma(T^*) = \overline{R\sigma(T)}$ (- complex conjug.), $R\sigma(T^*) = \overline{P\sigma(T)}$

Ex $X = \ell^2$, $T \to$, $T^* \leftarrow$ (not isom.!)

$R\sigma(\to) = \overline{P\sigma(\leftarrow)} = \{\lambda : |\lambda| < 1\}$ (isometry)

$A\sigma(\to) = \Pi$, so for $\to$, $R\sigma$ and $A\sigma$ are disjoint.

For $\leftarrow$: $R\sigma(\leftarrow) = \overline{P\sigma(\to)} = \emptyset$, so $A\sigma(\leftarrow) = \sigma(\leftarrow) = \mathbb{D}_1$

Analog. on $X = \ell^p$, $1 \le p < \infty$ and c_0 ($c_0' = \ell^1$)

$R\sigma(\to) = P\sigma(\leftarrow \text{ on } X') = \{\lambda : |\lambda| < 1\}$

$R\sigma(\leftarrow) = P\sigma(\to \text{ on } X') = \emptyset$

For isometries on Hilbert spaces one also has:

Prop. 3.6 For every isometry $T \in \mathcal{L}(H)$, H Hilbert, every 2 eigenvectors to different eigenvalues are orthogonal.

Proof Let $Tx = \lambda x$, $Ty = \mu y$, $\lambda \neq \mu$. We know $\lambda, \mu \in \mathbb{T}$ if eigenvalues (T isometry).

Then $\langle x, y\rangle = \langle Tx, Ty\rangle = \lambda\bar{\mu}\langle x, y\rangle \Rightarrow \langle x, y\rangle = 0$. ($Tx = \lambda x$, $Ty = \mu y$; $\lambda\bar{\mu} \neq 1$)

Rem. As we saw for $\to$ on ℓ^2, no EV is possible

Thm 3.7 (Spectrum of s.a. operators)

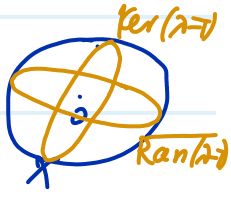
Let $T \in \mathcal{L}(H)$, H Hilbert, be s.a. ($T^* = T$). Then

(a) $\sigma(T) \subset \mathbb{R}$. In part., $\sigma(T) = A\sigma(T)$

(b) $P\sigma(T) = R\sigma(T)$ and $\ker(\lambda - T) = (\mathrm{Ran}(\lambda - T))^\perp$

(c) $\|T\| = r(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$

(d) in (a), any closed non-empty subset of $\mathbb{R}$ can be $\sigma(T)$.



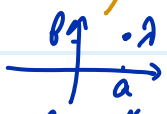
Rem. • $P\sigma(T) \subset \mathbb{R}$ easy: $Tx = \lambda x$, then

$$\lambda \cdot \|x\|^2 = \langle Tx, x\rangle \overset{T=T^*}{=} \langle x, Tx\rangle = \bar{\lambda}\|x\|^2, \quad \text{so } \lambda = \bar{\lambda} \text{ if } x \neq 0$$

• Eigenvectors to different EV are also orthogonal.

$Tx = \lambda x$, $Ty = \mu y$, $y \neq 0$, then $\lambda\langle x, y\rangle = \langle Tx, y\rangle \overset{T=T^*}{=} \langle x, Ty\rangle = \bar{\mu}\langle x, y\rangle$, so $\langle x, y\rangle = 0$ if $\lambda \neq \mu$

Proof of 3.7 (a) Let $\lambda = a + ib$, $b \neq 0$



We show. $\lambda - T$ is bdd from below (and hence $\lambda \notin \sigma(T)$ by Prop. 3.2)

$$\|(\lambda - T)x\|^2 = \langle ((a-T) + ib)x, ((a-T) + ib)x\rangle = \|(a-T)x\|^2 + ib\langle x, (a-T)x\rangle$$
$$+ \bar{i}b\langle (a-T)x, x\rangle + |b|^2\|x\|^2 = \|(a-T)x\|^2 + |b|^2\|x\|^2$$

($-i$; $= \langle x, (a-T)x\rangle$ by $T = T^*$ and $a \in \mathbb{R}$)

$$\geq |b|^2\|x\|^2$$

So $(\lambda - T)$ is bdd from below and $\lambda \notin \sigma(T)$. By Thm 3.4

$$(\mathbb{C} \setminus \mathbb{R}) \cap \partial\sigma(T) = \emptyset$$

- no boundary points in $\mathbb{C}\setminus\mathbb{R}$. Since $\sigma(T)$ is bdd,

$$(\mathbb{C}\setminus\mathbb{R}) \cap \sigma(T) = \emptyset,$$

i.e., $\sigma(T) \subset \mathbb{R}$

"in part." follows from $\sigma(T) = \partial\sigma(T)$.

(b) Let $\lambda \in \sigma(T)$. Then $\lambda \in \mathbb{R}$ by (a) and:

$$\ker(\lambda - T) \overset{\lambda - T \text{ is s.a}}{=} \ker(\lambda - T)^* = (\operatorname{Ran}(\lambda - T))^\perp$$

So if $\lambda - T$ is surj $\Rightarrow$ inj $\Rightarrow$ bij.

(c) $\|T\| = r(T)$ follows by normality, see Prop. 2.9.

$w := \sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \|T\|$ clear.

To show: $\geq$.

Observe: $\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle = 2\langle Tx, y\rangle + 2\langle Ty, x\rangle$

$$= 2\left(\langle Tx, y\rangle + \overline{\langle Tx, y\rangle}\right) = 4 \cdot \operatorname{Re}\langle Tx, y\rangle$$

By parallelogram law

$$4|\operatorname{Re}\langle Tx, y\rangle| \leq w \cdot (\|x+y\|^2 + \|x-y\|^2) = 2w(\|x\|^2 + \|y\|^2)$$

(def. of w; parallelogram law)

In part., $|\operatorname{Re}\langle Tx, y\rangle| \leq w$ $\forall x, y$ with $\|x\| = \|y\| = 1$

By multiplying x with appropriate $\lambda \in \mathbb{T}$,

$$|\langle Tx, y\rangle| \leq w \quad \forall x, y \text{ with } \|\cdot\| = 1,$$

i.e., $\|T\| \leq w$.

(d) $\longrightarrow$ K Take ℓ^2, multipl. with (a_j) dense in K.

(Recall: $M_a^* = M_{\bar{a}}$)

Rem 3.8 For $T \in \mathcal{L}(H)$ (H Hilbert), the set

$$W(T) := \{\langle Tx, x\rangle, \|x\|=1\}$$

is called the **numerical range of T** and the number

$$w(T) := \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

the **numerical radius of T**. They satisfy

- $W(T)$ is convex
- $\sigma(T) \subset \overline{W(T)}$

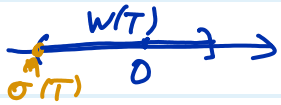
(no proof). In part., for s.a. T,

$$\sigma(T) \subset [\underbrace{\inf W(T), \sup W(T)}_{\text{set in } \mathbb{R}}]$$

$= W(T)$ up to end points by convexity

and one of the endpoints (the one with largest $|\cdot|$) is in $\sigma(T)$

by $r(T) = w(T)$



Rem. 3.9 (Further/alternative classification of the spectrum)

Often one considers the **continuous spectrum**

$$\sigma_c(T) := \{\lambda \in \sigma : \lambda - T \text{ inj., not surj, with dense range}\}$$

and the **rest spectrum**

$$\sigma_r(T) := \{\lambda \in \sigma : \lambda - T \text{ inj, range not dense}\}$$

Then one has

$$\sigma_r(T) = R\sigma(T) \setminus P\sigma(T)$$

$$\sigma_c(T) = A\sigma(T) \setminus (P\sigma(T) \cup R\sigma(T))$$

by prop. 3.2, $\lambda \in A\sigma(T) \Leftrightarrow \lambda - T$ not inj or range not closed

and the **disjoint decomp.**

$$\boxed{\sigma(T) = P\sigma(T) \cup \sigma_c(T) \cup \sigma_r(T)}$$

Ex T s.a $\Rightarrow \sigma_r(T) = \emptyset$ $\quad (\ker(\lambda - T) = (\operatorname{Ran} T)^{\perp})$

4 Spectrum of compact operators: Riesz' theory

Persons:

- Frigyes Riesz (1880–1956) – hungarian math., one of founders of FA
- Erik Ivar Fredholm (1866–1927) – swedish math. and math physicist
- Julius Schauder (1899–1943) – polish math., colleg of Banach.

Def 4.1 A lin. operator $T: X \to Y$ (Banach) is called **compact** if $\forall A \subset X$ bdd

$T(A)$ is rel. compact, i.e., $\overline{T(A)}$ is compact

Recall. in metric spaces, B is rel. comp $\Leftrightarrow$ $\forall$ seq. from B has a convergent subsequence (comp $\Leftrightarrow$ sequential comp.)

$\forall$ open cover has a finite subcover

Rem. 1) T comp. $\Leftrightarrow$ $T(B_X)$ rel. compact

B_X closed unit ball in X $\{x \in X : \|x\| \le 1\}$

$\Leftrightarrow$ $\forall (x_n) \subset X$ with $\|x_n\| \le 1$ $\exists$ subseq. (n_j) of $\mathbb{N}$ s.t. (Tx_{n_j}) conv.

(indeed, if $T(B_X)$ is rel. comp. then so is any subset of it, so $T(A)$ rel. comp. $\forall A \subset B_X$)

Attention: B_X itself is comp. $\Leftrightarrow \dim X < \infty$

2) T comp. $\Rightarrow T$ bdd since compact sets are bdd (why?)

Notation: ~~$\mathcal{K}(X,Y)$~~ $:= \{T : X \to Y \text{ comp.}\} \subset \mathcal{L}(X,Y)$

~~$\mathcal{K}(X)$~~ $:= \mathcal{K}(X,X) \subset \mathcal{L}(X)$

Ex 4.2 1) $\dim X < \infty \Rightarrow \mathcal{L}(X) = \mathcal{K}(X) = L(X)$ (lin. oper. on X)

T bdd

Heine-Borel: $T(A)$ bdd $\Rightarrow$ rel. comp

2) $\dim \operatorname{Ran} T < \infty \Rightarrow T$ comp (again Heine-Borel)

3) I comp $\Leftrightarrow \dim X < \infty$.

Thm 4.3 (Properties of $\mathcal{K}(X,Y)$)

(a) $\mathcal{K}(X,Y)$ is a closed lin. subspace of $\mathcal{L}(X,Y)$ and hence a Banach space itself.

(b) Let $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(Y,Z)$. If T or S is compact, then so is ST.

$X \xrightarrow{T} Y \xrightarrow{S} Z$ in part., $\mathcal{K}(X)$ is a (closed) ideal in $\mathcal{L}(X)$

T comp., S bdd $\Rightarrow TS$, ST comp.

Proof (a) lin. subspace: let $T, S \in \mathcal{K}(X,Y)$, $(x_n) \subset B_X$.

T comp $\Rightarrow \exists (n_j)$: (Tx_{n_j}) conv.

S comp., $(x_{n_j}) \subset B_X \Rightarrow \exists$ subseq. (m_j): (Sx_{m_j}) conv. (and (Tx_{m_j}) also)

So $((T+S)x_{m_j})$ conv.

closed Let $T_n \to T$, each T_n comp. To show: T comp.

Let $(x_n) \subset X$ with $\|x_n\| \le 1$ $\forall n$

T_1 comp. $\Rightarrow \exists$ subseq. $(x_{n,1})$: $(T_1 x_{n,1})$ conv.

T_2 comp. $\Rightarrow$ -"- $(x_{n,2})$ of $(x_{n,1})$: $(T_2 x_{n,2})$ conv.

etc.

$x_{1,1}$ $x_{2,1}$ $x_{3,1}$ $x_{2,1}$ $x_{1,2}$ $x_{3,2}$ ...

Consider $y_n := x_{n,n}$ - subseq. of each of $(x_{n,j})$ (and of (x_n)), ...

so $(T_j y_n)$ conv $\forall j$. To show: (Ty_n) conv.

Let $\varepsilon > 0$. Take j with $\|T - T_j\| < \varepsilon$ and observe:

$$\|Ty_n - Ty_m\| \le \underbrace{\|Ty_n - T_j y_n\|}_{\le \|T - T_j\| \, \|y_n\|,\ <\varepsilon,\ \le 1} + \|T_j y_n - T_j y_m\| + \underbrace{\|T_j y_m - Ty_m\|}_{\le \|T_j - T\| \cdot \|y_m\|,\ <\varepsilon,\ =1}$$

$$\le 2\varepsilon + \|T_j y_n - T_j y_m\| < 3\varepsilon \quad \text{for large } n,m \text{ since } (T_j y_m) \text{ conv.}$$

($\exists N: \forall n,m > N$)

So (Ty_n) is a Cauchy seq. $\Rightarrow$ conv and (Tx_n) has a conv subseq.

(b) $X \xrightarrow{T} Y \xrightarrow{S} Z$ Case 1: T comp.

A bdd $\Rightarrow TA$ rel. comp. $\Rightarrow STA$ rel. comp.

(T cont., S continuous)

Case 2 : S comp.. A bdd $\Rightarrow$ TA bdd $\Rightarrow$ STA rel comp.

(T bdd, S comp.)

Rem. In particular, limits of operators with finite dimensional range are compact. Converse (norm limit) false (counterexample by Enflo 1972)

Ex A multiplicator M_a on $l^p, c_0, .$ is compact $\Leftrightarrow a_j \to 0$ (exerc.)

(Note : $M_{1_X} = I$)

A multiplicator M_a on $L^p, C[\alpha,\beta]$ is compact $\Leftrightarrow a = 0$