$\boxed{\text{Step 1}}$: Decomposition into structured and random part

Recall: $L^2(X,\mu) = \underset{T=I}{\mathrm{Fix}\,T} \oplus \underset{\frac{1}{N}\sum_1^N T^n f \to 0}{\overline{\mathrm{Rg}(1-T)}}$ – von Neumann decomp.

The same for T^a, $a \in \mathbb{N}$!

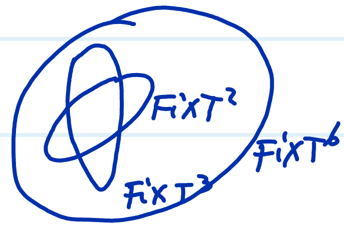
$$L^2(X,\mu) = \mathrm{Fix}\,T^a \oplus \overline{\mathrm{Rg}(1-T^a)}.$$

Property of $\mathrm{Fix}\,T^a$: $a | b \Rightarrow \mathrm{Fix}\,T^a \subset \mathrm{Fix}\,T^b$ ($T^b x = T^a \cdot \ldots \cdot T^a x$)

in part.,

$$\mathrm{Fix}\,T \subset \mathrm{Fix}\,T^2 \subset \mathrm{Fix}^{2\cdot 3} \subset \ldots \subset \mathrm{Fix}\,T^{n!} \subset \ldots$$

and $\forall a$, $\mathrm{Fix}\,T^a \subset \mathrm{Fix}\,T^{a!}$



So we have the decomp.

$$\boxed{L^2(X,\mu) = \underbrace{\overline{\bigcup_{a=1}^{\infty} \mathrm{Fix}\,T^a}}_{=: H_{rat}} \oplus \underbrace{\text{Rest}}_{= \bigcap_{a=1}^{\infty} \overline{\mathrm{Rg}(1-T^a)}}}$$

– rational spectrum decomposition

H_{rat} is called rational spectrum component

Rem. For measures it corresponds to the decomp. into the discrete part with only rational atoms and the rest: discrete part for irrat. atoms + continuous part

Properties of H_{rat}

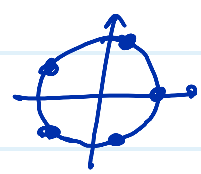
- H_{rat} is a closed lin. subspace:
  $$f \in \mathrm{Fix}\,T^a,\ g \in \mathrm{Fix}\,T^b \Rightarrow f+g \in \mathrm{Fix}\,T^{ab}$$
- $\boxed{H_{rat} = \overline{\mathrm{lin}}\{\underset{\text{eigenfcts}}{EF} \text{ to rational eigenvalues}\}}$

  $\supseteq$ if $Tf = \lambda f$ with $\lambda^a = 1$, then $T^a f = \lambda^a f = f$, i.e., $f \in \mathrm{Fix}\,T^a$

  $\subseteq$ follows from
  $$\mathrm{Fix}\,T^a = \ker(\lambda_1 - T) \oplus \ldots + \ker(\lambda_a - T) \text{ for } \lambda_1,\ldots,\lambda_a \text{ disjoint, } \lambda_1^a = \ldots = \lambda_a^a = 1.$$
  This holds $\forall$ unitary operator on a Hilbert space:

  Proof $\supseteq$ as above

  $\subseteq$ Let $f \in \mathrm{Fix}\,T^a$. By the spectral thm: WLOG $H = L^2(\mathbb{T},\nu)$,
  $(Tg)(z) = z g(z)$, $f = \mathbb{1}$.

  Now, $T^a f = f$ implies $z^a \mathbb{1} = \mathbb{1}$, i.e., $z^a = \mathbb{1}$ and $\mathrm{supp}\,\nu \subset \{\lambda_1,\ldots,\lambda_a\}$. Decompose
  $$f = \mathbb{1} = \mathbb{1}_{\{\lambda_1\}} + \ldots + \mathbb{1}_{\{\lambda_k\}}.$$
  Since $T\mathbb{1}_{\{\lambda_j\}} = \lambda_j \mathbb{1}_{\{\lambda_j\}}$, we have
  $$f \in \ker(\lambda_1 - T) + \ldots + \ker(\lambda_a - T).$$
  Moreover, eigenvectors to different eigenvalues are orthogonal.

  $Tf = \lambda_1 f,\ Tg = \lambda_2 g \Rightarrow \langle f,g\rangle = \langle Tf, Tg\rangle = \underbrace{\lambda_1\overline{\lambda_2}}_{\neq 1}\langle f,g\rangle$, so $\langle f,g\rangle = 0$. $\lambda_1,\lambda_2 \in \mathbb{T}$ since isom.
- $f_{rat} := P_{H_{rat}} f \geq 0$ for $f \geq 0$:

  $\geq$: Def. $f_{a!} := P_{\mathrm{Fix}\,T^{a!}} f$. By the mean erg. thm

$$f_{a!} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \underbrace{T^{a!n} f}_{\geq 0} \geq 0$$

Since $f_{rat} \underset{\uparrow}{=} L^2\text{-}\lim f_{a!}$, $f_{rat} \geq 0$.

(exer.): H Hilbert, • Y_n closed lin. subsp.
• $Y_n \subset Y_{n+1}$ $\Rightarrow$ $P_Y x = \lim_{n\to\infty} P_{Y_n} x$ $\forall x$.
• $Y := \overline{\bigcup Y_n}$

$>$: $0 < \int f \, d\mu = \langle f, 1\rangle \underset{\uparrow}{=} \langle f_{rat}, 1\rangle = \int f_{rat}\, d\mu$, so $f_{rat} \not\equiv 0$.

$f - f_{rat} \perp H_{rat} \ni 1$

not important here → • $\|f_{rat}\|_\infty \leq \|f\|_\infty$, in particular, f_{rat} is bdd if f is so
– follows from $f_{rat} = L^2\text{-}\lim f_{a!}$ and $f_{a!} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} \underbrace{T^{a!n} f}_{\|\cdot\|_\infty \leq \|f\|_\infty}$ as above.

• $\|f_{rat}\|_2 \leq \|f\|_2$ – the same reason.

Let now $f > 0$ and decompose
$$f = f_{rat} + f_{rest}, \quad f_{rat} \perp f_{rest}.$$

We have:
$$\langle T^n f, f\rangle = \langle T^n f_{rat}, f_{rat}\rangle + \langle T^n f_{rest}, f_{rest}\rangle + \underbrace{\langle \underbrace{T^n f_{rat}}_{\in H_{rat}}, f_{rest}\rangle}_{=0} + \underbrace{\langle \underbrace{T^n f_{rest}}_{\in Rest}, f_{rat}\rangle}_{=0}$$

We used here: H_{rat} and $Rest$ are T-inv.

H_{rat} – clear: $T^a g = g \Rightarrow T^a(Tg) = Tg$, so $Tg \in \operatorname{Fix} T^a$
$Rest$: $g = h - T^a h \Rightarrow Tg = Th - T^a(Th) \in Rg(I - T^a)$ } then appr. argument.

We show:
• $\frac{1}{N}\sum_{n=1}^{N} T^{p(n)} f_{rest} \to 0$ in L^2, so in part. $\langle \ldots, f_{rest}\rangle \to 0$.
• $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \langle T^{p(n)} f_{rat}, f_{rat}\rangle > 0$ (recall: $f_{rat} > 0$)

Step 2: The random part (Rest)

As for convergence, $\frac{1}{N}\sum_{n=1}^{N} T^{p(n)} f_{rest} \rightsquigarrow \frac{1}{N}\sum_{n=1}^{N} \lambda^{p(n)}$ in $L^2(\mathbb{T}, \nu)$ and
$\frac{1}{N}\sum_{n=1}^{N} \lambda^{p(n)} \to 0$ $\forall$ irrat. λ (Weyl).
Since $f \perp$ all eigenfcts to rat. eigenvalues and $T^n f \perp$ –"– $\forall n$ (why?),
ν has no rat. atoms and we have by dominated convergence
$$\frac{1}{N}\sum_{n=1}^{N} \lambda^{p(n)} \xrightarrow[N\to\infty]{} 0 \text{ in } L^2(\mathbb{T}, \nu).$$
Thus
$$\frac{1}{N}\sum_{n=1}^{N} T^{p(n)} f_{rest} \xrightarrow[N\to\infty]{} 0 \text{ in } L^2(X, \mu).$$

Step 3: The structured part (H_{rat})

Let $f \in H_{rat}$, $f > 0$.

Case 1 $f = f_a \in \operatorname{Fix} T^a$, i.e., $T^a f = f$. Then
$a \mid p(a)$
$a \mid p(2a)$ – recall: $p(0) = 0$!
$\vdots$
so $\langle T^{p(an)} f, f\rangle = \|f\|_2^2$. Therefore
$$\lim_{N\to\infty} \frac{1}{aN}\sum_{n=1}^{aN} \langle T^{p(n)} f, f\rangle \geq \overline{\lim_{N\to\infty}} \frac{1}{aN}\sum_{n=1}^{N} \|f\|_2^2 = \frac{1}{a}\|f\|_2^2 > 0$$
we know it exists

Case 2 $f \in H_{rat}$ arbitrary. Denote $d := \int f \, d\mu > 0$ and assume WLOG $\|f\|_2 \leq 1$ (recall: $\|f_{rat}\|_2 \leq \|f\|_2$).
Since $H_{rat} = \overline{\bigcup_n \operatorname{Fix} T^{n!}}$, $\exists a$: $\|f - f_a\|_2 < \frac{d^2}{4}$ for $f_a := P_{\operatorname{Fix} T^a} f$.

Thus we have
$$\langle T^n f, f\rangle = \langle T^n f_a, f_a\rangle + \underbrace{\langle T^n f_a, f - f_a\rangle}_{|\cdot| \leq \underbrace{\|T^n f_a\|_2}_{= \|f_a\|_2 \leq \|f\|_2 \leq 1} \|f - f_a\|_2 < \frac{d^2}{4}} + \overbrace{\langle T^n (f - f_a), f\rangle}^{|\cdot| \leq \|f\| \cdot \|f - f_a\| < \frac{d^2}{4}},$$

i.e.,

$$|\langle T^n f, f\rangle - \langle T^n f_a, f_a\rangle| < \frac{d^2}{2} \quad \forall n.$$

in part.,
$$|\langle T^{p(an)} f, f\rangle - \langle \underbrace{T^{p(an)} f_a}_{= f_a}, f_a\rangle| < \frac{d^2}{2} \quad \forall n$$

and therefore
$$\langle T^{p(an)} f, f\rangle \geq \|f_a\|_2^2 - \frac{d^2}{2} \underset{\text{CS (Cauchy-Schwarz)}}{\geq} \langle f_a, 1\rangle^2 - \frac{d^2}{2} = d^2 - \frac{d^2}{2} = d^2 > 0 \quad \forall n.$$

As in step 1 we obtain
$$\lim_{N\to\infty} \frac{1}{aN} \sum_{n=1}^{aN} \langle T^{p(n)} f, f\rangle \geq \frac{1}{a}\cdot\frac{d^2}{2} > 0$$
■

9. A tool: van der Corput lemma

$\boxed{\text{Thm } 9.1}$ (van der Corput lemma/trick)

Let H be Hilbert and $(u_n) \subset H$ bdd. Def. $\forall h \in \mathbb{N}$

$$\gamma_h := \overline{\lim_{N\to\infty}} \left|\frac{1}{N}\sum_{n=1}^{N} \langle u_{n+h}, u_n\rangle\right|$$

if $\frac{1}{H}\sum_{h=1}^{H} \gamma_h \xrightarrow[H\to\infty]{} 0$, then $\frac{1}{N}\sum_{n=1}^{N} u_n \xrightarrow[N\to\infty]{} 0$.

Rem. 1) $\langle u_{n+h}, u_n\rangle$ are sometimes called discrete derivatives along h:

if $u_n = e^{i\varphi_n}$, then
$$\langle u_{n+h}, u_n\rangle = e^{i(\varphi_{n+1} - \varphi_n)}$$
and $\varphi_{n+h} - \varphi_n$ is analog. to $\dfrac{f(x+h\cdot 1) - f(x)}{1}$.

2) Even more holds:
$$\boxed{\overline{\lim_{N\to\infty}} \left\|\frac{1}{N}\sum_{n=1}^{N} u_n\right\|^2 \leq \overline{\lim_{H\to\infty}} \frac{1}{H}\sum_{h=1}^{H} \gamma_h}$$
van der Corput's inequality

An alternative proof of step 2 (Rest) for polynomial recurrence / convergence without spectral Thm and without Weyl

Let $g \perp H_{rat}$. To show: $\frac{1}{N}\sum_{n=1}^{N} T^{p(n)} g \to 0$ in L^2 $\forall$ polyn. $p \in \mathbb{Z}[\cdot]$ with $\deg p \geq 1$

we assume T to be invert. if $p(n) < 0$ for ∞-many n ($\Leftrightarrow$ leading coeff < 0)

induction on $\deg p$

$\deg p = 1$ $\quad p(n) = an + b$.

Since $g \perp \mathrm{Fix}\, T^a$, by the mean ergodic thm
$$\frac{1}{N}\sum_{n=1}^{N} T^{an} g \to P_{\mathrm{Fix}T^a} g = 0 \quad \text{in } L^2$$

Now,
$$\frac{1}{N}\sum_{n=1}^{N} T^{an+b} g \underset{\text{if } b \geq 0}{=} T^b \Big(\underbrace{\frac{1}{N}\sum_{n=1}^{N} T^{an} g}_{\to 0}\Big) \to 0$$

otherwise take T^{an_0+b} out if $a > 0$, $b < 0$.

$\deg p$: $k-1 \to k$ $\quad$ Idea: use VdC (van der Corput) for $u_n = T^{p(n)} g$.

Let $h \in \mathbb{N}$ and observe
$$\langle u_{n+h}, u_n\rangle = \langle T^{p(n+h)} g, T^{p(n)} g\rangle \underset{T \text{ isometry}}{=} \langle T^{\overbrace{p(n+h) - p(n)}^{=: q_h(n)}} g, g\rangle$$

$q_h \in \mathbb{Z}[\cdot]$ with $\deg q_h = k-1$ (why?).

induction hypothesis:
$$\gamma_h := \overline{\lim_{N\to\infty}} \left|\frac{1}{N}\sum_{n=1}^{N} \langle u_{n+h}, u_n\rangle\right| = \overline{\lim_{N\to\infty}} \Big|\big\langle \underbrace{\frac{1}{N}\sum_{n}^{N} T^{q_h(n)} g}_{\to 0 \text{ by induction hyp.}}, g\big\rangle\Big| = 0 \quad \forall h$$

VdC: $\frac{1}{N}\sum_{n=1}^{N} T^{p(n)} g \to 0$ in $L^2(X,\mu)$. ■

Proof (Van der Corput)

WLOG let $\|u_n\| \le 1$ $\forall n$. Let $H \in \mathbb{N}$. We first show:

$$\left\| \frac{1}{N}\sum_{n=1}^{N} u_n - \frac{1}{N}\frac{1}{H}\sum_{n=1}^{N}\sum_{h=0}^{H-1} u_{n+h} \right\| \xrightarrow[N\to\infty]{} 0$$

replace u_n by $\frac{1}{H}\sum_{h=0}^{H} u_{n+h}$

Proof: For $N > H$

$$\left\| \frac{u_1+\dots+u_N}{N} - \frac{\frac{u_1+\dots+u_H}{H}+\dots+\frac{u_N+\dots+u_{N+H}}{H}}{N} \right\|$$

$\le$ [$u_H, \dots, u_N$ appear H times $\rightsquigarrow$ cancel]

$\le \frac{1}{N}\cdot 2\,[\|u_1\|+\dots+\|u_{H-1}\| + \|u_{N+1}\|+\dots+\|u_{N+H}\|]$

$\le \frac{2}{N}\cdot 2H \xrightarrow[N\to\infty]{} 0$ ■

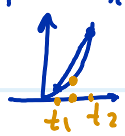
So it suffices to show:

$$\overline{\lim_{N\to\infty}} \underbrace{\left\| \frac{1}{NH}\sum_{n=1}^{N}\sum_{h=0}^{H-1} u_{n+h} \right\|}_{=:(*)}$$

can be arbitrarily small for large H

By the Δ-ineq.

$$\|(*)\| \le \frac{1}{N}\sum_{n=1}^{N}\left\| \frac{1}{H}\sum_{h=0}^{H-1} u_{n+h}\right\|.$$

Since the fct $t \mapsto t^2$  is convex $\left(\frac{t_1+t_2}{2}\right)^2 \le \frac{t_1^2+t_2^2}{2}$,

one has

$$\left(\frac{t_1+\dots+t_N}{N}\right)^2 \le \frac{t_1^2+\dots+t_N^2}{N},$$

so

$$\|(*)\|^2 \le \frac{1}{N}\sum_{n=1}^{N}\left\|\frac{1}{H}\sum_{h=0}^{H-1} u_{n+h}\right\|^2 = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{H^2}\sum_{h_1,h_2=0}^{H-1}\langle u_{n+h_1}, u_{n+h_2}\rangle$$

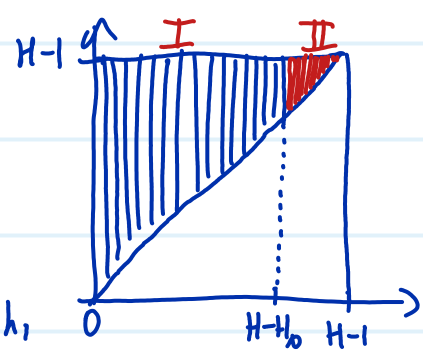
Therefore

$$\overline{\lim}\,\|(*)\|^2 \le \frac{1}{H^2}\sum_{h_1,h_2=0}^{H-1}\underbrace{\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle u_{n+h_1}, u_{n+h_2}\rangle\right|}_{=\gamma_{|h_1-h_2|}}$$

by shift invariance

We show:

$$\frac{1}{H^2}\sum_{h_1,h_2=0}^{H-1}\gamma_{|h_1-h_2|} \xrightarrow[H\to\infty]{} 0$$



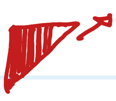
Let $\varepsilon > 0$ and take H_0: $\forall H \ge H_0$

$$\frac{1}{H}\sum_{h=0}^{H-1}\gamma_h < \varepsilon$$

Then (see picture)

$$\frac{1}{2}\frac{1}{H^2}\sum_{h_1,h_2=0}^{H-1}\gamma_{|h_1-h_2|} \le \frac{1}{H^2}\sum_{h_1=0}^{H-1}\sum_{h_2=h_1}^{H-1}\gamma_{h_2-h_1}$$

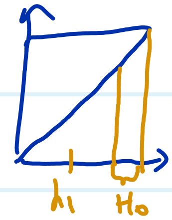
diagonal counted twice / 2



$$= \frac{1}{H}\sum_{h_1=0}^{H-H_0}\underbrace{\frac{1}{H}\sum_{h_2=h_1}^{H-1}\gamma_{h_2-h_1}}_{\frac{1}{H}\sum_{h=0}^{H-h_1-1}\gamma_h} + \frac{1}{H^2}\sum_{h_1=H-H_0+1}^{H-1}\sum_{h_2=h_1}^{H-1}\gamma_{h_2-h_1}$$

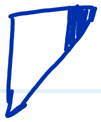
$$= \underbrace{\frac{1}{H}\sum_{h_1=0}^{H-H_0}\frac{1}{H}\sum_{h=0}^{H-h_1-1}\gamma_h}_{\mathrm{I}} + \underbrace{\frac{1}{H^2}\sum_{h_1=H-H_0+1}^{H-1}\sum_{h_2=h_1}^{H-1}\gamma_{h_2-h_1}}_{\mathrm{II}}$$

I. $\frac{1}{H}\sum_{h=0}^{H-h_1-1}\gamma_h \le \frac{1}{H-h_1-1}\sum_{h=0}^{H-h_1-1}\gamma_h < \varepsilon$



by $H-h_1 \ge \cancel{H}-\cancel{H}+H_0 = H_0$, $h_1 \le H-H_0$

so $I \le \frac{1}{H}\sum_{h_1=0}^{H-H_0}\varepsilon < \varepsilon$

II. 

$II \le \frac{1}{H^2}\cdot\frac{H_0^2}{2} \xrightarrow[H\to\infty]{} 0$, so $<\varepsilon$ for large H.

$\gamma_h = |\delta_h| \le 1 \ \forall h$ (Cauchy-Schwarz)

$\gamma_h = \lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle u_{n+h}, u_n\rangle\right|$

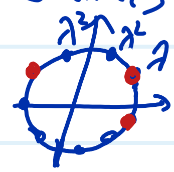
Altogether: $$\frac{1}{H^2}\sum_{h_1,h_2=0}^{H-1}\gamma_{|h_2-h_1|} \xrightarrow[H\to\infty]{} 0$$

and therefore $\frac{1}{N}\sum_{n=1}^{N}u_n \to 0$ ■

One (more) application of vdC: Thm of Weyl for polyn.

exerc./seminar: Show with vdC: $\forall p\in\mathbb{Z}[\cdot]$, $p\neq$ const, $\forall\lambda\in\mathbb{T}$ irrat., the polynomial orbit $(\lambda^{p(n)})_{n=1}^{\infty}$ is equidistr. in $\mathbb{T}$, i.e., (Weyl criterion of equidistribution):

$$\frac{1}{N}\sum_{n=1}^{N}\lambda^{p(n)m} \to 0 \quad \forall m\neq 0,\ m\in\mathbb{Z}$$



10. Roth Theorem and double convergence

Roth 1952 = Szemerédi for 3-term arithm. progressions:

$$C\subset\mathbb{N},\ \bar{d}(C)>0 \Rightarrow \exists\, a,n:\ a,\ a+n,\ a+2n\in C.$$

As above, we show:

$\boxed{\text{Thm 10.1}}$ Let (X,μ,T) be an MDS.

1) (Double conv.) $\forall f,g\in L^\infty$
$$\frac{1}{N}\sum_{n=1}^{N}T^nf\cdot T^{2n}g$$
conv. in L^2

2) (Double recurrence) $\forall f\in L^\infty$, $f>0$ — $f\ge 0$ and $\not\equiv 0$
$$\exists\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int f\cdot T^nf\cdot T^{2n}f\,d\mu > 0$$

in part., 2) $\Rightarrow$ Roth by Furstenberg's correspondence principle.

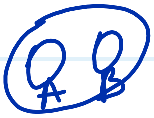
Idea As for polyn. conv./recurrence:

- A decomposition $L^2 = H_1\oplus H_2$ into T-invariant closed subspaces
- $\frac{1}{N}\sum_{n=1}^{N}T^nf\cdot T^{2n}g \xrightarrow{H} 0$ in L^2 if $f\in H_2$ or $g\in H_2$
- $P_{H_1}f>0$ and bdd for $f>0$, bdd and
$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int f\cdot T^nf\cdot T^{2n}f\,d\mu>0.$$

jacobs glicksberg de leeuw

10.1. Jacobs–Glicksberg–de Leeuw decomposition

Recall. T weakly mixing $\Leftrightarrow$ 1 is a simple eigenvalue, no other eigenvalues



$$\Leftrightarrow \frac{1}{N}\sum_{n=1}^{N} |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| \xrightarrow[N\to\infty]{} 0 \quad \forall A, B \text{ meas}$$

$$\Leftrightarrow \frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, g\rangle - \int f \cdot \int \bar{g}\,| \xrightarrow[N\to\infty]{} 0 \quad \forall f, g \in L^2$$

$f = 1_A, g = 1_B$ or appr. by simple fcts

$$\Leftrightarrow \frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, g\rangle| \longrightarrow 0 \quad \forall f \text{ with } \int f = 0, \ \forall g \in L^2$$

$f = \int f \cdot 1 + \underbrace{(f - \int f \cdot 1)}_{\int = 0}$

Def 10.2 We call $f \in L^2$ weakly mixing w.r.t. T if

$$\frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, g\rangle| \to 0 \quad \forall g \in L^2$$

Rem. 1) f w. mixing $\Rightarrow$ $\int f = 0$, i.e., $f \perp 1$

Take $g := 1$. $\langle T^n f, g\rangle = \int T^n f = \int f$, so $\frac{1}{N}\sum_{n=1}^{N} |\int f| \to 0$, so $\int f = 0$.

2) f is w. mixing $\Leftrightarrow$ $\frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle| \xrightarrow[N\to\infty]{} 0$.

Proof $(\Rightarrow)$ clear

$(\Leftarrow)$ $g := f$ ok, $g := T^k f$ ok $\forall k$ by $\langle T^n f, T^k f\rangle = \langle T^{n-k} f, f\rangle$ (for $n \geq k$) + shift invariance of Cesàro limit

So $g \in \operatorname{lin}\{T^k f, k \in \mathbb{N} \cup \{0\}\}$ ok by linearity

exerc.: $g \in \overline{\operatorname{lin}}\{\dots\}$ ok – approx. argument

$$|\langle T^n f, g\rangle| \leq |\langle T^n f, g_j\rangle| + \|f\|_2 \cdot \underbrace{\|g - g_j\|_2}_{\to 0}$$

For $g \perp \overline{\operatorname{lin}}\, \operatorname{Orb} f$ ok anyway: $\langle T^n f, g\rangle = 0 \ \forall n$ ∎

3) By Koopman–von Neumann Lemma:

$$f \text{ w. mixing} \Leftrightarrow \frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, g\rangle|^{2} \to 0 \quad \forall g \in L^2$$

$$\Leftrightarrow \frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle|^{2} \to 0.$$

Thm 10.3 (Jacobs–Glicksberg–de Leeuw (JGdL) decomposition)

For $H = L^2(X, \mu)$ the decomp. into T-invariant closed subspaces

$$\boxed{H = H_{kr} \oplus H_{wmix}}$$

holds, where

- $H_{kr} := \overline{\operatorname{lin}}\{f : Tf = \lambda f \text{ for some } \lambda \in \mathbb{T}\}$

  the Kronecker part (or: structured part)

- $H_{wmix} := \{f \text{ weakly mixing w.r.t. } T\}$

  the weakly mixing part (or: pseudorandom part)

Rem. (X, μ, T) is weakly mixing $\Leftrightarrow$ $H = \langle 1\rangle \oplus \{f \text{ weakly mixing}\}$

Proof $H_{kr} \perp H_{wmix}$: if $Tf = \lambda f$, g w. mixing $\Rightarrow$ $|\langle f, g\rangle| \underset{T \text{ isom}}{=} |\langle T^n f, T^n g\rangle| = |\langle f, T^n g\rangle| \xrightarrow{\text{Cesàro}} 0$, so $\langle f, g\rangle = 0$.

H_{kr} is closed lin. T-inv. subspace – clear

H_{wmix} is T-inv., lin subspace. For „closed" see below.

it suffices to show: $f \perp$ eigenfcts of T $\Rightarrow$ f is w. mixing.

Take $f \perp$ eigenfcts of T and rewrite

$$|\langle T^n f, f\rangle|^2 \underset{\text{use } |z|^2 = z\cdot\bar z}{=} \int_X \underline{(T^n f)(x)\cdot \bar f(x)}\, d\mu(x) \cdot \int_X \underline{(T^n \bar f)(y)}\, \underline{f(y)}\, d\mu(y)$$

$$\underset{\text{Fubini}}{=} \int_{X\times X} \underbrace{(T^n f \otimes T^n \bar f)}_{(T\times T)^n (f\otimes \bar f)}(x,y)\cdot (\bar f \otimes f)(x,y)\, d(\mu\times\mu)(x,y)$$

where $(f\otimes g)(x,y) := f(x)\cdot g(y)$

By von Neumann

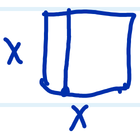
$$\frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle|^2 = \langle \underbrace{\frac{1}{N}\sum_{n=1}^{N} (T\times T)^n (f\otimes \bar f)}_{=: F_N}, f\otimes \bar f\rangle$$

$$\xrightarrow[N\to\infty]{} \langle \underbrace{P_{Fix(T\times T)} f\otimes \bar f}_{=: F}, f\otimes \bar f\rangle$$

We show: $F = 0$

Idea: use kernel operators + spectral thm for comp. s.a. operators

Def. K_N, K on $L^2(X,\mu)$ by



$$(K_N g)(x) := \int_X F_N(x,y)\, g(y)\, dy$$

$$(K g)(x) := \int_X F(x,y)\, g(y)\, dy.$$

Then:

- $\|K_N - K\| \underset{\text{FA1}}{\leq} \|F_N - F\|_{L^2} \to 0$ by von Neumann erg. thm
- $((T\times T)^n f\otimes \bar f)(x,y) = (T^n f)(x)\cdot (T^n \bar f)(y)$, so

$$(K_N g)(x) = \frac{1}{N}\sum_{n=1}^{N} \int_X (T^n f)(x)\, T^n \bar f(y)\, g(y)\, dy$$

$$= \frac{1}{N}\sum_{n=1}^{N} \langle g, T^n f\rangle (T^n f)(x), \quad \text{i.e.,}$$

$$\boxed{K_N g = \frac{1}{N}\sum_{n=1}^{N} \langle g, T^n f\rangle T^n f}$$ – finite dimensional operators

$\dim Rg(K_N) \leq N$

in part., K_N and K are compact

- K_N are s.a.,

self adjoint

$$\langle K_N g, h\rangle = \frac{1}{N}\sum_{n=1}^{N} \langle g, T^n f\rangle \langle T^n f, h\rangle = \langle g, K_N h\rangle$$

so also K is s.a.:

$$\langle K_N g, h\rangle = \langle g, K_N h\rangle$$

$$\langle K g, h\rangle \qquad \langle g, K h\rangle$$

- $KT = TK$:

$$KTg \leftarrow K_N T g = \frac{1}{N}\sum_{n=1}^{N} \langle \cancel{T} g, T^{\cancel{n}\, n-1} f\rangle T^n f = \frac{1}{N}\sum_{n=1}^{N} \langle g, T^{n-1} f\rangle T\, T^{n-1} f$$

$$= T\, \frac{1}{N}\sum_{n=1}^{N} \langle g, T^{n-1} f\rangle T^{n-1} f \to T K g$$

We show: λ is an eigenvalue of $K \Rightarrow \lambda = 0$

Let λ be such that $Kh = \lambda h$ for some $h \neq 0$.

Then $KT^n h = T^n K h = T^n \lambda h = \lambda T^n h$, i.e.,

$$\{T^n h,\ n \in \mathbb{N} \cup \{0\}\} \subset \ker(\lambda - K) =: Y$$

Assume that $\lambda \neq 0$, then $\dim Y < \infty$ – $K|_Y = \lambda \cdot I$

so $\text{lin}\{T^n h,\ n \in \mathbb{N} \cup \{0\}\}$ is finite dim., T-inv., so

$T|_{\text{lin}\dots}$ is a unitary matrix (isometry, all eigenvalues in $\mathbb{T}$, so 0 not eigenvalue)

Diagonalisation of unitary matrices: $\text{lin}\{T^n h, n\}$ is spanned by eigenfcts of T, so

$$h = h_1 + \dots + h_k, \quad Th_j = \lambda_j h_j, \quad \forall h_j \neq 0. \quad (h \neq 0)$$

But $f \perp$ eigenfcts of $T \Rightarrow \text{lin}\{f, Tf, \dots\} \perp$ eigenfcts of T

$\Rightarrow \underset{\text{contains } h:\ Kh = \lambda h}{\text{rg}\, K} \perp$ eigenfcts $\Rightarrow h \perp$ eigenfcts of T ↯ $(h=0)$.

So K does not have non-zero eigenvalues $\Rightarrow K = 0$

$\Rightarrow \underset{\text{kernel}}{F} = P_{\text{Fix}(T \times T)}(f \otimes \bar{f}) = 0$ (why?), spectral thm for comp. s.a. oper.

so $\frac{1}{N} \sum_{n=1}^{N} |\langle T^n f, f \rangle|^2 \to 0$ ■

10.2. The weakly mixing part and double convergence

$\boxed{\text{Prop. 10.4}}$ Assume that T is ergodic and f or g is w. mixing w.r.t. T.

Then $$\frac{1}{N} \sum_{n=1}^{N} T^n f \cdot T^{2n} g \longrightarrow 0 \text{ in } L^2.$$

Proof Let $u_n := T^n f \cdot T^{2n} g$

$$\langle u_{n+h}, u_n \rangle = \langle \underbrace{T^{n+h} f \cdot T^{2n+2h} g}_{T^n(T^h f\, T^{n+2h} g)}, \underbrace{T^n f \cdot T^{2n} g}_{T^n(f \cdot T^n g)} \rangle$$

$$= \int T^h f \cdot \bar{f} \cdot \underbrace{T^n (T^{2h} g \cdot \bar{g})}_{\xrightarrow{\text{Cesàro}} \int T^{2h} g \cdot \bar{g} \cdot \mathbb{1}} \, d\mu$$

$$\xrightarrow[\text{mean erg. thm}]{\text{Cesàro}} \underbrace{\int T^h f \cdot \bar{f} \, d\mu}_{\langle T^h f, f \rangle} \cdot \underbrace{\int T^{2h} g \cdot \bar{g} \, d\mu}_{\langle T^{2h} g, g \rangle}$$

If f is w. mixing, then

$$\gamma_h = \lim \left| \frac{1}{N} \sum_{n=1}^{N} \langle u_{n+h}, u_n \rangle \right| = \underbrace{|\langle T^h f, f \rangle|}_{\text{if } f \text{ w. mixing} \xrightarrow{\text{Cesàro}} 0} \cdot \underbrace{|\langle T^{2h} g, g \rangle|}_{\leq \|g\|^2} \xrightarrow{\text{Cesàro}} 0$$

VdC $\Rightarrow u_n \xrightarrow{\text{Cesàro}} 0$

exerc. the same for g w. mixing (show: g w. mixing for T^2) ■