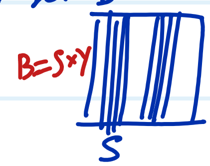
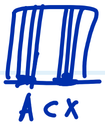
Proof of Lemma 11.2 idea: $\tilde f = f\cdot \mathbb{1}_B$ for some „good" cylinder set B



Step 1: Fibers where f is large:

$f>0 \Rightarrow \exists A\subset X$ 

with $\mu(A)>0$, $\exists\delta$: $\int f(x,y)\,dy > \delta\ \ \forall x\in A$

$\forall\delta$ $A_\delta := \{x\in X: \int_{x\text{-fiber}} f > \delta\}$, $A_\delta \nearrow$ as $\delta\searrow 0$, $\bigcup_\delta A_\delta = \{x: \int_{x\text{-fiber}} f>0\}$

if $\mu(A_\delta)=0\ \forall\delta$, then $\mu(\bigcup_\delta A_\delta)=0$, and by Fubini

$$\int f = \int_X\left(\int_{x\text{-fiber}} f\right)d\mu(x) = 0 \ \text{↯}$$

Step 2 Approximation with rel a p fcts

Let $\varepsilon_i\downarrow 0$ with $\sum\varepsilon_i < \frac{\mu(A)}{2}$ Take f_i rel. a.p with $\|f-f_i\|_2<\varepsilon_i$

For each ε_i $\exists\, g_{i,1},\dots,g_{i,\ell_i}$: $\forall n$, for a.e. $x\in X$

$$\|\tilde T^n f_i(x,\cdot) - g_{i,j}(x,\cdot)\|_2<\varepsilon_i \quad \text{for some } j=j(i,n,x)$$

Take additionally $g_{i,0}:=0$ (for new 0-fibers)

Then $\forall S\subset X$

$$(\ast)\quad \|\tilde T^n(f_i(x,\cdot)\mathbb{1}_S(x)) - g_{i,j}(x,\cdot)\|_2 < \varepsilon_i \quad \text{for some } j=j(i,n,x)$$

$\tilde T$: fibers → fibers

Step 3: Construction of good set S

idea: take fibers, where $f\sim f_i\ \forall i$ ($\sim$ dep. on ε_i)

$\|f-f_i\|_2<\varepsilon_i$ implies $\exists E_i\subset X$, $\mu(E_i)<\varepsilon_i$ („bad sets") s.t.

$$\|f(x,\cdot)-f_i(x,\cdot)\|_2 < \sqrt{\varepsilon_i}\quad \forall x\notin E_i$$

(Otherwise, if the measure of the set of x with [illegible] $\geq\sqrt{\varepsilon_i}$ is larger than ε_i,
then $\|f-f_i\|_2^2 = \int_X |f-f_i|^2 \underset{\text{Fubini}}{=} \int_X\left(\int_{x\text{-fiber}}|f-f_i|^2\right) \geq (\sqrt{\varepsilon_i})^2\cdot\mu(E_i) \geq \varepsilon_i^2$ ↯ $(\|f-f_i\|_2>\varepsilon_i)$)

Def $S := A\setminus\left(\bigcup_i E_i\right)$, $\mu(S)\geq \mu(A) - \sum \underbrace{\mu(E_i)}_{<\varepsilon_i} > \mu(A) - \sum_i\varepsilon_i \geq \frac{\mu(A)}{2}$

- $f\cdot\underbrace{\mathbb{1}_S}_{\text{identify with } \mathbb{1}_{S\times Y}} > 0$ because $\forall x\in S$ $\int_{x\text{-fiber}} f>\delta$ (Fubini: $\int f\cdot\mathbb{1}_S \geq \mu(S)\cdot\delta$)
- $f\cdot\mathbb{1}_S\leq f$ clear
- $f\cdot\mathbb{1}_S$ rel. alm. per:

  $\forall x\in S\ \ \forall i$ $\|f(x,\cdot)-f_i(x,\cdot)\|_2<\sqrt{\varepsilon_i}$ by constr. of S

  So: $\forall i\ \ \forall x\in \underline{\underline{X}}$ $\|f(x,\cdot)\mathbb{1}_S(x) - f_i(x,\cdot)\mathbb{1}_S(x)\|_2 < \sqrt{\varepsilon_i}$

Now, $\tilde{T}$ is measure pres., fibers $\to$ fibers, so
$$\|\tilde{T}^n(f\,\mathbb{1}_S) - \tilde{T}^n(f_i\,\mathbb{1}_S)\|_{L^2(x\text{-fiber})} < \sqrt{\varepsilon_i}$$
By $(*)$, $\forall \varepsilon_i$ $\exists g_{i,j}$ as above : $\forall n$, a.e. x
$$\|\tilde{T}(f\cdot \mathbb{1}_S) - g_{i,j}\|_{L^2(x\text{-fiber})} \underset{\Delta\text{-ineq.}}{<} \sqrt{\varepsilon_i} + \varepsilon_i$$
for some j. So $f\cdot\mathbb{1}_S$ is rel. alm. periodic. ■

Proof of Thm 11.1

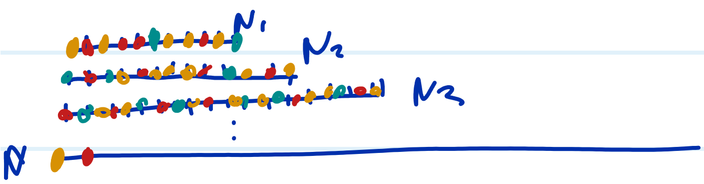
$\mathbb{N} = A_1 \cup \ldots \cup A_n \Rightarrow \exists j : A_j$ contains arb long AP

We use ~~finitary version of~~ $\boxed{\text{van der Waerden thm}}$:

$\underset{\text{length }(-1)}{\forall k}$ $\underset{\text{colours}}{\forall r}$ $\underset{\text{van der Waerden } k,r\text{-number}}{\exists W(k,r)}$: $\forall n \geq W(r,k)$ $\forall$ colouring of $\{1,\ldots,n\}$ into r colours, $\exists$ $\underbrace{\text{monochromatic}}_{\text{constant colour}}$ arithmetic progression of length $k+1$

Proof of finitary version assuming vdW

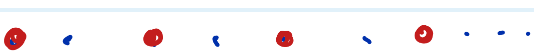
Assume, finitary version is false, i.e.,
$\exists r$ $\exists k$ $\exists N_n \nearrow \infty$ $\exists$ partition
$$\{1,\ldots,N_n\} = C_{n,1} \cup \ldots \cup C_{n,r}$$
without monochromatic arithm progr of length $k+1$.



Pigeonhole principle: $\exists$ subseq of (N_n): the colour of $n=1$ is constant
$\exists$ subseq. of this subseq. s.t. —"— $n=2$ —"—
etc

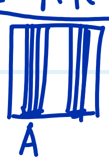
N_1^1 N_2^1 N_3^1 ... — colour of $n=1$ const
N_1^2 N_2^2 N_3^2 ... —"— $n=2$ —"—
N_1^3 N_2^3 N_3^3 ...

So we have a partition of $\mathbb{N}$ into r sets/colours $\overset{vdW}{\Longrightarrow}$ $\exists$ arithm. progr. of length $k+1$ having the same colour.



Since coordinatewise conv. of colours + arithm. progr is finite,
$\exists n : \{1,\ldots,N_n\}$ has an arithm progr. of length $k+1$ of this colour ↯ ■
finitary vdW

Step 1: MR for (X,μ,T) and good fibers



Let $f > 0$, $\underbrace{\|f\|_\infty \leq 1}_{\text{WLOG}}$, rel. alm. periodic (WLOG by Lemma 11.2)

Again, as in the proof of Lemma 11.2,
$\exists \delta > 0$ $\exists A \subset X$ with $\mu(A) > 0$ s.t. $\int_{x\text{-fiber}} f \geq \delta$ $\forall x \in A$

Take k. Let $\varepsilon = \varepsilon(\delta, k)$ to be chosen later.

f rel. a.p. $\quad \exists g_1, \dots, g_\ell : \ \forall n$, a.e. x

$$\| \tilde{T}^n f - g_i \|_{L^2(x\text{-fiber})} < \varepsilon \quad \text{for some } i.$$

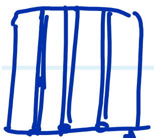
idea: view $g_1, \dots, g_\ell$ as colours.

Define $L := W(k, \ell)$. (MR_L) for (X, μ, T): $\exists \delta' > 0$.

$$\mu(\underbrace{A \cap T^{-n}A \cap \dots \cap T^{-Ln}A}_{=: E_n}) \geq \delta'$$

for many n. We assume: syndetic return times, i.e., such good n have proportion $\geq \eta > 0$ (Pos. upper density given by $\lim \frac{1}{N} \sum_1^N \dots > 0$ analogously, just longer formular)

(True for example for compact systems, Kronecker factor, circle rot...)



$E_n \subset A$

Step 2: VdW in the good fibers

Fix n good (as above). Look at $f, \tilde{T}^n f, \dots, \tilde{T}^{Ln} f$

Rel. a.p.: $\quad \forall m \in \{1, \dots, L\}$ for a.e. x

$\exists\, i = i(m, x, n) \in \{1, \dots, \ell\}$:

$$\| \tilde{T}^{mn} f - g_i \|_{L^2(x\text{-fiber})} < \varepsilon.$$

For a fixed x, this is a colouring of $\{0, \dots, L\}$ into ℓ colours.

VdW. $\exists\, a, d$ (dep. on x, n).

$a, a+d, \dots, a+kd$ corresp. to the same g_i.

In particular,

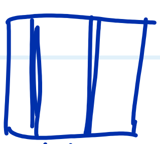
$$(**) \quad \| \tilde{T}^{(a+jd)n} f - \tilde{T}^{an} f \|_{L^2(x\text{-fiber})} < 2\varepsilon \quad \text{(all close to each other)}$$

Step 3: Fibers with the same progression

$\exists$ less than L possibilities for a, d.

Pigeonhole principle: $\exists\, E_n' \subset E_n$ with $\mu(E_n') \geq \frac{\mu(E_n)}{L^2}$ s.t.

$\forall x \in E_n'$ has the same a, d (dep. on n) (gap - why E_n' measur.)



$E_n' \subset E_n \subset A$

Step 4: Final estimation

By $(**)$ and Δ-ineq., for $x \in E_n'$

$$\int_{x\text{-fiber}} \tilde{T}^{an} f \cdots \tilde{T}^{(a+kd)n} f \geq \int_{x\text{-fiber}} (\tilde{T}^{an} f)^k - k \cdot 2\varepsilon$$

$a_i \quad a_j - a_1^j = a_1(a_2 - a_1) a_3 \dots a_j + a_1^2 (a_3 - a_1) \dots a_j + \dots + a_1^{j-1}(a_j - a_1)$

$$= \int_{\underbrace{T^{an}x}_{\in A}\text{-fiber}} f^k - 2k\varepsilon \geq \delta^k - 2k\varepsilon \geq \frac{\delta^k}{2} \quad \text{for } \varepsilon < \frac{\delta^k}{4k}$$

$\|f\|_\infty \leq 1$

$\int f \geq \delta \Rightarrow \int f^k \geq \delta^k$

(by Hölder ineq.: $|\int f| \leq \|f\|_p \underbrace{\|1\|_q}_{1}$, so $(\int f)^p \leq \int f^p$)

So by $f \geq 0$:

$$\int_X \cancel{\tilde{T}^{an}} f \cdots \tilde{T}^{(\cancel{a}+kd)n} f \geq \mu(E_n') \cdot \frac{\delta^k}{2} \geq \frac{\delta'}{L^2}\frac{\delta^k}{2},$$

i.e., $$\int_X f \cdot \tilde{T}^{dn} f \cdots \tilde{T}^{kdn} f \geq \frac{\delta' \delta^k}{2L^2}$$

Now: $d = d(n) \in \{1, \dots, L\}$

Observe: if $n \leq N$, then $n \cdot d(n) \leq NL$, and $\exists$ at most L possibilities for $m \leq NL$ to be written as $m = n \cdot d(n)$

So by $f \geq 0$ we have by the above

$$\sum_{m=1}^{NL} \int f \, \tilde{T}^m f \cdots \tilde{T}^{km} f \geq \eta \frac{N}{L} \frac{\delta' \delta^k}{2L^2}$$

proportion of good n s.t. $n\,d(n)$ as above

Hence $$\frac{1}{NL}\sum_{1}^{NL} \cdots \geq \frac{\eta \delta' \delta^k}{2L^4}$$ $\forall N \Longrightarrow$ $\lim \geq$ -"- also ∎

exerc

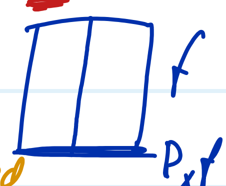
Rem. $\exists$ (longer) proof without van der Waerden

11.2 Weakly mixing extensions and sketch of the rest of the proof of Furstenberg/Szemerédi

Recall.
- (X, μ, T) is weakly mixing $\Longleftrightarrow \frac{1}{N}\sum_{n=1}^{N} \left| \int f \cdot T^n g - \int f \int g \right| \to 0$ $\forall f, g \in L^\infty$
- (X, μ, T) is not w. mixing $\Longrightarrow \exists$ nontrivial compact factor, not one-point
  dichotomy i.e., $\exists$ a.p. function $\neq$ const.

$\boxed{\text{Def}}$ $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is a weakly mixing extension of (X, μ, T) (or: w. mixing rel. to (X, μ, T)) if $\forall f, g \in L^\infty(\tilde{X}, \tilde{\mu})$ and for $P_X : L^2(\tilde{X}) \to L^2(X)$ def. by

$$(P_X f)(x) := \int_{x\text{-fiber}} f$$



one has

$$\frac{1}{N}\sum_{n=1}^{N} \left\| P_X(f \cdot \tilde{T}^n g) - P_X f \cdot T^n(P_X g) \right\|^{2}_{L^2(X)} \longrightarrow 0$$

2 can be deleted by Koopman-von Neumann's lemma

Here, P_X takes mean on fibers and is the conditional expectation onto the sub-σ-algebra of cylinder sets. (why?)

Rem. 1) (X,μ,T) is w mixing $\Leftrightarrow$ w mixing rel. to $\{\cdot\}$, viewing X as $\{\cdot\}\times X$ (one fiber) with
$$P_{\{\cdot\}}f = \int f\,d\mu$$
2) The same for compact extensions: (X,μ,T) comp. $\Leftrightarrow$ comp. extension of $\{\cdot\}$! exerc.

$\boxed{\text{Prop. 11.3}}$ (MR for weakly mixing extensions)

Let $(\tilde X,\tilde\mu,\tilde T)$ be $\boxed{\text{w. mixing extension}}$ of (X,μ,T), $k\geq 2$, $f_1,\dots,f_k\in L^\infty(\tilde X)$

Then
$$(\circledast)\quad \boxed{\left\|\frac1N\sum_{n=1}^N \tilde T^n f_1\cdot\ldots\cdot\tilde T^{kn}f_k - \frac1N\sum_{n=1}^N T^n(P_Xf_1)\cdot\ldots\cdot T^{kn}(P_Xf_k)\right\|^2_{L^2(\tilde X)}\longrightarrow 0},$$
where we identify fcts on X with fcts on $\tilde X$ constant on fibers

In particular, if (X,μ,T) has MR, then so does $(\tilde X,\tilde\mu,\tilde T)$

Proof Proof of „in particular". Assume $(\circledast)$. Then $\forall f>0,\ f\in L^\infty(\tilde X)$
$$\underbrace{\lim_{N\to\infty}\frac1N\sum_{n=1}^N\int f\cdot\tilde T^n f\cdot\ldots\cdot\tilde T^{kn}f}_{\text{difference with } \frac1N\sum_{n=1}^N\int P_Xf_0\cdot\ldots\cdot T^{kn}(P_Xf_0)\text{ goes to }0} = \lim_{N\to\infty}\frac1N\sum_{n=1}^N\int_{\tilde X} P_Xf\cdot T^nP_Xf\cdot\ldots\cdot T^{kn}P_Xf>0$$
$\tilde X$ replace by X since const. on fibers (Fubini)

$\int f\cdot\frac1N\sum_{n=1}^N\tilde T^n f\ldots\tilde T^{kn}f\sim\int f\cdot\frac1N\sum_{n=1}^N T^n(P_Xf)\ldots T^{kn}(P_Xf)=\int P_Xf\ldots$ because $f-P_Xf$ has 0 integrals on fibers $\Rightarrow\perp$ to any fct constant on fibers

since (X,μ,T) has MR by assumption. Here we used:
$$P_Xf>0,\quad P_Xf\text{ bdd}$$
(follows from Fubini) $\int_X P_Xf \underset{\text{Fubini}}{=}\int_{\tilde X}f>0$, bdd clear (why?)

Proof of $(\circledast)$

Assume WLOG $\|f_1\|_\infty\leq 1,\ldots,\|f_k\|_\infty\leq 1$

Since
$$\tilde T^nf_1\cdot\ldots\cdot\tilde T^{kn}f_k - T^n(P_Xf_1)\cdot\ldots\cdot T^{kn}(P_Xf_k) = \tilde T^n(f_1-P_Xf_1)\cdot\tilde T^{2n}f_2\cdots$$
$$+\ldots+T^nP_Xf_1\cdot\ldots\cdot T^{(k-1)n}P_Xf_{k-1}\cdot\tilde T^{kn}(f_k-P_Xf_k),$$
we can assume WLOG that $\exists\,\ell\in\{1,\dots,k\}$ with $\boxed{P_Xf_\ell=0}$

and show $\frac1N\sum_{n=1}^N\tilde T^nf_1\ldots\tilde T^{kn}f_k\longrightarrow 0$ in L^2

induction on k Basis: $k=1$: To show: $\frac1N\sum_{n=1}^N\tilde T^nf_1\to 0$ in L^2 if $P_Xf_1=0$

To use vdC, def. $u_n:=\tilde T^nf_1$ and observe
$$\langle u_n,u_{n+h}\rangle = \int\underbrace{\cancel{\tilde T^n}f_1\cdot\tilde T^{\cancel{n}+h}f_1}_{\tilde T\text{ isom.}} = \int f_1\cdot\tilde T^hf_1 \underset{\text{Fubini}}{=} \int_X P_X(f_1\cdot\tilde T^hf_1).$$
So
$$\gamma_h:=\lim_{N\to\infty}\left|\frac1N\sum_{n=1}^N\langle u_n,u_{n+h}\rangle\right|\leq\int_X|P_X(f_1\tilde T^hf_1)|\underset{\text{Cauchy-Schwarz}}{\leq}\|P_X(f_1\tilde T^hf_1)\|$$

vdC: $\frac{1}{N}\sum_{n=1}^{N} u_n \xrightarrow{\text{Cesàro}} 0$ by def. of w. mixing ext. (recall: $P_X f_1 = 0$) in L^2.

$\boxed{k-1 \rightsquigarrow k}$ Again we use vdC for $u_n := \tilde{T}^n f_1 \cdots \tilde{T}^{kn} f_k$.

$$\langle u_n, u_{n+h}\rangle = \int_{\tilde{X}} \cancel{\tilde{T}^n} f_1 \cdots \tilde{T}^{\cancel{kn}\,(k-1)n} f_k \cdot \tilde{T}^{\cancel{n+h}} f_1 \cdots \tilde{T}^{\cancel{k(n+h)}\,(k-1)n+kh} f_k$$

$$= \int_{\tilde{X}} (f_1 \cdot \tilde{T}^h f_1)\cdot \underbrace{\tilde{T}^n (f_2 \cdot \tilde{T}^{2h} f_2) \cdots \tilde{T}^{(k-1)n}(f_k \cdot \tilde{T}^{kh} f_k)}_{k-1 \text{ fcts, as in the induction hypothesis}}$$

By induction hypothesis,

$$\gamma_h := \lim \left|\frac{1}{N}\sum_{n=1}^{N} \langle u_n, u_{n+h}\rangle\right| = \lim \left|\int_{X} (f_1 \cdot \tilde{T}^h f_1)\cdot \frac{1}{N}\sum_{n=1}^{N} \tilde{T}^n (f_2 \cdot \tilde{T}^{2h} f_2) \cdots \tilde{T}^{(k-1)n}(f_k \cdot \tilde{T}^{kh} f_k)\right|$$

$$= \lim \left|\int (f_1 \cdot \tilde{T}^h f_1)\cdot \frac{1}{N}\sum_{n=1}^{N} \underbrace{\tilde{T}^n (P_X(f_2 \cdot \tilde{T}^{2h} f_2)) \cdots \tilde{T}^{(k-1)n} P_X(f_k \cdot \tilde{T}^{kh} f_k)}_{\text{const. on fibers}}\right|$$

$g - P_X g \perp$ const on fibers $\forall g$

$$= \lim \left|\int P_X (f_1 \cdot \tilde{T}^h f_1)\cdot \text{—''—}\right|$$

$$= \lim \left|\frac{1}{N}\sum_{n=1}^{N} \int P_X(f_1 \cdot \tilde{T}^h f_1) \cdots T^{(k-1)n} P_X(f_k \cdot \tilde{T}^{kh} f_k)\right|$$

So we have $\gamma_h \le \|P_X (f_\ell \cdot \tilde{T}^h f_\ell)\|_{L^1} \le \|P_X (f_\ell \cdot \tilde{T}^h f_\ell)\|_{L^2}$

$\|f_i\|_\infty \le 1$

w. mixing extension: $\frac{1}{N}\sum_{1}^{N} \|P_X(f_\ell \cdot \tilde{T}^h f_\ell) - \underbrace{P_X f_\ell \cdot T^h P_X f_\ell}_{0}\|_{L^2}^{2} \xrightarrow{\text{Cesàro}} 0$ ~~delete~~

vdC: $\frac{1}{N}\sum_{n=1}^{N} u_n \to 0$ ∎

$\boxed{\text{Rem.}}$ 1) (3) says that for w. mixing extensions, (X,μ,T) is characteristic for multiple averages (the limit of mult. averages does not change if one repl. each f_i by its projection $P_X f_i$)

2) Analogously, using vdC, one shows that rel. weak mixing implies rel. w. mixing of all orders, i.e. $\forall k$ $\forall f_0, \dots, f_k \in L^\infty(\tilde{X})$

$$\frac{1}{N}\sum_{n=1}^{N} \|P_X(f_0 \cdot \tilde{T}^n f_1 \cdots \tilde{T}^{kn} f_k) - P_X(f_0)\cdot T^n(P_X f_1) \cdots T^{kn}(P_X f_k)\|_{L^2(X)}^2 \to 0$$

Recall: w. mixing of order k was: $\forall A_0, \dots, A_k \subset X$ meas.

$$\frac{1}{N}\sum_{1}^{N} |\underbrace{\mu(A_0 \cap T^{-n} A_1 \cap \dots \cap T^{-kn} A_k)}_{\int 1_{A_0}\cdot T^n 1_{A_1} \cdots T^{kn} 1_{A_k}} - \underbrace{\mu(A_0)\cdots\mu(A_k)}_{\int 1_{A_0} \cdots \int 1_{A_k}}|^2 \to 0$$

exerc.

Relative setting of Tao 2011

For $f, g \in L^2(\tilde{X})$, $(\tilde{X}, \tilde{\mu}, \tilde{T})$ being an extension of (X,μ,T), the fct

$$\langle f, g\rangle_{L^2(\tilde{X}|X)} := P_X(f \cdot \bar{g})$$

on X is called relative (or conditional) scalar product of f and g.

We also write $\langle f,g\rangle_{rel}$.

Observe: $\langle f,g\rangle_{rel} \in L^1(X)$

$\int_X |P_X(f\cdot\bar g)| \le \int_X \int_{x\text{-fiber}} |f\cdot\bar g| = \int_{\tilde X} |f\cdot \bar g| = \|f\cdot\bar g\|_{L^1(\tilde X)} \le \|f\|_{L^2(\tilde X)}\cdot\|g\|_{L^2(\tilde X)}$

The rel. scalar product satisfies:

1) $\langle f,f\rangle_{rel} \ge 0$ and $\langle f,f\rangle_{rel} = 0 \Leftrightarrow f=0$

$P_X(f\bar f) = P_X(|f|^2)$

$\Rightarrow$: $\int \langle f,f\rangle_{rel} = \int_X\int_{x\text{-fiber}} f\cdot\bar f = \int_{\tilde X} |f|^2$

2) $\langle f,g\rangle_{rel} = \overline{\langle g,f\rangle_{rel}}$

3) $\langle f_1+f_2,g\rangle_{rel} = \langle f_1,g\rangle_{rel} + \langle f_2,g\rangle_{rel}$

$\langle \lambda f, g\rangle_{rel} = \lambda\langle f,g\rangle_{rel} \quad \forall \lambda\in L^\infty(X)$ (const on fiber)

$P_X(\lambda f\cdot\bar g)(x) = \int_{x\text{-fiber}} \lambda\cdot f\bar g = \lambda(x)\cdot\int_{x\text{-fiber}} f\cdot\bar g$

Moreover, it satisfies the rel Cauchy-Schwarz ineq.:

$$|\langle f,g\rangle_{rel}| \le \|f\|_{rel}\cdot\|g\|_{rel} \qquad \forall f,g\in L^2(\tilde X)$$

for the relative norm

$$\|f\|_{rel} := \sqrt{\langle f,f\rangle_{rel}} \in L^2(X)$$

$\boxed{\text{Def}}$ For an extension $(\tilde X,\tilde\mu,\tilde T)$ of (X,μ,T), $f\in L^2(\tilde X)$ is called rel. weakly mixing if

$$\frac1N\sum_{n=1}^N \|\langle \tilde T^n f, f\rangle_{rel}\|_{L^1} \longrightarrow 0,$$

and we write $f\in W(\tilde X|X)$.

Rem. 1) For $f\in L^\infty$, this is equiv. to $\frac1N\sum_1^N \|\ldots\|_{L^2}^2 \longrightarrow 0$

2) One can easily show using vdC that

$$f\in W(\tilde X|X) \Leftrightarrow \frac1N\sum_{n=1}^N \|\langle \tilde T^n f, g\rangle_{rel}\|_{L^1} \longrightarrow 0 \quad \forall g\in L^2(\tilde X)$$

in part., $(\tilde X,\tilde\mu,\tilde T)$ is a w mixing ext. of (X,μ,T) $\Leftrightarrow$ $\forall f\in L^2(\tilde X)$,

$$P_X f = 0 \text{ (a.e.)} \Rightarrow f\in W(\tilde X|X),$$

i.e., $\forall$ fct with rel. mean zero is rel. w. mixing.

in part.,

$$L^2(\tilde X) = \{\text{fcts constant on fibers}\} \oplus \{\text{rel. w. mixing fcts}\}$$

$f = P_X f + (f - P_X f)$

The general case:

$\boxed{\text{Thm } 11.4}$ (Relative jdLG-decomposition)

Let $(\tilde{X}, \tilde{\mu}, \tilde{T})$ be an extension of an invertible MDS (X, μ, T).
Then

$$\boxed{L^2(\tilde{X}) = \overline{A(\tilde{X}|X)} \overset{\perp}{\oplus} W(\tilde{X}|X)}$$

Proof (orthogonality)

Observation $\boxed{T P_X = P_X \tilde{T}}$

Proof of observ. Denote by $j: L^2(X) \to L^2(\tilde{X})$, $jf :=$ corr. fct constant on fibers

Then $P_X = j^*$ by $\int_{\tilde{X}} jf \cdot \bar{g} \underset{\text{Fubini}}{=} \int_X \int_{x\text{-fiber}} jf \cdot \bar{g} = \int_X f \cdot P_X \bar{g}$.

Factor property.
$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{T}} & \tilde{X} \\ \pi \downarrow & \circlearrowleft & \downarrow \pi \\ X & \xrightarrow{T} & X \end{array}$$
, π measure pres. $\pi \tilde{T} = T \pi$.

So $\forall f$ $f(\pi \tilde{T} \tilde{x}) = f(T \pi \tilde{x})$, so by $jf = f \circ \pi$

$$\tilde{T} j = j T \qquad f(\pi \tilde{T} \tilde{x}) = (jf)(\tilde{T} \tilde{x})$$

Taking adjoints:
$$P_X \tilde{T}^* = T^* P_X.$$

Multiplying:
$$T P_X \underbrace{\cancel{\tilde{T}^* \tilde{T}}}_{\tilde{T} \text{ isom.}} = \underbrace{\cancel{T T^*}}_{= I} P_X \tilde{T} \text{ by invertibility assumpt.}$$

■ of observ.

Take $f \in W(\tilde{X}|X)$ and $g \in A(\tilde{X}|X)$. We show:
$$\langle f, g \rangle_{rel} = 0 \quad (a.e.)$$

- then $\langle f, g \rangle \underset{\text{Fubini}}{=} \int_X \underbrace{P_X(f \bar{g})}_{\langle f, g \rangle_{rel}} = 0$.

$\forall \varepsilon > 0 \, \exists k \, \exists g_1, \dots, g_k \cdot \forall n$, for a.e. $x \in X$
$\| \tilde{T}^n g - g_j \|_{L^2(x\text{-fiber})} < \varepsilon$ for some $j \in \{1, \dots, k\}$

Let $\varepsilon > 0$. Since g is rel. alm. per.
$\exists k \; \exists g_1, \dots, g_k \in L^2(\tilde{X})$ s.t. $\forall n$

$$\tilde{T}^n g = \sum_{j=1}^{k} a_{n,j} g_j + r_n, \qquad (*)$$

where $a_{n,j}$ is a 0-1-fct const on fibers and

$$\| \| r_n \|_{L^2(\tilde{X}|X)} \|_{L^\infty(X)} \le \varepsilon$$

$$\| r_n \|^2_{L^2(\tilde{X}|X)} = P_X(r_n \cdot \bar{r}_n) = P_X(|r_n|^2) < \varepsilon$$

Thus we have
$$\| \langle f, g \rangle_{rel} \|_{L^1} \underset{\substack{\text{by observ} \\ + T \text{ isom.}}}{=} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \| \langle \tilde{T}^n f, \tilde{T}^n g \rangle_{rel} \|_{L^1}$$

$$\langle \tilde{T} f, \tilde{T} g \rangle_{rel} = P_X(\tilde{T}(f \cdot \bar{g})) = T P_X(f \cdot \bar{g})$$

$$\le \overline{\lim}\, \frac{1}{N} \sum_{n=1}^{N} \Big(\sum_{j=1}^{k} \| \underbrace{\cancel{\overline{a}_{n,j}}}_{\text{const on fibers}} \langle \tilde T^n f, g_j \rangle_{rel} \|_{L^1} + \| \underbrace{\langle \tilde T^n f, r_n \rangle_{rel}}_{\underset{\text{rel. CS}}{\le} \|r_n\|_{rel} \cdot \|\tilde T^n f\|_{rel}} \|_{L^1}$$

$$\le \sum_{j=1}^{k} \overline{\lim}\, \frac{1}{N} \sum_{1}^{N} \underbrace{\| \langle \tilde T^n f, g_j \rangle_{rel} \|_{L^1}}_{=0 \text{ when } f \in W(\tilde X | X)} + \varepsilon \cdot \|f\|_{L^2(\tilde X)}$$

0-1-fct

$$\| P_X (|\tilde T^n f|^2) \|_{L^1} \le \int_{\tilde X} \cancel{\tilde T^n} |f|^2 = \|f\|^2_{L^2}$$

$\forall \varepsilon$, so $\langle f, g \rangle_{rel} = 0$ a.e. ∎

Proof of decomp. (very rough sketch)

Let $f \in L^2(\tilde X) \setminus W(\tilde X | X)$. By orthogonality to show:

$$f \not\perp AP(\tilde X | X),$$

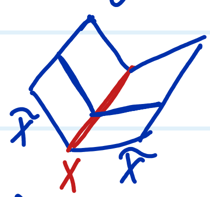
i.e., f correlates with some rel. a.p. fct.

Step 1 Construction of the operator $\mathcal{U}$

Def. the rel. product system $(\tilde X \times \tilde X, \tilde\mu \times_X \tilde\mu, \tilde T \times \tilde T)$ of $(\tilde X, \tilde\mu, \tilde T)$ with itself over (X, μ, T) by def. $\tilde\mu \times_X \tilde\mu$ by

$$\int_{\tilde X \times \tilde X} f \otimes g \, d(\tilde\mu \times_X \tilde\mu) := \int_X P_X f \cdot P_X g \, d\mu$$

$(f \otimes g)(\tilde x, \tilde y) := f(\tilde x) \cdot g(\tilde y)$



One shows: $\exists!$ such Borel prob. measure $\tilde\mu \times_X \tilde\mu$ on $\tilde X \times \tilde X$ and it is $(\tilde T \times \tilde T)$-inv.

The mean erg. thm for this system: $\exists (\tilde T \times \tilde T)$-inv. fct

$$H := \lim_{N \to \infty} \frac{1}{N} \sum_{1}^{N} \tilde T^n f \otimes \tilde T^n \bar f \quad \text{in } L^2(\mu \times_X \mu)$$

One shows: $\overline{H(\tilde x, \tilde y)} = H(\tilde y, \tilde x)$ $\tilde\mu \times_X \tilde\mu$-a.e.

For $x \in X$ def. the operator $\mathcal{U}_x$ on $L^2(x\text{-fiber})$ by

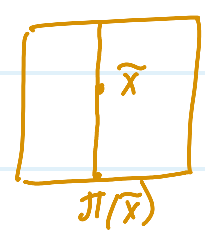
$$(\mathcal{U}_x g)(\tilde x) := \int_{x\text{-fiber}} H(\tilde x, \cdot)\, g(\cdot)$$

of $\tilde x$

in $\tilde x$

and $\mathcal{U}$ on $L^2(\tilde X)$ by

$$(\mathcal{U} g)(\tilde x) := \mathcal{U}_{\underbrace{\pi(\tilde x)}_{\text{proj. onto } x}} g$$



(i.e., $(\mathcal{U} g)\big|_{x\text{-fiber}} = \mathcal{U}_x (g|_{x\text{-fiber}})$)

One shows:

- $\mathcal{U}$ is s.a. and commutes with $\tilde T$
- $\langle \mathcal{U} f, f \rangle > 0$ (in fact, $\langle \mathcal{U} f, f \rangle = \overline{\lim}\, \frac{1}{N} \sum_{1}^{N} \| \langle \tilde T^n f, f \rangle_{rel} \|^2_{L^2}$)

- each U_x is s.a., comp. (even Hilbert-Schmidt, i.e., $\sum |\lambda_n|^2 < \infty$ eigenvalues)

Step 2 ~~Construction of the rel. a.p. fct correlating with f~~

Spectral thm for s.a (not necess comp.) operators: As for unitary oper.,

A s.a. with a cyclic vector ($\overline{\text{lin}}\{A^n x, n\in\mathbb{N}_0\}=H$) $\Rightarrow$ $\exists$ finite Borel measure ν (spectral measure) on $\mathbb{R}$ with $\sigma(A) = \operatorname{supp}\nu$ s.t.

$$A \underset{\text{isom.}}{\sim} M_{s\mapsto s} \text{ on } L^2(\mathbb{R},\nu)$$

if no cyclic vector, then decompose $H = \bigoplus H_n$ with cyclic vector

For a multiplicator, $\forall \varphi \in L^\infty(\mathbb{R})$ def. $\varphi(M_{s\mapsto s}) := M_{s\mapsto \varphi(s)}$ (for ex., $(M_{s\mapsto s})^2 = M_{s\mapsto s^2}$)

This leads to functional calculus (def. of $\varphi(A)$ for $\varphi \in L^\infty(\mathbb{R})$, A s.a.). s.t.

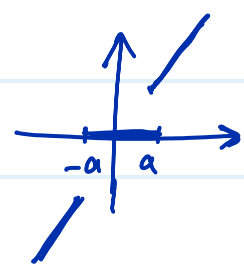
- $(\varphi\cdot\psi)(A) = \varphi(A)\cdot\psi(A)$ – multipl.
- $\overline{\varphi}(A) = (\varphi(A))^*$
- $\mathbb{1}(A) = I$, $(s\mapsto s)(A) = A$ $\rightsquigarrow$ natural def of polyn.
- $\varphi_n \to \varphi$ pointwise, (φ_n) unif. bdd, then $\varphi_n(A) \to \varphi(A)$ pointwise (as oper.)

exer.

Now, for $a\in\mathbb{R}$ consider

$$r_a(s) := s\cdot \mathbb{1}_{(-\infty,-a)\cup(a,\infty)}$$

(bdd on $\sigma(U)$)



One has: $r_a(s) \to (s\mapsto s)$ as $a\searrow 0$ pointwise

So by fct calculus

$$r_a(U) \longrightarrow U \text{ pointwise}$$

So $\exists a>0$: $\langle r_a(U)f, f\rangle > 0$ (recall: $\langle Uf,f\rangle>0$)

The last thing to show is:

$$g := r_a(U)f$$

is rel. a.p. (some work)

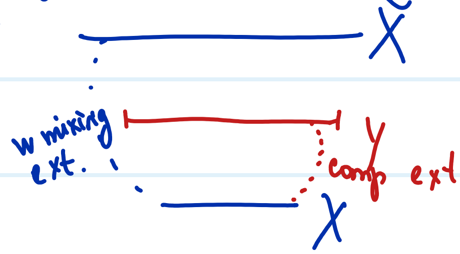
It uses $\operatorname{Orb}_{\tilde T} g = \{\underbrace{\tilde T^n g}_{\tilde T^n r_a(U) f}, n\in\mathbb{N}_0\} \subset \{r_a(U)\tilde T^n f, n\in\mathbb{N}_0\}$ (U commutes with $\tilde T$)

and $\mathbb{1}_{(-\infty,-a)\cup(a,\infty)}(U)_x$ = orth. projection onto $\overline{\text{lin}} \bigcup_{\lambda\notin(-a,a)} \ker(\lambda - U_x)$ on x-fiber (finite dim.)

Work: find finitely many "window fcts" indep. of the fiber.

$\boxed{\text{Cor. 11.5}}$ (Dychotomy between w. mixing and compact extensions)

Let $(\tilde{X}, \tilde{\mu}, \tilde{T})$ be an extension of (X,μ,T) and (X,μ,T) be invertible. If $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is not rel. w. mixing, then $\exists$ intermediate factor Y which is rel. compact over X (and $\neq X$). In part., if (X,μ,T) has MR, then either $(\tilde{X}, \tilde{\mu}, \tilde{T})$ has, or $\exists$ intermediate factor with MR

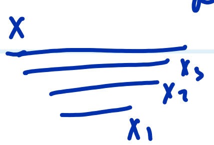


Rem. To show this, one needs to show that $\overline{A(\tilde{X}|X)}$ comes from a factor. (i.e., that $A(\tilde{X}|X) \cap L^\infty$ is dense in $\overline{A(\tilde{X}|X)}$ and a T-inv. conjug inv. subalgebra of L^∞)

Last step: Furstenberg tower

$\boxed{\text{Prop. 11.7}}$ (MR for limits of factors with MR)

Let $X_1, X_2, \dots$ be increasing chain of factors. If $\forall X_j$ has MR, then so does $\lim_j X_j$



$\lim_j X_j$ defined by the σ-alg. generated by $\bigcup \Sigma_j$ (σ-alg. for X_j)

(corresponds to increasing family of subspaces of $L^2(X,\mu)$, $\lim =$ pointwise limit of projections; proj. onto)

No proof here.

Strategy: Start with one-point factor (or Kronecker factor) — satisfy. (MR). ~~both~~ invertible!

If (X,μ,T) is rel. w. mixing to it — MR ✓

If not, $\exists X_2$ interm. factor comp. rel. to it. ($\Rightarrow$ MR)
And so on. We stop or $\exists$ chain of increasing factors $X_1, X_2, \dots$ Their limit is called isometric

We repeat — if (X,μ,T) not w. mixing rel. to it, $\exists$ intermediate factor comp. ext. of it etc.

Since $L^2(Y,\mu)$ is separable (σ-alg. is count. generated), one can apply „countable" lemma of Zorn to get

a weaker version: if $\forall$ count. chain has an upper bound, then $\exists$ max. element or $\exists$ an uncountable chain.

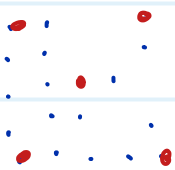
$\boxed{\text{Thm 11.8}}$ (Furstenberg-Zimmer structure theorem)

$\exists$ factor, called maximal isometric factor, which is a limit of increasing isom factors ($\Rightarrow$ has MR) so that (X,μ,T) is w. mixing relative to it.

$\boxed{\text{Cor. 11.9}}$ $\forall$ MDS has MR.

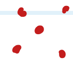
12 Further directions

① Multidim. Szemerédi (Furstenberg–Katznelson '78)



$C \subset \mathbb{N}_0^d$ with $\bar{d}(C) > 0$

def using boxes

Then $\forall F \subset \mathbb{N}_0^d$ finite  $\exists\, a, n : a + nF \subset C$

(for classical Szemerédi: $F = \{0, 1, \dots, k-1\}$, $d = 1$)

Similar Correspondence Principle:

(X, μ), $T_1, \dots, T_m$ commuting, μ-pres. Then $\forall f > 0$ bdd it suffices to show

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \int f \cdot T_1^n f \cdots T_m^n f \, d\mu > 0$$

② Polynomial Szemerédi / recurrence (Bergelson, Leibman 1996)

$\bar{d}(C) > 0 \implies \forall k, \forall p_1, \dots, p_k \in \mathbb{Z}[\cdot]$ with $p_j(0) = 0$

$\exists\, a, n : \quad a + p_1(n), \dots, a + p_k(n) \in C$

(e.g., $a + n^{17}$, $a + n^5 + n$)

$$\rightsquigarrow \quad \lim \frac{1}{N} \sum_{1}^{N} \int f\, T^{p_1(n)} f \cdots T^{p_k(n)} f > 0$$

③ Green–Tao Thm: $\mathbb{P}$ has (AP)

They showed: $\forall A \subset \mathbb{P}$ (primes) with positive rel. density has (AP)

④ Quantitative versions: How many AP? Where to find?

Green-Tao: $2^{2^{\cdot^{\cdot^{\cdot^{2^k}}}}}$

⑤ Convergence For L^2-conv. of

$$\frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdots T^{kn} f_k$$

one needs a different decomp (Host-Kra decomp. 2004) (rel. a.p. extensions are too large)

...

Open: Conjecture (Erdős–Turán 1936):

$$C = \{n_j\} \subset \mathbb{N} \text{ with } \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty \overset{?}{\implies} C \text{ has (AP)}$$