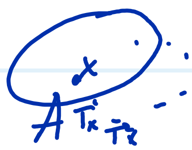
21.4.26

2. Recurrence and ergodicity

2.1 Recurrence

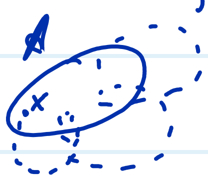
$\boxed{\text{Def. 2.1}}$ Let (X, μ, T) be an MDS and $A \subset X$ meas.



1) A is called <u>recurrent</u> if for a.e. $x \in A$ $\exists n \in \mathbb{N} : T^n x \in A$, or:

$$(1) \qquad \mu\Big(\underbrace{A \setminus \bigcup_{n=1}^{\infty} T^{-n}(A)}_{\text{points in } A \text{ which never come back to } A}\Big) = 0$$

2) A is called <u>infinitely recurrent</u> if for a.e. $x \in A$ $\exists (n_j) \subset \mathbb{N}$, $n_j \nearrow \infty$ s.t. $T^{n_j} x \in A$, or:



$$\mu\Big(A \setminus \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} T^{-n}(A)\Big) = 0$$

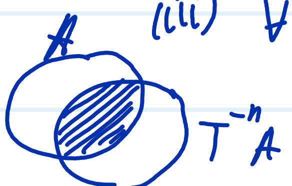
<u>Rem.</u> The def. is non-trivial only for A with $\mu(A) > 0$.

$\boxed{\text{Lemma 2.2}}$ For an MDS (X, μ, T), TFAE (The following assertions are equivalent)

(i) $\forall A \in \Sigma_X$ is recurrent

(ii) $\forall A \in \Sigma_X$ is infinitely recurrent

(iii) $\forall A \in \Sigma_X$ with $\mu(A) > 0$ $\exists n \in \mathbb{N}$: $\mu(A \cap T^{-n}(A)) > 0$



<u>Proof</u> (ii) $\Rightarrow$ (i) clear

(i) $\Rightarrow$ (ii) Take $k \in \mathbb{N}$. By (1) $A \subset \bigcup_{n=1}^{\infty} T^{-n}(A) \cup N$, N null set

Taking T^{-1} yield $T^{-1}(A) \subset \bigcup_{n=2}^{\infty} T^{-n}(A) \cup \underbrace{T^{-1}(N)}_{\text{null set}}$

$$T^{-(k-1)}(A) \subset \bigcup_{n=k}^{\infty} T^{-n}(A) \cup \underbrace{T^{-k}(N)}_{\text{null set}}$$

Again (1) implies

$$A \subset \bigcup_{n=k}^{\infty} T^{-n}(A) \cup \underbrace{(N \cup \ldots \cup T^{-k}(N))}_{\text{null set}},$$

so (ii).

(i) $\Rightarrow$ (iii) Assume $\mu(A \cap T^{-n}(A)) = 0$ $\forall n \in \mathbb{N}$. By (1)

$$A = \Big(A \cap \bigcup_{n=1}^{\infty} T^{-n}(A)\Big) \cup N_{\text{null set}}$$

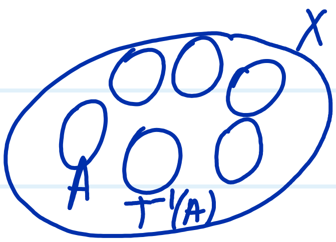
$$= \bigcup_{n=1}^{\infty} \underbrace{(A \cap T^{-n}(A))}_{\text{null set by assumption}} \cup N,$$

so $\mu(A) = 0$.

(iii) ⟹ (i) Let $A \in \Sigma_X$. Consider
$$B := A \setminus \bigcup_{k=1}^{\infty} T^{-k}(A) = A \cap \left[\bigcap_{k=1}^{\infty} T^{-k}(X \setminus A) \right]$$
We want to show: $\mu(B) = 0$.
Let $n \in \mathbb{N}$
$$B \cap T^{-n}(B) \subset A \cap \underbrace{T^{-n}(A) \cap \left[\bigcap^{\infty} T^{-k}(X\setminus A) \right]}_{=\emptyset} = \emptyset.$$
By (iii) $\mu(B) = 0$ ∎

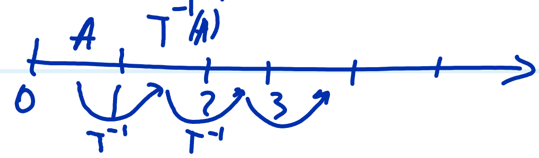
$\boxed{\text{Thm 2.3}}$ (Poincaré) ← Poincaré's recurrence Let (X, μ, T) be an MDS. Then $A \in \Sigma_X$ is ∞-recurrent.

Proof Let $A \subset X$ satisfy $\mu(A) > 0$ (recall: $\mu(A) = 0 \Rightarrow A$ ∞-recurr.)
By Lemma 2.2. it suffices to find n : $\mu(A \cap T^{-n}(A)) > 0$.
Assume not, so $\mu(A \cap T^{-n}A) = 0$ $\forall n$. Then $\forall k$

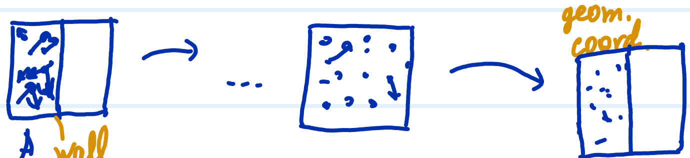


$$\mu(\underbrace{T^{-(n+k)}(A) \cap T^{-k}(A)}_{T^{-k}(T^{-n}(A) \cap A)}) = \mu(T^{-n}(A) \cap A) = 0 \quad \forall n,$$
i.e., the sets
$$A, T^{-1}(A), T^{-2}(A), \ldots$$
are pairwise disjoint. But they have the same measure, so
up to null sets
$$1 = \mu(X) \geq \sum_{n=0}^{\infty} \mu(A) = \infty \quad \lightning$$ ∎

Rem. Poincaré's recurrence thm is false for X with $\mu(X) = \infty$:
Take $X = \mathbb{R}$ with Leb., $Tx = x - 1$



Rem. Poincaré's thm has counterintuitive consequences:
Boltzmann's model. state space $X \subset [0,1]^{3n} \times \mathbb{R}^{3n} \subset \mathbb{R}^{6n}$
geom. coord — velocities

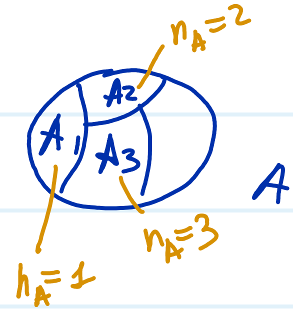


imagine: ∃ wall, you remove it, Poincaré: all particles will be on the left ∞-often.

Question How long should one wait?
Let (X, μ, T) be MDS, $A \subset X$ with $\mu(A) > 0$.

Def.
$$B_0 := \bigcap_{n=1}^{\infty} T^{-n}(X \setminus A)$$ — points which never visit A
$$B_1 := T^{-1}(A)$$ — points visiting A in 1 time step
$$\vdots$$
$$B_n := T^{-n}(A) \setminus \left[\bigcup_{k=1}^{n-1} T^{-k}(A) \right]$$ — points visiting A in n time steps but not before

We have:
- $X = B_0 \cup B_1 \cup \dots$
- all B_j are disjoint



Def. $A_n := A \cap B_n \quad \forall n$. By Poincaré

$$A = \bigcup_{n=1}^{\infty} A_n \quad \text{up to a null set}$$

Def. the return time fct

$$\begin{cases} n_A : A \to \mathbb{N} \\ n_A(x) = n \quad \text{if } x \in A_n \end{cases}$$

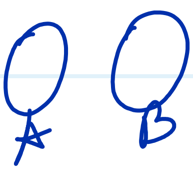
(on the null set any def.)

Question: What can we say about n_A?

Lemma 2.4 Let (X,μ,T) be MDS, let $A, B \in \Sigma_X$ and $n \in \mathbb{N}$.

Then:

$$\mu(B) = \sum_{k=1}^{n} \mu\Big(A \cap \Big[\bigcap_{j=1}^{k-1} T^{-j}(X \setminus A)\Big] \cap T^{-k}(B)\Big) + \mu\Big(\bigcap_{k=0}^{n-1} T^{-k}(X \setminus A) \cap T^{-n}(B)\Big).$$



Proof $n=1$

$$\mu(B) \underset{T\ \mu\text{-pres.}}{=} \mu(T^{-1}(B)) = \mu(A \cap T^{-1}(B)) + \mu((X \setminus A) \cap T^{-1}(B))$$

$n=2$ Do the same for the last term:

$$\mu(B) = \mu(A \cap T^{-1}(B)) + \underbrace{\mu\big(T^{-1}((X\setminus A) \cap T^{-1}(B))\big)}_{T \text{ is } \mu\text{-pres.}}$$

$$\mu(A \cap T^{-1}(X\setminus A) \cap T^{-2}(B)) + \mu((X \setminus A) \cap T^{-1}(X \setminus A) \cap T^{-2}(B))$$

etc. inductively (exerc.).

Def. the induced σ-algebra and the induced prob. measure on A "$\mu(A)>0$"

$$\Sigma_A := \{B \subset A,\ B \in \Sigma_X\} = A \cap \Sigma_X \subset \Sigma_X$$

$$\mu_A(B) := \frac{\mu(B)}{\mu(A)}, \quad B \in \Sigma_A$$

Thm. 2.5 Let (X,μ,T) be MDS, $A \subset X$ with $\mu(A) > 0$. Then:

$$\boxed{\int_A n_A \, d\mu_A = \frac{\mu\big(\bigcup_{n=0}^{\infty} T^{-n}(A)\big)}{\mu(A)}}$$

expected value of n_A

in part., $\boxed{\text{if } \bigcup_{n=0}^{\infty} T^{-n}(A) = X}$ up to a null set, then $\boxed{\int_A n_A \, d\mu_A = \frac{1}{\mu(A)}}$

a.e. point visits A at some time step

Proof Take $B := X$ in lemma 2.4:

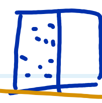
$$\mu(X)=\sum_{k=1}^{n}\mu\Big(A\cap\underbrace{\Big[\bigcap_{j=1}^{k-1}T^{-j}(X\setminus A)\Big]}_{=\bigcup_{j=k}^{\infty}A_j \text{ up to a null set}}\Big)+\mu\Big(\bigcap_{k=0}^{n-1}T^{-k}(X\setminus A)\Big)$$

$$\underset{A_j \text{ disjoint}}{=}\sum_{k=1}^{n}\sum_{j=k}^{\infty}\mu(A_j)+\mu\Big(\bigcap_{k=0}^{n-1}T^{-k}(X\setminus A)\Big),$$

i.e. $$\sum_{k=1}^{n}\sum_{j=k}^{\infty}\mu(A_j)=\mu(X)-\mu\Big(\bigcap_{k=0}^{n-1}T^{-k}(X\setminus A)\Big)=\mu\Big(\bigcup_{k=0}^{n-1}T^{-k}(A)\Big).$$

Take $n\to\infty$:
$$\mu\Big(\bigcup_{k=0}^{\infty}T^{-k}(A)\Big)=\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\mu(A_j)=\sum_{j=1}^{\infty}\sum_{k=1}^{j}\mu(A_j)=\sum_{j=1}^{\infty}\underbrace{j}_{n_A \text{ on } A_j}\cdot\mu(A_j)$$

$$=\mu(A)\cdot\sum_{j=1}^{\infty}j\cdot\frac{\mu(A_j)}{\mu(A)}=\mu(A)\cdot\int_A n_A\,d\mu_A \qquad \blacksquare$$

Rem. For Boltzmann's model:  $\mu(A)\sim\left(\frac{1}{2}\right)^n$

$n_A\sim 2^n$ if $\boxed{\bigcup_{n=0}^{\infty}T^{-n}(A)=X \text{ a.e.}}$ when does it hold?

2.2. Ergodicity

$\boxed{\text{Def 2.6}}$ Let (X,μ,T) be an MDS. A set $A\in\Sigma$ is called **invariant** if $T^{-1}(A)\subset A$ up to a null set, i.e.,
$$\mu\big(T^{-1}(A)\setminus A\big)=0.$$

$\boxed{\text{Lemma 2.7}}$ For an MDS (X,μ,T) and $A\in\Sigma$ TFAE: (the following ass. are equiv.)

(i) A is invariant

(ii) $A=T^{-1}(A)$ up to a null set $\quad \mu(A\,\Delta\, T^{-1}(A))=0$ symmetric difference 

(iii) $X\setminus A$ is invariant.

Proof (i) $\Rightarrow$ (ii) follows from $\mu(T^{-1}(A))=\mu(A)$

(ii) $\Rightarrow$ (iii) By (ii)
$$T^{-1}(X\setminus A)\supset X\setminus A,$$ but the same measure, so $=$ up to a null set $\blacksquare$

(iii) $\Rightarrow$ (i) symmetry $A\leftrightarrow X\setminus A$

Rem. A inv., $\mu(A)>0$ $\Rightarrow$ decomp. of X into 2 systems
$$(A,\Sigma_A,\mu_A,T) \text{ and } (X\setminus A,\Sigma_{X\setminus A},\mu_{X\setminus A},T)$$

$\boxed{\text{Def 2.8}}$ An MDS (X,μ,T) is called **ergodic** if
$$A \text{ inv.} \Rightarrow \mu(A)\in\{0,1\}$$
holds $\forall A\in\Sigma_X$.

Rem. 1) Thus erg. systems are indecomposable

2) $\exists$ MDS without ergodic subsystems:

$\boxed{\text{Ex}}$ $X=[0,1]$, Leb, $T=\mathrm{id}$

$\boxed{\text{Prop. 2.9}}$ For an MDS (X,μ,T) TFAE:

(i) (X,μ,T) is ergodic

(ii) $\forall A\subset X$ with $\mu(A)>0$

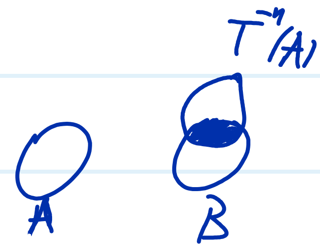
$$\underbrace{\bigcap_{n=0}^{\infty}\bigcup_{k\geq n} T^{-k}(A)}_{\text{points from } X \text{ visiting } A\ \infty\text{-often}} = X \quad \text{up to a null set}$$

(iii) $\forall A\subset X$ with $\mu(A)>0$

$$\underbrace{\bigcup_{n=0}^{\infty} T^{-n}(A)}_{-\parallel-\ \text{at least once}} = X \quad \text{up to a null set}$$

(iv) $\forall A,B\subset X$ with $\mu(A),\mu(B)>0$ $\exists n\geq 1$:

$$\mu(T^{-n}(A)\cap B)>0$$



Proof (i)$\Rightarrow$(ii) Let $A\subset X$ with $\mu(A)>0$. Def.

$$B_n := \bigcup_{k\geq n} T^{-k}(A)$$

Then: $T^{-1}(B_n)=\bigcup_{k=n+1}^{\infty} T^{-k}(A)\subset B_n$, i.e., B_n is invariant.

hence $\mu(B_n)\geq\mu(A)>0$, by ergodicity $\mu(B_n)=1$ $\forall n$,

so $\mu\left(\bigcap_{n=1}^{\infty} B_n\right)=1$.

(ii)$\Rightarrow$(iii) clear

(iii)$\Rightarrow$(iv) Let A,B with $\mu(A)>0$. Assume

$$\mu(T^{-n}(A)\cap B)=0 \quad \forall n\geq 1.$$

Then also $\overbrace{\bigcup_{n=1}^{\infty}\left(T^{-n}(A)\cap B\right)}^{B\cap\left(\bigcup_{n=1}^{\infty}T^{-n}(A)\right)}$ is a null set

$$\bigcup_{n=1}^{\infty} T^{-n}(A) = T^{-1}\Big(\underbrace{\bigcup_{n=0}^{\infty} T^{-n}(A)}_{=X \text{ up to a null set by (iii)}}\Big) = X \quad \text{up to a null set},$$

so $B=B\cap\left(\bigcup_{n=1}^{\infty} T^{-n}(A)\right)$ is a null set.

(iv)$\Rightarrow$(i) Let A be invariant, i.e., $T^{-1}(A)\subset A$ up to a null set,

Then also $T^{-n}(A)\subset A$ $-\parallel-$ $\forall n$.

For $B:=X\setminus A$ we have

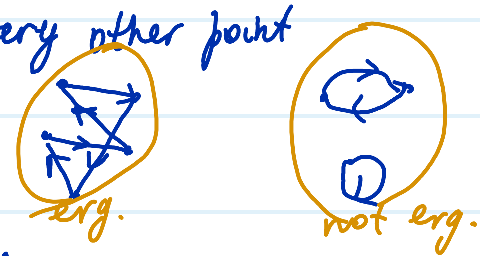
$$B\cap T^{-n}(A)\subset B\cap(A\cup \overset{\text{null set}}{N}) = \underbrace{(B\cap A)}_{\emptyset}\cup\underbrace{(B\cap N)}_{\text{null set}},$$

so $\mu(B\cap T^{-n}(A))=0$ $\forall n$. By (iv), $\mu(A)=0$ or $\underbrace{\mu(B)=0}_{\mu(A)=1}$. ■

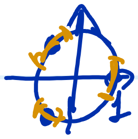
$\boxed{\text{Ex. 2.10}}$ 1) Finite systems: X finite, μ rescaled counting measure, $T: X \to X$ bij. Then:

T erg. $\iff$ $\forall$ point visits every other point

$\iff$ T is one cycle



2) Rotation on $\mathbb{T}$: $(\mathbb{T}, \text{leb}, a)$ rescaled



Case 1 a rational, i.e. $a^{n_0} = 1$ for some $n_0 \in \mathbb{N}$

Then the set

$$U_\varepsilon(1) \cup U_\varepsilon(a) \cup \ldots \cup U_\varepsilon(a^{n_0-1})$$

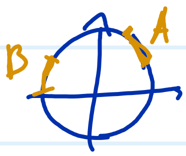
is a-inv. for small ε and has measure $\in (0,1)$

So $(\mathbb{T}, \text{leb}, a)$ is not erg.

Case 2 a irrational. We show later: $(\mathbb{T}, \text{leb}, a)$ is ergodic.

just a preview:

First, $\text{Orb}(a) = \{a, a^2, a^3, \ldots\}$ is dense in $\mathbb{T}$ exerc



Take A, B open intervals. Let n_1 be s.t. $a^{n_1} \in A$

n_2: $a^{n_2} \in B$, WLOG $n_2 < n_1 + 1$ without loss of gener.

Then $a^{n_2} \in \underbrace{B \cap T^{n_2 - n_1 - 1}(A)}_{\text{open interval}}$,

so $\mu(B \cap T^{n_2 - n_1 - 1}(A)) > 0$.

More generally, we will show later:

$\boxed{\text{Prop. 2.11}}$ Let (G, a) be a group rotation with Haar measure. TFAE:

(i) (G, Haar, a) is ergodic

(ii) $\text{Orb}\, a := \{a^n, n \in \mathbb{N}\}$ is dense in G.

(for open sets the same argument, one needs: U open, $\neq \emptyset \Rightarrow \text{Haar}(U) > 0$.
True because G compact, so $G = \bigcup_{j=1}^{m} g_j U$ and hence $\mu(U) \geq \frac{1}{m}$)

$\boxed{\text{Ex}}$ (Bernoulli shifts)

We show that the Bernoulli shifts are ergodic by showing the foll. stronger property:

$\boxed{\text{Prop 2.12}}$ Let $(X, \mu, T) = (\{0, \ldots, k-1\}^{\mathbb{N}}, \mu, \leftarrow)$ or $(\{0, \ldots, k-1\}^{\mathbb{Z}}, \mu, \leftarrow)$ the one- or two-sided Bernoulli shift. Then product measure

$$(*) \quad \boxed{\mu(T^{-n}(A) \cap B) \longrightarrow \mu(A) \cdot \mu(B)} \quad \forall A, B \in \Sigma.$$

In part., (X, μ, T) is ergodic

Rem. The property $(*)$ is called mixing (or strong mixing) and is much stronger than ergodicity (more soon)

Proof of Prop. 2.12 Last assertion follows from Prop. 2.9

We first check (*) for cylinder sets

Step 1 Let A, B be cylinders:
$$A = \dots \times \underbrace{\{a_1\}}_{j_1 \text{ position}} \times \dots \times \underbrace{\{a_k\}}_{j_k \text{ pos.}} \times \dots$$
$$B = \dots \times \underbrace{\{b_1\}}_{m_1 \text{ position}} \times \dots \times \underbrace{\{b_\ell\}}_{m_\ell \text{ pos.}} \times \dots$$

For $n > m_\ell - j_1$,
$$T^{-n}(A) \cap B = \dots \times \underbrace{\{b_1\}}_{m_1} \times \dots \underbrace{\{b_\ell\}}_{m_\ell} \times \dots \underbrace{\{a_1\}}_{j_1+n} \times \dots \underbrace{\{a_k\}}_{j_k+n} \dots,$$
so
$$\mu(T^{-n}(A) \cap B) = \underbrace{p(b_1) \cdots p(b_\ell)}_{\mu(B)} \cdot \underbrace{p(a_1) \cdots p(a_k)}_{\mu(A)} = \mu(A) \cdot \mu(B)$$

Step 2 Let $A = \bigcup_{j=1}^{m} A_j$, A_j disjoint cylinders up to null sets

$B = \bigcup_{i=1}^{\ell} B_i$, B_i —"— disjoint cylinders

We have
$$T^{-n}(A) \cap B = \bigcup_{i=1}^{\ell} \bigcup_{j=1}^{m} \left(T^{-n}(A_j) \cap B_i\right),$$
$$T^{-n}\Big(\bigcup_{j=1}^{m} A_j\Big) \cap B = \bigcup_{j=1}^{m} \left(T^{-n}(A_j) \cap B\right)$$

so
$$\mu(T^{-n}(A) \cap B) = \sum_{i=1}^{\ell} \sum_{j=1}^{m} \underbrace{\mu(T^{-n}(A_j) \cap B_i)}_{\text{cylinders};\ = \mu(A_j) \cdot \mu(B_i) \text{ for large } n}$$
$$= \sum_{i=1}^{\ell} \sum_{j=1}^{m} \mu(A_j) \cdot \mu(B_i)$$
$\exists n_0 : \forall n \geq n_0$ by step 1
$$= \sum_j \mu(A_j) \cdot \sum_i \mu(B_i) = \mu(A) \cdot \mu(B)$$

Step 3 Let A, B be arbitrary measur. sets, $\varepsilon > 0$.

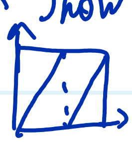
Since $\bigcup$ cylinders generate the σ-algebra, $\exists A_1, B_1$ as in step 2:
$$\mu(A \triangle A_1) < \varepsilon, \quad \mu(B \triangle B_1) < \varepsilon,$$
then $\mu(T^{-n}(A) \triangle T^{-n}(A_1)) < \varepsilon \quad \forall n$ (why?)

Take $n \geq n_0$, n_0 from step 2 for A_1, B_1.
$$T^{-n}(A) \cap B = T^{-n}(A_1) \cap B_1 \text{ up to a set of measure } \leq 2\varepsilon$$
and $\mu(A) \cdot \mu(B) = \mu(A_1) \cdot \mu(B_1)$ up to an error $\leq 2\varepsilon$

$|ab - cd| \leq |a-c| \cdot |b| + |c||b-d|$

So
$$\left|\mu(T^{-n}(A) \cap B) - \mu(A) \cdot \mu(B)\right| \leq \underbrace{\left|\mu(T^{-n}(A_1) \cap B_1) - \mu(A_1) \cdot \mu(B_1)\right|}_{=0\,! \text{ for } n \geq n_0}$$
up to an error $\leq 4\varepsilon$ ∎

Exerc. Show that the doubling map and the tent map are mixing.




$\boxed{\text{Thm 2.13}}$ (Kac)

Let (X,μ,T) be an ergodic MDS and $A\subset X$ with $\mu(A)>0$. Then the expected return time to A satisfies

$$\int_A n_A \, d\mu_A = \frac{1}{\mu(A)}$$

Proof By Prop. 2.9 $\bigcup_{n=0}^{\infty} T^{-n}(A) = X$ up to a null set. The assertion follows from Thm 2.5 ∎

$\boxed{\text{Ex}}$ (Recurrence in random literature)

The book of Nietzsche „Thus spoke Zarathustra" contains ~ 680.000 characters. Assume, Snoopy types randomly on his typewriter.
letters, points, space charact. etc.

Claim: he types Nietzsche's book ∞-often with prob. 1.

$N := 680.000$, Typewriter: say 90 characters, $p=\frac{1}{90}$ (equally probable)

$(X,\mu,T) = B(\underbrace{p,\dots,p}_{90 \text{ times}})$, so $X = \{0,\dots,89\}^{\mathbb{N}}$, $T \leftarrow$,
Bernoulli shift

μ product measure.

Let Book $= R_1 \cdot \ldots \cdot R_N$ (sequence of characters)

$A := \{(x_j) : x_1 = R_1, \dots, x_N = R_N\}$ - cylinder

$$\mu(\{x : \exists k \in \underset{\mathbb{N}\cup\{0\}}{\mathbb{N}_0} : \underbrace{x_{k+1} = R_1, \dots, x_{k+N} = R_N}_{T^k x \in A}\}) = \mu\left(\bigcup_{k=0}^{\infty} T^{-k}(A)\right)$$

$= 1$ by ergodicity of the Bernoulli shift

∞-often corresponds to $\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} T^{-k}(A)\right) = 1$ - holds by Prop. 5.9 ∎

2.3. Koopman operator and its spectrum

Let (X,μ,T) be an MDS. For $1\leq p<\infty$ def. $E := L^p(X,\mu)$ and $T: E\to E$ by
the same letter

$$(Tf)(x) := f(Tx), \quad x\in X,$$

– the Koopman operator induced by the MDS (or by the transformation T)

Properties of T on $L^p(X,\mu)$

- T is lin., $\|T\|=1$ and even T is isometry:

$$\|Tf\|_p^p = \int_X |f(Tx)|^p \, d\mu \underset{\text{Lemma 1.8}}{=} \int |f(x)|^p \, d\mu = \|f\|_p^p$$

- $T\mathbb{1}_A = \mathbb{1}_{T^{-1}(A)}$ $\forall A$ measur., in part., $T\mathbb{1} = \mathbb{1}$:
  $(T\mathbb{1}_A)(x) = \mathbb{1}_A(Tx) = \begin{cases} 1 & \text{if } Tx \in A \ \ x \in T^{-1}(A) \\ 0 & \text{otherw.} \end{cases} = \mathbb{1}_{T^{-1}(A)}(x)$
- T is <u>positive</u>, i.e., $f \geq 0 \Rightarrow Tf \geq 0$
- T is multiplicative: $T(f \cdot g) = Tf \cdot Tg$ – whenever $f \cdot g \in L^p$
  respects conjugation: $T(\bar{f}) = \overline{Tf}$
- $|Tf| = T|f|$ – <u>lattice homomorphism</u>

Rem. $\forall$ operator on L^p with these properties is a Koopman operator for some μ-pres transformation (no proof)

$\boxed{\text{Prop.}}$ If (X, μ, T) is invertible, then the Koopman operator is invertible (in part., unitary for $p = 2$)

Proof Let $S: X \to X$ be μ-pres. with $S \circ T = id$, $T \circ S = id$ a.e.

Claim: The Koopman operator S is the inverse of T:

$$((T \cdot S)f)(x) = f(\underbrace{T(Sx)}_{= x \text{ a.e.}}) = f(x) \quad \text{a.e.},$$

the same for $S \cdot T$. So T is invertible on L^p ∎

Rem. 1) The converse is not true in general:

$\boxed{\text{Ex}}$ T: $\{\cdot, *\} \to \{\cdot, *\}$ not invert., $X = \{\cdot, *\}$, $\Sigma = \{\emptyset, \{\cdot, *\}\}$,
but $L^2(X, \mu) = \text{lin}\{\mathbb{1}\}$, so Koopm. operator is I, so invertible.

2) The converse holds if (X, μ) is a „<u>standard probability space</u>" (no proof) isom. to $([0,1], \text{Leb})$

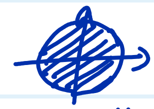
Question How does $\sigma(T)$ look like?

Recall: $\sigma(T) := \{\lambda \in \mathbb{C} : \underbrace{\lambda I - T}_{\lambda - T} \text{ not invert.}\}$ – the <u>spectrum</u> of T

$P\sigma(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ not injective}\}$ – <u>point spectrum</u> of T

Recall (FA) T unitary (isom., invert.) $\Rightarrow \sigma(T) \subset \mathbb{T}$

T isometric, not invert. $\Rightarrow \sigma(T) = \{z : |z| \leq 1\}$



T isom. $\Rightarrow$ eigenvalues $\subset \mathbb{T}$

Moreover, for Koopman oper. $1 \in P\sigma(T)$ since $T\mathbb{1} = \mathbb{1}$

$\boxed{\text{Prop. 2.14}}$ Let (X, μ, T) be an MDS with Koopm. oper. T on $L^p(X, \mu)$, $p \in [1, \infty)$. Def.

$$\text{Fix}\, T := \{f \in L^p : Tf = f\}.$$

TFAE:
(i) (X, μ, T) is $\boxed{\text{ergodic}}$
(ii) $\dim \text{Fix}\, T = 1$, i.e., 1 is a simple eigenvalue with $\boxed{\text{Fix}\, T = \mathbb{C}\mathbb{1}}$

in this case $\forall$ eigenfct of T satisf. $|f| = \text{const}$ a.e.

Proof last assertion (provided (i)$\Leftrightarrow$(ii))

$$Tf = \lambda f, \ |\lambda| = 1 \Rightarrow T|f| = |Tf| = |f|, \text{ so } |f| \in \mathrm{Fix}\,T = \mathbb{C}\cdot\mathbb{1}.$$

(i)$\Rightarrow$(ii) let $f \in \mathrm{Fix}\,T$, i.e., $f(Tx) = f(x)$ a.e.

Then also $\mathrm{Re} f, \mathrm{Im} f \in \mathrm{Fix}\,T$, so WLOG let f be real valued

Let $c \in \mathbb{R}$ arb.

Claim: $A := \{x : f(x) \le c\}$ is T-inv.

indeed, $T^{-1}A = \{x : \underbrace{f(Tx)}_{=f(x) \text{ a.e.}} \le c\} = A$ up to a null set

(X,μ,T) erg. $\Rightarrow \mu(A) = 0$ or $= 1$, i.e., f const. a.e. (why?)

(ii)$\Rightarrow$(i) let A be T-inv., i.e., $T^{-1}A = A$ up to a null set

Then $T\mathbb{1}_A = \mathbb{1}_{T^{-1}A} = \mathbb{1}_A$, i.e., $\mathbb{1}_A \in \mathrm{Fix}\,T$

By (ii) $\mathbb{1}_A = 0$ a.e. or $\mathbb{1}_A = 1$ a.e., i.e., $\mu(A) = 0$ or 1. ■

Rem. If (X,μ,T) is not erg., then there can be many fix fcts and many eigenfcts $\boxed{\text{Ex}}$ $T = I \Rightarrow \forall f \in \mathrm{Fix}\,T$

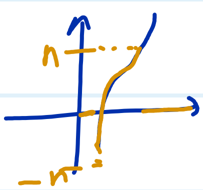
But: $\boxed{\text{Lemma 2.15}}$ **(Bounded EF (eigenfcts) are dense in EF)**

$\forall \lambda \in P\sigma(T)$, $\ \mathrm{Ker}(\lambda - T) \cap L^\infty(X,\mu)$ is dense in $\mathrm{Ker}(\lambda - T)$.

Proof Step 1: $\lambda = 1$.

let $f \in \mathrm{Fix}\,T$, so $Tf = f$. Consider

$$f_n(x) := \begin{cases} f(x) & \text{if } |f(x)| \le n \\ 0 & \text{otherw.,} \end{cases}$$



so $f_n = f \cdot \mathbb{1}_{\{|f| \le n\}}$

Claim: $Tf_n = f_n$. Indeed, we have a.e.

$$(Tf_n)(x) = f_n(Tx) = \begin{cases} f(Tx) = f(x) = f_n(x) & \text{if } \overbrace{|f(Tx)|}^{|f(x)| \text{ a.e.}} \le n \\ 0 = f_n(x) & \text{otherwise} \end{cases}$$

So $f_n \in \mathrm{Fix}\,T$ and

$$\|f_n - f\|_p = \int_{\{|f| > n\}} |f(x)|^p \, dx \xrightarrow[n \to \infty]{} 0$$

Step 2: λ arbitr.

Let $Tf = \lambda f$, $|\lambda| = 1$. As before, $|Tf| = T|f| = |f|$, so

$$T\mathbb{1}_{\{|f| \le n\}} = \mathbb{1}_{\{x : \underbrace{|Tf|}_{=|f| \text{ a.e.}}(x) \le n\}} = \mathbb{1}_{\{|f| \le n\}}$$

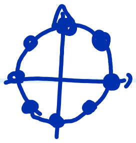
and therefore $f_n := f \cdot \mathbb{1}_{\{|f| \le n\}}$ is a bdd eigenfct (if not 0) by

$$T(\underbrace{f \cdot \mathbb{1}_{\{|f| \le n\}}}_{f_n}) \underset{\text{multipl. of } T}{=} Tf \cdot T\mathbb{1}_{\{|f| \le n\}} = \underbrace{\lambda f \cdot \mathbb{1}_{\{|f| \le n\}}}_{f_n}$$

and f_n appr. f as above as $n \to \infty$. ■

$\boxed{\text{Thm 2.16}}$ **(Point spectrum of Koopman operators)**

Let $(X, \mu; T)$ be an MDS with Koopm. oper. T on $L^p(X,\mu)$, $p \in [1,\infty)$.



(a) $P\sigma(T) \subset \mathbb{T}$ is a union of subgroups of $\mathbb{T}$ and is indep. of p.

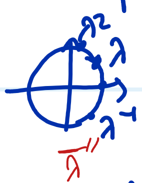
(b) $(X,\mu;T)$ is erg. $\iff$ $\forall \lambda \in P\sigma(T)$ is simple and generated by a unimodular ($|\cdot|=1$) eigenfct f_λ.

$\ker(\lambda - T) = \mathbb{C}\cdot f_\lambda$, $|f_\lambda| = 1$

In this case $P\sigma(T)$ is a subgroup of $\mathbb{T}$

Proof (a) independence of p follows from Lemma 2.15:

$\exists$ EF in $L^p \iff \exists$ EF in L^∞



it remains to show: $\lambda \in P\sigma \implies \lambda^n \in P\sigma \quad \forall n \in \mathbb{Z}$

By Lemma 2.15 $\exists f_\lambda$ bdd, $\neq 0$ with $Tf_\lambda = \lambda f_\lambda$

since $f_\lambda^n \in L^\infty \quad \forall n \in \mathbb{N}_0$

$\neq 0!$ $\quad T(f_\lambda^n) \underset{\text{multipl.}}{=} (Tf_\lambda)^n = \lambda^n \cdot f_\lambda^n$,

so $\lambda^n \in P\sigma(T)$. Moreover, $\overline{f_\lambda}^n \in L^\infty$, $\neq 0$, and

$$(T\overline{f_\lambda}^n) = (T\overline{f_\lambda})^n = \overline{\lambda}^n \cdot \overline{f_\lambda}^n$$

and $\overline{\lambda}^n = \lambda^{-n} \in P\sigma(T)$.

(b) Prop. 2.14: $(X,\mu;T)$ erg. $\iff$ $\operatorname{Fix} T = \mathbb{C}\cdot\mathbb{1}$, so $\Leftarrow$ holds.

$\implies$ Let $\lambda \in P\sigma(T)$ and $f \neq 0$. $Tf = \lambda f$. Then $T|f| = |f|$, so $|f| = c$ a.e. for some $c \neq 0$.

Def. $f_\lambda := \frac{f}{c}$, then $|f_\lambda| = 1$ a.e. and $Tf_\lambda = \lambda f_\lambda$.

Assume $\exists g_\lambda$ such, so $|g_\lambda| = 1$ a.e., $Tg_\lambda = \lambda g_\lambda$

then $$T\Big(\underbrace{\frac{f_\lambda}{g_\lambda}}_{|\cdot|=1 \text{ a.e.}}\Big) = \frac{Tf_\lambda}{Tg_\lambda} = \frac{\cancel{\lambda} f_\lambda}{\cancel{\lambda} g_\lambda} = \frac{f_\lambda}{g_\lambda}.$$

By erg. $\frac{f_\lambda}{g_\lambda} = \text{const}$, i.e., $\ker(\lambda - T)$ is 1-dim.

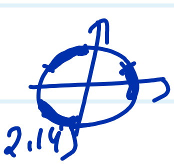
Now „in this case": We know: $f_{\lambda_1} f_{\lambda_2}$ is an eigenfct to $\lambda_1 \cdot \lambda_2$ by multipl., and $\neq 0$ by $|\cdot| = 1$, so $\lambda_1, \lambda_2 \in P\sigma \implies \lambda_1 \cdot \lambda_2 \cdot P\sigma$

Moreover, $\overline{f_\lambda}$ is EF to $\overline{\lambda} = \lambda^{-1}$, so $\lambda \in P\sigma \implies \lambda^{-1} \in P\sigma$. ■

$\boxed{\text{Ex}}$ (Rotation on $\mathbb{T}$)

$\boxed{\text{Prop. 2.17}}$ $(\mathbb{T}, a)$ erg. $\iff$ a irrat., i.e., $a^n \neq 1 \quad \forall n \neq 0$

in this case $P\sigma(T) = \{a^n, n \in \mathbb{Z}\}$



Proof $\implies$ done before (a rat. $\implies \exists$ many a-inv. sets)

$\Leftarrow$ We show: $\operatorname{Fix} T = \mathbb{C}\,\mathbb{1}$ on $L^2(\mathbb{T}, \text{Leb})$ (then use Prop. 2.14)

Let $f \in \operatorname{Fix} T$, i.e., $Tf = f$, $f \in L^2(\mathbb{T})$

- The fcts $\chi_n(z) = z^n$, $n \in \mathbb{Z}$ build an ONB of $L^2(\mathbb{T})$

Recall: $\{e^{2\pi i \cdot}\}$ is an ONB of $L^2[0,1]$

- $(T\chi_n)(z) = \chi_n(az) = a^n z^n = a^n \cdot \chi_n(z)$,
  so χ_n is a unimodular EF to eigenvalue a^n.

Let now $f = \sum_{n=-\infty}^{\infty} c_n \chi_n$ be Fourier repres. of f (ONB-repres.)

Then $Tf = \sum_{n=-\infty}^{\infty} c_n a^n \chi_n$

$Tf = f$ means $c_n = c_n a^n$ $\forall n \in \mathbb{Z}$. By $a^n \neq 1$ $\forall n \neq 0$
we have $c_n = 0$ $\forall n \neq 0$, i.e., $f = c_0 \cdot \underbrace{\chi_0}_{1}$ - const.

exerc.: $P_\sigma = \{a^n, n \in \mathbb{Z}\}$, where $\supset$ is done above. ∎

Rem. Analog. one shows:

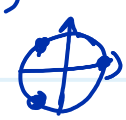
$(\mathbb{T}^d, \underbrace{a}_{(a_1,\dots,a_d)})$ erg $\iff$ $a_1, \dots, a_d$ are $\mathbb{Z}$-indep., i.e.

$$a_1^{n_1} \cdot \ldots \cdot a_d^{n_d} = 1 \iff n_1 = \dots = n_d = 0$$

For the proof one needs that $\chi_{n_1,\dots,n_d}(z) := z_1^{n_1} \cdot \ldots \cdot z_d^{n_d}$ build an ONB of $L^2(\mathbb{T}^d)$.

Rem. if $(\mathbb{T}, a)$ is not erg., so $a^n = 1$ for some n, then

$$\sigma(T) = P_\sigma(T) = \{1, a, \dots, a^{n-1}\}$$



and $T = I\big|_{\text{lin}(\chi_{kn}, k \in \mathbb{Z})} \overset{\perp}{\oplus} a \cdot I\big|_{\text{lin}(\chi_{kn+1}, k\in\mathbb{Z})} \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} a^{n-1} \cdot I\big|_{\text{lin}\{\chi_{kn+n-1}, k \in \mathbb{Z}\}}$

3. Classical ergodic theorems

3.1. Cesàro limit

Boltzmann's erg. hypothesis:

$$\frac{1}{N} \sum_{n=1}^{N} (T^n f)(x) = \frac{1}{N} \sum_{n=1}^{N} f(T^n x) \to \int f \, d\mu$$

"time average = space average"

Here we have $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N}$ - Cesàro limit

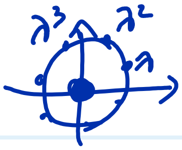
How can you make nonconv. sequences converge? 0 1 0 1 0 1 .. $\to \frac{1}{2}$

Def Let $(a_n) \subset \mathbb{C}$. We say that (a_n) is Cesàro convergent to $a \in \mathbb{C}$, write $C\text{-}\lim a_n = a$ or $a_n \xrightarrow{\text{Cesàro}} a$, if

$$\frac{a_1 + \dots + a_N}{N} = \frac{1}{N} \sum_{n=1}^{N} a_n \longrightarrow a.$$

Ex 1) $0, 1, 0, 1, .. \to 1/2$
$0, 1, 1, 0, 1, 1, .. \to 2/3$
(a_n) periodic with period d $\Rightarrow$ $C\text{-}\lim a_n = \frac{a_1 + \dots + a_d}{d}$.

2) Let $\lambda \in \mathbb{T}$. Then

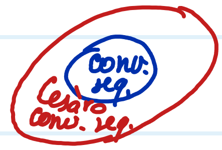


$\lambda \neq 1: \frac{1}{N}\sum_{n=1}^{N}\lambda^n = \frac{\lambda}{N}\sum_{n=0}^{N-1}\lambda^n = \frac{\lambda}{N}\cdot\frac{1-\lambda^N}{1-\lambda} \xrightarrow[N\to\infty]{} 0$ $\quad |\cdot| \le 2$, if $\lambda \neq 1$

$\lambda = 1: \quad \frac{1}{N}\sum_{n=1}^{N}\lambda^n = 1$

Properties of the Cesàro limit

1) $a_n \to a \implies a_n \xrightarrow{\text{Cesàro}} a$ exerc.



2) $\exists (a_n)$ bdd, not Cesàro conv.:

$0, 1, \dots, 1, 0, \dots\dots\dots, 0, 1, \dots \dots \dots 1$

3) $C\text{-}\lim a_{n+1} = C\text{-}\lim a_n$ (shift invariance) exerc.

4) Attention: $C\text{-}\lim a_n \cdot C\text{-}\lim b_n \neq C\text{-}\lim a_n b_n$ in general:

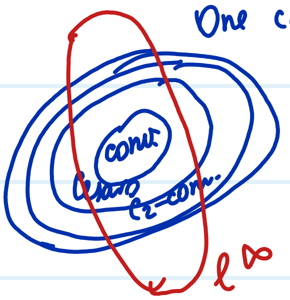
Reason: shift invariance:
a_n: 0 1 0 1 0 1 … $\xrightarrow{\text{Ces}} 1/2$
b_n: 1 0 1 0 1 0 … $\xrightarrow{\text{Ces}} 1/2$
$a_n b_n$: 0 0 0 0 0 0 … $\to 0 \neq 1/4$

Rem. 1) $a_n \xrightarrow{\text{Cesàro}} a \iff s_N := \frac{1}{N}\sum_{j=1}^{N} a_j \to a$

One can define C-C-lim. (write C_2-lim) : $s_N \xrightarrow{\text{Ces.}} a$ etc.

Then $\{C_n\text{-conv. seq.}\} \supsetneq \{C_{n-1}\text{-conv. seq.}\}$ exerc.

but $\bigcup_n \{C_n\text{-conv. seq.}\} \not\supset \ell^\infty$



2) Analog. one def Cesàro convergence for functions and operator:

$f_n \xrightarrow{\text{Cesàro}} f \overset{\text{def}}{\iff} \frac{1}{N}\sum_{n=1}^{N} f_n \to f$

$T_n \xrightarrow{\text{Cesàro}} T \overset{\text{def}}{\iff} \frac{1}{N}\sum_{n=1}^{N} T_n \to T$ – in norm/strong/weak oper. top.

$S_n \to S$ if $S_n x \to S x \ \forall x$

if $\langle S_n x, y\rangle \to \langle S x, y\rangle \ \forall x, y$

3.2. Von Neumann's mean ergodic theorem

Let (X,μ,T) be an MDS. Def. for $N \in \mathbb{N}$

$$S_N f := \frac{1}{N}\sum_{n=1}^{N} T^n f \quad \text{for } f \in L^1(X,\mu)$$

Let $H := L^2(X,\mu)$ $\ (\subset L^1(X,\mu)$ by Cauchy-Schwarz: $\int|f| \le \sqrt{\int |f|^2})$

$\boxed{\text{Thm 3.1}}$ (Von Neumann)
Let (X,μ,T) be an MDS and $f \in L^2(X,\mu)$. Then

$$\frac{1}{N}\sum_{n=1}^{N} T^n f \xrightarrow[N\to\infty]{} Pf$$

in L^2, where P is the orthogonal projection onto $\mathrm{Fix}\,T \subset L^2$.

Rem. Thm. 3.1 is called the mean ergodic theorem, Mittelergodensatz
„mean" because of L^2-conv. (not Cesàro limit)

Proof (Riesz)

Step 1: Orthogonal decomposition

We show: $\boxed{L^2 = \mathrm{Fix}\,T \overset{\perp}{\oplus} \overline{Rg(1-T)}}$ – von Neumann decomp.

1) $\mathrm{Fix}\,T \perp \overline{Rg(1-T)}$

Let $f \in \mathrm{Fix}\,T$, i.e., $Tf = f$. To show: $f \perp (g - Tg)$ $\forall g$

$\langle f, g - Tg\rangle = \langle f, g\rangle - \langle \underset{Tf}{f}, Tg\rangle = \langle f, g\rangle - \langle \cancel{T}f, \cancel{T}g\rangle \underset{T\text{ isom.}}{=} \langle f, g\rangle - \langle f, g\rangle = 0.$

2) Let $f \perp Rg(1-T)$. To show: $f \in \mathrm{Fix}\,T$.

in part., $f \perp (f - Tf)$, i.e.,

$0 = \langle f, f - Tf\rangle = \langle f, f\rangle - \langle f, Tf\rangle = \|f\|^2 - \langle f, Tf\rangle$, so

$\langle f, Tf\rangle = \|f\|^2 \underset{T\text{ isom.}}{=} \|f\| \cdot \|Tf\|$

Equality in C-S: $Tf = c \cdot f$ for $c \in \mathbb{C}$.

So $\|f\|^2 = \langle f, cf\rangle = \bar{c} \cdot \|f\|^2$ and $c = 1$, i.e., $f \in \mathrm{Fix}\,T$

Use decomp. $L^2 = M \oplus M^\perp$ for closed subspace M.

Step 2: Cesàro convergence

Let $f = f_1 + f_2$, $f_1 \in \mathrm{Fix}\,T$, $f_2 \in \overline{Rg(1-T)}$

To show: $S_N f = \frac{1}{N}\sum_{n=1}^{N} T^n f \to f_1$, i.e., $S_N f_1 \to f_1$, $S_N f_2 \to 0$

1) if $f \in \mathrm{Fix}\,T$, then $S_N f = f \to f$

2) $f = g - Tg$

$S_N f := \frac{1}{N}\sum_{n=1}^{N} T^n f = \frac{1}{N}\sum_{n=1}^{N}(T^n g - T^{n+1} g) \underset{\text{telesc. sum}}{=} \frac{Tg - T^{N+1}g}{N} \xrightarrow[N\to\infty]{} 0$ $\quad \|\cdot\|_2 \le 2 \cdot \|g\|_2$

So: conv. to 0 on $Rg(1-T)$.

Since $\|S_N\| \le \frac{1}{N}\sum_{n=1}^{N} \underbrace{\|T^n\|}_{=1} \le 1$ $\forall N$, one has

conv. to 0 on $\overline{Rg(1-T)}$

Recall: (T_n) unif. bdd: $\|T_n\| \le M$ $\forall n$, $T_n x \to Tx$ $\forall x \in D$, D dense. $\Rightarrow \forall x$:

WLOG $T = 0$

$\|T_n f\| \le \underbrace{\|T_n g\|}_{\to 0} + \underbrace{\|T_n(f-g)\|}_{\le M\cdot\|f-g\| \le M\cdot\varepsilon} \le (M+1)\varepsilon$ for large n. ■

choose $g \in D$, $\|f - g\| < \varepsilon$

Rem. The proof works $\forall$ isometry on a Hilbert space.

The mean erg. thm holds $\forall$ contraction on a Hilbert space (exerc.), also $\forall$ power bdd operator ($\sup_n \|T^n\| < \infty$) on a reflexive Banach space and even more general operators.

Def The projection P is called the mean ergodic projection of T 'mittelergodische Projektion' in German

Properties of P

1) $\int Pf\,d\mu = \int f\,d\mu$

$$\int S_N f\,d\mu = \frac{1}{N}\sum_{n=1}^{N} \underbrace{\int \cancel{T^n} f\,d\mu}_{\int f\,d\mu} = \int f\,d\mu \quad (T \text{ isom.})$$

$$\left|\int S_N f\,d\mu - \int Pf\,d\mu\right| \le \int |S_N f - Pf|\,d\mu \underset{C\text{-}S}{\le} \|S_N f - Pf\|_2 \xrightarrow{\text{von Neumann}} 0$$

2) $P(f\cdot Pg) = Pf\cdot Pg$:

$$\underset{T \text{ multipl.}}{S_N(f\cdot Pg)} = \frac{1}{N}\sum_{n=1}^{N} T^n f\cdot \cancel{T^n} \underbrace{Pg}_{\in \operatorname{Fix} T} = S_N f\cdot Pg$$

3) $Pf = \mathbb{E}(f|\Sigma')$ - condit. expectation of f, exerc.

where $\Sigma' := \{A\in\Sigma : A \text{ } T\text{-inv.}\}$

Recall: $\forall f\in L^1(X,\Sigma,\mu)\ \exists!\ g\in L^1(X,\Sigma',\mu)$ s.t. $\int_A f\,d\mu = \int_A g\,d\mu$ $\forall A\in\Sigma'$!

exerc.: $f\in \operatorname{Fix} T \iff f$ is Σ'-measurable

$\boxed{\text{Thm. 3.2}}$ (Characterisation of ergodicity)

Let (X,μ,T) be an MDS with Koopman oper. T on L^2 and the mean erg. projection P. TFAE:

(i) (X,μ,T) is ergodic

(ii) $\dim \operatorname{Fix} T = 1$

(iii) $\dim(\operatorname{Fix} T\cap L^\infty) = 1$

(iv) $\frac{1}{N}\sum_{n=1}^{N} T^n f \to \int f\,d\mu\cdot \mathbb{1}$ in L^2 $\forall f\in L^2$, i.e.

$$Pf = \int f\,d\mu\cdot\mathbb{1}$$

(v) $\frac{1}{N}\sum_{n=1}^{N} \langle T^n f, g\rangle \to \int f\,d\mu\cdot\int \bar{g}\,d\mu$ $\quad\forall f,g\in L^2$

(vi) $\frac{1}{N}\sum_{n=1}^{N} \mu(T^{-n}A\cap B) \to \mu(A)\mu(B)$ $\quad\forall A,B\in\Sigma$

(vii) $\frac{1}{N}\sum_{n=1}^{N} \mu(T^{-n}A\cap A) \to \mu(A)^2$ $\quad\forall A\in\Sigma.$

Rem. (vi) says that $T^{-n}A$ is (in mean and asymptotically) everywhere to find

Proof (i)$\iff$(ii)$\iff$(iii) done before

(ii)$\Rightarrow$(iv) We know: $Pf\in\operatorname{Fix} T$, so $Pf = c\cdot\mathbb{1}$ and

$c = \int Pf = \int f$ + von Neumann

(iv)$\Rightarrow$(v) $\frac{1}{N}\sum_{n=1}^{N}\langle T^n f, g\rangle = \langle \underbrace{\frac{1}{N}\sum_{n=1}^{N} T^n f}_{\to \int f\,d\mu}, g\rangle \to \int f\,d\mu\cdot\int\bar{g}\,d\mu$

(v)$\Rightarrow$(vi) For $A,B\in\Sigma$ def. $f:=\mathbb{1}_A$, $g:=\mathbb{1}_B$. Then

$$\langle T^n f, g\rangle = \int \mathbb{1}_{T^{-n}A}\cdot\mathbb{1}_B\,d\mu = \mu(T^{-n}A\cap B) \xrightarrow{\text{Ces.}} \int f\int\bar{g} = \mu(A)\cdot\mu(B)$$

(vi) $\Rightarrow$ (vii) clear :)

(vii) $\Rightarrow$ (i) Let $A \in \Sigma$ be T-inv. By (vii)

$$\underbrace{\frac{1}{N}\sum_{n=1}^{N} \mu(\underbrace{T^{-n}A \cap A}_{\mu(A)})}_{\mu(A)} \longrightarrow \mu(A)^2$$

$\mu(A) = \mu(A)^2$, so $\mu(A) \in \{0,1\}$ ■