if (X,μ,T) is invertible, then the Koopman operator T on $L^2(X,\mu)$ is also invertible, so unitary.

Since (iii) holds for $f=\mathbb{1}$, by linearity we have to show:

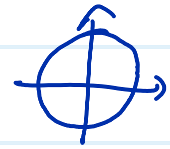
$$f \perp \mathbb{1} \Rightarrow \frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, g\rangle| \longrightarrow 0 \quad \forall g \in L^2$$

Let f satisfy $\int f\,d\mu = 0$

Fact 1. Spectral theorem for unitary operators

Let U be a unitary operator on a Hilbert space H having a cyclic vector, i.e., $\exists x_0 \in H$:

$$\overline{\text{lin}}\{U^n x_0,\ n\in\mathbb{Z}\} = H.$$



Then $\exists$ Borel probability measure ν on $\mathbb{T}$ s.t.

$$(H,U) \overset{\text{isom.}}{\sim} (L^2(\mathbb{T},\nu), M_z),$$

where M_z is the multiplicator def. by $(M_z f)(z) := z f(z)$.

Moreover, $x_0 \sim \mathbb{1}$ under this isomorphism. function $z \mapsto z$

Consider now $H := \overline{\text{lin}}\,\text{Orb}_{\mathbb{Z}} f := \overline{\text{lin}}\{T^n f,\ n\in\mathbb{Z}\}$. On H, the Koopman operator $T|_H$ is still unitary and has a cyclic vector f.

Spectral thm: $\exists$ prob. measure (Borel) ν on $\mathbb{T}$ s.t.

$$\boxed{\begin{array}{c}(H,T) \overset{\text{isom.}}{\sim} (L^2(\mathbb{T},\nu), M_z) \\ f \sim \mathbb{1}\end{array}}$$

We have $\langle T^n f, f\rangle \overset{\text{def. of isom.}}{=} \int_{\mathbb{T}} z^n \, d\nu =: \hat{\nu}(n)$ – n-th Fourier coefficient of ν.

Fact 2: Wiener's lemma

Let ν be a (Borel) prob. measure on $\mathbb{T}$. Then

$$\boxed{\frac{1}{N}\sum_{n=1}^{N} |\hat{\nu}(n)|^2 \longrightarrow \sum_{\substack{a \text{ atom} \\ \text{of } \nu \\ (a\in\mathbb{T})}} \nu(\{a\})^2}$$

Since $f \perp \mathbb{1}$, (iv) implies now

$$\frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle|^2 \longrightarrow 0$$

because ν has no atoms ($\nu(\{a\})>0 \Leftrightarrow \mathbb{1}_{\{a\}} \not\equiv 0$, so an eigenfct of M_z: $M_z \mathbb{1}_{\{a\}} = a\mathbb{1}_{\{a\}}$ and M_z has no eigenvalues since $f \perp \mathbb{1} \Rightarrow \overline{\text{lin}}\{T^n f,\ n\in\mathbb{Z}\} \perp \mathbb{1}$)

Koopman–von Neumann: $\exists J \subset \mathbb{N}$ of density 1: $|\underbrace{\langle T^n f, f\rangle}_{\text{old}}|^2 \xrightarrow[n\in J]{n\to\infty} 0$, so $\Updownarrow$ $\langle T^n f, f\rangle \xrightarrow[n\in J]{n\to\infty} 0$

(again K–vN):

$$\frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle| \longrightarrow 0$$

So (iii) holds for $g := f$.

For $g := T^k f$ for $k \in \mathbb{Z}$

$$\langle T^n f, g\rangle = \langle T^n f, T^k f\rangle \overset{T \text{ unitary}}{=} \langle T^{n-k} f, f\rangle,$$

so $\frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, g\rangle| \to 0$ by shift invariance of C-lim.

By linearity, (iii) holds $\forall g \in \text{lin}\, \text{Orb}_{\mathbb{Z}}(f) \xrightarrow[\text{by approx.}]{} \forall g \in \overline{\text{lin}}\, \text{Orb}_{\mathbb{Z}}(f)$ exer.

Now, for $g \perp \text{Orb}_{\mathbb{Z}}(f)$, $\langle T^n f, g\rangle = 0 \quad \forall n$.

So by linearity, (iii) holds $\forall g$ ∎

$\boxed{\text{Prop. 4.7}}$ Two further equivalent conditions to those in Thm. 4.6 are

(i') $\forall f, g \ \exists (n_j) \subset \mathbb{N} \nearrow : \langle T^{n_j} f, g\rangle \xrightarrow[j\to\infty]{} \int f \cdot \int \overline{g}$

(v) $(X \times X, \mu \times \mu, T \times T)$ is ergodic.

In part., one can remove "density 1" requirement in the def. of weak mixing.

<u>Proof</u> (iii) ⇒ (i') clear (Koopman–von Neumann)

(i') ⇒ (iv) Assume $f \perp \mathbb{1}$, $Tf = \lambda f$. (recall: eigenfcts to different eigenvalues are orthogonal for isometries. $Tf = \lambda_1 f$, $Tg = \lambda_2 g$, $\langle f, g\rangle = \langle Tf, Tg\rangle = \lambda_1 \overline{\lambda_2} \langle f, g\rangle$)

By (i') for $g := f$

$$\langle T^{n_j} f, f\rangle = \lambda^{n_j} \|f\|^2 \xrightarrow[j\to\infty]{} |\textstyle\int f|^2 = 0$$

for some (n_j), i.e., $\|f\| = 0$, so (iv) holds.

(v) ⇒ (iv) Assume $Tf = \lambda f$. Then

$$(T \times T)(f, \overline{f})(x, y) = (Tf)(x) \cdot \underbrace{(T\overline{f})(y)}_{\overline{f}(Ty)} = \cancel{\lambda} f(x) \cdot \cancel{\overline{\lambda}}\, \overline{f}(y)$$

$$= (f, \overline{f})(x, y) \quad \text{a.e.}$$

By (v), $(f, \overline{f})$ is constant, so f is constant.

(ii) ⇒ (v) We show more:

$(X \times Y, \mu \times \nu, T \times S)$ is erg. $\forall$ erg. (Y, ν, S).

Let $A_1, B_1 \in \Sigma_X$, $A_2, B_2 \in \Sigma_Y$. Then

$$\frac{1}{N}\sum_{n=1}^{N} (\mu \times \nu)\Big(\underbrace{(T \times S)^{-n}(A_1 \times A_2) \cap (B_1 \times B_2)}_{(T^{-n}A_1 \times S^{-n}A_2) \cap (B_1 \times B_2) = (T^{-n}A_1 \cap B_1) \times (S^{-n}A_2 \cap B_2)}\Big)$$

$$= \frac{1}{N}\sum_{n=1}^{N} \mu(T^{-n}A_1 \cap B_1) \cdot \nu(S^{-n}A_2 \cap B_2)$$

$$= \frac{1}{N}\sum_{n=1}^{N} [\mu(T^{-n}A_1 \cap B_1) - \mu(A_1)\mu(B_1)]\, \nu(S^{-n}A_2 \cap B_2)$$

$$+ \mu(A_1)\mu(B_1) \cdot \frac{1}{N}\sum_{n=1}^{N} \nu(S^{-n}A_2 \cap B_2)$$

$$=: \mathrm{I}_N + \mathrm{II}_N$$

$$\mathrm{II}_N \xrightarrow[N\to\infty]{\text{Charact. of ergodicity}} \mu(A_1) \cdot \mu(B_1)\, \nu(A_2)\, \nu(B_2) = (\mu \times \nu)(A_1 \times A_2) \cdot (\mu \times \nu)(B_1 \times B_2)$$

right limit!

$$|I_N| \le \frac{1}{N}\sum_{n=1}^{N} |\mu(T^{-n}A_1\cap B_1) - \mu(A_1)\cdot\mu(B_1)| \xrightarrow[N\to\infty]{} 0$$ by (ii)

So $I_N + II_N \to (\mu\times\nu)(A_1\times A_2)\cdot(\mu\times\nu)(B_1\times B_2)$.
Since sets of the form $A_1\times A_2$ induce the product σ-algebra,
$(X\times Y, \mu\times\nu, T\times S)$ is ergodic. (charact. of ergodicity) ■

Rem. We showed that also

(V') $(X\times Y, \mu\times\nu, T\times S)$ is erg. $\forall$ erg. (Y,ν,S)

is equiv. to (i)–(V) (since every weakly mixing system is erg., i.e., (V') $\Rightarrow$ (V)).

Alternative proof of (iV) $\Rightarrow$ (V)

Assume that $(X\times X, \mu\times\mu, T\times T)$ is not erg. Then $\exists f\in L^2(X\times X,\mu\times\mu)$ $(T\times T)$-inv. (i.e., $(T\times T)f = f$) and not constant.

Consider
$$f_1(x,y) := f(x,y) + \overline{f(y,x)}$$
$$f_2(x,y) := i\,(f(x,y) - \overline{f(y,x)})$$

Properties of f_1, f_2

- $f_j(x,y) = \overline{f_j(y,x)}$
- $f = \frac{f_1 - if_2}{2}$, in part., $\exists j : f_j \ne$ const.
- f_j are $(T\times T)$-inv.:
$$((T\times T)f_1)(x,y) = f_1(Tx,Ty) = \underbrace{f(\cancel{T}x,\cancel{T}y)}_{((T\times T)f)(x,y) = f(x,y)} + \overline{f(\cancel{T}y,\cancel{T}x)} = f_1(x,y),$$
analogously f_2.

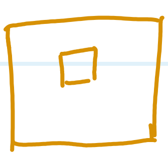
So we can assume WLOG that
$$f(x,y) = \overline{f(y,x)}$$
(otherwise take f_1 or f_2 instead of f).

Moreover, we can assume WLOG
$$\int_{X\times X} f \, d(\mu\times\mu) = 0$$
(otherwise replace f by $f - \int f\cdot \mathbb{1}$) (check: $\int f = \overline{\int f}$ using)

Def. $F: L^2(X,\mu) \to L^2(X,\mu)$ by
$$(Fg)(x) := \int_X f(x,y)\,g(y)\,d\mu(y).$$

We have:

- F is compact (see FA 1)
- F is s.a. by
$$(F^*g)(x) = \int \underbrace{\overline{f(y,x)}}_{=f(x,y) \text{ by assumption a.e.}} g(y)\,d\mu(y) \quad \text{exerc.}$$
$$= (Fg)(x) \quad \text{a.e.}$$



- $F \ne 0$ because $f \not\equiv 0$

Spectral thm for compact s.a. operators (FA 1):
$\exists \lambda \ne 0$, $\lambda \in P\sigma(F)$, $\dim(\operatorname{Ker}(\lambda - F)) < \infty$

Denote $V_\lambda := \operatorname{Ker}(\lambda - F)$. We show:
$$TV_\lambda \subset V_\lambda$$

Let $g \in V_\lambda$, i.e., $Fg = \lambda g$. Then
$$(F(Tg))(x) = \int_X \underbrace{f(x,y)}_{f(Tx,Ty)} \underbrace{(Tg)(y)}_{g(Ty)} \, d\mu(y) = \int_X f(Tx,y)\,g(y)\,d\mu(y)$$

$$= \underbrace{(Fg)}_{=\lambda\cdot g}(Tx) = \lambda\cdot(Tg)(x),$$

so $Tg\in V_\lambda$, i.e., V_λ is T-inv.

consider $T|_{V_\lambda}$ - a matrix $\Rightarrow$ $\exists g\in V_\lambda$ an eigenvector for $T|_{V_\lambda}$, so for T.

By (iv), $g=\text{const}$, say $g=c\neq 0$. But then

$Fg=\lambda g$ implies $\cancel{c}\int f(x,y)\,d\mu(y)=\lambda\cancel{c}$ for a.e. x

By Fubini, $\lambda=\int_{X\times X} f=0$ ↯ $\blacksquare$

Mixing of all orders

$\boxed{\text{Def}}$ (X,μ,T) is called mixing of order k if

$$\mu(A_0\cap T^{-n_1}A_1\cap\ldots\cap T^{-n_k}A_k)\xrightarrow[\substack{n_1\to\infty\\ n_2-n_1\to\infty\\ \vdots\\ n_k-n_{k-1}\to\infty}]{}\mu(A_0)\mu(A_1)\cdot\ldots\cdot\mu(A_k)$$

$\forall A_0,\ldots,A_k\subset X$ measur.

08

Rem. Mixing = mixing of order 1.

Open question (Furstenberg ~1970)

Mixing $\overset{?}{\Rightarrow}$ mixing of all orders

Rem. 1) Even $\mu(A\cap T^{-n}B\cap T^{-2n}C)\longrightarrow\mu(A)\mu(B)\mu(C)$ is open for mixing systems

2) We will see later: true for weak mixing (and w. mixing of all orders) for $n, 2n, \ldots, kn$ $\forall k$.

So: mixing : natural and very difficult

weak mixing: unnatural but easy to check

$$\begin{tikzcd}
\underset{(X,\mu,T)}{\text{MDS}} \arrow[dr, shift left] \arrow[rr, shift right, "\text{„topological models"}"'] & & \underset{(X,T)}{\text{TDS}} \arrow[ll, shift right, "\text{Krylov-Bogolyubov}"'] \arrow[dl, shift right] \\
& \text{Koopman oper. } T \text{ on } L^2(X,\mu) \text{ or } C(X) \arrow[ul, shift left] \arrow[ur, shift right] &
\end{tikzcd}$$

„topological models": $\forall$ MDS is (meas. theor.) isomorphic to a TDS with some T-inv. prob. measure.

Jewett-Krieger thm: MDS erg. $\Rightarrow \exists$ TDS uniquely erg.

II Multiple ergodic thms and application to combinatorics

5. Motivation: Szemerédi's thm

We study the following property of a set $C\subset\mathbb{N}$:

(AP) $\forall k$ $\exists \underset{\text{beginning}}{a}, \underset{\text{step}}{n}$: $\underbrace{a, a+n, \ldots, a+kn}_{k+1 \text{ is length}} \in C$ $\quad (AP_k)$ $\quad \otimes \cdot \otimes \times \otimes \cdot \otimes \times \cdot \,..$

- existence of arbitr. long arithm. progressions.

$\boxed{\text{Thm 5.1}}$ (Van der Waerden, 1927)

$\mathbb{N} = C_1 \cup \dots \cup C_m \Rightarrow \exists j : C_j$ has (AP)

($\mathbb{N}$ coloured with finitely many colours $\Rightarrow \exists$ $\underbrace{\text{monochromatic}}_{\text{of the same colour}}$ arithm. progr. of arb. length)

$\boxed{\text{Ex}}$ (No ∞-long $\underset{\text{arithm. progr.}}{\text{AP}}$ in general)

$\underbrace{1, \dots, 10}_{C_1}, \underbrace{11, \dots, 100}_{C_2}, \underbrace{101 \dots 1000}_{C_1}, \dots$

Rem. Thm 1 is an example of a result in Ramsey theory: $\mathbb{N}$ has structure ($\exists$ arb long AP), coloured with finitely many colours $\Rightarrow$ $\exists$ colour with the same structure

$\boxed{\text{Thm 5.2}}$ (Szemerédi, 1975)

$\forall C \subset \mathbb{N}$ with $\bar{d}(C) > 0$ satisfies (AP)

(C „not too small" $\Rightarrow$ C has structure)

Recall: $\bar{d}(C) := \overline{\lim_{N \to \infty}} \frac{|C \cap \{1, \dots, N\}|}{N}$

Rem. 1) Szemerédi $\Rightarrow$ van der Waerden

$\mathbb{N} = C_1 \cup \dots \cup C_m \Rightarrow \exists C_j$ with $\bar{d}(C_j) > 0$

exerc.: $\bar{d}(C_1 \cup C_2) \leq \bar{d}(C_1) + \bar{d}(C_2)$

2) Szemerédi solved a conjecture of Erdös, Turan from 1936

3) Green and Tao showed in 2004 that $\underset{\text{primes}}{\mathbb{P}}$ satisfy (AP)

$\exists$ several proofs of Szemerédi:

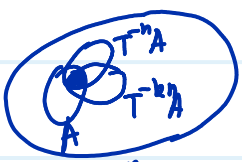
- Furstenberg 1977 - erg theory proof
- Gowers 2002 - Fourier analysis proof („higher order Fourier analysis")
- Gowers + 4, 2004 - graph theory proof
- ⋮

Story not finished yet?

Furstenberg: translation into erg. theoretic language + solution of the translation.

$\boxed{\text{Thm 5.3}}$ (Furstenberg, multiple recurrence 1977)

Let (X, μ, T) be an MDS, $A \subset X$ be with $\mu(A) > 0$. Then



$$\forall k \quad \underline{\lim}_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) > 0$$

'multiple recurrence

Rem. 1) Furstenberg $\Leftrightarrow$ Szemerédi, but we need only $\Rightarrow$

2) $\exists n : \mu(A \cap \dots \cap T^{-kn}A) > 0$ already suffices for Szemerédi

6. Furstenberg's correspondence principle

$\boxed{\text{Prop. 6.1}}$ Multiple recurrence (for a fixed k) $\Rightarrow$ Szemerédi (for this k)

Tools from functional analysis

1) Weak* topology / convergence on E', E Banach space:

$$\varphi_n \xrightarrow{\text{weak}^*} \varphi \overset{\text{def}}{\iff} \varphi_n(x) \to \varphi(x) \quad \forall x \in E$$

(n ↝ for nets)

Subbasis of top.: $U_{\varphi_0, x_0, \varepsilon} := \{\varphi : |\varphi(x_0) - \varphi_0(x_0)| < \varepsilon\}$

2) $(\ell^1)' = \ell^\infty$ and weak* conv. on ℓ^∞ is coordinate wise conv.:

$$(t_n) \xrightarrow{\text{weak}^*} t \iff (t_n)_j \to t_j \;\; \forall j \quad \text{exer.} \quad t_j = \langle \underset{\ell^\infty = (\ell^1)'}{t}, \underset{\in \ell^1}{e_j} \rangle$$

3) $(C(K))' = \mathcal{M}(K)$ – Borel measures on K with (K comp)

$$\varphi(f) = \int f \, d\mu \quad (\text{Riesz' thm})$$

Weak* convergence corresp. to conv. on sets:

$$\mu_n \xrightarrow{\text{weak}^*} \mu \iff \mu_n(A) \to \mu(A) \quad \forall A \text{ Borel}$$

exerc. $\mu(A) = \int \mathbb{1}_A \, d\mu$, then approx.

4) Thm of Banach–Alaoglu. The closed unit ball in E' is weak* comp. in part., if $\underbrace{E \text{ is separable}}_{\text{true for us}}$, then $\forall$ bdd seq. in E' has a weak*-conv. subseq.

Proof of Prop. 6.1: Let $C \subset \mathbb{N}$ with $\bar{d}(C) > 0$. To construct: (X, μ, T), A with $\mu(A) > 0$

$$C \rightsquigarrow \mathbb{1}_C \in \ell^\infty(\mathbb{Z}) \qquad \{1,2,5,\dots\} \rightsquigarrow$$

Let $T: \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be the left shift

$$T((t_j)_{j=-\infty}^{\infty}) := (t_{j+1})_{j=-\infty}^{\infty}$$

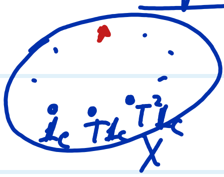
Def. $$X := \overline{\{T^k \mathbb{1}_C, \, k \in \mathbb{Z}\}}^{\text{weak}^*} \subset \ell^\infty(\mathbb{Z})$$

- $\forall x \in X$ is a 0-1-seq. (since coordinate wise limits of such)
- X is weak* comp. (since closed in the comp. unit ball of $\ell^\infty(\mathbb{Z}) = (\ell^1(\mathbb{Z}))'$ – Banach-Alaoglu)

Let $T \circlearrowleft$ on X (invertible)

Consider $A := \{(t_j) \in X : t_0 = 1\}$ – cylinder set.

Properties of A:



- A is closed and open $(X \setminus A = \{(t_j) \in X : t_0 = 0\})$
- $\boxed{T^m \mathbb{1}_C \in A \iff m \in C}$:

$$T^m \mathbb{1}_C \in A \iff (\mathbb{1}_C)_m = 1 \iff m \in C$$

- C satisf. $(AP_k) \iff \exists a, n : \underbrace{a, a+n, \dots, a+kn \in C}$

$$\iff \exists a, n : \underbrace{T^a \mathbb{1}_C}_{\text{such elements are dense}} \in \underbrace{A \cap T^{-n}(A) \cap \dots \cap T^{-kn}(A)}_{\text{open}} \qquad T^a \mathbb{1}_C \in A, \dots, T^{a+kn} \mathbb{1}_C \in A$$

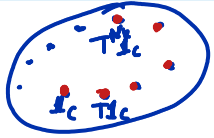
$$\iff \exists n : A \cap T^{-n}A \cap \dots \cap T^{-kn}A \neq \emptyset$$

in part., for (AP_k) it suffices to find μ on X (prob., T-inv.) s.t. $\mu(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) > 0$ for some n.

Construction of μ Positive upper density: $\exists N_k \nearrow \infty$ s.t.

$$\underbrace{\frac{|C \cap \{1, \dots, N_k\}|}{N_k}}_{=: (*)_k} \to \bar{d}(C) > 0$$

$$(*)_k = \frac{1}{N_k} \sum_{n=1}^{N_k} \underset{n \in C \iff T^n \mathbb{1}_C \in A}{\mathbb{1}_C(n)} = \underbrace{\frac{1}{N_k} \sum_{n=1}^{N_k} \delta_{\{T^n \mathbb{1}_C\}}}_{=: \mu_k}(A) \text{ – Borel prob. meas. on } X \; (\in \text{unit ball of } C(X)')$$



Banach–Alaoglu:

$\exists\, \mu_{k_j} \to \mu$ for some $\mu \in \mathcal{M}(X)$

Properties of μ

- μ is a prob measure ($\mu \geq 0 \Leftrightarrow \int f\, d\mu \geq 0 \;\forall f \in C(X)$ positive; $\mu(X) = 1$ by $\mu_{k_j}(X) = \int 1\, d\mu_{k_j} = 1$, $\mu_{k_j}(X) \to \mu(X)$)
- μ is T-inv.:

$$|\mu_k(T^{-1}B) - \mu_k(B)| = \frac{1}{N_k}\left|\sum_{n=1}^{N_k} \left(\delta_{\{T^{n+1}1_c\}}(B) - \delta_{\{T^n 1_c\}}(B)\right)\right|$$

$$= \frac{1}{N_k}\left|\delta_{\{T^{N_k+1}1_c\}}(B) - \delta_{\{T1_c\}}(B)\right| \xrightarrow[j\to\infty]{} 0,$$

so $\mu(T^{-1}B) = \mu(B)$

- $\mu(A) = \lim_{j\to\infty} \mu_{k_j}(A) = \bar{d}(C) > 0$

Thus mult recurrence of order k for (X,μ,T) implies

$$\mu(A \cap T^{-n}A \cap \ldots \cap T^{-kn}A) > 0$$

for many n ($\lim \frac{1}{N}\sum_{n=1}^{N} \ldots > 0$), and in part., $A \cap T^{-n}A \cap \ldots \cap T^{-kn}A \neq \emptyset$ ■

Reformulation of mult. recurrence

Let (X,μ,T) be an MDS and $f > 0$. Then $\forall k$ ($f \geq 0$, $f \not\equiv 0$ (for example, 1_A for A with $\mu(A) > 0$))

$$(MR_k) \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int f \cdot T^n f \cdot \ldots \cdot T^{kn} f\, d\mu > 0$$

mult. recurrence of order k

for $f = 1_A$ this is $\mu(A \cap T^{-n}A \cap \ldots \cap T^{-kn}A)$

Idea

- Prove it for "easy" systems/fcts
- Show that an arbitrary system/fct is a "combination" of "easy" ones

Rem. The limit in (MR_k) exists:

Host–Kra, 2005: $\forall f_1, \ldots, f_k$ bdd

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot \ldots \cdot T^{kn} f_k$$ – multiple convergence (of order k)

conv. in L^2 ($k=1$: von Neumann's erg. thm)

(take $f_1 = \ldots = f_k = f$ and scalar product with f)

7. Reduction to erg. measures and invert. transf.

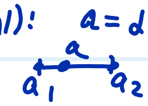
We first show that μ in Furstenberg's construction can be chosen to be ergodic.

$\boxed{\text{Thm. 7.1}}$ (Krein–Milman)

For $M \subset E'$ (E Banach), M convex and weak*-closed,

$$M = \overline{\mathrm{conv}}^{\,weak^*}\{\text{extremal points of } M\}$$

Here, $a \in M$ is extremal if $\nexists\, a_1, a_2 \in M$, $a_1 \neq a_2$, $\nexists\, \alpha \in (0,1)$: $a = \alpha a_1 + (1-\alpha)a_2$



We apply Krein–Milman to

$$\mathcal{M}_T(X) := \{\text{all } T\text{-inv. prob. measures on } X\}$$

for a TDS (X,T)

- $\mathcal{M}_T(X) \subset$ unit ball of $(C(X))'$
- $\mathcal{M}_T(X)$ is convex: μ_1,μ_2 T-inv., prob meas., $\lambda \in (0,1)$, then $\lambda \underbrace{\mu_1(T^{-1}A)}_{\mu_1(A)} + (1-\lambda)\underbrace{\mu_2(T^{-1}A)}_{\mu_2(A)} = \lambda\mu_1(A) + (1-\lambda)\mu_2(A)$
- $\mathcal{M}_T(X)$ is weak*-closed, so weak* comp. by Banach-Alaoglu

$\mu_n \xrightarrow{weak^*} \mu \iff \int f\,d\mu_n \to \int f\,d\mu \quad \forall f \in C(X)$

$$\int f\,d\mu_n = \int Tf\,d\mu_n \to \int Tf\,d\mu,$$

so μ is T-inv.

So we can apply Krein-Milman. But what are the extremal points?

Prop. 7.2 μ is an extremal point of $\mathcal{M}_T(X)$
$\iff (X,\mu,T)$ is erg.

Proof ($\Rightarrow$) Assume, μ is not erg.: $\exists B$: $T^{-1}B = B$ up to a null set
$T^{-1}(X\setminus B) = X\setminus B$ —"—
with $\mu(B) \in (0,1)$

Def. μ_1,μ_2 by

$$\left.\begin{aligned} \mu_1(A) &:= \frac{\mu(A\cap B)}{\mu(B)} \\ \mu_2(A) &:= \frac{\mu(A\cap(X\setminus B))}{\mu(X\setminus B)} \end{aligned}\right\} \text{prob. meas. on } X$$

We have:

- μ_1,μ_2 are T-inv.:
$$\mu_1(T^{-1}A) = \frac{\mu(T^{-1}A \cap B)}{\mu(B)} \overset{T^{-1}B \text{ up to null set}}{=} \frac{\mu(\cancel{T^{-1}}(A\cap B))}{\mu(B)} = \mu_1(A)$$
The same for μ_2.
- $\mu(A) = \mu(A\cap B) + \mu(A \cap (X\setminus B)) = \mu(B)\mu_1(A) + \underbrace{\mu(X\setminus B)}_{1-\mu(B)}\mu_2(A)$,
so $\mu = \mu(B)\mu_1 + (1-\mu(B))\mu_2$ ↯ (μ not extremal)

($\Leftarrow$) Let μ be erg., i.e.,
$$\frac{1}{N}\sum_{n=1}^{N} T^n f \xrightarrow{\|\cdot\|_{L^2(\mu)}} \int f\,d\mu \cdot \mathbb{1} \quad \forall f \in C(X) \quad (*)$$

Assume $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for some $\mu_1,\mu_2 \in \mathcal{M}_T(X)$, $\lambda \in (0,1)$.

Then $\mu_1 \le \frac{1}{\lambda}\mu$ and
$$\|g\|_{L^2(\mu_1)} = \left(\int |g|^2\,d\mu_1\right)^{1/2} \le \frac{1}{\sqrt{\lambda}}\cdot\left(\int |g|^2\,d\mu\right)^{1/2} = \frac{1}{\sqrt{\lambda}}\|g\|_{L^2(\mu)},$$
i.e., conv. w.r.t. $\|\cdot\|_{L^2(\mu)}$ implies conv. w.r.t. $\|\cdot\|_{L^2(\mu_1)}$.

By $(*)$ this implies
$$\int f\,d\mu_1 \overset{\mu_1 \text{ T-inv.}}{=} \underbrace{\int \left(\frac{1}{N}\sum_{n=1}^{N} T^n f\right) d\mu_1}_{\langle \frac{1}{N}\sum T^n f, \mathbb{1}\rangle_{L^2(\mu_1)}} \xrightarrow{(*)} \int f\,d\mu \quad \forall f \in C(X),$$
so $\mu_1 = \mu$ and then also $\mu_2 = \mu$, so μ is extr. point ■

Recall: Furstenberg's correspondence principle:
$$X = \overline{\{T^k \mathbb{1}_C,\ k \in \mathbb{Z}\}}^{weak^*}, \quad T \circlearrowleft, \quad A = \{x \in X : x_0 = 1\}$$
μ: $\mu(A) = \bar{d}(C) > 0$

Prop. 7.2 + Krein-Milman: $\mu \in \overline{\text{conv}}^{weak^*}\{\nu \text{ T-inv. prob., ergodic}\}$

So: $\exists \nu \in \mathcal{M}_T(X)$ erg.:
$$\nu(A) \ge \frac{\mu(A)}{2} = \frac{\bar{d}(C)}{2} > 0$$
(if not, then $\nu(A) \le \frac{\mu(A)}{2}$ $\forall$ erg. ν, so $\forall$ conv. comb. and limit, so $\mu(A) \le \frac{\mu(A)}{2}$ ↯)

So, if we show Furstenberg mult. recurrence for erg. systems, then $\exists$ erg. ν: $\nu(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) > 0$, so Szemerédi.

<u>We will assume: μ is erg.</u> (But mult. recurrence does hold for arb systems)

<u>Reduction to invertible transf./systems</u>

Recall: (X,μ,T) in Furstenberg's corresp. principle is invertible and it suffices to prove MR for ergodic invert. systems.

$\exists$ way to show that MR (mult. recurrence) for invert. (erg) systems $\Rightarrow$ MR for all (erg) systems

Let (X,μ,T) be an MDS.

<u>Claim</u> $\exists$ an <u>invertible extension</u> $(\tilde{X},\tilde{\mu},\tilde{T})$, i.e., invertible system s.t. $\exists \pi: \tilde{X} \to X$ meas. pres. s.t. $\pi\tilde{T} = T\pi$

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{T}"] \arrow[d, "\pi"'] & \tilde{X} \arrow[d, "\pi"] \\ X \arrow[r, "T"] & X \end{tikzcd}$$

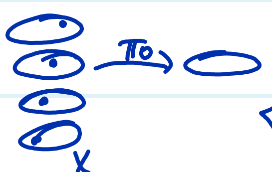
<u>Proof</u> (under assumption $TX = X$)

Consider

- $\tilde{X} := \{(x_k) \in X^{\mathbb{Z}} : x_{k+1} = Tx_k \ \forall k \in \mathbb{Z}\}$ – „two-sided orbits" $\exists$ by $TX = X$
- $\tilde{T} \leftarrow$ left shift $\tilde{T}((x_k)) = (x_{k+1})$
- $\tilde{\Sigma}$ the smallest σ-algebra s.t. the projections $\pi_n: \tilde{X} \to X$, $\pi_n((x_k)) := x_n$ are measurable $\forall n \leq 0$
- ν is def. by
  $$\nu(\{x : x_{n_1} \in A_1, \dots, x_{n_k} \in A_k\}) := \mu(T^{-n_1}A_1 \cap \dots \cap T^{-n_k}A_k)$$
  for $n_1 < \dots < n_k \leq 0$ $\forall A_1, \dots, A_k \subset X$ meas.
  – extend to $\tilde{\Sigma}$ by Carathéodory's thm

Exerc Show:

- $\tilde{\Sigma}$ is invariant under the right shift
- $(\tilde{X},\tilde{\mu},\tilde{T})$ is an extension of (X,μ,T) for $\pi := \pi_0$



This extension is called natural invertible extension of (X,μ,T).

<u>Rem.</u> if (X,μ,T) is erg., then also $(\tilde{X},\tilde{\mu},\tilde{T})$ is erg. (no proof)

exerc. MR for a system $\Rightarrow$ MR for each factor, so MR for invert. (erg.) systems $\Rightarrow$ MR for all (erg.) systems (use invert. ext.)

8. A warm up: polynomial convergence and recurrence

$\boxed{\text{Thm } 8.1}$ (<u>Furstenberg–Sárközy</u> 1977)

Let $C \subset \mathbb{N}$ with $\bar{d}(C) > 0$, $p \in \mathbb{Z}[\cdot]$ with $p(0) = 0$. Then
$$\exists a, n: \ a, a + p(n) \in C$$
(e.g., $a, a+n^2 \in C$)

$\boxed{\text{Ex}}$ (<u>$p(0) = 0$ cannot be dropped</u>)

$C := 2\mathbb{N}$, $d(C) = \frac{1}{2}$, $p(n) := 2n+1$. Then
$$\nexists a, n: \ a, a+2n+1 \in 2\mathbb{N}$$

A small modification of Furstenberg's corresp. principle implies (exerc) that Thm 8.1 follows from

$\boxed{\text{Thm 8.2}}$ Let (X,μ,T) be an MDS, $A\subset X$ with $\mu(A)>0$, $p\in\mathbb{Z}[\cdot]$ with $p(0)=0$. Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-p(n)}A)>0$$

Considering $f:=1_A$: $\mu(A\cap T^{-p(n)}A)=\int T^{p(n)}f\cdot f\,d\mu=\langle T^{p(n)}f,f\rangle$

We show:

$\boxed{\text{Thm 8.3}}$ Let $p\in\mathbb{Z}[\cdot]$, (X,μ,T) be an MDS.

1) (Polynomial convergence)
$\forall f\in L^2$ $\frac{1}{N}\sum_{n=1}^{N}T^{p(n)}f$ conv. in L^2 (here we assume T to be invert. if $p(n)<0$ for ∞-many n)

2) (Polynomial recurrence)
if $p(0)=0$, $f\geq 0$, then

$$\underbrace{\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle T^{p(n)}f,f\rangle}_{\text{correlation of the limit in 1 with } f}>0$$

Rem. 2) $\Rightarrow$ Furstenberg–Sarközy (exerc.)

Proof: polyn. convergence

WLOG T invert. (exerc: use invert. extension)

Take $H:=\underbrace{\overline{\text{lin}}\{T^nf:n\in\mathbb{Z}\}}_{\text{is } T\text{-invariant}}$. Then f is a cyclic vector for $T|_H$,

so by the spectral thm

$\exists$ spectral measure ν on $\mathbb{T}$

s.t. $T^nf \overset{\text{isom}}{\sim}$ multiplicator with $\lambda\mapsto\lambda^n$ on $L^2(\mathbb{T},\nu)$

$f\sim 1$.

So conv. of $\frac{1}{N}\sum_{n=1}^{N}T^{p(n)}f$ in $L^2(X,\mu)$ corresp. to conv. of $\lambda\mapsto\frac{1}{N}\sum_{n=1}^{N}\lambda^{p(n)}$ in $L^2(\mathbb{T},\nu)$

Case 1 λ irrat.: $\lambda^n\neq 1$ $\forall n\neq 0$

$\boxed{\text{Thm}}$ (Weyl) $(\lambda^{p(n)})$ is equidistr. in $\mathbb{T}$ $\forall$ irrat. $\lambda\in\mathbb{T}$ $\forall p\in\mathbb{Z}[\cdot]$

Proof comes soon, sorry (we had it for $p(n)=n$)

Assume it.

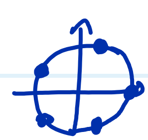
Recall: Weyl's equidistr. criterion:

(λ_n) is equid. in $\mathbb{T}$ $\Leftrightarrow$ $\frac{1}{N}\sum_{n=1}^{N}f(\lambda_n)\to\int f$ $\forall f\in C(\mathbb{T})$ (for Lebesgue measure)

$\Leftrightarrow$ $\frac{1}{N}\sum_{n=1}^{N}\lambda_n^k\to 0$ $\forall k\in\mathbb{Z}$, $k\neq 0$

So in part. $\frac{1}{N}\sum_{n=1}^{N}\lambda^{p(n)}\to 0$ (take $k:=1$)

Case 2 λ rat., say $\lambda^a=1$.

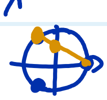


Observe: $p(n+a)=p(n)+a\cdot\underbrace{\text{something}}_{\text{dep. on } n}$,

$(n+a)^k=n^k+a\cdot\text{something}$

so $\lambda^{p(n)}$ is a-periodic by

$$\lambda^{p(n+a)}=\lambda^{p(n)}\cdot(\underbrace{\lambda^a}_{1})^{\text{something}}$$

$\boxed{\text{Ex}}$ $a=3$, $p(n)=n^2$  n^2 mod 3: $1,1,0,1,1,0,1,1,0\ldots$

So we have:

$$\frac{1}{N}\sum_{n=1}^{N}\lambda^{p(n)} = [N = ka+r] = \underbrace{\frac{1}{ka+r}\sum_{n=1}^{ka}\lambda^{p(n)}}_{\frac{k}{ka+r}\sum_{n=1}^{a}\lambda^{p(n)}} + \underbrace{\frac{1}{N}\sum_{n=ka+1}^{ka+r}\lambda^{p(n)}}_{|\cdot|\le a}$$

$r \in \{0,\dots,a-1\}$, $\quad \frac{k}{ka+r} \xrightarrow{N\to\infty} \frac{1}{a}$, $\quad \frac{1}{N}\sum_{n=ka+1}^{ka+r}\lambda^{p(n)} \xrightarrow{N\to\infty} 0$

$$\xrightarrow{N\to\infty} \frac{1}{a}\sum_{n=1}^{a}\lambda^{p(n)} =: c(\lambda)$$

Lebesgue dominated conv.:

$$\frac{1}{N}\sum_{n=1}^{N}\lambda^{p(n)} \to g \quad \text{in } L^2(\mathbb{T},\nu),$$

where

$$g(\lambda) := \begin{cases} c(\lambda) & \text{if } \lambda \text{ rat. and } \underline{\text{atom}} \text{ of } \nu \\ 0 & \text{otherwise} \end{cases}$$

Domin. conv.: $g_n \to g$ a.e., $|g_n| \le h \in L^1$ $\Rightarrow$ $g_n \xrightarrow{\|\cdot\|_1} g$, and for unif. bdd fcts: (recall: $f\sim\mathbb{1}$) $\|g_n - g\|_2^2 = \int |g_n-g|^2 \le \int |g_n - g| \cdot \underbrace{\|g_n-g\|_\infty}_{\text{bdd}} \to 0$ ■

Rem. in particular, $\lim \frac{1}{N}\sum_{1}^{N} T^{p(n)} f = 0$ if the spectral measure ν does not have any rational atoms, i.e., if $T|_H$ does not have any rat. eigenvalues

atom λ of ν $\Longleftrightarrow$ $\mathbb{1}_{\{\lambda\}}$ eigenfct of M_z

(For ex., if $f \perp \mathbb{1}$ and T is w. mixing – or, in general, if $f \perp$ all eigenfcts to rat. eigenvalues)

Proof: pol. recurrence To show: if $f > 0$, then

$$\left\langle \lim \frac{1}{N}\sum_{1}^{N} T^{p(n)} f, f \right\rangle = \lim \frac{1}{N}\sum_{n=1}^{N} \langle T^{p(n)} f, f\rangle > 0$$

– positive correlation with f