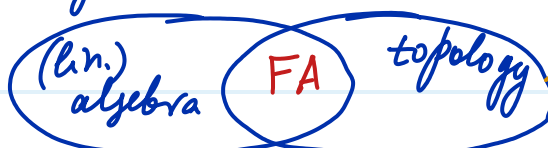


Functional Analysis II

WS 2026/27
Leipzig

- FA:
- Born - beginning of 20. century, founded by Hilbert, Schmidt, Riesz, Banach, von Neumann
 - Few axioms/def. $\Rightarrow$ strong results, beautiful abstract theory
 - Applications to many areas such as Fourier analysis & PDE, math. physics, numerics, measure theory, stochastics, function theory, number theory...
 - ∞ -dim. analysis



extra structure:
open sets, convergence,...

first name: „top algebra“

Objects: lin. spaces and lin. maps (functionals / operator)

my FA 2: more with focus on spectral theory

I introduction to spectral theory

1. spectrum and resolvent

Notation: • if nothing else said, X always a Banach space over $\mathbb{C}$

$$\lambda - T := \lambda I - T$$

Recall: $A \in M_{d \times d}$

$\lambda \in \sigma(A) \Leftrightarrow \lambda$ is an eigenvalue of A , i.e., $\exists x \neq 0: Ax = \lambda x$
 $(\Rightarrow) \lambda - A$ is not inj.

$(\Rightarrow) \lambda - A$ is not surj

so one has

$$\lambda \notin \sigma(A) \Leftrightarrow \exists (\lambda - A)^{-1}$$

and $\exists$ at most d -many eigenvalues.

Def. 1.1

Let $T \in \mathcal{L}(X)$. The resolvent set of T is
 'lin. bdd operators on X '

$$\rho(T) := \{ \lambda \in \mathbb{C} : \exists (\lambda - T)^{-1} \in \mathcal{L}(X) \}$$

The spectrum of T is $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not bijective}\}$
 (odd inverse then)

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

$\lambda \in \sigma(T)$ are called spectral values of T . The point spectrum is

$$\begin{aligned} P_\sigma(T) &:= \{\lambda \in \mathbb{C} : \lambda - T \text{ not inj}\} \\ &= \{\lambda \in \mathbb{C} : \exists x \neq 0 : Tx = \lambda x\} \end{aligned}$$

$\lambda \in P_\sigma(T)$ are called eigenvalues (EV) of T and corresponding $x \neq 0$ eigenvectors of T to the EV λ .

The resolvent of T in $\lambda \in \rho(T)$ is the operator

$$R(\lambda, T) := (\lambda - T)^{-1} \in \mathcal{L}(X)$$

and the resolvent map is the map

$$\begin{aligned} \lambda &\mapsto R(\lambda, T) \\ \rho(T) &\rightarrow \mathcal{L}(X) \end{aligned}$$



Rem. 1) $P_\sigma(T) \subset \sigma(T)$, but $\neq$ in general
 2) $P_\sigma(T) = \emptyset$ is possible

Prop. 1.2 (Resolvent identity)

Let $\lambda, \mu \in \rho(T)$. Then

$$R(\lambda, T) - R(\mu, T) = (\mu - \lambda) R(\lambda, T) R(\mu, T)$$

Attention:
 $\lambda, \mu, \mu - \lambda$
 (different order)

Proof $(\lambda - T) R(\lambda, T) = I$ by def.

$$\lambda R(\lambda, T) - T R(\lambda, T) = I \quad | \cdot R(\mu, T) \text{ on the right}$$

$$\mu R(\mu, T) - T R(\mu, T) = I \quad | \cdot R(\lambda, T) \text{ on the left}$$

$$(\mu - \lambda) R(\lambda, T) R(\mu, T) = R(\lambda, T) - R(\mu, T)$$

Rem. in part, $R(\lambda, T)$ and $R(\mu, T)$ commute
 Of course, they also commute with T by $(\lambda - T) R(\lambda, T) = R(\lambda, T) (\lambda - T) = I$

Lemma + Def 1.3 (Neumann series)

Let $T \in \mathcal{L}(X)$. if the series $\sum_{n=0}^{\infty} T^n$ (called Neumann series)
 normed conv., then

$$\exists (I - T)^{-1} = \sum_{n=0}^{\infty} T^n$$

in part., this holds if $\|T\| < 1$ and X is complete (Banach).

Rem. $T^n = \underbrace{T \circ T \circ \dots \circ T}_{n \text{ times}}$

Proof By assumption, $\|T^n\| \xrightarrow{n \rightarrow \infty} 0$ (why?)
For $N \in \mathbb{N}$

$$\underbrace{(I + T + \dots + T^N)}_{\sum_{n=0}^{\infty} T^n} (1 - T) = 1 - T^{N+1} \xrightarrow{\text{telesc. sum}} 0 = (1 - T) \underbrace{(I + T + \dots + T^N)}_{\sum_{n=0}^{\infty} T^n},$$

so $\sum_{n=0}^{\infty} T^n (1 - T) = I = (1 - T) \sum_{n=0}^{\infty} T^n$

if $\|T\| < 1$, then by $\|T^n\| \leq \|T\|^n$, $\sum_{n=0}^{\infty} T^n$ conv absolutely
if X is Banach, so is $\mathcal{L}(X)$, so $\sum_{n=0}^{\infty} T^n$ conv. (why?)

Prop. 1.4 (Representation of resolvent)

For $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$ one has
 $\lambda \in \rho(T)$ and

$$R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

in part., $\sigma(T)$ is a bounded subset of $\mathbb{C}$ with

$$\sigma(T) \subset \{\lambda : |\lambda| \leq \|T\|\}$$

Proof let $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$. Then $\lambda \neq 0$ and

$$\lambda - T = \lambda \left(I - \underbrace{\frac{T}{\lambda}}_{\| \cdot \| < 1} \right),$$

so by Lemma 1.3

$$\exists (\lambda - T)^{-1} = \frac{1}{\lambda} \left(I - \frac{T}{\lambda} \right)^{-1} = \frac{1}{\lambda} \cdot \sum_{n=0}^{\infty} \left(\frac{T}{\lambda} \right)^n = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

Neumann series

Rem. For $|\lambda| \leq \|T\|$, the representation holds as soon as
the series converges.

Thm 1.5 (Properties of spectrum and resolvent)

let $T \in \mathcal{L}(X)$.

(a) $\rho(T)$ is open and the resolvent map is continuous

(b) The resolvent map is holomorphic on $\rho(T)$

(c) $\sigma(T) \neq \emptyset$.

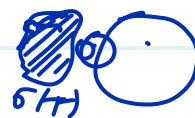
in particular, $\sigma(T)$ is always a compact, nonempty subset of $\mathbb{C}$



Proof (a) Let $\lambda_0 \in \rho(T)$ and λ with

$$|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, T)\|}$$

$R(\lambda_0, T) \neq 0$ since 0 not invert.



Observe $\lambda - T = (\lambda - \lambda_0) + (\lambda_0 - T) = \underbrace{(\lambda - \lambda_0)R(\lambda_0, T)}_{\|\cdot\| < 1} + I \underbrace{(\lambda_0 - T)}_{\text{inv.}}$

Neumann series (Lemma 1.3)

$$\exists R(\lambda, T) = R(\lambda_0, T) \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, T)^n = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, T)^{n+1}$$

in part.,

power series representation!

$$\begin{aligned} \|R(\lambda, T) - R(\lambda_0, T)\| &= \left\| \sum_{n=1}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, T)^{n+1} \right\| \\ &\leq \sum_{n=1}^{\infty} \left(|\lambda_0 - \lambda| \|R(\lambda_0, T)\| \right)^n \cdot \|R(\lambda_0, T)\| \\ &= \frac{|\lambda_0 - \lambda| \|R(\lambda_0, T)\|^2}{1 - |\lambda - \lambda_0| \|R(\lambda_0, T)\|} \xrightarrow{\lambda \rightarrow \lambda_0} 0 \end{aligned}$$

$$\sum_{n=1}^{\infty} q^n = \frac{q}{1-q}$$

$\Rightarrow R(\cdot, T)$ is cont.

(b) Resolvent identity.

$$\frac{R(\lambda+h, T) - R(\lambda, T)}{h} = - \underbrace{R(\lambda+h, T) R(\lambda, T)}_{\xrightarrow{h \rightarrow 0} R(\lambda, T) \text{ by continuity}} \rightarrow -R(\lambda, T)^2,$$

so $\exists \boxed{R(\lambda, T)' = -R^2(\lambda, T)}$

(c) Assume $\sigma(T) = \emptyset$, i.e., $\rho(T) = \mathbb{C}$ and by (b), $R(\cdot, T)$ is holom. on the whole $\mathbb{C}$. Idea: use Liouville's thm

We first show: $R(\cdot, T)$ is bdd in $\mathbb{C}$



Let $|\lambda| \geq 2\|T\|$. Then $R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ and

$$\|R(\lambda, T)\| \leq \sum_{n=0}^{\infty} \frac{\|T\|^n}{|\lambda|^{n+1}} \leq \sum_{n=0}^{\infty} \frac{\|T\|^n}{2^{n+1} \|T\|^{n+1}} = \frac{1}{\|T\|}$$

• Since $R(\cdot, T)$ is contin. on $\mathbb{C}$, it is bdd inside the circle, so bdd everywhere.

Moreover, $\boxed{\|R(\lambda, T)\| \xrightarrow{|\lambda| \rightarrow \infty} 0}$

By $\lambda R(\lambda, T) = I + TR(\lambda, T)$, so $\|\lambda R(\lambda, T)\| \leq C$ for some C and $\forall \lambda$,
 bdd in λ

therefore $\|R(\lambda, T)\| \leq \frac{C}{|\lambda|} \rightarrow 0$.

Take $\varphi \in (\mathcal{L}(X))'$. Def.

$\varphi: \mathcal{L}(X) \rightarrow \mathbb{C}$
lin, bdd

$$f(\lambda) := \varphi(R(\lambda, T)), \quad f: \mathbb{C} \rightarrow \mathbb{C}$$

• f is holom:

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = \varphi\left(\frac{R(\lambda, T) - R(\mu, T)}{\lambda - \mu}\right) \xrightarrow{\text{thm}} \varphi(-R^2(\mu, T))$$

• f is bdd:

$$|f(\lambda)| \leq \|\varphi\| \cdot \|R(\lambda, T)\|$$

bdd on $\mathbb{C}$

Liouville's thm: $f \equiv \text{const.}$ Which const?

$$|f(\lambda)| \leq \|\varphi\| \cdot \|R(\lambda, T)\| \xrightarrow{|\lambda| \rightarrow \infty} 0,$$

i.e., $f \equiv 0$. So

$$\varphi(R(\lambda, T)) = 0 \quad \forall \varphi \in (\mathcal{L}(X))', \quad \forall \lambda \in \mathbb{C}$$

H.-B.: $R(\lambda, T) = 0 \quad \forall \lambda \in \mathbb{C} \quad \nRightarrow \quad (0 \text{ not invert.})$

Rem. By the power series repr. of the resolvent from the above proof, $R(\cdot, T)$ is ∞ -often diff'ble with

$$R(\lambda, T)^{(n)} = (-1)^n n! R(\lambda_0, T)^{n+1}$$

(no details)

Ex 1.6 1) $X = \mathbb{C}^d \Rightarrow \sigma(T) = P_\sigma(T)$,
at most d -many points, 

$\forall \lambda_0 \in \sigma(T)$ is an isolated singularity and a pole of the resolvent, i.e., $\exists k = k(\lambda_0)$:

$$(\lambda - \lambda_0)^k R(\lambda, T)$$

is holom. in a neighb. $U(\lambda_0)$ of λ_0 . 

Proof: exerr. (use jordan normal form)

2) left shift $T: \ell^1 \rightarrow \ell^1$ (or $\ell_0 \rightarrow \ell_0$, $\ell^p \rightarrow \ell^p$, $p \in [1, \infty)$).

$$T(t_1, t_2, \dots) = (t_2, t_3, \dots)$$

$$\|T\| = 1 \Rightarrow \sigma(T) \subset \{\lambda : |\lambda| \leq 1\}.$$

why?

We show. " $\supset$ ".

Let $\lambda: |\lambda| < 1$. Then $\lambda \in P_\sigma(T)$ by the equivalence

$$(t_2, t_3, \dots) = (\lambda t_1, \lambda t_2, \dots)$$

$$\Leftrightarrow t_2 = \lambda t_1, \quad t_3 = \lambda t_2 = \lambda^2 t_1, \dots$$

$$\Leftrightarrow (t_1, t_2, \dots) = c \cdot (1, \lambda, \lambda^2, \dots) \in \ell^1$$

in part.,

$$\dim \ker(\lambda - T) = 1$$

$$\forall \lambda: |\lambda| < 1$$



So $P_\sigma(T) = \{\lambda \mid |\lambda| < 1\} \subset \sigma(T)$
 $\sigma(T)$ closed $\Rightarrow \sigma(T) = \{\lambda \mid |\lambda| \leq 1\}$ - the closed unit disc

Rem. 1) On ℓ^∞ , $P_\sigma(T) = \sigma(T) = \overline{\mathbb{D}}_1$ - why?
 2) By modifying T (and X) one can achieve $\dim \ker(\lambda - T) = \infty$ (exerr. sheet)

3) Multiplicators

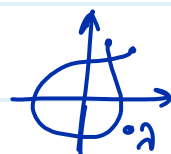
3a) $X = C[0,1]$ $Tf := g \cdot f$ for a fixed $g \in C[0,1]$

Then $\|T\| = \|g\|_\infty$ ($\leq$ clear, $\geq$ $f \equiv 1 \cdot \|Tf\|_\infty = \|g\|_\infty$)

We show:

$$\sigma(T) = g([0,1]) = \{g(s) \mid s \in [0,1]\}$$

• $\lambda \notin g([0,1])$. Denote $\delta := \text{dist}(\lambda, \underbrace{g([0,1])}_{\text{closed}}) > 0$
 so the multiplier by $\frac{1}{\lambda - g}$ is $R(\lambda, T)$.



$$R(\lambda, T)f = \frac{1}{\lambda - g} f \quad \neq 0 \text{ and cont.}$$

• $\lambda \in g([0,1])$, say $\lambda = g(s_0)$. Observe

$$((\lambda - T)f)(s_0) = \lambda f(s_0) - \underbrace{g(s_0)}_{\lambda} f(s_0) = 0$$

i.e., $\text{Ran}(\lambda - T) \subset \{h \mid h(s_0) = 0\} \neq C[0,1]$,
 so $\lambda - T$ is not surj.

So $\sigma(T) = g([0,1])$.

What is $P_\sigma(T)$?

Observe: $Tf = \lambda f \Leftrightarrow gf = \lambda f \Leftrightarrow (\lambda - g)f = 0$.

So $P_\sigma(T) = \{\lambda \mid g|_{[a,b]} \equiv \lambda \text{ for some } [a,b] \subset [0,1]\}$

indeed: $\subset$ if $f \neq 0$, $\exists [a,b] \cdot f(s) \neq 0 \forall s \in [a,b]$.

$\supset$ take f



in part., $P_\sigma(T) = \emptyset$ possible (take $g(s) := s$)

3b) Sequence spaces: $X = \ell^1$ (or c_0 or ℓ^p):

$T(t_1, t_2, \dots) := (a_1 t_1, a_2 t_2, \dots)$ for a fixed bdd seq. (a_j) .

Then $\|T\| = \|a\|_\infty$ (why?) and

$$\sigma(T) = \overline{\{a_j \mid j \in \mathbb{N}\}} \quad (\text{exerr.})$$

$$\left(\frac{1}{\lambda - a_j}\right)_{j=1}^\infty \in \ell^\infty$$

3c) Fcts on $K \subset \mathbb{C}$:

Let $K \subset \mathbb{C}$ be comp. (bdd and closed).

Consider $X := C(K)$ -Banach, T given by



$$(Tf)(z) = z f(z).$$

As in 3a), $\sigma(T) = \{z, z \in K\} = K$.

Here, $\lambda \in \rho(T) \Leftrightarrow \lambda$ is isolated in K :

$$z f(z) = \lambda f(z) \Leftrightarrow f(z) = 0 \text{ or } z = \lambda$$

Rem. 3b) and 3c) imply:

cont. $\Rightarrow \lambda$ is isolated

$\forall K \subset \mathbb{C}$ comp., $\neq \emptyset$, $\exists X, T \in \mathcal{L}(X)$ with $\sigma(T) = K$

By taking ℓ^2 instead of ℓ^1 in 3b), X can be taken to be Hilbert.

So in part., no further properties of $\sigma(T)$ generally hold — any comp. $\neq \emptyset$ set is possible.

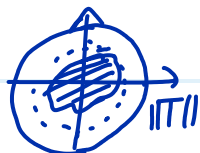
For ex., Cantor set or Sierpiński carpet



But $\sigma(T) \subset \{\lambda: |\lambda| \leq \|T\|\}$ can be often sharpened, so

$$\sup \{|\lambda|: \lambda \in \sigma(T)\} < \|T\| \text{ possible}$$

smallest radius of the circle at 0 surrounding $\sigma(T)$



Def 1.7 The number

$$r(T) := \max \{|\lambda|: \lambda \in \sigma(T)\}$$

is called the spectral radius of T .

We know: $r(T) \leq \|T\|$. Is there a better formula?

Lemma 1.8 Let $(a_n) \subset \mathbb{R}$ with

$$0 \leq a_{n+m} \leq a_n \cdot a_m \quad \forall n, m$$

$$a_n = 2^n$$

Then

$$\exists \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \inf \sqrt[n]{a_n}$$

Proof Let $\varepsilon > 0$ and N with $\sqrt[N]{a_N} < a + \varepsilon$ for

$$a := \inf \sqrt[n]{a_n}$$

Def. $\beta := \max \{a_1, \dots, a_N\}$ and write

$$n = k \cdot N + r, \quad 1 \leq r \leq N$$

Then

$$\sqrt[n]{a_n} = a_{kN+r}^{1/n} \leq (a_N^k \cdot a_r)^{1/n} \leq (a + \varepsilon)^{\frac{kN}{n}} \cdot \beta^{1/n}$$

assumpt.

$$a_N^k < (a + \varepsilon)^N$$

$$\leq (a + \varepsilon) \cdot (a + \varepsilon)^{-\frac{r}{n}} \cdot \beta^{1/n} \leq a + 2\varepsilon \text{ for large } n.$$

$$\frac{kN}{n} = 1 - \frac{r}{n}$$

$$\xrightarrow{n \rightarrow \infty} 1$$

$$\text{So: } a \leq \underline{\lim} \sqrt[n]{a_n} \leq \overline{\lim} \sqrt[n]{a_n} \leq a + 2\varepsilon \quad \forall \varepsilon,$$

$$\text{i.e., } a = \inf \sqrt[n]{a_n}.$$

Observe now: $a_n := \|T^n\|$ satisf. $0 \leq a_{n+m} \leq a_n \cdot a_m$! $\|T^{n+m}\| \leq \|T^n\| \|T^m\|$

Thm 1.9 For $T \in \mathcal{L}(X)$,

$$r(T) = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$$

Proof: $\exists \lim \|T^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}}$ by Lemma 1.8.

Part 1 $r(T) \leq r$ or, equivalently, $\forall \lambda$ with $|\lambda| > r$, $\lambda \in \rho(T)$.

Take λ with $|\lambda| > r$.

We show:

$$\sum_{n=0}^{\infty} \frac{T^n}{\lambda^n} \text{ conv.} \quad (*)$$

Then $\exists (\lambda - T)^{-1} = \frac{1}{\lambda} \cdot \sum_{n=0}^{\infty} \frac{T^n}{\lambda^n}$ (Neumann series)

Observe: $\lim_{n \rightarrow \infty} \|(\frac{T}{\lambda})^n\|^{\frac{1}{n}} = \frac{1}{|\lambda|} \cdot \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = \frac{r}{|\lambda|} =: q < 1$,

so $\forall \varepsilon > 0$ $\|(\frac{T}{\lambda})^n\| \leq (q + \varepsilon)^n$ for large n .

Take ε s.t. $q + \varepsilon < 1$ — (*) follows ($\sum$ conv absolutely)

Part 2 $r(T) \geq r$, i.e., $\exists \lambda \in \sigma(T)$ with $|\lambda| \geq r$

Assume it is false, i.e., $r(T) < r$.

Let $\varphi \in (\mathcal{L}(X))'$ and consider

$$f(\lambda) := \varphi(R(\lambda, T))$$

$$f: \{\lambda: |\lambda| > r(T)\} =: D \rightarrow \mathbb{C}$$

We know: $\bullet$ f is holom. in D (see 1.5 and its proof)

$\bullet$ On $D_0 := \{\lambda: |\lambda| > r\}$ the series

$$\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

conv. (see (*) in part 1), so

$$f(\lambda) = \varphi\left(\underbrace{\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}}_{\text{Neumann series}}\right) = \sum_{n=0}^{\infty} \underbrace{\frac{\varphi(T^n)}{\lambda^{n+1}}}_{\varphi \in (\mathcal{L}(X))'} - \text{Laurent series!}$$

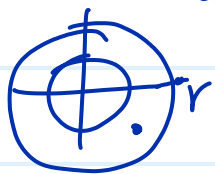
Recall (function theory):

f holom. in a rchy $\Rightarrow f$ has a unique Laurent series repres in D .



So: $f(\lambda) = \sum_{n=0}^{\infty} \frac{\varphi(T^n)}{\lambda^{n+1}}$ holds on the whole D !

Take now $\mu \in D \setminus D_0$.



The series $\sum_{n=0}^{\infty} \frac{\varphi(T^n)}{\mu^{n+1}}$ conv. $\Rightarrow \left(\frac{\varphi(T^n)}{\mu^{n+1}} \right) = \left(\varphi \left(\frac{T^n}{\mu^{n+1}} \right) \right)$ is bdd in n . $\forall \varphi$

Principle of uniform bdd'ness:

$$\gamma := L(X), \gamma_n := \frac{T^n}{\mu^{n+1}}$$

$$\text{Def } S_n \cdot \gamma' \rightarrow \left(\begin{array}{l} S_n \varphi := \varphi / \gamma_n \end{array} \right)$$

$$\text{Then } \|S_n\| = \|\gamma_n\|$$

why? (K-B)

$$\left(\frac{T^n}{\mu^{n+1}} \right)_{n=1}^{\infty} \text{ is bdd:}$$

$$\exists c: \left\| \frac{T^n}{\mu^{n+1}} \right\| \leq c \quad \forall n,$$

$$\text{i.e., } \|T^n\|^{1/n} \leq (c|\mu|)^{1/n} \cdot |\mu|$$

$$r = \lim_{n \rightarrow \infty} \|T^n\|^{1/n} \leq |\mu| \quad \text{as } n \rightarrow \infty \quad \underline{1}$$



20.10 Remark The formula

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$$

is not easy to use to compute $r(T)$. Often it is used to say something about (T^n) if $r(T)$ is known. For example, $r(T) < 1$, say $q := r(T)$, then

$$\|T^n\| \leq (q + \varepsilon)^n \text{ for large } n$$

Taking $q_1 < 1$, $\|T^n\| \rightarrow 0$ exponentially fast

Prop. 10 (Spectral mapping theorem for the resolvent)

Let $T \in \mathcal{L}(X)$. Then $\forall \lambda \in \rho(T)$

$$\sigma(R(\lambda, T)) = \frac{1}{\lambda - \sigma(T)}$$

Formally,

$$R(\lambda, T) = (\lambda - T)^{-1}, \quad \sigma(R(\lambda, T)) = (\lambda - \sigma(T))^{-1}$$

$$\text{i.e., } \sigma(R(\lambda, T)) = \left\{ \frac{1}{\lambda - z} : z \in \sigma(T) \right\}$$

Proof Step 1 Reduction to $\sigma(S^{-1}) = \frac{1}{\sigma(S)}$.

Def. $S := \lambda - T$. Then

$$R(\lambda, T) = S^{-1}$$

$$\mu \in \sigma(T) \Leftrightarrow \mu - \lambda \text{ not invert.} \quad (\Rightarrow) \quad (\mu - \lambda) + \underbrace{(\lambda - T)}_S \text{ not invert.} \quad (\Rightarrow) \quad \lambda - \mu \in \sigma(S)$$

Step 2 To show: $\sigma(S^{-1}) = \frac{1}{\sigma(S)}$

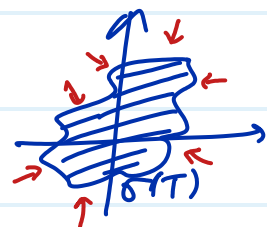
$$\text{Observe } \mu - S^{-1} = \mu \left(S - \frac{1}{\mu} \right) S^{-1}.$$

So $\mu = S^{-1}$ invert $\Leftrightarrow S = \frac{1}{\mu}$ invert. $\Leftrightarrow \frac{1}{\mu} \in \rho(S)$
 and $\mu \in \sigma(S^{-1}) \Leftrightarrow \frac{1}{\mu} \in \sigma(S)$

Cor 1.11 (Lower bound for the resolvent)

For $\lambda \in \rho(T)$

$$\|R(\lambda, T)\| \geq \frac{1}{\text{dist}(\lambda, \sigma(T))}$$



in part, $\|R(\lambda, T)\| \xrightarrow{\lambda \rightarrow \sigma(T)} \infty$

Proof $\|R(\lambda, T)\| \geq r(R(\lambda, T)) = \sup\{|\mu|, \mu \in \sigma(R(\lambda, T))\}$

Prop 1.4

$$= \frac{1}{\inf\{|\lambda - z|, z \in \sigma(T)\}} = \frac{1}{\text{dist}(\lambda, \sigma(T))}$$

Rem. Recall from the proof of Thm 1.5 that $\|R(\lambda, T)\| \xrightarrow{|\lambda| \rightarrow \infty} 0$ at least linearly: $\|R(\lambda, T)\| \leq \frac{C}{|\lambda|}$ for $|\lambda| \geq 2\|T\|$.

Cor 1.11 shows that $\exists c_2$. $\|R(\lambda, T)\| \geq \frac{c_2}{|\lambda|}$ for $|\lambda|$ large (why?)
 So precisely linear decay at ∞ .

2. Spectrum of adjoint operators

Def 2.1 Let $T \in \mathcal{L}(X, Y)$. The adjoint (or dual, or Banach space dual) operator $T': Y' \rightarrow X'$ is def. by

$$\begin{matrix} X & \xrightarrow{T} & Y \\ X' & \xleftarrow{T'} & Y' \end{matrix}$$

$$(T'\varphi)(x) := \varphi(Tx) \quad \forall x \in X, \forall \varphi \in Y'$$

$$\begin{matrix} X & \xrightarrow{T} & Y \\ T'\varphi & \searrow & \swarrow \varphi \\ & \mathbb{C} & \end{matrix}$$

Rem. 1) Linearity in x and in φ is clear. By

$$|\varphi(Tx)| \leq \|\varphi\| \cdot \|Tx\| \leq \|\varphi\| \cdot \|T\| \cdot \|x\|$$

we have $T'\varphi \in X'$ with $\|T'\varphi\| \leq \|\varphi\| \cdot \|T\|$, so

$$T' \in \mathcal{L}(Y', X')$$

with $\|T'\| \leq \|T\|$. We will see: " $=$ ".

2) The operator T' is called predual operator of T .

Ex 2.2 1) $X = Y = \ell^p$, $p \in [1, \infty)$, T_e the left shift

$$T_e(t_1, t_2, \dots) := (t_2, t_3, \dots)$$

$$T_e((t_j)) = (t_{j+1})$$

We identify X' with ℓ^q , $\frac{1}{p} + \frac{1}{q} = 1$, via $\varphi = (s_j) \in \ell^q$, $\varphi((t_j)) := \sum_{j=1}^{\infty} s_j t_j$
 Observe

$$\varphi(T_e x) = \sum_{j=1}^{\infty} s_j t_{j+1} = \sum_{j=2}^{\infty} s_{j-1} t_j$$

Thus, $T'_l(s_1, s_2, \dots) = (0, s_1, s_2, \dots)$ - the right shift (on ℓ^q)

So $T'_l = T_r$, $T_r \rightarrow$.

Analogously, $T'_r = T_l$ (on ℓ^q)

Observe: for shifts on ℓ^2 , $T_l T_r = I$ but $T_r T_l \neq I$ - only one-sided inverse here

Rem. Attention: T'_l is isomorphic to T_r on ℓ^q since living on a different space (of functionals on ℓ^p)

2) $X = Y = L^p[0,1]$, $1 \leq p < \infty$, T multiplier with $a \in L^\infty[0,1]$:
 $Tf = af$

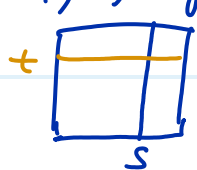
Again, identify X' with $L^q[0,1]$, $\frac{1}{p} + \frac{1}{q} = 1$ via $g(f) = \int gf$

Then

$$g(Tf) = \int g \cdot a \cdot f = \int (ag) f,$$

i.e., $T'g = ag$ - multipl. with the same a , but on L^q .

3) $X = \mathbb{C}^d$, $Tx = Ax$, $A = (a_{ij})_{i,j=1}^d$. Then T' is given by (a_{ji}) (again, identifying X' with X) exerc.

4) Integral operator T on $L^2[0,1]$ with kernel $K \in L^\infty([0,1]^2)$
 $(Tf)(s) = \int_0^1 K(s,t) f(t) dt$ (why well defined?)

exerc T' is the integral operator with kernel K' with $K'(s,t) := K(t,s)$

Prop. 2.3 (Properties of adjoint oper.)

(a) $(S+T)' = S' + T'$, $(\lambda T)' = \lambda T'$

(b) $\|T'\| = \|T\|$

(c) $(ST)' = T' S'$ $\forall T \in \mathcal{L}(X,Y)$ $\forall S \in \mathcal{L}(Y,Z)$

(d) $\text{Ker } T' = (\text{Ran } T)^\perp := \{\varphi \in Y' : \varphi|_{\text{Ran } T} = 0\}$.

in particular, T' injective $\Leftrightarrow \text{Ran } T$ is dense

(e) The bidual operator T'' satisfies $T''|_X = T$ under the identification of X with $i(X)$ for $i: X \rightarrow X''$, $i(x)\varphi := \varphi(x)$.

Proof (a) clear

(b) $\|T'\| = \sup_{\|\varphi\| \leq 1} \|T'\varphi\| = \sup_{\|\varphi\| \leq 1} \sup_{\|x\| \leq 1} |\varphi(Tx)| = \|T\|$.
 $\sup_{\|\varphi\| \leq 1} |\varphi(Tx)| = \|Tx\|$ by H.-B.

(c) $((ST)'\varphi)(x) = \varphi(STx) = (S'\varphi)(Tx) = (T'S'\varphi)(x)$ $(\geq \text{by H.B.})$

(d) $T'\varphi = 0 \Leftrightarrow (T'\varphi)(x) = 0 \forall x \Leftrightarrow \varphi|_{\text{Ran } T} = 0$ "in part." $\Leftrightarrow$ clear (continuity of φ)
if $\text{Ran } T \neq Y$, then $\exists \varphi \neq 0$ with $\varphi|_{\text{Ran } T} = 0$

(e) $T''(i(x))(\varphi) = i(x)(T'\varphi) = (T'\varphi)(x) = \varphi(Tx) = i_y(Tx)(\varphi)$
def of i def of T'
 $T': Y' \rightarrow X'$ $T'': X'' \rightarrow Y''$

Rem. By (e), an operator $S \in \mathcal{L}(Y', X')$ is an adjoint operator
 $\Leftrightarrow S'(X) \subset Y$ (again under identif. of X with $i(X)$ and Y with $i_Y(Y)$)
 $\Rightarrow$ is exactly (e)
 $\Leftarrow$ take $T := S': X \rightarrow Y$ (restriction) and check: $T' = S$ (again, (e)).

[Ex] Recall: $C_0' \cong \ell^1$, $C_0'' \cong \ell^\infty$. Consider $S \in \mathcal{L}(\ell^4)$ def. by
 $S(t_1, t_2, \dots) := (\sum_{j=1}^{\infty} t_j, 0, 0, \dots)$ (why well def.?)

Then $S' \in \mathcal{L}(\ell^\infty)$, $(S'(s_1, s_2, \dots))(t_1, t_2, \dots) = (s_1, s_2, \dots) (\sum_{j=1}^{\infty} t_j, 0, 0, \dots)$
 $= s_1 \cdot \sum_{j=1}^{\infty} t_j = \sum_{j=1}^{\infty} s_1 t_j$,
 so $S'(s_1, s_2, \dots) = (s_1, s_1, s_1, \dots)$
 hence $S'(C_0) \not\subset C_0$, S is not an adjoint operator

Rem. in many books T^* denotes the (Banach space) adjoint of T .
 We use T^* for the Hilbert space adjoint:

Def. 2.4 For $T \in \mathcal{L}(H)$, the (Hilbert space) adjoint $T^* \in \mathcal{L}(H)$ is
 def. by $\langle Tx, y \rangle = \langle x, T^*y \rangle \quad \forall x, y \in H$ (*)

Rem. 1) Well defined by Riesz repr. thm: For y fixed,
 $\varphi(x) := \langle Tx, y \rangle$ defines $\varphi \in X'$, Riesz: $\exists! z =: T^*y$: $\langle Tx, y \rangle = \langle x, z \rangle$

2) Recall: H' is isom. to H via the conjugate lin. isometric isomorphism $\Phi: H \rightarrow H'$, $(\Phi x)(y) := \langle y, x \rangle$ (again Riesz).

So (*) means: $(\Phi y)(Tx) = \Phi(T^*y)(x) \quad \forall x, y$, i.e.,
 $(T^*\Phi y)(x) = \Phi(T^*y)(x)$

$$T'\Phi = \Phi T^*, \quad \text{so}$$

$$T^* = \Phi^{-1} T' \Phi$$

in this course, $\langle x, y \rangle$ is linear in x and conj. linear in y .

[Ex. 2.5] 0) $H = \ell^2$, $(a_{ij})^* = (\overline{a_{ji}})$ (exerr.)

1) $T_e^* = T_e$ on ℓ^2 by $\sum_{j=1}^{\infty} t_{j+1} \overline{s_j} = \sum_{j=2}^{\infty} \overline{s_{j-1}} t_j$, and $T_r^* = T_r$

2) For multipl. operator M_a from Ex. 4.6 2) on $L^2[0,1]$.
 Then $\langle af, g \rangle = \langle f, \overline{ag} \rangle \quad \forall f, g$ implies $M_a^* = M_{\overline{a}}$

3) For the integral op. with kernel k from Ex. 2.3 3) on $L^2[0,1]$,
 T^* is -1- $k^*(s, t) = \overline{k(t, s)}$ (exerr.)

Prop. 2.6 (Properties of the Hilbert space adjoint)

(a) $(S+T)^* = S^* + T^*$, $(\lambda T)^* = \overline{\lambda} T^*$

(b) $(ST)^* = T^* S^*$

(c) $\|T^*\| = \|T\|$

(d) $T^{**} = T$

(e) $\|TT^*\| = \|T^*T\| = \|T\|^2$

(f) $\text{Ker } T^* = (\text{Ran } T)^\perp$. in part., T^* is inj $\Leftrightarrow \text{Ran } T$ is dense

(g) $\text{Ker } T = (\text{Ran } T^*)^\perp$ -1- T is inj $\Leftrightarrow \text{Ran } T^*$ is dense

Proof (a), (d) clear (d): $\langle Tx, y \rangle = \langle x, T^*y \rangle = \overline{\langle T^*y, x \rangle}$
 $\langle Ty, x \rangle = \overline{\langle y, T^*x \rangle}$

(b), (c), (f) as for T' exerc.

(g) follows from (f) for T^* using (d)

(e) $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \leq \|T^*T\| \cdot \|x\|^2 \quad \forall x,$

so $\|T\|^2 \leq \|T^*T\|$ holds by (c)

So $\|T^*T\| = \|T\|^2 \quad \forall T$ This for T^* implies $\|TT^*\| = \|T^*\| = \|T\|^2$

Rem. Analog, one def. $T^*: H_2 \rightarrow H_1$ for $T: H_1 \rightarrow H_2$ by

$$\langle Tx, y \rangle_{H_2} = \langle x, T^*y \rangle_{H_1}.$$

All the above properties hold. Now we focus on spectral theory,
 $H_1 = H_2$ in this course.

Def 2.7 $T \in \mathcal{L}(H)$ is called

- normal if $TT^* = T^*T$
- self adjoint (s.a.) if $T^* = T$
- unitary if $T^* = T^{-1}$

For numbers: $\uparrow$
 $\bar{a} = a$ - real number
 $\bar{a} = \frac{1}{a}$ - π -unit circle

Rem. s.a. or unitary $\Rightarrow$ normal

Ex 2 Multiplicators

By $(Ma)^* = M\bar{a}$ (on ℓ^2 or L^2), see above, so each mult. is normal

Now, M_a is s.a. $\Leftrightarrow a = \bar{a}$, i.e., a is real valued (a.s. if on L^2)

unitary $\Leftrightarrow a^{-1} = \bar{a}$, i.e., a is π -valued (—)



Ex shift on ℓ^2 is not normal: $T_\ell^* = T_r$, $T_\ell \cdot T_r = I$, but $T_r T_\ell \neq I$

$$(T_r T_\ell)(t_1, t_2, \dots) = (0, t_1, t_2, \dots)$$

Prop. 2.9 (Spectral radius of normal operators)

Let $T \in \mathcal{L}(H)$ be normal, then
 (complex) Hilbert

$$r(T) = \|T\|$$

in part., $\sigma(T) = \{0\} \Leftrightarrow T = 0$

As for Banach spaces, all Hilbert spaces are over $\mathbb{C}$ if not said otherwise.

Proof We first show $\|T^2\| = \|T\|^2$

By Prop. 2.6 (e) applied to T^2

$$\|T^2\|^2 = \|T^2 \cdot (T^2)^*\| = \|(T \cdot T^*)^2\| \quad (***)$$

Claim: S s.a. $\Rightarrow \|S^2\| = \|S\|^2$ normal

Proof of claim $\|Sx\|^2 = \langle Sx, Sx \rangle = \langle S^2x, x \rangle \leq \|S^2\| \|x\|^2,$

so $\|S\|^2 \leq \|S^2\|$, but $\geq$ always holds, so " $=$ ". $\blacksquare$ -claim

Apply claim to $S := T \cdot T^*$ ($S^* = T^{**}T^* = T \cdot T^*$) to continue (***)

$$\|T^2\|^2 = \dots = \|T \cdot T^*\|^2 \stackrel{\text{Prop. 2.6 (e)}}{=} \|T\|^4,$$

so $\|T^2\| = \|T\|^2.$

Induction (T^n normal $\forall n$): $\|T^{2^n}\| = \|T\|^{2^n} \quad \forall n$, so

$$r(T) = \lim_{k \rightarrow \infty} \|T^{2^k}\|^{\frac{1}{2^k}} = \|T\|.$$

$\boxed{\in X} (r(T) < \|T\|): T = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}, \sigma(T) = \{0\}, \|T\| = N$

Here: $T^2 = 0$, so $\|T^n\| = 0 \forall n \geq 2$.

Thm. 2.10 (Spectrum of adjoint operators)

For $T \in \mathcal{L}(X)$ Banach, $\boxed{\sigma(T') = \sigma(T)}$. If X is Hilbert, then $\boxed{\sigma(T^*) = \overline{\sigma(T)}} = \{\bar{\lambda} : \lambda \in \sigma(T)\}$. *complex conjugation, not closure*

Proof Part 1: X Banach To show: $\lambda - T$ invert. $\Leftrightarrow (\lambda - T')' = (\lambda - T)'$ invert

By $S := \lambda - T$, enough to show: T inv. $\Leftrightarrow T'$ inv. (so $\lambda = 0$) $\forall T$

$\Rightarrow$ We show: $(T^{-1})' = (T')^{-1}$. Indeed,

$$T'(T^{-1})' = (T' \cdot T)' = I' = I \quad \text{and} \quad (T^{-1})' \cdot T' = (T \cdot T^{-1})' = I' = I$$

$\Leftarrow$ Recall: $i: X \rightarrow X''$, $i(x)(\varphi) = \varphi(x)$. i is isometry and $X \cong i(X)$.

We write $X \subset X''$ (identify X with $i(X)$).

We first show: $\exists c > 0$.

$$(\text{***}) \quad \|Tx\| \geq c\|x\| \quad \forall x \in X.$$

By $\Rightarrow$ applied to T' , we know: T'' is invert. So.

$$\|Tx\| = \|T''x\| \geq \frac{\|(T'')^{-1}T''x\|}{\|(T'')^{-1}\|} = \frac{\|x\|}{\|(T'')^{-1}\|} = c \cdot \|x\|$$

by $T''|_X = T$

for $c := \frac{1}{\|(T'')^{-1}\|}$, so (***)

By (***):

- T is inj.
- $\text{Ran } T$ is closed: indeed, if $Tx_n \rightarrow y$, then (Tx_n) is Cauchy $\Rightarrow (x_n)$ Cauchy $\Rightarrow \exists x = \lim x_n$ $\Rightarrow Tx = y$.

it remains to show: $\text{Ran } T$ is dense ($=) = X$ since closed)

By Prop 2.3 (d), $\text{Ker } T' = (\text{Ran } T)^\perp$, so $\varphi|_{\text{Ran } T} = 0 \Rightarrow \varphi = 0$. *since T' invert*

M.-B.: $\overline{\text{Ran } T} = X$.

So T is inj + surj = bij.