Proof of double convergence (in Thm 10.1)

To show: $\frac{1}{N}\sum_{n=1}^{N} T^n f \cdot T^{2n} g$ conv. in $L^2(X,\mu)$.

Decompose $f = f_{kr} + f_{wmix}$

$g = g_{kr} + g_{wmix}$.

By the above, we can assume $f = f_{kr}$, $g = g_{kr}$ (for f_{wmix} or g_{wmix} $\lim = 0$)

Density argument + linearity. WLOG $Tf = \lambda_1 f$, $Tg = \lambda_2 g$. Then

$$\frac{1}{N}\sum_{n=1}^{N} T^n f \cdot T^{2n} g = \underbrace{\frac{1}{N}\sum_{n=1}^{N} (\lambda_1 \lambda_2^2)^n}_{\text{conv. to } \begin{cases} 1 & \text{if } \lambda_1 \cdot \lambda_2^2 = 1 \\ 0 & \text{otherwise} \end{cases}} f \cdot g$$

conv. ∎

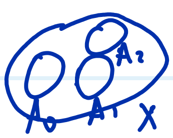
Rem. in particular,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f \cdot T^{2n} g = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f_{kr} \cdot T^{2n} g_{kr}$$

One says: the Kronecker part is characteristic for double averages.

$\boxed{\text{Thm } 10.5}$ (Weak mixing implies weak mixing of all orders)

Let (X,μ,T) be weakly mixing. Then (X,μ,T) is weakly mixing of all orders, i.e., $\forall k \geq 1$ $\forall A_0, \ldots, A_k$ measurable



$$\frac{1}{N}\sum_{n=1}^{N} \left|\mu(A_0 \cap T^{-n}(A_1) \cap \ldots \cap T^{-kn} A_k) - \mu(A_0) \cdots \mu(A_k)\right| \xrightarrow[N\to\infty]{} 0$$

in part., multiple recurrence holds for weakly mixing systems.

Proof As before it suffices to show: $\forall f_0, \ldots, f_k \in L^\infty(X,\mu)$

$$\frac{1}{N}\sum_{n=1}^{N} \left|\int f_0 \cdot T^n f_1 \cdots T^{kn} f_k - \int f_0 \cdots \int f_k\right| \xrightarrow[N\to\infty]{} 0.$$

(take $f_0 := 1_{A_0}, \ldots, f_k := 1_{A_k}$)

Step 1 We show first that

$$\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdots T^{kn} f_k \xrightarrow[N\to\infty]{} \int f_1 \cdots \int f_k \, 1 \text{ in } L^2.$$

induction on k:

$k=1$: von Neumann's mean erg. thm. + ergodicity ($P_{Fix} f = \int f \, 1$)
recall: weak mixing $\Rightarrow$ ergodicity

$k-1 \to k$ Decompose $f_1 = c1 + \tilde{f}_1$, $\tilde{f}_1 \perp 1$.

For $f_1 = c \cdot 1$.

$$\Big\| \underbrace{c \cdot \frac{1}{N}\sum_{n=1}^{N} T^{2n} f_2 \cdots T^{kn} f_k}_{= T^n\left(c \frac{1}{N}\sum_{n=1}^{N} T^n f_2 \cdots T^{(k-1)n} f_k\right)} - \underbrace{c \int f_2 \cdots \int f_k 1}_{= T^n(\ldots) \text{ since } T1 = 1} \Big\| \underset{T \text{ isom.}}{=} |c| \cdot \Big\| \frac{1}{N}\sum_{n=1}^{N} T^n f_2 \cdots T^{(k-1)n} f_k - \int f_2 \cdots \int f_k 1 \Big\|$$

$\xrightarrow[N\to\infty]{} 0$ by induction hypothesis

So we can assume WLOG $f_1 \perp 1$, i.e., $\int f_1 = 0$.

Recall: T w. mixing $\Rightarrow L^2 = \langle 1 \rangle \oplus H_{wmix}$, ($H_{kr} = \langle 1 \rangle$ – 1 simple eigenvalue no other eigenvalues)
so f_1 is weakly mixing.
idea: use van der Corput, so def.

$$u_n := T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k$$

Compute

$$\langle u_{n+h}, u_n \rangle = \int T^{\cancel{n}}(T^h f_1 \cdot \bar{f}_1) \cdots T^{\cancel{kn}\,(k-1)n}(T^{kh} f_k \cdot \bar{f}_k)\, d\mu$$

$$\underset{T\ \mu\text{-pres.}}{=} \int (T^h f_1 \bar{f}_1) \cdot \underbrace{T^n(T^{2h} f_2 \cdot \bar{f}_2) \cdots T^{(k-1)n}(T^{kh} f_k \cdot \bar{f}_k)}_{\text{as in the induction hypothesis}}\, d\mu$$

So by induction hypothesis:

$$\Big|\frac{1}{N}\sum_{n=1}^N \langle u_{n+h}, u_n \rangle\Big| \xrightarrow[N\to\infty]{} |\langle T^h f_1, f_1 \rangle|\, |\langle T^{2h} f_2, f_2 \rangle| \cdots |\langle T^{kh} f_k, f_k \rangle| =: \gamma_h$$

We have

$$\frac{1}{N}\sum_{h=1}^N \gamma_h \underset{CS}{\leq} \underbrace{\frac{1}{N}\sum_{h=1}^N |\langle T^h f_1, f_1 \rangle|}_{\to 0 \text{ since } f_1 \text{ is w.mixing}} \|f_2\|_2^2 \cdots \|f_k\|_2^2 \xrightarrow[N\to\infty]{} 0.$$

VdC: $\frac{1}{N}\sum_{n=1}^N u_n \to 0$ in L^2.

<u>Step 2</u> We now show

$$\frac{1}{N}\sum_{n=1}^N \Big| \int f_0 \cdot T^n f_1 \cdots T^{kn} f_k \Big|^2 \to \Big|\int f_0\Big|^2 \cdots \Big|\int f_k\Big|^2$$

indeed,

$$\Big|\int_X f_0 \cdots T^{kn} f_k\Big|^2 \underset{|z|^2 = z\cdot\bar{z}}{=} \int_X f_0 \cdots T^{kn} f_k \cdot \int_X \bar{f}_0 \cdots T^{kn} \bar{f}_k$$

$$= \int_{X\times X} (f_0 \otimes \bar{f}_0)\,(T\times T)^n (f_1 \otimes \bar{f}_1) \cdots (T\times T)^{kn} (f_k \otimes \bar{f}_k)$$

recall: $(f\otimes g)(x,y) = f(x)\cdot g(y)$

$$\xrightarrow{\text{Cesàro}}_{\text{step 1}} \int_{X\times X} f_0 \otimes \bar{f}_0 \cdots \int_{X\times X} f_k \otimes \bar{f}_k = \Big|\int f_0\Big|^2 \cdots \Big|\int f_k\Big|^2$$

if we show that $(X\times X, \mu\times\mu, T\times T)$ is also weakly mixing. This is the case by

$$(\mu\times\mu)\big((T\times T)^{-n}\underbrace{(A_1\times A_2)\cap(B_1\times B_2)}_{\text{such sets induce the product } \sigma\text{-algebra on } X\times X}\big) = (\mu\times\mu)\big((T^{-n}A_1 \cap B_1)\times(T^{-n}A_2 \cap B_2)\big)$$

$$= \mu(T^{-n}A_1 \cap B_1)\cdot \mu(T^{-n}A_2 \cap B_2)$$

$$\xrightarrow[\substack{n\to\infty \\ n\in J}]{} \mu(A_1)\,\mu(B_1)\cdot\mu(A_2)\cdot\mu(B_2)$$

density 1 (intersection of two density 1 sets)

$$= (\mu\times\mu)(A_1\times B_1)\cdot(\mu\times\mu)(A_2\cdot B_2).$$

<u>Step 3</u> it suffices to show that $\forall (a_n) \subset \mathbb{C}$ $\forall a \in \mathbb{C}$

$$\left.\begin{aligned} &\frac{1}{N}\sum_{n=1}^N a_n \to a \\ &\frac{1}{N}\sum_{n=1}^N |a_n|^2 \to |a|^2 \end{aligned}\right\} \Rightarrow \frac{1}{N}\sum_{n=1}^N |a_n - a| \to 0.$$

Since $t \mapsto t^2$ is convex, we have

$$\Big(\tfrac{1}{N}\sum_{n=1}^{N}|a_n-a|\Big)^2 \underset{\text{convexity}}{\le} \tfrac{1}{N}\sum_{n=1}^{N}|a_n-a|^2 \underset{|z|^2=z\bar z}{=} \tfrac{1}{N}\sum_{n=1}^{N}(a_n-a)(\bar a_n-\bar a)$$

$$= \underbrace{\tfrac{1}{N}\sum_{n=1}^{N}|a_n|^2}_{\to|a|^2} - a\underbrace{\tfrac{1}{N}\sum_{n=1}^{N}\bar a_n}_{\to\bar a} - \bar a\underbrace{\tfrac{1}{N}\sum_{n=1}^{N}a_n}_{\to a} + \underbrace{\tfrac{1}{N}\sum_{n=1}^{N}|a|^2}_{=|a|^2}$$

$$\xrightarrow[N\to\infty]{} \cancel{|a|^2} - \cancel{a\bar a} - \cancel{\bar a a} + \cancel{|a|^2} = 0$$

Steps 1+2+3 together:

$$\tfrac{1}{N}\sum_{n=1}^{N}\Big|\int f_0\cdots T^{kn}f_k - \int f_0\cdots\int f_k\Big| \xrightarrow[N\to\infty]{} 0$$

Multiple recurrence statement („In particular"):
let $f_0=\dots=f_k=f>0$, bdd. Then

$$\tfrac{1}{N}\sum_{n=1}^{N}\int f\cdot T^n f\cdots T^{kn}f \longrightarrow \Big(\int f\Big)^{k+1}>0 \quad \forall k.$$

(here we only need step 1) ∎

Rem. As mentioned above, the following question is open since 1949 (Rokhlin):

T mixing $\overset{?}{\Longrightarrow}$ $\forall k\ \forall A_0,\dots,A_k$ meas.
$$\mu(A_0\cap T^nA_1\cap\dots\cap T^{kn}A_k)\longrightarrow\mu(A_0)\cdots\mu(A_k)$$

(even for $k=2$ still open)

10.3. Kronecker part and double recurrence

it ~~remains to show~~ up to implication for Roth (= Szemerédi for 3-term arithm. progressions) that

$f>0$, f bdd $\Rightarrow$ $P_{kr}f>0$, bdd

$$\lim_{N\to\infty}\tfrac{1}{N}\sum_{n=1}^{N}\int f\cdot T^nf\,T^{2n}f\,d\mu>0$$

if $f>0$, f bdd, $f\in H_{kr}=\overline{\mathrm{lin}}\{\text{eigenfcts of } T \underbrace{\text{to eigenvalues in } \mathbb{T}}_{\text{automatic since } T \text{ isom}}\}$

Recall: $L^2(X,\mu)=H_{kr}\overset{\perp}{\oplus}H_{wmix}$

H_{wmix}: $f:\ \tfrac{1}{N}\sum_{n=1}^{N}T^nf\,T^{2n}g\to0$ in L^2 $\forall g$

Observation: if $Tf=\lambda f$, $\lambda\in\mathbb{T}$, then

$$\mathrm{Orb}(f):=\{T^nf,\ n\in\mathbb{N}_0\}=\{\lambda^n,\ n\in\mathbb{N}_0\}\cdot f\subset\mathbb{T}\cdot f.$$

$\mathbb{N}_0 = \{0,1,2,\dots\}$

in particular, $\mathrm{Orb}\,f$ is rel compact (closure is comp.) in $L^2(X,\mu)$.

$\boxed{\text{Def}}$ f is called almost periodic if $\overline{\mathrm{Orb}f}\subset L^2$ is compact.

$\boxed{\text{Prop. 10.6}}$ 1) f is almost periodic $\Longleftrightarrow$ $\forall\varepsilon>0$ the set

$$M_\varepsilon:=\{n:\ \|T^nf-f\|_2<\varepsilon\}\subset\mathbb{N}$$

is syndetic (or has bounded gaps), i.e.,

$\exists L \in \mathbb{N} : M_\varepsilon \cap [n, n+L] \neq \emptyset \quad \forall n \in \mathbb{N}$

dep. on ε (M_ε intersects every interval of length L)

2) $AP(X) := \{$almost periodic fcts$\}$ is a T-inv., closed lin subspace of $L^2(X,\mu)$.

Proof 1) $\Rightarrow$ Let f be alm. periodic, i.e., $Orb(f)$ is rel. compact. Let $\varepsilon > 0$ and cover $Orb(f)$ with ε-balls. These balls also cover $\overline{Orb(f)}$ (why?), so by compactness

$$\exists k \; \exists n_1, \dots, n_k : \forall n \; \underbrace{\|T^n f - T^{n_j} f\|}_{= \|T^{n-n_j} f - f\| \text{ if } n \geq n_j} < \varepsilon \text{ for some } j = j(n).$$

in particular $\forall n \geq \max\{n_1, \dots, n_k\}$

$n - n_1 \in M_\varepsilon$ or $n - n_2 \in M_\varepsilon$ .. or $n - n_k \in M_\varepsilon$.

So all gaps in M_ε are $\leq \max\{n_1, .., n_k\}$

$\Leftarrow$ Let $\varepsilon > 0$. We show: $\exists L \; \bigcup_{n=0}^{L} U_\varepsilon(T^n f) \supset Orb(f)$

Let L be the largest gap in M_ε ($\exists$ by assumption). Then

$$\forall n > L \quad n \in M_\varepsilon \text{ or } n-1 \in M_\varepsilon \text{ or} .. \; n - L \in M_\varepsilon,$$

i.e., $\|T^n f - f\| < \varepsilon$ or $\underbrace{\|T^{n-1} f - f\|}_{\|T^n f - Tf\|} < \varepsilon$ or ... $\underbrace{\|T^{n-L} f - f\|}_{\|T^n f - T^L f\|} < \varepsilon$,

so
$$Orb(f) \subset \bigcup_{n=0}^{L} U_\varepsilon(T^n f)$$

exerc. · In a complete metric space, a set is rel. compact $\Leftrightarrow$ $\forall \varepsilon > 0 \; \exists$ finitely many ε-balls covering this set.

For $\Leftarrow$ use that a set is rel. comp. $\Leftrightarrow$ $\forall$ sequence has a Cauchy subseq.

2) • T-inv. follows from 1): $Orb(Tf) \subset Orb(f)$ + def. or

$$\|T^n T f - Tf\| = \|T^n f - f\|,$$
(T isom)

so $M_\varepsilon(Tf) = M_\varepsilon(f)$

• lin. subspace: exerc. (use: comp. = $\forall$ seq. has a conv. subseq.)

$Orb(f_1 + f_2) = \{T^n f_1 + T^n f_2, \; n \in \mathbb{N}_0\}$

• closed: assume $f_j \to f$, $\forall f_j \in AP(X)$.

Let $\varepsilon > 0$ $\exists j$: $\|f - f_j\| < \varepsilon/3$. Observe:

$$\|T^n f - f\| \leq \|T^n f_j - f_j\| + \underbrace{\|T^n f_j - T^n f\|}_{< \varepsilon/3 \text{ (isometry)}} + \overbrace{\|f - f_j\|}^{< \varepsilon/3}$$

$$\leq \|T^n f_j - f_j\| + \tfrac{2}{3}\varepsilon,$$

so $\underbrace{M_{\varepsilon/3}(f_j)}_{\text{bdd gaps}} \subset M_\varepsilon(f)$, so $M_\varepsilon(f)$ has bdd gaps ■

Cor. 10.7 $\boxed{AP(X) = H_{kr}}$

Proof $\Leftarrow$ if f is EF (eigenfct): $Tf = \lambda^n f$, so

$\cancel{\mathrm{Orb} f} = \overline{\underbrace{\{\lambda^n, n \in \mathbb{N}_0\}}_{\subset \mathbb{T}\text{ comp.}}}\ f$ - comp.

Since $AP(X)$ is a closed lin. subspace, $H_{kr} \subset AP(X)$.

$\Rightarrow$ We know: $AP(X) \supset H_{kr}$. it suffices to show

$$\underbrace{AP(X) \cap H_{kr}^{\perp}}_{\text{closed lin. subspaces}} = \{0\}.$$

Assume $\exists f \neq 0$, $f \in AP(X)$ and $\underbrace{\text{weakly mixing}}_{f \in H_{kr}^{\perp}\ (\text{Jdl}6)}$.

WLOG assume $\|f\| = 1$.

Let $\varepsilon \in (0,1)$. Then $\exists L$: $M_\varepsilon := \{n : \|T^n f - f\|_2 < \varepsilon\}$ intersects $\forall$ interval of length L.

For $n \in M_\varepsilon$.

$$\langle T^n f, f\rangle = \underbrace{\langle T^n f - f, f\rangle}_{\substack{|\cdot| \le \varepsilon \\ (CS)}} + \underbrace{\|f\|^2}_{=1}, \text{ so } |\langle T^n f, f\rangle| \geq 1 - \varepsilon$$

So $\forall N$:

$$\frac{1}{LN}\sum_{n=1}^{LN} |\langle T^n f, f\rangle| = \frac{1}{LN}\left[\sum_{n=1}^{L} + \sum_{n=L+1}^{2L} + \dots + \sum_{n=LN-L+1}^{LN}\right]$$

$$\geq \frac{1}{L\cancel{N}} \cdot \cancel{N}(1-\varepsilon) \not\to 0 \quad ↯ \ (f \text{ not w. mixing}) \quad \blacksquare$$

$\boxed{\text{Prop. 10.8}}$ (**Multiple recurrence for almost periodic fcts**)

Let f be bdd, > 0, almost periodic. Then

$$\lim \frac{1}{N}\sum_{1}^{N} \int f \cdot T^n f \cdots T^{kn} f \, d\mu > 0 \quad \forall k.$$

Proof: We show it for $k=2$, rest analog. (exerc.)

WLOG $\|f\|_\infty < 1$. Let $\varepsilon > 0$. The set of n:

$$\|T^n f - f\|_2 < \varepsilon$$

is syndetic. Take such n. Then

$$\|T^{2n} f - f\| \leq \underbrace{\|T^{\cancel{2}n} f - \cancel{T^n} f\|}_{< \varepsilon\ (\text{isom.})} + \underbrace{\|T^n f - f\|}_{< \varepsilon} < 2\varepsilon.$$

So

$$\left|\int f\, T^n f \cdot T^{2n} f - \int f^3\right| \leq \left|\int f (T^n f - f) T^{2n} f\right| + \left|\int f^2 (T^{2n} f - f)\right|$$

$$\leq \|T^n f - f\|_2 + \|T^{2n} f - f\|_2 < 3\varepsilon.$$

Denote $\delta := \int f^3 d\mu > 0$. Choosing $\varepsilon < \frac{\delta}{6}$

$$\int f \cdot T^n f\, T^{2n} f > \delta - 3 \cdot \frac{\delta}{6} = \frac{\delta}{2}$$

for a syndetic set S of n. Take L: $S \cap \{n, \dots, n+L\} \neq \emptyset$ $\forall n$.

Then

$$\frac{1}{NL}\sum_{n=1}^{NL} \int f \cdot T^n f \cdot T^{2n} f > \frac{1}{\cancel{N}L} \cdot \cancel{N} \frac{\delta}{2} = \frac{\delta}{2L} \quad \forall N$$

exerc.: $\lim \frac{1}{N}\sum^{N} \int f \cdot T^n f \cdot T^{2n} f \geq \frac{\delta}{2L}$ $\blacksquare$

($\forall N$ belongs to some $[N_0 L, (N_0+1)L)$)

$\boxed{\text{Rem.}}$ 1) One can show: TFAE:

(i) $L^2(X) = H_{kr}$ (so $\forall f$ is alm periodic)

(ii) (X, μ, T) is isomorphic to a rotation on a compact Abelian metrisable group.

Such systems are called <u>compact</u>

2) We showed: $L^2 = \{\text{almost periodic fcts}\} \overset{\perp}{\oplus} \{\text{weakly mixing fcts}\}$,

so if (X, μ, T) is <u>not</u> w. mixing, then $\exists$ non-const. alm. per fcts,

or: $\exists$ <u>compact subsystem</u> $(X, \Sigma', \mu|_{\Sigma'}, T)$ (in which $\forall$ L^2-fct is alm. periodic)

T-inv. sub-σ-alg. generated by EFs (smallest s.t. $\forall$ EF is measurable)

This subsystem is called <u>Kronecker factor</u>

subsystem

The very last thing to show for Roth are the properties:

$$(*) \quad \begin{cases} f > 0 \implies P_{H_{kr}} f > 0 \\ f \in L^\infty \implies P_{H_{kr}} f \in L^\infty \end{cases}$$

Here one shows first

$$\boxed{P_{H_{kr}} f = \mathbb{E}(f \mid \Sigma')}$$

cond expect — the smallest T-inv. sub-σ-alg. s.t. $\forall$ EF is Σ'-meas.

given f, $\exists!$ Σ'-meas. g s.t. $\int_A f = \int_A g \quad \forall A \in \Sigma'$

Such Σ' exists $\forall$ subspace $Y \subset L^2$ of the form $Y = \overline{D}$ for some $D \subset L^\infty$ conj. invariant, T-inv. subalgebra of L^∞ containing $\mathbb{1}$.

$f, g \in D \Rightarrow f \cdot g \in D$

(For $Y = H_{kr}$, $D = \text{lin}\{\text{bdd eigenfcts}\}$)

Now, $(*)$ are clear:

$f \geq 0 \implies \int_A P_{H_{kr}} f = \int_A f \geq 0 \quad \forall A \in \Sigma'$, so $P_{H_{kr}} f \geq 0$

$f > 0 \implies P_{H_{kr}} f \geq 0$ and $\int_X P_{kr} f = \int_X f > 0$, so $P_{H_{kr}} f > 0$

$f \in L^\infty \implies \left|\int_A P_{H_{kr}} f\right| = \left|\int_A f\right| \leq \mu(A) \cdot \|f\|_\infty \quad \forall A \in \Sigma'$,

so $\|P_{H_{kr}} f\|_\infty \leq \|f\|_\infty$

So Roth is „proved" (and double recurrence).

11. Szemerédi for $k \geq 3$

<u>problem:</u> The decomposition into compact (a.p.) and weakly mixing part (JdLG) is not good for $k \geq 3$

Reason $\frac{1}{N}\sum_{n=1}^{N} T^n f\, T^{2n} g \cdot T^{3n} h \not\to 0$ possible if say f is weakly mixing. So the Kronecker part is no more characteristic.
The relevant part is larger (found by Host–Kra in 2006, for convergence and recurrence)

$\boxed{\text{Ex } 11.1}$ (Generalised eigenfcts)

Def Let (X,μ,T) be ergodic

- f is called a generalised EF of T of order 2 if

$$Tf = g\cdot f \quad \text{for } g \text{ with } Tg = \underset{\mathbb{T}}{\underset{\cap}{\lambda}} g \quad \text{(so } g \text{ is an eigenfct)}$$
$$\text{and } |f| \equiv 1$$

(here, $|g| = \text{const}$, say $|g| = 1$, then $|f| = \text{const}$ by ergodicity)

- Analogously, f is a gener. EF of order k if

$$T f = \underset{\nwarrow \text{ gener. EF of order } k-1}{g} f \quad \text{and} \quad |f| = 1.$$

Observation: Let f be a gener. EF of order 2, so $|f| = 1$ and $Tf = gf$, $Tg = \lambda g$, $\lambda \in \mathbb{T}$.

Then.
$$T^2 f = T(g\cdot f) = \lambda g \cdot g\, f = \lambda g^2 f$$
$$T^3 f = T(\lambda g^2 f) = \lambda \lambda^2 g^2 \cdot g f = \lambda^3 g^3 f$$
$$\vdots$$
$$T^n f = \lambda^{1+\dots+(n-1)} g^n \cdot f = \lambda^{\frac{n(n-1)}{2}} \cdot g^n \cdot f$$

We show. 1) $\langle T^n f, f\rangle = 0 \ \forall$ $\int g^n = 0$ (then f is weakly mixing if all $g^n \perp 1$)

2) $T^n(f^3) \cdot T^{2n}(\bar f^3) \cdot T^{3n} f = f$

and in part.
$$\frac{1}{N}\sum_{n=1}^{N} T^n(f^3)\, T^{2n}(\bar f^3)\, T^{3n} f \not\to 0 \text{ in } L^2$$
– The Kronecker part is no more characteristic

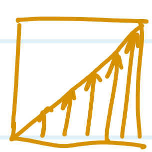
Proof 1) $\int T^n f \cdot \bar f \, d\mu = \lambda^{\frac{n(n-1)}{2}} \int g^n \underbrace{f \cdot \bar f}_{=1} = 0.$

2) $T^n(f^3)\, T^{2n}(\bar f^3) \cdot T^{3n} f = (T^n f)^3 \cdot (T^{2n} \bar f)^3\, T^{3n} f$

$$= \lambda^{\frac{3n(n-1)}{2} - \frac{6n(2n-1)}{2} + \frac{3n(3n-1)}{2}} \cdot \cancel{g^{3n-6n+3n}} \cdot f^{3-3+1}$$
$$= \lambda^{\frac{\cancel{3n^2} - \cancel{3n} - \cancel{12n^2} + \cancel{6n} + \cancel{9n^2} - \cancel{3n}}{2}} \cdot f = f \quad \blacksquare$$

A concrete example:

$X = (\mathbb{R}/\mathbb{Z})^2$, $T(x,y) = (x+\alpha, x+y)$ – skew shift



Consider $\underset{[0,1]^2}{}f(x,y) := e^{2\pi i y}$ and $g(x,y) := e^{2\pi i x}$

$(Tf)(x,y) = f(x+\alpha, x+y) = e^{2\pi i (x+y)} = g(x,y)\, f(x,y)$

$$(Tg)(x,y) = g(x+\alpha, x+y) = e^{2\pi i\alpha}\cdot g(x,y),$$

$|f| = |g| = 1$ clear

So g is an EF and f is a gener. EF of order 2.

Moreover,
$$\int g^n = \int e^{2\pi i n x}\,dx = 0 \quad \forall n \neq 0$$

$\Rightarrow f$ is weakly mixing, but $T^n(f^3)\cdot T^{2n}(\bar{f}^3)\,T^{3n}f = f \overset{\text{Cesàro}}{\not\to} 0$.

$\boxed{\text{Remark}}$ Also the „generalized Kronecker part" (Abramov part)

$$\overline{\text{lin}}\{\text{gener. EF of } T \text{ of all orders}\}$$

is not characteristic for $k \geq 3$:

$\boxed{\text{Ex}}$ (Furstenberg 1990). For the Heisenberg system

$$X = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \Big/ \overbrace{\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}}^{=:\Gamma} \text{ - comp.} \qquad Tg\Gamma = ag\Gamma$$

T rotation by $\underbrace{\begin{pmatrix} 1 & \alpha & \gamma \\ & 1 & \beta \\ & & 1 \end{pmatrix}}_{a}$, μ Haar, $\exists f$ weakly mixing with

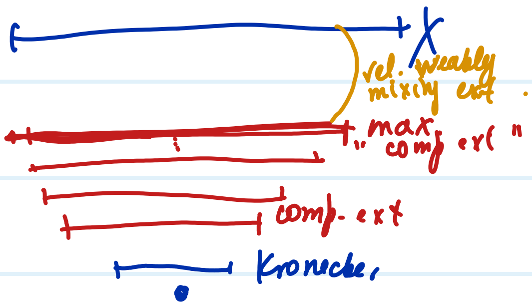
$$T^n f_1\; T^{2n} f_2 \cdot T^{3n} f \overset{\text{Cesàro}}{\not\to} 0$$

(for some f_1, f_2), and $\nexists$ non-trivial gener. EFs of order ≥ 2,

i.e., Kronecker = Abramov — without proof

Furstenberg's idea (Furstenberg–Katznelson 1978): Study MR (mult. recurr.) for larger and larger systems

idea:
- Start with a system with MR (for ex, 1-point-system or Kronecker)
- Prove MR for „good" extensions of it: relatively compact and relatively w mixing extensions.
- Build a tower of „good" factors, and show: one achieves X



First: What is a subsystem / factor

$\boxed{\text{Ex}}$ (Skew shift)

$(\mathbb{R}/\mathbb{Z})^2$, $\alpha \in \mathbb{R}/\mathbb{Z}$ irrat. $T(x,y) = (x+\alpha, x+y)$.

The fiber over x $(= \{x\} \times \mathbb{R}/\mathbb{Z}) \xrightarrow{T}$ fiber over $R_\alpha x = x+\alpha$

So for projection $\pi_1 : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}/\mathbb{Z}$
$\pi_1(x,y) := x$

$$\pi_1(T(x,y)) = R_\alpha \pi_1(x,y)$$

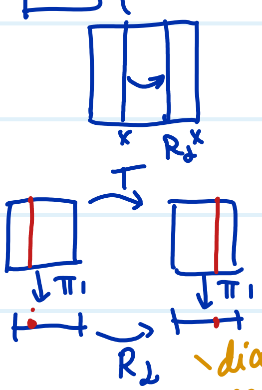


diagram commutes

So $(\mathbb{R}/\mathbb{Z}, R_\alpha)$ is a factor of $((\mathbb{R}/\mathbb{Z})^2, T)$

image under a measure-pres. map respecting the transf.

We say: the skew shift is an extension of the circle rotation.

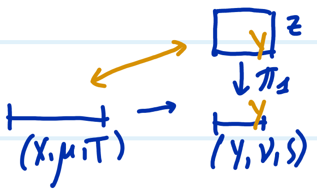
Second view $\quad (\mathbb{R}/\mathbb{Z}, R_\alpha) \overset{\text{isom}}{\sim} \left((\mathbb{R}/\mathbb{Z})^2, \Sigma', T\right),$

T-inv. sub-σ-alg. of „cylinder sets" $\square\,\mathbb{R}/\mathbb{Z}$, A

so fibers $\leftrightarrow$ base points
correspond.

2 views of factors: a smaller system or a smaller (T-inv.) σ-alg.

Rohlin's theorem: nice (complete sep. metric space, Borel meas. or comp. space)
$\forall (X,\mu,T)$ $\forall$ factor (Y,ν,S), (X,μ,T) can be written as a product



$$X = Y \times Z,$$
something
$$T(y,z) = (Sy, *)$$

Here, $\mu_\square$ projects onto ν under π_1 and one can use Fubini

Also here, the sub-σ-alg. on Y ~ cylinder sets on $\square$.

In general, for nice systems TFAE:
complete sep. metric space or compact spaces
Borel measures

(i) (Y,ν,S) is a factor of (X,μ,T), i.e., $\exists \pi: X \to Y$ measure pres. s.t.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\pi"] \\ Y \arrow[r, "S"'] & Y \end{tikzcd}$$

(ii) (Y,ν,S) is isomorphic to (X,Σ',μ,T) for a T-inv. sub-σ-alg. Σ' of Σ

(iii) $L^\infty(Y,\nu)$ is isom. to a lin. subalgebra $\mathcal{A}$ of $L^\infty(X,\mu)$

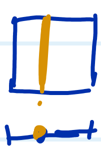
In this case, Σ' = smallest T-inv. sub-σ-alg. of Σ s.t. $\forall f \in \mathcal{A}$ is Σ'-meas.
With Rohlin, Σ' can be identified with a family of cylinder sets and X with $Y \times Z$ for some Z.

$\boxed{Ex}$ 1) Let (X,Σ,μ,T) is a (not necess. ergodic) MDS. Consider

$$\mathcal{A} := \operatorname{Fix} T \cap L^\infty(X,\mu)$$

- all T-inv. bdd fcts. (Recall: $\mathcal{A}$ is dense in $\operatorname{Fix} T$)
$\mathcal{A}$ is an algebra: $f,g \in \mathcal{A} \Rightarrow f\cdot g \in \mathcal{A}$
The corresponding Σ' is

$$\Sigma' = \{\text{T-inv. sets}\}$$
$A \in X:\ T^{-1}A = A$ up to a null set



The corresp. decomp. of the measure μ is called ergodic decomp.
More precisely, $\quad \mu = \int_Y \nu_y \, d\nu(y)$
ergodic

2) For (X,μ,T) consider

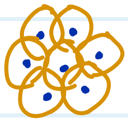
$$\mathcal{A} := \operatorname{lin}\{EF\} \cap L^\infty$$

(also an algebra) $\rightsquigarrow$ Kronecker factor

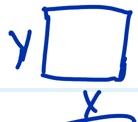
3) Also $\quad \mathcal{A} := \operatorname{lin}\{\text{gener. } EF\} \cap L^\infty$
is an algebra (exerc), so it comes from a factor (Abramov factor)

11.1 Relatively compact extensions and multiple recurrence

Recall: f is almost periodic if $\{T^n f, n \geq 0\}$ is rel. compact, i.e.,
$\forall \varepsilon > 0 \ \exists \ell \ \exists g_1, \dots, g_\ell : \forall n \ \|T^n f - g_j\| < \varepsilon$ for some $j = j(n)$.
finitely many ε-balls cover the orbit



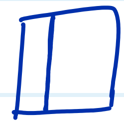
Let $(\tilde{X}, \tilde{T})$ be an extension of (X, T)



$\tilde{X} = X \times Y$
$\tilde{T}(x,y) = (Tx, \text{something})$

Def $f \in L^2(\tilde{X})$ is called almost periodic rel. to X if $\forall \varepsilon > 0$
$\exists \ell : \exists g_1, \dots, g_\ell \in L^2(\tilde{X}) : \forall n$ for a.e. x

$$\|\tilde{T}^n f(x,\cdot) - g_j(x,\cdot)\|_{L^2(\text{fiber of } x)} < \varepsilon$$



for some $j = j(x,n)$. Notation. $f \in AP(\tilde{X}|X)$ alm. per rel. to x

Rem. $AP(\tilde{X}|X)$ is a lin. subspace.

- cf clear (take $c \cdot g_1, \dots, c \cdot g_\ell$ to ε/c)
- $f+g$: consider $(f_\ell + g_j)_{\ell,j}$, where $f_1, \dots, f_\ell$ corr to f and $\varepsilon/2$
  $g_1, \dots, g_m$ — " — g — " —

Def $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is a compact extension of (X, μ, T) (or: compact rel. to (X, μ, T))
if $AP(\tilde{X}|X)$ is dense in $L^2(\tilde{X})$.

Rem. 1) $AP(\tilde{X}|X) \cap L^\infty(\tilde{X})$ is an algebra. $\forall \varepsilon > 0$ consider $(f_i \cdot g_j)_{i,j}$
for $f, g \in AP(\tilde{X}|X) \cap L^\infty$

$$\|\tilde{T}^n (f \cdot g)(x,\cdot) - (f_i \, g_j)(x,\cdot)\|_{L^2(\text{fiber over } x)} \leq$$

$$\leq \underbrace{\|(\tilde{T}^n f)(x,\cdot) - f_i(x,\cdot)\|_{L^2(\dots)}}_{< \varepsilon} \underbrace{\|g(x,\cdot)\|_{L^\infty(\text{fiber})}}_{\leq \|g\|_\infty \text{ a.e.}} +$$

$$+ \underbrace{\|f_i(x,\cdot)\|_{L^\infty(\dots)}}_{\leq \|f_i\|_\infty \text{ a.e.}} \cdot \underbrace{\|(g - g_j)(x,\cdot)\|_{L^2(\dots)}}_{< \varepsilon}$$

$$\leq \varepsilon \, (\|g\|_\infty + \underbrace{\|f_i\|_\infty}_{\leq 2\|f\|_\infty \text{ (why?) wlog}}) \leq \varepsilon \, (\|g\|_\infty + 2\|f\|_\infty)$$

Also $\boxed{L^\infty(X) \text{ is a subalgebra of } AP(\tilde{X}|X)}$ - take $\forall \varepsilon$
bdd meas. fcts constant on fibers
constant fcts $f_1, \dots, f_\ell$ with distance $< \varepsilon$

2) $AP(\tilde{X}|X)$ is in general not closed

So we have a chain of factors

$$\tilde{X} \longrightarrow Y \longrightarrow X$$

Y corresp. to $\overline{AP(\tilde{X}|X)}$

if $\tilde{X} = Y$, then $\tilde{X}$ is rel. comp. extension

$\boxed{\text{Ex}}$ (Skew shift is a comp. extension of circle rotation)

$T(x,y) = (x+\alpha, x+y)$

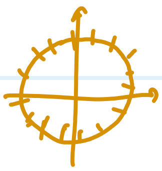
Fact: $f_\ell(x,y) := e^{2\pi i \ell x}$, $h_m(x,y) := e^{2\pi i m y}$ $\{f_\ell h_m, \ell, m \in \mathbb{Z}\}$ forms an ONB of $L^2([0,1]^2)$

Enough to show: $\forall f_\ell\ \forall h_m$ are rel. a.p. (subalg. + lin subsp.) exerc.

- $\forall f_\ell$ is EF $\Rightarrow$ a.p $\Rightarrow$ a.p. rel. to everything
- $\forall h_m$ is a second order EF

$$T^n h_m = \lambda^{m\frac{(n-1)n}{2}} f_m^n h_m,$$

so $(T^n h_m)(x,\cdot) = \underbrace{\overset{\lambda = e^{2\pi i \alpha}}{\lambda^{\frac{m(n-1)n}{2}}} e^{2\pi i n m x}}_{\text{indep. of } y \text{ element of } \mathbb{T}} \underbrace{h_m(x,\cdot)}_{\text{indep. of } x}$



So $\forall \varepsilon > 0$ take $\lambda_1, \dots, \lambda_\ell$ ε-dense in $\mathbb{T}$,
then $\forall x$ $(T^n h_m)(x,\cdot)$ is ε-closed to $\lambda_j h_m$ for some j.

So the skew shift is a comp. ext. of the rotation by α.

__Rem.__ The skew shift is not comp.
For $f = h_1 = e^{2\pi i y}$
$$(T^n f)(x,y) = c(n,\alpha) \cdot e^{2\pi i n x} e^{2\pi i y}, \text{ so}$$
$$T^n f \perp T^m f \quad \forall n \neq m \quad (\text{why?})$$
So $d(T^n f, T^m f) = \sqrt{2}$ $\forall n, m$ – no compactness!

$\boxed{\text{Thm 11.1}}$ (Relatively compact extensions preserve MR)

if (X,μ,T) has MR (mult. rec. $\forall k$) and $(\tilde{X},\tilde{\mu},\tilde{T})$ is comp. extension of (X,μ,T), then $(\tilde{X},\tilde{\mu},\tilde{T})$ has MR.

$\boxed{\text{Lemma 11.2}}$ Let $f > 0$, bdd. Then $\exists \tilde{f}$ with $0 < \tilde{f} \le f$, $\tilde{f}$ rel. almost periodic.

In particular, it suffices to show MR for rel. alm. periodic fcts.

$$\int f \cdot T^n f \cdots T^{kn} f \ge \int \tilde{f} \cdot T^n \tilde{f} \cdots T^{kn} \tilde{f}$$